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Nonlocal Changing-Sign Perturbation Tempered Fractional Sub-Diffusion Model with Weak Singularity

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Abstract: In this paper, we study the existence of positive solutions for a changing-sign perturbation tempered fractional model with weak singularity which arises from the sub-diffusion study of anomalous diffusion in Brownian motion. By two-step substitution, we first transform the higher-order sub-diffusion model to a lower-order mixed integro-differential sub-diffusion model, and then introduce a power factor to the non-negative Green function such that the linear integral operator has a positive infimum. This innovative technique is introduced for the first time in the literature and it is critical for controlling the influence of changing-sign perturbation. Finally, an a priori estimate and Schauder's fixed point theorem are applied to show that the sub-diffusion model has at least one positive solution whether the perturbation is positive, negative or changing-sign, and also the main nonlinear term is allowed to have singularity for some space variables.

Keywords: nonlocal boundary condition; tempered fractional equation; sign-changing perturbation; Schauder's fixed point theorem

MSC: 34A08; 34A25; 47H14

Citation: Zhang, X.; Chen, J.; Chen, P.; Li, L.; Wu, Y. Nonlocal Changing-Sign Perturbation Tempered Fractional Sub-Diffusion Model with Weak Singularity. *Fractal Fract.* **2024**, *8*, 337. https://doi.org/10.3390/fractalfract8060337

Academic Editors: Jordan Hristov, Yuri Dimitrov, Venelin Todorov and Slavi Georgiev

Received: 10 May 2024
Revised: 28 May 2024
Accepted: 30 May 2024
Published: 5 June 2024

1. Introduction

The particle's random walks in Brownian motion is governed by the following diffusion equation [1]:

$$\partial_t f(z,t) = \partial_z^2 f(z,t),$$

where $f(z,t)$ is the particle jump density function. In anomalous diffusion, the mean square variance sometimes grows faster to create a super-diffusion and sometimes spreads slower to form sub-diffusion than in the Gaussian process; thus, anomalous diffusion in Brownian motion exhibits a long-range dependence characteristic, which can be modeled by the fractional diffusion equation

$$\partial_t^\alpha f(z,t) = \partial_z^\beta f(z,t),$$

where the fractional space derivative of order $0 < \beta < 2$ corresponds to heavy-tailed power law particle jumps $P[J > z] \approx z^{-\beta}$ in the Lèvy flight, and the fractional time derivative of order $0 < \alpha \leq 1$ describes the heavy-tailed power law waiting time $P[W > t] \approx t^{-\alpha}$ between jumps. This indicates that the solution of the anomalous diffusion equation exhibits a feature of heavy tails, that is, it falls off like a power law decay at infinity with a time lag. However, in most cases, a more rapid exponential decay is much more favored. In order to find a strategy to temper the power law decay, Sabzikar et al. [2] introduced an exponential factor $e^{-\lambda|z|}$ into the particle jump density, and gave a Fourier transform form for the tempered probability density function $f(z,t)$ and tempered fractional derivative operators $\partial^{\beta,\lambda}_{\pm,z} f(z,t)$

$$\mathcal{F}[f](z,t) = e^{-[pB^{\beta,\lambda}_{+}(z)+qB^{\beta,\lambda}_{-}(z)]\mu t}, \quad \mathcal{F}[\partial^{\beta,\lambda}_{\pm,z} f](z,t) = B^{\beta,\lambda}_{\pm}(z)\mathcal{F}[f](z,t),$$

where $p = 1-q, 0 \le p \le 1$, μ is constant and

$$B^{\beta,\lambda}_{\pm}(z) = \begin{cases} (\lambda \pm zi)^{\beta} - \lambda^{\beta}, & 0 < \beta < 1, \\ (\lambda \pm zi)^{\beta} - \lambda^{\beta} - \pm\beta\lambda^{\beta-1}zi, & 1 < \beta < 2, \end{cases} \tag{1}$$

which leads to the following tempered fractional equation in anomalous diffusion:

$$\partial_t f(z,t) = (-1)^k C_T\{p\partial^{\beta,\lambda}_{+,z} + q\partial^{\beta,\lambda}_{-,z}\} f(z,t), \beta \in (k-1,k), k = 1,2.$$

The point source of solutions for this equation is tempered stable probability densities satisfying semi-heavy tails with transition from power law to Gaussian. This flexible model with an exponential decay over long time scales has many advantages, for example, in the real physical system, a Gaussian stochastic process without sharp cutoff can be captured by a tempered Lévy flight [3]. The probability densities of the tempered Lévy flight can be controlled by the tempered fractional diffusion model [4]. In addition, the tempered diffusion model has been shown to have important applications in geophysics [5,6], finance [7] and Laguerre polynomials [8]. In finance, the price fluctuations with semi-heavy tails can be simulated by the tempered stable process, which resemble a power law over a moderate time but converge to a Gaussian process over a long time [9]. In other words, the decay follows a power law over short time scales, but eventually follows the exponential rule over long time scales. In the following, we give the strictly mathematical definition of the tempered fractional:

Definition 1 ([10])**.** *Let $x : (0,+\infty) \to \mathbb{R}$, then the tempered fractional derivative is defined by*

$$\mathbb{D}^{\alpha,\lambda}_t x(t) = e^{-\lambda t}\mathscr{D}_t{}^{\alpha}(e^{\lambda t}x(t)),$$

where

$$\mathscr{D}_t{}^{\alpha} x(t) = \frac{d^n}{dt^n}\left({}_0I^{n-\alpha}_t x(t)\right) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}x(s)ds, \ \ n = [\alpha]+1$$

is the Riemann–Liouville fractional derivative and

$${}_0I^{n-\alpha}_t x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds$$

denotes the Riemann–Liouville fractional integral operator.

