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Further Hermite–Hadamard-Type Inequalities for Fractional Integrals with Exponential Kernels

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Abstract: This paper introduces new versions of Hermite–Hadamard, midpoint- and trapezoid-type inequalities involving fractional integral operators with exponential kernels. We explore these inequalities for differentiable convex functions and demonstrate their connections with classical integrals. This paper validates the derived inequalities through a numerical example with graphical representations and provides some practical applications, highlighting their relevance to special means. This study presents novel results, offering new insights into classical integrals as the fractional order β approaches 1, in addition to the fractional integrals we examined.

Keywords: fractional integrals with exponential kernels; Hermite–Hadamard inequality; midpoint-type inequalities; trapezoid-type inequalities; convex functions

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Citation: Li, H.; Meftah, B.; Saleh, W.; Xu, H.; Kiliçman, A.; Lakhdari, A. Further Hermite–Hadamard-Type Inequalities for Fractional Integrals with Exponential Kernels. *Fractal Fract.* **2024**, *8*, 345. https://doi.org/10.3390/fractalfract8060345

Academic Editors: Milton Ferreira, Maria Manuela Fernandes Rodrigues and Nelson Felipe Loureiro Vieira

Received: 27 April 2024
Revised: 22 May 2024
Accepted: 30 May 2024
Published: 7 June 2024

1. Introduction and Preliminaries

Convexity is a fundamental concept in both pure and applied mathematics that serves as a basis for optimization, inequality, and problem-solving in several fields of study. A function $\mathfrak{g}$ is considered convex on the interval $[\mathfrak{v},\mathfrak{w}]$, if for all z_1, $z_2 \in [\mathfrak{v},\mathfrak{w}]$ and $\varkappa \in [0,1]$, the following inequality holds:

$$\mathfrak{g}(\varkappa z_1 + (1-\varkappa)z_2) \le \varkappa\mathfrak{g}(z_1) + (1-\varkappa)\mathfrak{g}(z_2).$$

One of the most intriguing results associated with the concept of convexity is the Hermite–Hadamard inequality (see [1,2]). This inequality has garnered significant attention and is widely recognized for its utility in related fields. For a convex function $\mathfrak{g}$ on the interval $[\mathfrak{v},\mathfrak{w}]$, we have

$$\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) \le \tfrac{1}{\mathfrak{w}-\mathfrak{v}}\int_{\mathfrak{v}}^{\mathfrak{w}} \mathfrak{g}(z)\,dz \le \tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}. \tag{1}$$

In [3], Kirmaci obtained the following result connected with the left-hand side of (1) as follows:

Theorem 1 ([3]). *Let $\mathfrak{g} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I such that $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$ (Lebesgue integrable), where $\mathfrak{v}, \mathfrak{w} \in I$ with $\mathfrak{v} < \mathfrak{w}$. If $|\mathfrak{g}'|$ is convex on $[\mathfrak{v}, \mathfrak{w}]$, then the following inequality holds:*

$$\left| \mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \tfrac{1}{\mathfrak{w}-\mathfrak{v}} \int\limits_{\mathfrak{v}}^{\mathfrak{w}} \mathfrak{g}(z)\, dz \right| \leq \tfrac{\mathfrak{w}-\mathfrak{v}}{8} \left[|\mathfrak{g}'(\mathfrak{v})| + |\mathfrak{g}'(\mathfrak{w})| \right].$$

A refinement of the above result was provided by Kadakal in [4] as follows:

Theorem 2 ([4]). *Let $\mathfrak{g} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I such that $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$, where $\mathfrak{v}, \mathfrak{w} \in I$ with $\mathfrak{v} < \mathfrak{w}$. If $|\mathfrak{g}'|$ is convex on $[\mathfrak{v}, \mathfrak{w}]$, then the following inequality holds:*

$$\left| \mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \tfrac{1}{\mathfrak{w}-\mathfrak{v}} \int\limits_{\mathfrak{v}}^{\mathfrak{w}} \mathfrak{g}(z)\, dz \right| \leq \tfrac{\mathfrak{w}-\mathfrak{v}}{24} \left[|\mathfrak{g}'(\mathfrak{v})| + 4\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right| + |\mathfrak{g}'(\mathfrak{w})| \right].$$

Dragomir and Agarwal presented the subsequent result associated with the right-hand side of (1) in [5].

Theorem 3 ([5]). *Let $\mathfrak{g} : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o, $\mathfrak{v}, \mathfrak{w} \in I^o$ with $\mathfrak{v} < \mathfrak{w}$. If $|\mathfrak{g}'|$ is convex on $[\mathfrak{v}, \mathfrak{w}]$, then the following inequality holds:*

$$\left| \tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2} - \tfrac{1}{\mathfrak{w}-\mathfrak{v}} \int\limits_{\mathfrak{v}}^{\mathfrak{w}} \mathfrak{g}(z)\, dz \right| \leq \tfrac{\mathfrak{w}-\mathfrak{v}}{8} \left[|\mathfrak{g}'(\mathfrak{v})| + |\mathfrak{g}'(\mathfrak{w})| \right].$$

In [6], Kavurmacı et al. provided a refinement of the above result as follows:

Theorem 4 ([6]). *Let $\mathfrak{g} : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o such that $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$, where $\mathfrak{v}, \mathfrak{w} \in I$ with $\mathfrak{v} < \mathfrak{w}$. If $|\mathfrak{g}'|$ is convex on $[\mathfrak{v}, \mathfrak{w}]$, then the following inequality holds:*

$$\left| \tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2} - \tfrac{1}{\mathfrak{w}-\mathfrak{v}} \int\limits_{\mathfrak{v}}^{\mathfrak{w}} \mathfrak{g}(z)\, dz \right| \leq \tfrac{\mathfrak{w}-\mathfrak{v}}{12} \left[|\mathfrak{g}'(\mathfrak{v})| + \left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right| + |\mathfrak{g}'(\mathfrak{w})| \right].$$

Fractional calculus expands upon conventional calculus by incorporating derivatives and integrals of non-integer order. It is utilized in various scientific disciplines to accurately represent non-local and non-Markovian phenomena. Various fractional integral operators have been developed to enhance the modeling of different phenomena in science and engineering. One of the most frequently seen operators is the Riemann–Liouville operators, which can be defined in the following manner.

Definition 1 ([7]). *Let $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$. The left and right Riemann–Liouville fractional integrals of order $\beta > 0$ are defined by*

$$\begin{aligned} J^{\beta}_{\mathfrak{v}^+}\mathfrak{g}(\varkappa) &= \tfrac{1}{\Gamma(\beta)} \int\limits_{\mathfrak{v}}^{\varkappa} (\varkappa - z)^{\beta-1} \mathfrak{g}(z) dz, \quad \varkappa > \mathfrak{v}, \\ J^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\varkappa) &= \tfrac{1}{\Gamma(\beta)} \int\limits_{\varkappa}^{\mathfrak{w}} (z - \varkappa)^{\beta-1} \mathfrak{g}(z) dz, \quad \mathfrak{w} > \varkappa, \end{aligned}$$

respectively, where $\Gamma(\beta) = \int\limits_{0}^{\infty} e^{-\nu} \nu^{\beta-1} d\nu$ is the gamma function, and $J^0_{\mathfrak{v}^+}\mathfrak{g}(z) = J^0_{\mathfrak{w}^-}\mathfrak{g}(z) = \mathfrak{g}(z)$.

In [8], Sarikaya established the analogue of inequality (1) for Riemann–Liouville fractional integrals as follows.

Theorem 5 ([8])**.** *Let* $\mathfrak{g} : [\mathfrak{v}, \mathfrak{w}] \to \mathbb{R}$ *be a positive function such that* $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$ *with* $0 \le \mathfrak{v} < \mathfrak{w}$*. If* $\mathfrak{g}$ *is convex on* $[\mathfrak{v}, \mathfrak{w}]$*, then for* $\beta > 0$ *the following inequalities hold:*

$$\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) \le \tfrac{\Gamma(\beta+1)}{2(\mathfrak{w}-\mathfrak{v})^{\beta}}\left[J^{\beta}_{\mathfrak{v}+}\mathfrak{g}(\mathfrak{w}) + J^{\beta}_{\mathfrak{w}-}\mathfrak{g}(\mathfrak{v})\right] \le \tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2},$$

where $J^{\beta}_{\nu-}$ *and* $J^{\beta}_{\nu+}$ *denote the left and right Riemann–Liouville integrals, respectively.*

In [9], Sarikaya and Yıldırım provided another version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals.

Theorem 6 ([9])**.** *Let* $\mathfrak{g} : [\mathfrak{v}, \mathfrak{w}] \to \mathbb{R}$ *be a positive function such that* $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$ *with* $0 \le \mathfrak{v} < \mathfrak{w}$*. If* $\mathfrak{g}$ *is convex on* $[\mathfrak{v}, \mathfrak{w}]$*, then for* $\beta > 0$*, the following inequalities hold:*

$$\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) \le \tfrac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{w}-\mathfrak{v})^{\beta}}\left[J^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}) + J^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right] \le \tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}.$$

In the same paper, the authors derived the Riemann–Liouville fractional midpoint-type inequalities as follows:

Theorem 7 ([9])**.** *Let* $\mathfrak{g} : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I *such that* $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$*, where* $\mathfrak{v}, \mathfrak{w} \in I$ *with* $\mathfrak{v} < \mathfrak{w}$*. If* $|\mathfrak{g}'|$ *is convex on* $[\mathfrak{v}, \mathfrak{w}]$*, then for* $\beta > 0$*, the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \tfrac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{w}-\mathfrak{v})^{\beta}}\left[J^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}) + J^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right| \\ &\le \tfrac{\mathfrak{w}-\mathfrak{v}}{4(\beta+1)}\left[|\mathfrak{g}'(\mathfrak{v})| + |\mathfrak{g}'(\mathfrak{w})|\right].\end{aligned}$$

In [10], Özdemir et al. provided the following trapezoid-type inequalities via Riemann–Liouville fractional integrals.

Theorem 8 ([10])**.** *Let* $\mathfrak{g} : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I *such that* $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$*, where* $\mathfrak{v}, \mathfrak{w} \in I$ *with* $\mathfrak{v} < \mathfrak{w}$*. If* $|\mathfrak{g}'|$ *is convex on* $[\mathfrak{v}, \mathfrak{w}]$*, then for* $\beta > 0$*, the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2} - \tfrac{2^{\beta-1}\Gamma(\beta+1)}{(\mathfrak{w}-\mathfrak{v})^{\beta}}\left[J^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}) + J^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right| \\ &\le \tfrac{\mathfrak{w}-\mathfrak{v}}{4}\left[\tfrac{(\beta+1)(\beta+2)-2}{2(\beta+1)(\beta+2)}\left[|\mathfrak{g}'(\mathfrak{v})| + |\mathfrak{g}'(\mathfrak{w})|\right] + \tfrac{\beta}{(\beta+2)}\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right].\end{aligned}$$

Remark 1. *Setting* $\beta = 1$ *in Theorem 8 yields Theorem 4.*

For additional studies on fractional integral inequalities involving Riemann–Liouville operators, readers may refer to [11–15] and the references cited therein.

In [16], Ahmad et al. introduced the following new fractional integrals with exponential kernels.

Definition 2 ([16])**.** *Let* $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$*. The fractional integrals* $\mathcal{I}^{\beta}_{\mathfrak{v}+}$ *and* $\mathcal{I}^{\beta}_{\mathfrak{w}-}$ *of order* $\beta \in (0, 1)$ *are defined, respectively, by*

$$\mathcal{I}^{\beta}_{\mathfrak{v}+}\mathfrak{g}(\varkappa) = \frac{1}{\beta}\int_{\mathfrak{v}}^{\varkappa} \exp\left(-\frac{1-\beta}{\beta}(\varkappa - z)\right)\mathfrak{g}(z)\,dz, \quad \varkappa > \mathfrak{v},$$

$$\mathcal{I}^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\varkappa) = \frac{1}{\beta}\int_{\varkappa}^{\mathfrak{w}} \exp\Big(-\frac{1-\beta}{\beta}(z-\varkappa)\Big)\mathfrak{g}(z)\,dz, \quad \mathfrak{w} > \varkappa.$$

In the following, for the sake of simplicity, we adopt the following notation: $A = \frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})$.

Remark 2. *From Definition 2, one can derive the following statements.*

1. $\lim\limits_{\beta\to 1}\mathcal{I}^{\beta}_{\mathfrak{v}^+}\mathfrak{g}(\varkappa) = \int_{\mathfrak{v}}^{\varkappa}\mathfrak{g}(z)\,dz$ *and* $\lim\limits_{\beta\to 1}\mathcal{I}^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\varkappa) = \int_{\varkappa}^{\mathfrak{w}}\mathfrak{g}(z)\,dz.$
2. $\lim\limits_{\beta\to 0}\mathcal{I}^{\beta}_{\mathfrak{v}^+}\mathfrak{g}(\varkappa) = \lim\limits_{\beta\to 0}\mathcal{I}^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\varkappa) = \mathfrak{g}(\varkappa).$

In the same paper, the authors provided the Hermite–Hadamard inequality as well as trapezoid-type inequalities via the newly introduced fractional integrals.

Theorem 9 ([16])**.** *Let $\mathfrak{g} : [\mathfrak{v}, \mathfrak{w}] \to \mathbb{R}$ be a positive function such that $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$ with $0 \le \mathfrak{v} < \mathfrak{w}$. If $\mathfrak{g}$ is convex on $[\mathfrak{v}, \mathfrak{w}]$, then for $\beta \in (0, 1)$, the following inequalities hold:*

$$\mathfrak{g}\Big(\frac{\mathfrak{v}+\mathfrak{w}}{2}\Big) \le \frac{1-\beta}{2(1-\exp(-A))}\Big[\mathcal{I}^{\beta}_{\mathfrak{v}^+}\mathfrak{g}(\mathfrak{w}) + \mathcal{I}^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\mathfrak{v})\Big] \le \frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}.$$

Theorem 10 ([16])**.** *Let $\mathfrak{g} : [\mathfrak{v}, \mathfrak{w}] \to \mathbb{R}$ be a differentiable function on $[\mathfrak{v}, \mathfrak{w}]$ such that $\mathfrak{g}' \in L[\mathfrak{v}, \mathfrak{w}]$ with $0 \le \mathfrak{v} < \mathfrak{w}$. If $|\mathfrak{g}'|$ is convex, then we have*

$$\begin{aligned}&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2} - \frac{1-\beta}{2(1-\exp(-A))}\Big[\mathcal{I}^{\beta}_{\mathfrak{v}^+}\mathfrak{g}(\mathfrak{w}) + \mathcal{I}^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\mathfrak{v})\Big]\right| \\ &\le \frac{\mathfrak{w}-\mathfrak{v}}{2A}\tanh\Big(\frac{A}{4}\Big)\big[|\mathfrak{g}'(\mathfrak{v})| + |\mathfrak{g}'(\mathfrak{w})|\big].\end{aligned}$$

On the other hand, Wu et al. in their publication [17] introduced midpoint-type inequalities for the same operator in the following manner:

Theorem 11 ([17])**.** *Let $\mathfrak{g} : [\mathfrak{v}, \mathfrak{w}] \to \mathbb{R}$ be a differentiable function on $[\mathfrak{v}, \mathfrak{w}]$ such that $\mathfrak{g}' \in L[\mathfrak{v}, \mathfrak{w}]$ with $0 \le \mathfrak{v} < \mathfrak{w}$. If $|\mathfrak{g}'|$ is convex, then we have*

$$\begin{aligned}&\left|\mathfrak{g}\Big(\frac{\mathfrak{v}+\mathfrak{w}}{2}\Big) - \frac{1-\beta}{2(1-\exp(-A))}\Big[\mathcal{I}^{\beta}_{\mathfrak{v}^+}\mathfrak{g}(\mathfrak{w}) + \mathcal{I}^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\mathfrak{v})\Big]\right| \\ &\le \frac{\mathfrak{w}-\mathfrak{v}}{2}\left(\frac{1}{2} - \frac{\tanh\big(\frac{A}{4}\big)}{A}\right)\big[|\mathfrak{g}'(\mathfrak{v})| + |\mathfrak{g}'(\mathfrak{w})|\big].\end{aligned}$$

In [18], the authors explored another variant of the Hermite–Hadamard inequality as well as Hermite–Hadamard-type inequalities for left-sided fractional integrals with exponential kernels. Furthermore, Zhou et al. [19] established a series of results related to three-point quadrature formulas for twice-differentiable convex functions. However, the most intriguing work related to this type of operators was performed by Yuan et al. in [20]. In this study, the authors conducted a parametric analysis on the same operator previously addressed by Ahmad et al. [16] and Wu et al. [17], and they were able to establish the following result, from which they deduced the error bounds of several quadrature rules.