On the other hand, we also notice that, if $\lambda = 0$, the tempered fractional derivative reduces to the Riemann–Liouville and Caputo fractional derivatives, which implies that the tempered fractional derivative is one of the generalized forms of the fractional derivatives, that is, in a mathematical sense, tempered fractional calculus extends the theory of classical fractional calculus such as the Riemann–Liouville, Caputo fractional derivatives, etc. [11–24] to a more general case. In a recent work [10], the existence of a positive solution for a singular tempered fractional turbulent flow model for a porous medium with order $\alpha \in (0,1]$, $\beta \in (1,2]$ was established.

$$\begin{cases} \mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p\left(\mathbb{D}_t^{\beta,\lambda}z(t)\right)\right) = f(t,z(t)), \ t \in (0,1), \\ z(0) = 0, \ \mathbb{D}_t^{\beta,\lambda}z(0) = 0, \ z(1) = \int_0^1 e^{-\lambda(1-t)}z(t)dt. \end{cases} \tag{2}$$

It has been proven that the tempered turbulent flow model (2) has at least one positive solution satisfying

$$k_1 e^{-\lambda t}t^{\beta-1} \leq w(t) \leq k_2 e^{-\lambda t}t^{\beta-1},$$

where $k_1, k_2 > 0$ are constants. Ledesma et al. [25] considered a boundary value problem with tempered fractional derivatives and oscillating term

$$\begin{cases} \mathbb{D}_{b^-}^{\alpha,\lambda}\left({}^C\mathbb{D}_{a^+}^{\alpha,\lambda}z(t)\right) = \mu\varrho(t)f(z(t)), \ t \in (a,b), \\ z(a) = z(b) = 0, \end{cases} \tag{3}$$

where $\frac{1}{2} < \alpha < 1$, $\lambda > 0$ and $\mu \in \mathbb{R}$; $\varrho \in L^\infty(a,b)$, $\mathbb{D}_{b^-}^{\alpha,\lambda}$ and ${}^C\mathbb{D}_{a^+}^{\alpha,\lambda}$ are tempered left Riemann–Liouville and right Caputo fractional derivatives, respectively. By using a variational principle due to Ricceri, the existence of infinitely many weak solutions was established provided that the nonlinear term f has a suitable oscillating behavior either at the origin or at infinity.

In this paper, we focus on a sub-diffusion model in anomalous diffusion which possesses a changing-sign perturbation

$$\begin{cases} -\mathbb{D}_t^{\alpha,\lambda}z(t) = f\left(t, e^{\lambda t}z(t), \mathbb{D}_t{}^{\beta,\lambda}z(t)\right) + \kappa(t) \\ \mathbb{D}_t{}^{\beta,\lambda}z(0) = 0, \ \mathbb{D}_t{}^{\beta,\lambda}z(1) = \int_0^1 e^{-\lambda(1-t)}\mathbb{D}_t{}^{\beta,\lambda}z(t)dt, \end{cases} \tag{4}$$

where $1 < \alpha \leq 2, 0 < \beta < 1$ and $1 < \alpha - \beta \leq \beta + 1$, $f : [0,1] \times (0,\infty) \times (0,\infty) \to (0,\infty)$ is continuous which implies that the nonlinearity may be singular in some space variables; $\kappa \in C([0,1], R)$ may be sign-changing.

As far as we know, although many analysis methods, such as the spaces theories [26–31], iterative techniques [32–34], smooth theories [35–37], operator methods [38–41], upper–lower solution methods [42–44], the moving sphere method [45] and critical point theories [46–49], have been employed to study various nonlinear problems, little work has been carried out for changing-sign problems except for [50,51]. In particular, no work has been carried out for the changing-sign perturbation tempered fractional equation with singularity for space variables and a nonlocal boundary condition. The present paper is the first work for the sub-diffusion model in anomalous diffusion processes with a changing-sign perturbation.

2. Basic Definitions and Preliminaries

Firstly, we recall some properties of the Riemann–Liouville fractional calculus.

Lemma 1 ([10])**.**

(1) *Let* $g(t) \in L^1[0,1] \cap C[0,1], \gamma > 0$, *then*

$${}_0I_t^\gamma \mathscr{D}_t{}^\gamma(g(t)) = g(t) + b_1t^{\gamma-1} + b_2t^{\gamma-2} + \cdots + b_mt^{\gamma-m},$$

where $b_i \in R, i = 1,2,3,\ldots,m, (m = [\gamma]+1)$.