Theorem 12 ([19])**.** *Let $\mathfrak{g} : [\mathfrak{v}, \mathfrak{w}] \to \mathbb{R}$ be a differentiable function on $[\mathfrak{v}, \mathfrak{w}]$ such that $\mathfrak{g}' \in L[\mathfrak{v}, \mathfrak{w}]$ with $0 \le \mathfrak{v} < \mathfrak{w}$. If $|\mathfrak{g}'|$ is convex, then we have for $0 \le \lambda \le 1$*

$$\begin{aligned}&\left|(1-\lambda)\mathfrak{g}\Big(\frac{\mathfrak{v}+\mathfrak{w}}{2}\Big) + \lambda\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2} - \frac{1-\beta}{2(1-\exp(-A))}\Big(\mathcal{I}^{\beta}_{\mathfrak{v}^+}\mathfrak{g}(\mathfrak{w}) + \mathcal{I}^{\beta}_{\mathfrak{w}^-}\mathfrak{g}(\mathfrak{v})\Big)\right| \\ &\le \frac{\mathfrak{w}-\mathfrak{v}}{2}\left(\frac{1-\lambda}{2} + \frac{\tanh\big(\frac{A}{4}\big)}{A}\right)\big[|\mathfrak{g}'(\mathfrak{v})| + |\mathfrak{g}'(\mathfrak{w})|\big].\end{aligned}$$

In this study, we present new versions of Hermite–Hadamard and Hermite–Hadamard-type inequalities by employing fractional operators with exponential kernels that differ from the ones previously established in Theorems 9–11. The results we obtained are analogous to those established in Theorems 6 and 8; although, it is important to mention that our findings are just generalizations of classical inequalities. The study is supported by graphical representations that confirm the validity of the acquired results.

The organization of this paper is as follows: Section 1 presents some previous results and useful definitions pertinent to our study. Section 2 is divided into three subsections. In the Section 2.1, we provide a new version of the Hermite–Hadamard inequality for fractional integrals with exponential kernels. The Section 2.2 is dedicated to the study of the midpoint inequality for the same type of integrals. The Section 2.3 establishes several trapezoid-type inequalities for the same operators. Section 3 provides a numerical example with graphical representations that justify the accuracy of our outcomes, while Section 4 presents some applications of the obtained results to special means. The study concludes with a summary of the key findings in Section 5.

2. Main Results

In this section, we initiate our study by establishing a novel version of the Hermite–Hadamard inequality for the new fractional integrals introduced in [16]. Subsequently, we introduce two new identities, from which we derive several new midpoint- and trapezoid-type inequalities for differentiable convex functions.

2.1. Hermite–Hadamard Inequality

In this part, we present a new version of the Hermite–Hadamard inequality specifically tailored for fractional integrals with exponential kernels.

Theorem 13. *Let $\mathfrak{g} : [\mathfrak{v}, \mathfrak{w}] \to \mathbb{R}$ be a positive function such that $\mathfrak{g} \in L[\mathfrak{v}, \mathfrak{w}]$ with $0 \leq \mathfrak{v} < \mathfrak{w}$. If $\mathfrak{g}$ is convex on $[\mathfrak{v}, \mathfrak{w}]$, then for $\beta \in (0,1)$, the following inequalities hold:*

$$\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) \leq \frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}) + \mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right] \leq \frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}. \tag{2}$$

Proof. From the convexity of $\mathfrak{g}$, we have

$$\mathfrak{g}\left(\tfrac{u+v}{2}\right) \leq \frac{\mathfrak{g}(u)+\mathfrak{g}(v)}{2}. \tag{3}$$

By performing the variable changes $u = \frac{2-\varkappa}{2}\mathfrak{v} + \frac{\varkappa}{2}\mathfrak{w}$ and $v = \frac{\varkappa}{2}\mathfrak{v} + \frac{2-\varkappa}{2}\mathfrak{w}$, inequality (3) yields

$$2\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) \leq \mathfrak{g}\left(\tfrac{2-\varkappa}{2}\mathfrak{v} + \tfrac{\varkappa}{2}\mathfrak{w}\right) + \mathfrak{g}\left(\tfrac{\varkappa}{2}\mathfrak{v} + \tfrac{2-\varkappa}{2}\mathfrak{w}\right). \tag{4}$$

Multiplying both sides of (4) by $\exp\left(-\frac{1}{2}A\varkappa\right)$ and then integrating the resulting inequality with respect to $\varkappa$ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{4\left(1-\exp\left(-\frac{1}{2}A\right)\right)}{A}\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) \\
\leq&\int_0^1 \exp\left(-\tfrac{1}{2}A\varkappa\right)\left[\mathfrak{g}\left(\tfrac{2-\varkappa}{2}\mathfrak{v} + \tfrac{\varkappa}{2}\mathfrak{w}\right) + \mathfrak{g}\left(\tfrac{\varkappa}{2}\mathfrak{v} + \tfrac{2-\varkappa}{2}\mathfrak{w}\right)\right] d\varkappa \\
=&\int_0^1 \exp\left(-\tfrac{1}{2}A\varkappa\right)\mathfrak{g}\left(\tfrac{2-\varkappa}{2}\mathfrak{v} + \tfrac{\varkappa}{2}\mathfrak{w}\right) d\varkappa + \int_0^1 \exp\left(-\tfrac{1}{2}A\varkappa\right)\mathfrak{g}\left(\tfrac{\varkappa}{2}\mathfrak{v} + \tfrac{2-\varkappa}{2}\mathfrak{w}\right) d\varkappa \\
=&\frac{2}{\mathfrak{w}-\mathfrak{v}}\left[\int_{\mathfrak{v}}^{\frac{\mathfrak{v}+\mathfrak{w}}{2}} \exp\left[-\tfrac{1-\beta}{\beta}(z-\mathfrak{v})\right]\mathfrak{g}(z)dz + \int_{\frac{\mathfrak{v}+\mathfrak{w}}{2}}^{\mathfrak{w}} \exp\left[-\tfrac{1-\beta}{\beta}(\mathfrak{w}-z)\right]\mathfrak{g}(z)dz\right]
\end{aligned}$$

$$=\frac{2\beta}{\mathfrak{w}-\mathfrak{v}}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right].$$

As a result, we obtain

$$\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\leq\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]. \tag{5}$$

Furthermore, we have for $\varkappa\in[0,1]$

$$\mathfrak{g}\left(\frac{2-\varkappa}{2}\mathfrak{v}+\frac{\varkappa}{2}\mathfrak{w}\right)\leq\frac{2-\varkappa}{2}\mathfrak{g}(\mathfrak{v})+\frac{\varkappa}{2}\mathfrak{g}(\mathfrak{w})$$

and

$$\mathfrak{g}\left(\frac{\varkappa}{2}\mathfrak{v}+\frac{2-\varkappa}{2}\mathfrak{w}\right)\leq\frac{\varkappa}{2}\mathfrak{g}(\mathfrak{v})+\frac{2-\varkappa}{2}\mathfrak{g}(\mathfrak{w}).$$

By adding the above inequalities, we obtain

$$\mathfrak{g}\left(\frac{2-\varkappa}{2}\mathfrak{v}+\frac{\varkappa}{2}\mathfrak{w}\right)+\mathfrak{g}\left(\frac{\varkappa}{2}\mathfrak{v}+\frac{2-\varkappa}{2}\mathfrak{w}\right)\leq\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w}). \tag{6}$$

Multiplying both sides of (6) by $\exp\left(-\frac{1}{2}A\varkappa\right)$ and then integrating the resulting inequality with respect to $\varkappa$ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}{\mathfrak{w}-\mathfrak{v}}[\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})]\\
&\geq\int_{0}^{1}\exp\left(-\frac{1}{2}A\varkappa\right)\left[\mathfrak{g}\left(\frac{2-\varkappa}{2}\mathfrak{v}+\frac{\varkappa}{2}\mathfrak{w}\right)+\mathfrak{g}\left(\frac{\varkappa}{2}\mathfrak{v}+\frac{2-\varkappa}{2}\mathfrak{w}\right)\right]d\varkappa\\
&=\int_{0}^{1}\exp\left(-\frac{1}{2}A\varkappa\right)\mathfrak{g}\left(\frac{2-\varkappa}{2}\mathfrak{v}+\frac{\varkappa}{2}\mathfrak{w}\right)d\varkappa+\int_{0}^{1}\exp\left(-\frac{1}{2}A\varkappa\right)\mathfrak{g}\left(\frac{\varkappa}{2}\mathfrak{v}+\frac{2-\varkappa}{2}\mathfrak{w}\right)d\varkappa\\
&=\frac{2}{\mathfrak{w}-\mathfrak{v}}\left[\int_{\mathfrak{v}}^{\frac{\mathfrak{v}+\mathfrak{w}}{2}}\exp\left[-\frac{1-\beta}{\beta}(z-\mathfrak{v})\right]\mathfrak{g}(z)dz+\int_{\frac{\mathfrak{v}+\mathfrak{w}}{2}}^{\mathfrak{w}}\exp\left[-\frac{1-\beta}{\beta}(\mathfrak{w}-z)\right]\mathfrak{g}(z)dz\right]\\
&=\frac{2\beta}{\mathfrak{w}-\mathfrak{v}}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right],
\end{aligned}$$

that is

$$\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\leq\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}. \tag{7}$$

The desired result is obtained by combining inequalities (5) and (7).

The proof is completed. □

Remark 3. *For $\beta\to1$, we have*

$$\lim_{\beta\to1}\frac{1-\beta}{2(1-\exp(-A))}=\frac{1}{2(\mathfrak{w}-\mathfrak{v})}.$$

Thus, making β tend to 1, inequality (2) will be reduced the classical Hermite–Hadamard inequality (1).

2.2. Midpoint-Type Inequalities

Here, we focus on the study of the midpoint inequality, extending it to apply to fractional integrals with exponential kernels.

Lemma 1. *Let* $\mathfrak{g} : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $\mathcal{I}^{\circ}$, $\mathfrak{v}, \mathfrak{w} \in \mathcal{I}^{\circ}$ *with* $\mathfrak{v} < \mathfrak{w}$, *and* $\mathfrak{g}' \in L[\mathfrak{v}, \mathfrak{w}]$. *Then, for* $\beta \in (0,1)$, *the following equality holds:*

$$\begin{aligned}
&\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \tfrac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}) + \mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right] \\
=& \tfrac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\int_{0}^{\frac{1}{2}}(1-\exp(-A\varkappa))\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa\right. \\
&\left. - \int_{\frac{1}{2}}^{1}(1-\exp(-A(1-\varkappa)))\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa\right].
\end{aligned}$$

Proof. Let

$$\begin{aligned}
\mathcal{L}_1 =& \int_{0}^{\frac{1}{2}}(1-\exp(-A\varkappa))\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa, \\
\mathcal{L}_2 =& \int_{\frac{1}{2}}^{1}(1-\exp(-A(1-\varkappa)))\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa.
\end{aligned}$$

Integrating by parts $\mathcal{L}_1$, we obtain

$$\begin{aligned}
\mathcal{L}_1 =& \tfrac{1}{\mathfrak{w}-\mathfrak{v}}(1-\exp(-A\varkappa))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\Big|_{\varkappa=0}^{\varkappa=\frac{1}{2}} \\
& - \tfrac{A}{\mathfrak{w}-\mathfrak{v}}\int_{0}^{\frac{1}{2}}(\exp(-A\varkappa))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa \\
=& \tfrac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \tfrac{1-\beta}{\beta}\int_{0}^{\frac{1}{2}}\exp(-A\varkappa)\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa \\
=& \tfrac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \tfrac{1-\beta}{\beta(\mathfrak{w}-\mathfrak{v})}\int_{\mathfrak{v}}^{\frac{\mathfrak{v}+\mathfrak{w}}{2}}\exp\left(-\tfrac{1-\beta}{\beta}(u-\mathfrak{v})\right)\mathfrak{g}(u)du \\
=& \tfrac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \tfrac{1-\beta}{\mathfrak{w}-\mathfrak{v}}\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}).
\end{aligned} \tag{8}$$

Similarly, we obtain

$$\begin{aligned}
\mathcal{L}_2 =& \tfrac{1}{\mathfrak{w}-\mathfrak{v}}(1-\exp(-A(1-\varkappa)))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\Big|_{\varkappa=\frac{1}{2}}^{\varkappa=1} \\
& + \tfrac{A}{\mathfrak{w}-\mathfrak{v}}\int_{\frac{1}{2}}^{1}\exp(-A(1-\varkappa))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa \\
=& -\tfrac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) + \tfrac{1-\beta}{\beta}\int_{\frac{1}{2}}^{1}\exp(-A(1-\varkappa))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa \\
=& -\tfrac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\tfrac{1}{2}A\right)\right)\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) + \tfrac{1-\beta}{\beta(\mathfrak{w}-\mathfrak{v})}\int_{\frac{\mathfrak{v}+\mathfrak{w}}{2}}^{\mathfrak{w}}\exp\left(-\tfrac{1-\beta}{\beta}(\mathfrak{w}-u)\right)\mathfrak{g}(u)du
\end{aligned}$$

$$= -\frac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\frac{1}{2}A\right)\right)\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)+\frac{1-\beta}{\mathfrak{w}-\mathfrak{v}}\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w}). \tag{9}$$

Making the difference between (8) and (9), and then multiplying the resulting equality by $\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}$, we obtain the desired result. □

Theorem 14. *Let* $\mathfrak{g}:[\mathfrak{v},\mathfrak{w}]\to\mathbb{R}$ *be a differentiable function on* $[\mathfrak{v},\mathfrak{w}]$ *such that* $\mathfrak{g}'\in L[\mathfrak{v},\mathfrak{w}]$ *with* $0\le\mathfrak{v}<\mathfrak{w}$. *If* $|\mathfrak{g}'|$ *is convex, then we have*

$$\begin{aligned}&\left|\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right|\\
\le&\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\frac{8-8A+3A^2}{8A^2}+\frac{A-2}{2A^2}\exp\left(-\frac{1}{2}A\right)\right)\left(|\mathfrak{g}'(\mathfrak{v})|+|\mathfrak{g}'(\mathfrak{w})|\right)\right.\\
&\left.+2\left(\frac{A^2-8}{8A^2}+\frac{A+2}{2A^2}\exp\left(-\frac{1}{2}A\right)\right)\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right].\end{aligned}$$

Proof. From Lemma 1, properties of modulus, and the convexity of $|\mathfrak{g}'|$, we have

$$\begin{aligned}&\left|\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right|\\
\le&\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\int_0^{\frac{1}{2}}|1-\exp(-A\varkappa)||\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})|d\varkappa\right.\\
&\left.+\int_{\frac{1}{2}}^{1}|1-\exp(-A(1-\varkappa))||\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})|d\varkappa\right]\\
\le&\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\int_0^{\frac{1}{2}}(1-\exp(-A\varkappa))\left((1-\varkappa)|\mathfrak{g}'(\mathfrak{v})|+\varkappa\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right)d\varkappa\right.\\
&\left.+\int_{\frac{1}{2}}^{1}(1-\exp(-A(1-\varkappa)))\left((1-\varkappa)\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|+\varkappa|\mathfrak{g}'(\mathfrak{w})|\right)d\varkappa\right]\\
=&\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\int_0^{\frac{1}{2}}(1-\exp(-A\varkappa))(1-\varkappa)d\varkappa\right)|\mathfrak{g}'(\mathfrak{v})|\right.\\
&+\left(\int_0^{\frac{1}{2}}(1-\exp(-A\varkappa))\varkappa\,d\varkappa+\int_{\frac{1}{2}}^{1}(1-\exp(-A(1-\varkappa)))(1-\varkappa)d\varkappa\right)\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\\
&\left.+\left(\int_{\frac{1}{2}}^{1}(1-\exp(-A(1-\varkappa)))\varkappa\,d\varkappa\right)|\mathfrak{g}'(\mathfrak{w})|\right]\\
=&\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\frac{8-8A+3A^2}{8A^2}+\frac{A-2}{2A^2}\exp\left(-\frac{1}{2}A\right)\right)\left(|\mathfrak{g}'(\mathfrak{v})|+|\mathfrak{g}'(\mathfrak{w})|\right)\right.\\
&\left.+2\left(\frac{A^2-8}{8A^2}+\frac{A+2}{2A^2}\exp\left(-\frac{1}{2}A\right)\right)\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right],\end{aligned}$$