(2) *If* $u \in L^1(0,1), \alpha > \beta > 0$, *then*

$${}_0I_t^\alpha {}_0I_t^\beta u(t) = {}_0I_t^{\alpha+\beta}u(t), \ \mathscr{D}_t{}^\beta {}_0I_t^\alpha u(t) = {}_0I_t^{\alpha-\beta}u(t), \ \mathscr{D}_t{}^\beta {}_0I_t^\beta u(t) = u(t).$$

(3) *If $\rho > 0, \mu > 0$, then*

$$\mathscr{D}_t^{\rho} t^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)} t^{\mu-\rho-1}.$$

The work space of this paper is the Banach space $E = C[0,1]$ which is equipped with the usual maximum norm $||x|| = \max_{0 \le t \le 1} |x(t)|$. Let

$$P = \{x \in E : x(t) \ge 0, t \in [0,1]\},$$

then, P is a normal cone.

Lemma 2. *Suppose $\kappa(t)$ is a positive continuous function in $[0,1]$; then, the linear tempered fractional equation*

$$\begin{cases} -\mathbb{D}_t^{\alpha-\beta,\lambda} z(t) = \kappa(t), \\ z(0) = 0, \;\; z(1) = \int_0^1 e^{-\lambda(1-t)} z(t) dt, \end{cases} \tag{5}$$

has the unique positive solution

$$\varphi(t) = \int_0^1 H(t,s)\kappa(s) ds, \tag{6}$$

where

$$H(t,s) = \begin{cases} \dfrac{[(\alpha-\beta)(1-s)^{\alpha-\beta-1}(\alpha-\beta-1+s)t^{\alpha-\beta-1} - (\alpha-\beta)(\alpha-\beta-1)(t-s)^{\alpha-\beta-1}]e^{-\lambda(t-s)}}{(\alpha-\beta-1)\Gamma(\alpha-\beta+1)}, \\ \qquad s \le t; \\ \dfrac{(\alpha-\beta)(1-s)^{\alpha-\beta-1}(\alpha-\beta-1+s)}{(\alpha-\beta-1)\Gamma(\alpha-\beta+1)} t^{\alpha-\beta-1} e^{-\lambda(t-s)}, \; t \le s, \end{cases} \tag{7}$$

is the Green function of (5).

Clearly, $H(t,s)$ is non-negative and continuous for $(t,s) \in [0,1] \times [0,1]$.
Make the transformation

$$z(t) = e^{-\lambda t} I^{\beta}(e^{\lambda t} y(t)), \quad y(t) \in C[0,1],$$

then, by [10], the singular perturbation tempered fractional Equation (4) can be converted to the following equivalent mixed integro-differential tempered fractional equation:

$$\begin{cases} -\mathbb{D}_t^{\alpha-\beta,\lambda} y(t) = f(t, I^{\beta}(e^{\lambda t} y(t)), y(t)) + \kappa(t), \\ y(0) = 0, \;\; y(1) = \int_0^1 e^{-\lambda(1-t)} y(t) dt. \end{cases} \tag{8}$$

As a result, a function y is a solution of (8) if and only if it is a solution of the following integral equation:

$$y(t) = \int_0^1 H(t,s)\Big[f(s, I^{\beta}(e^{\lambda s} y(s)), y(s)) + \kappa(s)\Big] ds.$$

Now, let $y(t) = t^{\alpha-\beta-1} x(t)$; then, we can rewrite the above integral equation:

$$x(t) = \int_0^1 t^{\beta+1-\alpha} H(t,s)\Big[f(s, I^{\beta}(e^{\lambda s} s^{\alpha-\beta-1} x(s)), s^{\alpha-\beta-1} x(s)) + \kappa(s)\Big] ds.$$

Let

$$H^*(t,s) = t^{\beta+1-\alpha} H(t,s),$$

then, we consider the fixed point of the following operator:

$$(Tx)(t) = \int_0^1 H^*(t,s)\Big[f(s, I^\beta(e^{\lambda s} s^{\alpha-\beta-1} x(s)), s^{\alpha-\beta-1} x(s)) + \kappa(s)\Big] ds.$$

We use the following hypothesis in this paper.

(**C1**) There exist a constant $\theta \in (0, \frac{\beta}{\alpha-\beta-1})$ and $\omega_1, \omega_2 \in P, \omega_1 \not\equiv 0, \omega_2 \not\equiv 0$ such that

$$\frac{t^{\theta(\alpha-\beta-1)}\omega_1(t)}{(x+y)^\theta} \le f(t,x,y) \le \frac{t^{\theta(\alpha-\beta-1)}\omega_2(t)}{(x+y)^\theta}, \ \ (x,y) \in [0,\infty) \times (0,\infty), \ \ t \in (0,1).$$

Let $\varphi(t)$ be the unique solution of Equation (5) and rewrite $\varphi(t)$ as

$$\varphi(t) = \int_0^1 H^*(t,s)\kappa(s) ds,$$

and define functions

$$\phi(t) = \int_0^1 H^*(t,s)\omega_1(s) ds, \ \ \psi(t) = \int_0^1 H^*(t,s)\omega_2(s) ds.$$

For the above the functions φ, ϕ and ψ, we denote

$$\begin{aligned} \varphi_* &= \min_{0\le t\le 1} \varphi(t), \quad \varphi^* = \sup_{0\le t\le 1} \varphi(t), \\ \phi_* &= \min_{0\le t\le 1} \phi(t), \quad \phi^* = \max_{0\le t\le 1} \phi(t), \\ \psi_* &= \min_{0\le t\le 1} \psi(t), \quad \psi^* = \max_{0\le t\le 1} \psi(t). \end{aligned}$$

Noticing

$$\phi(t) = \int_0^1 H^*(t,s)\omega_1(s) ds \ge \int_0^1 \frac{(\alpha-\beta)(1-s)^{\alpha-\beta-1}(\alpha-\beta-1+s)}{(\alpha-\beta-1)\Gamma(\alpha-\beta+1)} e^{-\lambda}\omega_1(s) ds,$$

then, it follows from (**C1**) that $0 < \phi_* \le \phi^* \le \psi_* \le \psi^*$.