where we have used

$$\begin{aligned}
&\int_0^{\frac{1}{2}}(1-\exp(-A\varkappa))(1-\varkappa)d\varkappa\\
=&\int_{\frac{1}{2}}^{1}(1-\exp(-A(1-\varkappa)))\varkappa\, d\varkappa\\
=&\left(\varkappa+\tfrac{1}{A}\exp(-A\varkappa)\right)(1-\varkappa)\Big|_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}}\left(\varkappa+\tfrac{1}{A}\exp(-A\varkappa)\right)d\varkappa\\
=&\tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{A}\exp\left(-\tfrac{1}{2}A\right)\right)-\tfrac{1}{A}+\left(\tfrac{1}{2}\varkappa^2-\tfrac{1}{A^2}\exp(-A\varkappa)\right)\Big|_0^{\frac{1}{2}}\\
=&\tfrac{1}{4}+\tfrac{1}{2A}\exp\left(-\tfrac{1}{2}A\right)-\tfrac{1}{A}+\tfrac{1}{8}-\tfrac{1}{A^2}\exp\left(-\tfrac{1}{2}A\right)+\tfrac{1}{A^2}\\
=&\tfrac{8-8A+3A^2}{8A^2}+\tfrac{A-2}{2A^2}\exp\left(-\tfrac{1}{2}A\right)
\end{aligned}\tag{10}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}}(1-\exp(-A\varkappa))\varkappa\, d\varkappa\\
=&\int_{\frac{1}{2}}^{1}(1-\exp(-A(1-\varkappa)))(1-\varkappa)d\varkappa\\
=&\left(\varkappa+\tfrac{1}{A}\exp(-A\varkappa)\right)\varkappa\Big|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\left(\varkappa+\tfrac{1}{A}\exp(-A\varkappa)\right)d\varkappa\\
=&\tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{A}\exp\left(-\tfrac{1}{2}A\right)\right)-\left(\tfrac{1}{2}\varkappa^2-\tfrac{1}{A^2}\exp(-A\varkappa)\right)\Big|_0^{\frac{1}{2}}\\
=&\tfrac{A^2-8}{8A^2}+\tfrac{A+2}{2A^2}\exp\left(-\tfrac{1}{2}A\right).
\end{aligned}\tag{11}$$

The proof is completed. □

Corollary 1. *In Theorem 14, using the convexity of $|\mathfrak{g}'|$, we obtain*

$$\begin{aligned}
&\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\tfrac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\
\leq&\tfrac{\mathfrak{w}-\mathfrak{v}}{4A\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left((A-2)+2\exp\left(-\tfrac{1}{2}A\right)\right)\left[|\mathfrak{g}'(\mathfrak{v})|+|\mathfrak{g}'(\mathfrak{w})|\right].
\end{aligned}$$

Remark 4. *By making β tend to 1, we find*

1. $\lim\limits_{\beta\to 1}\frac{1-\beta}{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{\mathfrak{w}-\mathfrak{v}}.$
2. $\lim\limits_{\beta\to 1}\frac{8-8\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})+3\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+\left(4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)-8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{16\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{12}.$
3. $\lim\limits_{\beta\to 1}\frac{\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2-8+4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})+2\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{8\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{12}.$
4. $\lim\limits_{\beta\to 1}\frac{\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-2\right)+2\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{4\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{8}.$

Corollary 2. *In Theorem 14, letting β tend to 1, we obtain*

$$\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\tfrac{1}{\mathfrak{w}-\mathfrak{v}}\int\limits_{a}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\tfrac{\mathfrak{w}-\mathfrak{v}}{12}\left[\left|\mathfrak{g}'(\mathfrak{v})\right|+\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|+\left|\mathfrak{g}'(\mathfrak{w})\right|\right].$$

This represents a new result analogous to that established by Kadakal in Corollary 1 from [4].

Remark 5. *By tending β to 1, Corollary 1 will be reduced to Theorem 2.2 from [3].*

Theorem 15. *Let $\mathfrak{g}:[\mathfrak{v},\mathfrak{w}]\to\mathbb{R}$ be a differentiable function on $(\mathfrak{v},\mathfrak{w})$ such that $\mathfrak{g}'\in L[\mathfrak{v},\mathfrak{w}]$ with $0\leq\mathfrak{v}<\mathfrak{w}$. If $|\mathfrak{g}'|^{q}$ is convex where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}&\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\tfrac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right|\\&\leq\tfrac{\mathfrak{w}-\mathfrak{v}}{2}\left(\int\limits_{0}^{\frac{1}{2}}\left(\tfrac{1-\exp(-A\varkappa)}{\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\right)^{p}d\varkappa\right)^{\frac{1}{p}}\left[\left(\tfrac{3|\mathfrak{g}'(\mathfrak{v})|^{q}+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^{q}+3|\mathfrak{g}'(\mathfrak{w})|^{q}}{8}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. From Lemma 1, properties of modulus, Hölder's inequality, and the convexity of $|\mathfrak{g}'|^{q}$, we have

$$\begin{aligned}&\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\tfrac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right|\\&\leq\tfrac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\int\limits_{0}^{\frac{1}{2}}|1-\exp(-A\varkappa)|^{p}d\varkappa\right)^{\frac{1}{p}}\left(\int\limits_{0}^{\frac{1}{2}}\left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|^{q}d\varkappa\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int\limits_{\frac{1}{2}}^{1}|1-\exp(-A(1-\varkappa))|^{p}d\varkappa\right)^{\frac{1}{p}}\left(\int\limits_{\frac{1}{2}}^{1}\left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|^{q}d\varkappa\right)^{\frac{1}{q}}\right]\\&\leq\tfrac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\int\limits_{0}^{\frac{1}{2}}(1-\exp(-A\varkappa))^{p}d\varkappa\right)^{\frac{1}{p}}\left[\left(\int\limits_{0}^{\frac{1}{2}}\left((1-\varkappa)|\mathfrak{g}'(\mathfrak{v})|^{q}+\varkappa\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^{q}\right)d\varkappa\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int\limits_{\frac{1}{2}}^{1}\left((1-\varkappa)\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^{q}+\varkappa|\mathfrak{g}'(\mathfrak{w})|^{q}\right)d\varkappa\right)^{\frac{1}{q}}\right]\\&=\tfrac{\mathfrak{w}-\mathfrak{v}}{2}\left(\int\limits_{0}^{\frac{1}{2}}\left(\tfrac{1-\exp(-A\varkappa)}{\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\right)^{p}d\varkappa\right)^{\frac{1}{p}}\left[\left(\tfrac{3|\mathfrak{g}'(\mathfrak{v})|^{q}+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\tfrac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^{q}+3|\mathfrak{g}'(\mathfrak{w})|^{q}}{8}\right)^{\frac{1}{q}}\right].\end{aligned}$$

The proof is completed. □

Corollary 3. *In Theorem 15, using the convexity of $|\mathfrak{g}'|^{q}$, we obtain*

$$\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\tfrac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right|$$

$$\leq \frac{\mathfrak{w}-\mathfrak{v}}{2}\left(\int\limits_0^{\frac{1}{2}}\left(\frac{1-\exp(-A\varkappa)}{1-\exp\left(-\frac{1}{2}A\right)}\right)^p d\varkappa\right)^{\frac{1}{p}}\left[\left(\frac{7|\mathfrak{g}'(\mathfrak{v})|^q+|\mathfrak{g}'(\mathfrak{w})|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{g}'(\mathfrak{v})|^q+7|\mathfrak{g}'(\mathfrak{w})|^q}{16}\right)^{\frac{1}{q}}\right].$$

Remark 6. *By making β tend to 1, we get*

$$\lim_{\beta\to1}\frac{1-\exp\left(-\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\varkappa\right)}{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\varkappa.$$

Thus, we obtain

$$\lim_{\beta\to1}\int\limits_0^{\frac{1}{2}}\left(\frac{1-\exp\left(-\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\varkappa\right)}{1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}\right)^p d\varkappa=\int\limits_0^{\frac{1}{2}}(2\varkappa)^p d\varkappa=\frac{1}{2(p+1)}.$$

Corollary 4. *In Theorem 15, letting β tend to 1, then we obtain*

$$\begin{aligned}&\left|\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int\limits_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\\&\leq\frac{\mathfrak{w}-\mathfrak{v}}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{g}'(\mathfrak{v})|^q+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+3|\mathfrak{g}'(\mathfrak{w})|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 5. *In Corollary 3, letting β tend to 1, we obtain*

$$\left|\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int\limits_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\frac{\mathfrak{w}-\mathfrak{v}}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|\mathfrak{g}'(\mathfrak{v})|^q+|\mathfrak{g}'(\mathfrak{w})|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{g}'(\mathfrak{v})|^q+7|\mathfrak{g}'(\mathfrak{w})|^q}{8}\right)^{\frac{1}{q}}\right].$$

Theorem 16. *Let $\mathfrak{g}:[\mathfrak{v},\mathfrak{w}]\to\mathbb{R}$ be a differentiable function on $[\mathfrak{v},\mathfrak{w}]$ such that $\mathfrak{g}'\in L[\mathfrak{v},\mathfrak{w}]$ with $0\leq\mathfrak{v}<\mathfrak{w}$. If $|\mathfrak{g}'|^q$ is convex where $q\geq1$, then we have*

$$\begin{aligned}&\left|\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\&\leq\frac{\mathfrak{w}-\mathfrak{v}}{2}\left(\frac{A-2+2\exp\left(-\frac{1}{2}A\right)}{2A\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\right)^{1-\frac{1}{q}}\\&\times\left[\left(\frac{8-8A+3A^2+4(A-2)\exp\left(-\frac{1}{2}A\right)}{8A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{v})|^q+\frac{A^2-8+4(A+2)\exp\left(-\frac{1}{2}A\right)}{8A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{A^2-8+4(A+2)\exp\left(-\frac{1}{2}A\right)}{8A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+\frac{8-8A+3A^2+4(A-2)\exp\left(-\frac{1}{2}A\right)}{8A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{w})|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. From Lemma 1, properties of modulus, power mean integral inequality, and the convexity of $|\mathfrak{g}'|^q$, we have

$$\begin{aligned}&\left|\mathfrak{g}\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\&\leq\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\int\limits_0^{\frac{1}{2}}|1-\exp(-A\varkappa)|d\varkappa\right)^{1-\frac{1}{q}}\right.\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_0^{\frac{1}{2}} |1-\exp(-A\varkappa)| \left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|^q d\varkappa \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{2}}^{1} |1-\exp(-A(1-\varkappa))| d\varkappa \right)^{1-\frac{1}{q}} \\
&\times \left(\int_{\frac{1}{2}}^{1} |1-\exp(-A(1-\varkappa))| \left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|^q d\varkappa \right)^{\frac{1}{q}} \Bigg] \\
\leq & \frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left(\int_0^{\frac{1}{2}} (1-\exp(-A\varkappa)) d\varkappa \right)^{1-\frac{1}{q}} \\
&\times \Bigg[\left(\int_0^{\frac{1}{2}} (1-\exp(-A\varkappa)) \left((1-\varkappa)\left|\mathfrak{g}'(\mathfrak{v})\right|^q + \varkappa\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q \right) d\varkappa \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{2}}^{1} (1-\exp(-A(1-\varkappa))) \left((1-\varkappa)\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q + \varkappa\left|\mathfrak{g}'(\mathfrak{w})\right|^q \right) d\varkappa \right)^{\frac{1}{q}} \Bigg] \\
= & \frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left(\frac{A-2+2\exp\left(-\frac{1}{2}A\right)}{2A} \right)^{1-\frac{1}{q}} \\
&\times \Bigg[\left(\frac{8-8A+3A^2+4(A-2)\exp\left(-\frac{1}{2}A\right)}{8A^2} \left|\mathfrak{g}'(\mathfrak{v})\right|^q + \frac{A^2-8+4(A+2)\exp\left(-\frac{1}{2}A\right)}{8A^2} \left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q \right)^{\frac{1}{q}} \\
&+ \left(\frac{A^2-8+4(A+2)\exp\left(-\frac{1}{2}A\right)}{8A^2} \left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q + \frac{8-8A+3A^2+4(A-2)\exp\left(-\frac{1}{2}A\right)}{8A^2} \left|\mathfrak{g}'(\mathfrak{w})\right|^q \right)^{\frac{1}{q}} \Bigg],
\end{aligned}
$$

where we have used (10) and (11). The proof is achieved. □

Corollary 6. *In Theorem 16, using the convexity of* $|\mathfrak{g}'|^q$, *we obtain*

$$
\begin{aligned}
&\left| \mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right) - \frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v}) + \mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w}) \right] \right| \\
\leq & \frac{\mathfrak{w}-\mathfrak{v}}{2} \left(\frac{A-2+2\exp\left(-\frac{1}{2}A\right)}{2A\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \right)^{1-\frac{1}{q}} \\
&\times \Bigg[\left(\frac{8-16A+7A^2+(12A-8)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left|\mathfrak{g}'(\mathfrak{v})\right|^q + \frac{A^2-8+4(A+2)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left|\mathfrak{g}'(\mathfrak{w})\right|^q \right)^{\frac{1}{q}} \\
&+ \left(\frac{A^2-8+4(A+2)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left|\mathfrak{g}'(\mathfrak{v})\right|^q + \frac{8-16A+7A^2+(12A-8)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} \left|\mathfrak{g}'(\mathfrak{w})\right|^q \right)^{\frac{1}{q}} \Bigg].
\end{aligned}
$$

Remark 7. *By making* β *tend to* 1, *we find*

1. $\displaystyle \lim_{\beta\to 1} \frac{\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-2+2\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{2\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)} = \frac{1}{4}.$
2. $\displaystyle \lim_{\beta\to 1} \frac{8-8\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)+3\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-2\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{8\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)} = \frac{1}{6}.$

3. $\displaystyle\lim_{\beta\to 1}\frac{\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2-8+4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})+2\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{8\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{12}.$
4. $\displaystyle\lim_{\beta\to 1}\frac{8-16\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)+7\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+\left(12\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)-8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{16\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{5}{24}.$
5. $\displaystyle\lim_{\beta\to 1}\frac{\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2-8+4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})+2\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{16\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{24}.$

Corollary 7. *In Theorem 16, letting β tend to* 1*, we obtain*

$$\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\frac{\mathfrak{w}-\mathfrak{v}}{8}\left[\left(\frac{2|\mathfrak{g}'(\mathfrak{v})|^q+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+2|\mathfrak{g}'(\mathfrak{w})|^q}{3}\right)^{\frac{1}{q}}\right].$$

Corollary 8. *In Corollary 6, letting β tend to* 1*, then we obtain*

$$\left|\mathfrak{g}\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\frac{\mathfrak{w}-\mathfrak{v}}{8}\left[\left(\frac{5|\mathfrak{g}'(\mathfrak{v})|^q+|\mathfrak{g}'(\mathfrak{w})|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{g}'(\mathfrak{v})|^q+5|\mathfrak{g}'(\mathfrak{w})|^q}{6}\right)^{\frac{1}{q}}\right].$$

2.3. Trapezoid-Type Inequalities

Now, we establish several trapezoid-type inequalities for the same fractional integral operators, further enriching our theoretical framework.