Our main tool used in deriving our results is the following Schauder fixed point theorem.

Lemma 3 (Schauder fixed point theorem)**.** *Let Ω be a closed bounded convex subset of a Banach space E. Assume that $T : \Omega \to \Omega$ is compact. Then, T has at least one fixed point in Ω.*

3. Positive Case for $\varphi(t)$

Theorem 1. *Assume that* (**C1**) *holds and $\varphi_* \ge 0$. Then, the singular perturbation tempered fractional equation with nonlocal boundary condition* (4) *has at least one positive solution.*

Proof. Firstly, choose a constant

$$R > \max\left\{1, \phi_*^{-\frac{1}{1-\theta}}\left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)^{\frac{\theta}{1-\theta}}, (\psi^* + \varphi^*)^{\frac{1}{1-\theta}}\right\}$$

and take the closed convex set of P

$$\Omega = \{x \in P : \frac{1}{R} \le x(t) \le R, \ \ t \in [0,1]\}.$$

Noticing that $1 < \alpha - \beta \leq \beta + 1$, we have

$$\begin{aligned}\frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)R} &\leq I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{\alpha-\beta-1}e^{\lambda s}x(s)ds \\ &\leq \frac{e^{\lambda}Rt^{\beta}}{\Gamma(\beta+1)} \leq \frac{e^{\lambda}Rt^{\alpha-\beta-1}}{\Gamma(\beta+1)} \leq \frac{e^{\lambda}R}{\Gamma(\beta+1)},\end{aligned} \tag{9}$$

for $\theta \in (0, \frac{\beta}{\alpha-\beta-1})$ and $s \in (0,1)$; it follows from (**C1**) and (9) that

$$\begin{aligned}&\frac{\omega_1(s)}{R^{\theta}\left(1+\frac{e^{\lambda}}{\Gamma(\beta+1)}\right)^{\theta}} \\ &\leq \frac{s^{\theta(\alpha-\beta-1)}\omega_1(s)}{(s^{\alpha-\beta-1}x(s)+I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)))^{\theta}} \\ &\leq f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s)) \\ &\leq \frac{s^{\theta(\alpha-\beta-1)}\omega_2(s)}{(s^{\alpha-\beta-1}x(s)+I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)))^{\theta}} \leq \omega_2(s)R^{\theta}.\end{aligned} \tag{10}$$

Consequently, by (9) and (10), one has

$$\begin{aligned}(Tx)(t) &= \int_0^1 H^*(t,s)\Big[f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s)) + \kappa(s)\Big]ds \\ &= \int_0^1 H^*(t,s)f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s))ds + \varphi(t).\end{aligned} \tag{11}$$

Thus, $T : \Omega \to E$ is a completely continuous operator.

In what follows, we show that T maps the closed convex set Ω into Ω. In fact, for any given $x \in \Omega$ and $t \in [0,1]$, from (9)–(11), one has

$$\begin{aligned}(Tx)(t) &= \int_0^1 H^*(t,s)f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s))ds + \varphi(t) \\ &\geq \frac{1}{R^{\theta}\left(1+\frac{e^{\lambda}}{\Gamma(\beta+1)}\right)^{\theta}}\int_0^1 H^*(t,s)\omega_1(s)ds \\ &\geq \frac{\phi_*}{R^{\theta}\left(1+\frac{e^{\lambda}}{\Gamma(\beta+1)}\right)^{\theta}} \\ &= \frac{\phi_* R^{1-\theta}}{R\left(1+\frac{e^{\lambda}}{\Gamma(\beta+1)}\right)^{\theta}} \\ &\geq \frac{1}{R},\end{aligned}$$

and

$$\begin{aligned}(Tx)(t) &= \int_0^1 H^*(t,s)f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s))ds + \varphi(t) \\ &\leq R^{\theta}\int_0^1 H^*(t,s)\omega_2(s)ds + \varphi^* \\ &\leq \psi^* R^{\theta} + \varphi^* \leq (\psi^* + \varphi^*)R^{\theta} \leq R.\end{aligned}$$

Thus, T maps the closed convex set Ω into Ω. According to Schauder's fixed point theorem, T has a fixed point $x^* \in \Omega$ and, hence, the singular perturbation tempered fractional equation with a nonlocal boundary condition (4) has at least one positive solution. □

4. Negative Case for $\varphi(t)$

Theorem 2. *Assume that* (**C1**) *holds and* $\varphi^* \leq 0$. *If*

$$\varphi_* \geq \left(\frac{\phi_* \theta^2}{\psi^{*\theta} \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^{\theta}} \right)^{\frac{1}{1-\theta^2}} \left(1 - \frac{1}{\theta^2}\right), \tag{12}$$

then, the singular perturbation tempered fractional equation with nonlocal boundary condition (4) *has at least one positive solution.*

Proof. For this case, we only need to find the suitable $0 < r < R$ such that $T : \Omega \to \Omega$, where

$$\Omega = \{x \in P : r \leq x(t) \leq R, \ \ t \in [0,1]\}.$$

Similar to (9), we have

$$\begin{aligned} &\frac{\Gamma(\alpha-\beta)t^{\alpha-1}r}{\Gamma(\alpha)} \\ &\leq I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)) \\ &\leq \frac{e^{\lambda}Rt^{\beta}}{\Gamma(\beta+1)} \\ &\leq \frac{e^{\lambda}R}{\Gamma(\beta+1)}, \end{aligned} \tag{13}$$

and then

$$\begin{aligned} &\frac{\omega_1(s)}{R^{\theta}\left(1 + \frac{e^{\lambda}}{\Gamma(\beta+1)}\right)^{\theta}} \\ &\leq f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s)) \\ &\leq \frac{\omega_2(s)}{r^{\theta}}. \end{aligned} \tag{14}$$

Consequently, for any given $x \in \Omega$ and $t \in [0,1]$, from (14), one has

$$\begin{aligned} (Tx)(t) &= \int_0^1 H^*(t,s) f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s))ds + \varphi(t) \\ &\geq \frac{1}{R^{\theta}\left(1 + \frac{e^{\lambda}}{\Gamma(\beta+1)}\right)^{\theta}} \int_0^1 H^*(t,s)\omega_1(s)ds + \varphi_* \\ &\geq \frac{\phi_*}{R^{\theta}\left(1 + \frac{e^{\lambda}}{\Gamma(\beta+1)}\right)^{\theta}} + \varphi_*, \end{aligned} \tag{15}$$

and

$$\begin{aligned} (Tx)(t) &= \int_0^1 H^*(t,s) f(s, I^{\beta}(e^{\lambda s}s^{\alpha-\beta-1}x(s)), s^{\alpha-\beta-1}x(s))ds + \varphi(t) \\ &\leq \frac{1}{r^{\theta}} \int_0^1 H^*(t,s)\omega_2(s)ds \\ &\leq \frac{\psi^*}{r^{\theta}}. \end{aligned} \tag{16}$$

From (15) and (16), $T : \Omega \to \Omega$ holds provided that the following inequalities are valid:

$$\frac{\phi_*}{R^{\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^{\theta}} + \varphi_* \geq r, \quad \frac{\psi^*}{r^{\theta}} \leq R. \tag{17}$$

In order to show that (17) holds, let

$$R = \frac{\psi^*}{r^\theta}.$$

Clearly, to ensure that the inequalities (17) are true, we only need to find a suitable $r > 0$ such that

$$R = \frac{\psi^*}{r^\theta} > r, \quad \frac{\phi_* r^{\theta^2}}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} + \varphi_* \geq r,$$

i.e.,

$$0 < r < \psi^{*\frac{1}{1+\theta}}, \quad \varphi_* \geq r - \frac{\phi_*}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} r^{\theta^2}. \tag{18}$$

Now, let

$$h(x) = x - \frac{\phi_*}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} x^{\theta^2}, \quad x \in (0, \infty).$$

Obviously,

$$h'(x) = 1 - \frac{\theta^2 \phi_*}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta x^{1-\theta^2}},$$

which implies that $h(x)$ takes the minimum value

$$h(\vartheta) = \left(\frac{\phi_* \theta^2}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{1-\theta^2}} \left(1 - \frac{1}{\theta^2}\right),$$

when

$$\vartheta = \left(\frac{\theta^2 \phi_*}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{1-\theta^2}}.$$

Let $r = \vartheta$; since $\phi_* \leq \psi^*, 0 < \theta^2 < 1, 1 + \frac{1}{\Gamma(\beta+1)} > 1$, from (12), we obtain

$$\begin{aligned} r = \vartheta &= \left(\frac{\theta^2 \phi_*}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{1-\theta^2}} \\ &< \left(\frac{\psi^*}{\psi^{*\theta}}\right)^{\frac{1}{1-\theta^2}} \\ &= \psi^{*\frac{1}{1+\theta}}, \end{aligned}$$

and

$$\begin{aligned} \varphi_* &\geq \left(\frac{\phi_* \theta^2}{\psi^{*\theta}\left(1 + \frac{1}{\alpha\Gamma(\alpha)}\right)^\theta}\right)^{\frac{1}{1-\theta^2}} \left(1 - \frac{1}{\theta^2}\right) \\ &= r - \frac{\phi_*}{\psi^{*\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} r^{\theta^2}. \end{aligned}$$

Therefore, if

$$r = \vartheta, \; R = \frac{\psi^*}{\vartheta^\theta},$$

then (18) is true. Thus, by mean of Schauder's fixed point theorem, T has a fixed point $x^* \in \Omega$ and, hence, the singular perturbation tempered fractional Equation (4) with nonlocal boundary condition has at least one positive solution. □

5. Changing-Sign Case for $\varphi(t)$

In order to establish the existence of a positive solution for the changing-sign case, we firstly consider the following equation:

$$x^{1-\theta^2}(\psi^* + \varphi^* x^\theta)^{1+\theta} = \frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}. \tag{19}$$