Lemma 2. *Let* $\mathfrak{g}: I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable function on* $\mathcal{I}^\circ$, $\mathfrak{v},\mathfrak{w}\in\mathcal{I}^\circ$ *with* $\mathfrak{v}<\mathfrak{w}$, *and* $\mathfrak{g}'\in L[\mathfrak{v},\mathfrak{w}]$. *Then, for* $\beta\in(0,1)$ *the following equality holds*

$$\begin{aligned}&\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\\&=\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\int_0^{\frac{1}{2}}\left(\exp\left(-\tfrac{1}{2}A\right)-\exp(-A\varkappa)\right)\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa\right.\\&\left.-\int_{\frac{1}{2}}^{1}\left(\exp\left(-\tfrac{1}{2}A\right)-\exp(-A(1-\varkappa))\right)\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa\right].\end{aligned}$$

Proof. Let

$$\begin{aligned}\mathcal{K}_1&=\int_0^{\frac{1}{2}}\left(\exp\left(-\tfrac{1}{2}A\right)-\exp(-A\varkappa)\right)\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa,\\\mathcal{K}_2&=\int_{\frac{1}{2}}^{1}\left(\exp\left(-\tfrac{1}{2}A\right)-\exp(-A(1-\varkappa))\right)\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa.\end{aligned}$$

Integrating by parts $\mathcal{K}_1$, we obtain

$$\begin{aligned}\mathcal{K}_1=&\frac{1}{\mathfrak{w}-\mathfrak{v}}\left(\exp\left(-\tfrac{1}{2}A\right)-\exp(-A\varkappa)\right)\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\Big|_{\varkappa=0}^{\varkappa=\frac{1}{2}}\\&-\frac{A}{\mathfrak{w}-\mathfrak{v}}\int_0^{\frac{1}{2}}(\exp(-A\varkappa))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\frac{1}{2}A\right)\right)\mathfrak{g}(\mathfrak{v}) - \frac{1-\beta}{\beta}\int_{0}^{\frac{1}{2}} \exp(-A\varkappa)\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa \\
&= \frac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\frac{1}{2}A\right)\right)\mathfrak{g}(\mathfrak{v}) - \frac{1-\beta}{\beta(\mathfrak{w}-\mathfrak{v})}\int_{\mathfrak{v}}^{\frac{\mathfrak{v}+\mathfrak{w}}{2}} \exp\left(-\frac{1-\beta}{\beta}(u-\mathfrak{v})\right)\mathfrak{g}(u)du \\
&= \frac{1}{\mathfrak{w}-\mathfrak{v}}\left(1-\exp\left(-\frac{1}{2}A\right)\right)\mathfrak{g}(\mathfrak{v}) - \frac{1-\beta}{\mathfrak{w}-\mathfrak{v}}\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}).
\end{aligned} \tag{12}
$$

Similarly, we obtain

$$
\begin{aligned}
\mathcal{K}_2 &= \frac{1}{\mathfrak{w}-\mathfrak{v}}\left(\exp\left(-\frac{1}{2}A\right)-\exp(-A(1-\varkappa))\right)\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\Big|_{\varkappa=\frac{1}{2}}^{\varkappa=1} \\
&\quad + \frac{A}{\mathfrak{w}-\mathfrak{v}}\int_{\frac{1}{2}}^{1} \exp(-A(1-\varkappa))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa \\
&= \frac{1}{\mathfrak{w}-\mathfrak{v}}\left(\exp\left(-\frac{1}{2}A\right)-1\right)\mathfrak{g}(\mathfrak{w}) + \frac{1-\beta}{\beta}\int_{\frac{1}{2}}^{1} \exp(-A(1-\varkappa))\mathfrak{g}((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})d\varkappa \\
&= \frac{1}{\mathfrak{w}-\mathfrak{v}}\left(\exp\left(-\frac{1}{2}A\right)-1\right)\mathfrak{g}(\mathfrak{w}) + \frac{1-\beta}{\beta(\mathfrak{w}-\mathfrak{v})}\int_{\frac{\mathfrak{v}+\mathfrak{w}}{2}}^{\mathfrak{w}} \exp\left(-\frac{1-\beta}{\beta}(\mathfrak{w}-u)\right)\mathfrak{g}(u)du \\
&= \frac{1}{\mathfrak{w}-\mathfrak{v}}\left(\exp\left(-\frac{1}{2}A\right)-1\right)\mathfrak{g}(\mathfrak{w}) + \frac{1-\beta}{\mathfrak{w}-\mathfrak{v}}\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w}).
\end{aligned} \tag{13}
$$

Making the difference between (12) and (13), and then multiplying the resulting equality by $\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}$, we obtain the desired result. ☐

Theorem 17. *Let $\mathfrak{g}:[\mathfrak{v},\mathfrak{w}]\to\mathbb{R}$ be a differentiable function on $[\mathfrak{v},\mathfrak{w}]$ such that $\mathfrak{g}'\in L[\mathfrak{v},\mathfrak{w}]$ with $0\le\mathfrak{v}<\mathfrak{w}$. If $|\mathfrak{g}'|$ is convex, then we have*

$$
\begin{aligned}
&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2} - \frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}) + \mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right| \\
&\le \frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\frac{A-1}{A^2} - \frac{3A^2+4A-8}{8A^2}\exp\left(-\frac{1}{2}A\right)\right)\left(|\mathfrak{g}'(\mathfrak{v})|+|\mathfrak{g}'(\mathfrak{w})|\right)\right. \\
&\quad \left. + 2\left(\frac{1}{A^2} - \frac{A^2+4A+8}{8A^2}\exp\left(-\frac{1}{2}A\right)\right)\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right].
\end{aligned}
$$

Proof. From Lemma 2, properties of modulus, and convexity of $|\mathfrak{g}'|$, we have

$$
\begin{aligned}
&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2} - \frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}}\mathfrak{g}(\mathfrak{v}) + \mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}}\mathfrak{g}(\mathfrak{w})\right]\right| \\
&\le \frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\int_{0}^{\frac{1}{2}}\left(\left|\exp\left(-\frac{1}{2}A\right)-\exp(-A\varkappa)\right|\right)\left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|d\varkappa\right. \\
&\quad \left. + \int_{\frac{1}{2}}^{1}\left|\exp\left(-\frac{1}{2}A\right)-\exp(-A(1-\varkappa))\right|\left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|d\varkappa\right] \\
&\le \frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\Bigg[\int_{0}^{\frac{1}{2}}\left|\exp\left(-\frac{1}{2}A\right)-\exp(-A\varkappa)\right|\left((1-\varkappa)|\mathfrak{g}'(\mathfrak{v})|+\varkappa\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right)d\varkappa
\end{aligned}
$$

$$
\begin{aligned}
&+\int\limits_{\frac{1}{2}}^{1}\left|\exp\left(-\tfrac{1}{2}A\right)-\exp(-A(1-\varkappa))\right|\left((1-\varkappa)\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|+\varkappa\left|\mathfrak{g}'(\mathfrak{w})\right|\right)d\varkappa\Bigg]\\
=&\tfrac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\Bigg[\int\limits_{0}^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\tfrac{1}{2}A\right)\right)\left((1-\varkappa)\left|\mathfrak{g}'(\mathfrak{v})\right|+\varkappa\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right)d\varkappa\\
&+\int\limits_{\frac{1}{2}}^{1}\left(\exp(-A(1-\varkappa))-\exp\left(-\tfrac{1}{2}A\right)\right)\left((1-\varkappa)\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|+\varkappa\left|\mathfrak{g}'(\mathfrak{w})\right|\right)d\varkappa\Bigg]\\
=&\tfrac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\Bigg[\left(\int\limits_{0}^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\tfrac{1}{2}A\right)\right)(1-\varkappa)d\varkappa\right)\left|\mathfrak{g}'(\mathfrak{v})\right|\\
&+\left(\int\limits_{0}^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\tfrac{1}{2}A\right)\right)\varkappa\,d\varkappa\right.\\
&\left.+\int\limits_{\frac{1}{2}}^{1}\left(\exp(-A(1-\varkappa))-\exp\left(-\tfrac{1}{2}A\right)\right)(1-\varkappa)d\varkappa\right)\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\\
&+\left(\int\limits_{\frac{1}{2}}^{1}\left(\exp(-A(1-\varkappa))-\exp\left(-\tfrac{1}{2}A\right)\right)\varkappa\,d\varkappa\right)\left|\mathfrak{g}'(\mathfrak{w})\right|\Bigg]\\
=&\tfrac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\tfrac{A-1}{A^2}-\tfrac{3A^2+4A-8}{8A^2}\exp\left(-\tfrac{1}{2}A\right)\right)\left(\left|\mathfrak{g}'(\mathfrak{v})\right|+\left|\mathfrak{g}'(\mathfrak{w})\right|\right)\right.\\
&\left.+2\left(\tfrac{1}{A^2}-\left(\tfrac{A^2+4A+8}{8A^2}\right)\exp\left(-\tfrac{1}{2}A\right)\right)\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|\right],
\end{aligned}
$$