Lemma 4. *If $0 < \theta < 1$, then Equation (19) possesses a unique positive solution ϑ in $(0,\infty)$. Moreover,*

$$\vartheta \in \left(0, \left(\frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{2+2\theta}}\right).$$

Proof. Let

$$\mu := \left(\frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{2+2\theta}}, \quad h(x) = \frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} - x^{1-\theta^2}(\psi^* + \varphi^* x^\theta)^{1+\theta},$$

then

$$h(0) = \lim_{x \to 0} h(x) = \frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} > 0. \tag{20}$$

Since $0 < \theta < 1, \ \psi^* \geq \phi_*$, one has

$$0 < \mu = \left(\frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{2+2\theta}} < \psi^{*\frac{1}{1+\theta}},$$

which yields

$$\mu^{(1+\theta)^2} < \psi^{*1+\theta}.$$

Hence,

$$\begin{aligned} h(\mu) &= \mu^{2+2\theta} - \mu^{1-\theta^2}(\psi^* + \varphi^* \mu^\theta)^{1+\theta} \\ &= \mu^{1-\theta^2}\left[\mu^{(1+\theta)^2} - (\psi^* + \varphi^* \mu^\theta)^{1+\theta}\right] \\ &< \mu^{1-\theta^2}\left[\mu^{(1+\theta)^2} - \psi^{*1+\theta}\right] < 0, \end{aligned} \tag{21}$$

Moreover,

$$\begin{aligned} h'(x) = &-(1-\theta^2)x^{-\theta^2}(\psi^* + \varphi^* x^\theta)^{1+\theta} - \varphi^* \theta x^{1-\theta^2}(\psi^* + \varphi^* x^\theta)^\theta x^{\theta-1} \\ &< 0, \ \ x \in (0,\infty), \end{aligned} \tag{22}$$

which implies that $h(x)$ is strictly decreasing in $(0,\infty)$. Thus, it follows from (20)–(22) that Equation (19) possesses a unique positive solution $\vartheta \in (0,\mu)$. □

Theorem 3. *Suppose that* (**C1**) *holds and* $\varphi_* \leq 0, \ \varphi^* \geq 0$. *If*

$$\varphi_* \geq \vartheta - \frac{\phi_* \vartheta^{2\theta}}{(\psi^* + \varphi^* \vartheta^\theta)^\theta \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}, \tag{23}$$

where ϑ *is the solution of Equation* (19), *then the singular perturbation tempered fractional Equation* (4) *with nonlocal boundary condition has at least one positive solution.*

Proof. If $\varphi_* \leq 0, \ \varphi^* \geq 0$, we still need to seek for suitable $0 < r < R$ such that $T : \Omega \to \Omega$, where

$$\Omega = \{x \in P : r \leq x(t) \leq R, \ t \in [0,1]\}.$$

For any given $x \in \Omega$, it follows from (13) and (14), $\varphi_* \leq 0, \ \varphi^* \geq 0$ that

$$\begin{aligned}(Tx)(t) &= \int_0^1 H^*(t,s) f(s, I^\beta(e^{\lambda s} s^{\alpha-\beta-1} x(s)), s^{\alpha-\beta-1} x(s)) ds + \varphi(t) \\ &\geq \frac{1}{R^\theta \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)^\theta} \int_0^1 H^*(t,s) \omega_1(s) ds + \varphi_* \\ &\geq \frac{\phi_*}{R^\theta \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)^\theta} + \varphi_*,\end{aligned} \tag{24}$$

and

$$\begin{aligned}(Tx)(t) &= \int_0^1 H^*(t,s) f(s, I^\beta(e^{\lambda s} s^{\alpha-\beta-1} x(s)), s^{\alpha-\beta-1} x(s)) ds + \varphi(t) \\ &\leq \frac{1}{r^\theta} \int_0^1 H^*(t,s) \omega_2(s) ds \leq \frac{\psi^*}{r^\theta} + \varphi^*.\end{aligned} \tag{25}$$

By (24) and (25), to guarantee $T : \Omega \to \Omega$, it is sufficient to find $0 < r < R$ such that

$$\frac{\phi_*}{R^\theta \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} + \varphi_* \geq r, \quad \frac{\psi^*}{r^\theta} + \varphi^* \leq R. \tag{26}$$

To perform this, take

$$R = \frac{\psi^*}{r^\theta} + \varphi^*.$$

This is equivalent to finding an $r > 0$ satisfying

$$\frac{\psi^*}{r^\theta} + \varphi^* > r, \ \ \varphi_* \geq r - \frac{\phi_* r^{2\theta}}{(\psi^* + \varphi^* r^\theta)^\theta \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}. \tag{27}$$

Let

$$h(x) = x - \frac{\phi_* x^{2\theta}}{(\psi^* + \varphi^* x^\theta)^\theta \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}.$$

Then

$$\begin{aligned}h'(x) &= 1 - \frac{\phi_* \left[\theta^2 x^{\theta^2-1} (\psi^* + \theta^2 \varphi^* x^\theta)^\theta - \varphi^* x^{\theta^2} (\psi^* + \varphi^* x^\theta)^{\theta-1} x^{\theta-1}\right]}{(\psi^* + \varphi^* x^\theta)^{2\theta} \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} \\ &= 1 - \frac{\phi_* \theta^2 x^{\theta^2-1}}{(\psi^* + \varphi^* x^\theta)^\theta \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} \left[1 - \frac{\varphi^* x^\theta}{\psi^* + \varphi^* x^\theta}\right] \\ &= 1 - \frac{\phi_* \psi^* \theta^2 x^{\theta^2-1}}{(\psi^* + \varphi^* x^\theta)^{\theta+1} \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta},\end{aligned}$$

and, hence,

$$\lim_{x\to 0} h'(x) = -\infty, \quad \lim_{x\to\infty} h'(x) = 1.$$

Thus, there exists ϑ such that

$$h'(\vartheta) = 1 - \frac{\phi_* \psi^* \theta^2 \vartheta^{\theta^2-1}}{(\psi^* + \varphi^* \vartheta^\theta)^{\theta+1}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} = 0,$$

i.e., ϑ solves Equation (19).

$$x^{1-\theta^2}(\psi^* + \varphi^* x^\theta)^{1+\theta} = \frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}.$$

It follows from Lemma 4 that ϑ is unique and

$$\vartheta \in \left(0, \left(\frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{2+2\theta}}\right).$$

On the other hand,

$$h''(r) = \frac{(1-\theta^2)\theta^2 \phi_* \psi^* r^{\theta^2-2}}{(\psi^* + \varphi^* r^\theta)^{\theta+1}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} + \frac{(1+\theta)\theta^3 \varphi^* \phi_* \psi^* r^{\theta^2+\theta-2}}{(\psi^* + \varphi^* r^\theta)^{\theta+2}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta} > 0,$$

which implies that $h(r)$ reaches its minimum value at ϑ, that is

$$h(\vartheta) = \min_{r\in(0,\infty)} h(r).$$

Take $r = \vartheta$, assumption (23) ensures the following inequality holds:

$$\varphi_* \geq r - \frac{\phi_* r^{2\theta}}{(\psi^* + \varphi^* r^\theta)^\theta \left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}.$$

Next, we verify the inequality $R > r$. It follows from

$$R = \frac{\psi^*}{\vartheta^\theta} + \varphi^*, \quad \vartheta^{1-\theta^2}(\psi^* + \varphi^* \vartheta^\theta)^{1+\theta} = \frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta},$$

that

$$\vartheta^{1+\theta} R^{1+\theta} = \frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta},$$

that is

$$R = \frac{1}{\vartheta}\left(\frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{1+\theta}}.$$

Since

$$0 < \vartheta < \left(\frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^\theta}\right)^{\frac{1}{2+2\theta}},$$

one has

$$R > \left(\frac{\theta^2 \psi^* \phi_*}{\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^{\theta}} \right)^{\frac{1}{2+2\theta}} > \vartheta = r.$$

Hence, we have found a constant $r = \vartheta$ such that (26) holds. According to Schauder's fixed point theorem, the singular perturbation tempered fractional Equation (4) with nonlocal boundary condition has at least one positive solution. □

6. Example

In anomalous diffusion, the mean square variance sometimes grows slower than Gaussian to form a sub-diffusion which can be modeled by the time tempered fractional diffusion equation. In this section, we only give an example for the most complex case, i.e., with uncertain changing-sign perturbations, to demonstrate the application of our main results.

Example 1. *Consider the sub-diffusion model (4) in anomalous diffusion with a changing-sign perturbation $\kappa(t)$ for the case $\alpha = \frac{3}{2}, \beta = \frac{1}{4}, \lambda = 2$,*

$$\begin{cases} -\mathbb{D}_t^{\frac{3}{2},2} z(t) = \dfrac{100.5 t^{\frac{1}{12}}}{e^{2t}\left(e^{2t} z(t) + |\mathbb{D}_t^{\frac{1}{4},2} z(t)|\right)^{\frac{1}{3}}} + \kappa(t) \\ \mathbb{D}_t^{\frac{1}{4},2} z(0) = 0, \ \ \mathbb{D}_t^{\frac{1}{4},2} z(1) = \displaystyle\int_0^1 e^{-2(1-t)} \mathbb{D}_t^{\frac{1}{4},2} z(t) dt, \end{cases} \tag{28}$$

where

$$\kappa(t) = \begin{cases} -\dfrac{1}{5} e^{-2t}, & t \in \left[0, \dfrac{1}{2}\right), \\ \dfrac{1}{5} e^{-2t+1}, & t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

Conclusion The sub-diffusion model (28) has at least one positive solution.

Proof. Let $\omega_1(t) = 100 e^{-2t}, \ \omega_2(t) = 101 e^{-2t}$ and

$$f(t,x,y) = \frac{100.5 t^{\frac{1}{12}}}{e^{2t}(x+y)^{\frac{1}{3}}}, \ \ (x,y) \in [0,\infty) \times (0,\infty), \ \ t \in (0,1),$$

then

$$\frac{100 t^{\frac{1}{12}} e^{-2t}}{(x+y)^{\frac{1}{3}}} \le f(t,x,y) \le \frac{101 t^{\frac{1}{12}} e^{-2t}}{(x+y)^{\frac{1}{3}}}, (x,y) \in [0,\infty) \times (0,\infty), \ \ t \in (0,1).$$

Thus, (**C1**) holds. On the other hand, by Lemma 2, we obtain

$$H(t,s) = \begin{cases} \dfrac{5\left[(1-s)^{\frac{1}{4}}(\frac{1}{4}+s) t^{\frac{1}{4}} - \frac{1}{4}(t-s)^{\frac{1}{4}}\right] e^{-2(t-s)}}{\Gamma(\frac{9}{4})}, & s \le t; \\ \dfrac{5(1-s)^{\frac{1}{4}}(\frac{1}{4}+s)}{\Gamma(\frac{9}{4})} t^{\frac{1}{4}} e^{-2(t-s)}, & t \le s, \end{cases} \tag{29}$$

then

$$H^*(t,s) = \begin{cases} \dfrac{5\left[(1-s)^{\frac{1}{4}}(\frac{1}{4}+s) - \frac{1}{4}t^{-\frac{1}{4}}(t-s)^{\frac{1}{4}}\right]e^{-2(t-s)}}{\Gamma(\frac{9}{4})}, & s \le t; \\ \dfrac{5(1-s)^{\frac{1}{4}}(\frac{1}{4}+s)}{\Gamma(\frac{9}{4})}e^{-2(t-s)}, & t \le s. \end{cases} \tag{30}$$

Now, we compute φ^*, φ_* and ϕ_*, ψ^*. Note that

$$\varphi(t) = \int_0^1 H^*(t,s)\kappa(s)ds = \frac{1}{\Gamma(\frac{9}{4})}\begin{cases} -\dfrac{5}{9}e^{-2t} + \dfrac{1}{25}e^{-2t}t, & 0 \le t < \dfrac{1}{2}, \\ \dfrac{5}{9}e^{-2t+1} + \dfrac{1}{25}e^{-2t+1}t, & \dfrac{1}{2} \le t \le 1, \end{cases}$$

and, hence,

$$\varphi_* = \frac{-\frac{5}{9}}{\Gamma(\frac{9}{4})} = -0.4903, \quad \varphi^* = \frac{5}{9\Gamma(\frac{9}{4})} + \frac{1}{50\Gamma(\frac{9}{4})} = 0.5080.$$

Similarly, we have

$$\phi(t) = \int_0^1 H^*(t,s)\omega_1(s)ds = \frac{2500}{9\Gamma(\frac{9}{4})}e^{-2t} - \frac{20}{\Gamma(\frac{9}{4})}e^{-2t}t,$$

$$\psi(t) = \int_0^1 H^*(t,s)\omega_2(s)ds = \frac{2525}{9\Gamma(\frac{9}{4})}e^{-2t} - \frac{101}{5\Gamma(\frac{9}{4})}e^{-2t}t,$$

and, consequently,

$$\phi_* = \min_{0\le t\le 1}\phi(t) = 30.8, \quad \phi^* = \max_{0\le t\le 1}\phi(t) = 245.2,$$
$$\psi_* = \min_{0\le t\le 1}\psi(t) = 27.472, \quad \psi^* = \max_{0\le t\le 1}\phi(t) = 247.6.$$

Thus, Equation (19) becomes

$$x^{\frac{8}{9}}(247.6 + 0.5080x^{\frac{1}{3}})^{\frac{4}{3}} = \frac{\frac{1}{9}\times 247.6 \times 30.8}{(1+1.1033)^{\frac{1}{3}}} = 661.3405. \tag{31}$$

which has a unique solution $\vartheta = 0.3814$. Hence,

$$\vartheta - \frac{\phi_*\vartheta^{2\theta}}{(\psi^* + \varphi^*\vartheta^{\theta})^{\theta}\left(1 + \frac{1}{\Gamma(\beta+1)}\right)^{\theta}} = 0.3814 - \frac{0.3814^{\frac{2}{3}}\times 30.8}{(247.6 + 0.5080\times 0.3814^{\frac{1}{3}})^{\frac{1}{3}}(1+1.1033)^{\frac{1}{3}}}$$
$$= -1.6310 < \varphi_* = -0.4903.$$

It follows from Theorem 3 that the sub-diffusion model (28) has at least one positive solution. □

7. Conclusions

The tempered fractional derivative is a more flexible alternative to classical fractional derivatives, which can model random walks with semi-heavy tails and stable probability densities for the particle jumps, i.e., the transition has a semi-long-range dependence in a long time range as a power law but is like the Gaussian in a short time range. In this paper, by using the technique of two-step substitution, the higher-order sub-diffusion model is converted to a lower-order mixed integro-differential sub-diffusion model, and then, by introducing a power factor, we derive that the linear integral operator we define has a positive infimum; this innovative technique is introduced for the first time in the literature and it is critical for controlling the influence of changing-sign perturbation. Then, an a priori estimate and Schauder's fixed point theorem are applied to prove that the sub-diffusion model has at least one positive solution whether the perturbation is positive, negative or changing-sign. In particular, the main nonlinear term is allowed to have singularity for some space variables. In the end, we shall also address that the perturbation in this paper only depends on a time variable; if it relies on both time variables and space variables, then further study will become more challenging and interesting.

Author Contributions: Investigation, P.C. and L.L.; Writing—original draft, X.Z. and J.C.; Writing—review & editing, Y.W. All authors have read and agreed to the published version of the manuscript.

Funding: The authors are supported financially by the Natural Science Foundation of Shandong Province of China (ZR2022AM015), and an ARC Discovery Project Grant.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.

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