where we have used

$$
\begin{aligned}
&\int\limits_{0}^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\tfrac{1}{2}A\right)\right)(1-\varkappa)d\varkappa\\
&=\left(-\tfrac{1}{A}\exp(-A\varkappa)-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)(1-\varkappa)\Big|_{0}^{\frac{1}{2}}\\
&\quad-\int\limits_{0}^{\frac{1}{2}}\left(\tfrac{1}{A}\exp(-A\varkappa)-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)d\varkappa\\
&=\tfrac{A-1}{A^2}-\tfrac{3A^2+4A-8}{8A^2}\exp\left(-\tfrac{1}{2}A\right),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&\int\limits_{0}^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\tfrac{1}{2}A\right)\right)\varkappa\,d\varkappa\\
&=\left(-\tfrac{1}{A}\exp(-A\varkappa)-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)\varkappa\Big|_{0}^{\frac{1}{2}}\\
&\quad-\int\limits_{0}^{\frac{1}{2}}\left(\tfrac{1}{A}\exp(-A\varkappa)-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)d\varkappa\\
&=\tfrac{1}{A^2}-\tfrac{A^2+4A+8}{8A^2}\exp\left(-\tfrac{1}{2}A\right),
\end{aligned}
\tag{15}
$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left(\exp(-A(1-\varkappa))-\exp\left(-\tfrac{1}{2}A\right)\right)(1-\varkappa)d\varkappa\\
=&\left(\tfrac{1}{A}\exp(-A(1-\varkappa))-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)(1-\varkappa)\Big|_{\frac{1}{2}}^{1}\\
&+\int_{\frac{1}{2}}^{1}\left(\tfrac{1}{A}\exp(-A(1-\varkappa))-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)d\varkappa\\
=&\tfrac{1}{A^2}-\tfrac{A^2+8A+8}{8A^2}\exp\left(-\tfrac{1}{2}A\right)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left(\exp(-A(1-\varkappa))-\exp\left(-\tfrac{1}{2}A\right)\right)\varkappa\, d\varkappa\\
=&\left(\tfrac{1}{A}\exp(-A(1-\varkappa))-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)\varkappa\Big|_{\frac{1}{2}}^{1}\\
&-\int_{\frac{1}{2}}^{1}\left(\tfrac{1}{A}\exp(-A(1-\varkappa))-\varkappa\exp\left(-\tfrac{1}{2}A\right)\right)d\varkappa\\
=&\tfrac{A-1}{A^2}-\tfrac{3A^2+4A-8}{8A^2}\exp\left(-\tfrac{1}{2}A\right).
\end{aligned} \tag{17}$$

The proof is completed. □

Corollary 9. *In Theorem 17, using the convexity of $|\mathfrak{g}'|$, we obtain*

$$\begin{aligned}
&\left|\tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\tfrac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\
\leq&\tfrac{\mathfrak{w}-\mathfrak{v}}{4A\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(2-(2+A)\exp\left(-\tfrac{1}{2}A\right)\right)\left[|\mathfrak{g}'(\mathfrak{v})|+|\mathfrak{g}'(\mathfrak{w})|\right].
\end{aligned}$$

Remark 8. *For $\beta\to 1$, Corollary 9 reduces to Theorem 3 established by Dragomir and Agarwal in [5].*

Remark 9. *By making β tend to 1, we find*

1. $\lim\limits_{\beta\to 1}\frac{1-\beta}{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{\mathfrak{w}-\mathfrak{v}}.$
2. $\lim\limits_{\beta\to 1}\frac{8\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-1\right)-\left(3\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)-8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{16\mathfrak{v}^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{5}{48}.$
3. $\lim\limits_{\beta\to 1}\frac{8-\left(\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)+8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{8\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{24}.$
4. $\lim\limits_{\beta\to 1}\frac{2-\left(2+\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{4\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{8}.$

Corollary 10. *In Theorem 17, letting β tend to 1, we obtain*

$$\left|\tfrac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\tfrac{1}{\mathfrak{w}-\mathfrak{v}}\int_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\tfrac{\mathfrak{w}-\mathfrak{v}}{48}\left[5|\mathfrak{g}'(\mathfrak{v})|+2\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|+5|\mathfrak{g}'(\mathfrak{w})|\right].$$

This represents a new result analogous to that established by Kavurmacı et al. in [6].

Theorem 18. *Let $\mathfrak{g}:[\mathfrak{v},\mathfrak{w}]\to\mathbb{R}$ be a differentiable function on $(\mathfrak{v},\mathfrak{w})$ such that $\mathfrak{g}'\in L[\mathfrak{v},\mathfrak{w}]$ with $0\le\mathfrak{v}<\mathfrak{w}$. If $|\mathfrak{g}'|^q$ is convex where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}
&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\
&\le\frac{\mathfrak{w}-\mathfrak{v}}{2}\left(\int_0^{\frac{1}{2}}\left(\frac{\exp(-A\varkappa)-\exp\left(-\frac{1}{2}A\right)}{1-\exp\left(-\frac{1}{2}A\right)}\right)^p d\varkappa\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{3|\mathfrak{g}'(\mathfrak{v})|^q+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+3|\mathfrak{g}'(\mathfrak{w})|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 2, properties of modulus, Hölder's inequality and the convexity of $|\mathfrak{g}'|^q$, we have

$$\begin{aligned}
&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\
&\le\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\int_0^{\frac{1}{2}}\left|\exp\left(-\frac{1}{2}A\right)-\exp(-A\varkappa)\right|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})|^q d\varkappa\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left|\exp\left(-\frac{1}{2}A\right)-\exp(-A(1-\varkappa))\right|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})|^q d\varkappa\right)^{\frac{1}{q}}\right]\\
&\le\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\int_0^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\frac{1}{2}A\right)\right)^p d\varkappa\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left((1-\varkappa)|\mathfrak{g}'(\mathfrak{v})|^q+\varkappa\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^1\left((1-\varkappa)\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+\varkappa|\mathfrak{g}'(\mathfrak{w})|^q\right)d\varkappa\right)^{\frac{1}{q}}\right]\\
&=\frac{\mathfrak{w}-\mathfrak{v}}{2}\left(\int_0^{\frac{1}{2}}\left(\frac{\exp(-A\varkappa)-\exp\left(-\frac{1}{2}A\right)}{1-\exp\left(-\frac{1}{2}A\right)}\right)^p d\varkappa\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\frac{3|\mathfrak{g}'(\mathfrak{v})|^q+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+3|\mathfrak{g}'(\mathfrak{w})|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The proof is completed. □

Corollary 11. *In Theorem 18, using the convexity of $|\mathfrak{g}'|^q$, we obtain*

$$\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|$$

$$\leq \frac{\mathfrak{w}-\mathfrak{v}}{2}\left(\int_0^{\frac{1}{2}}\left(\frac{\exp(-A\varkappa)-\exp\left(-\frac{1}{2}A\right)}{1-\exp\left(-\frac{1}{2}A\right)}\right)^p d\varkappa\right)^{\frac{1}{p}}\left[\left(\frac{7|\mathfrak{g}'(\mathfrak{v})|^q+|\mathfrak{g}'(\mathfrak{w})|^q}{16}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{g}'(\mathfrak{v})|^q+7|\mathfrak{g}'(\mathfrak{w})|^q}{16}\right)^{\frac{1}{q}}\right].$$

Remark 10. *By making β tend to 1, we get*

$$\lim_{\beta\to 1}\frac{\exp\left(-\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\varkappa\right)-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=1-2\varkappa.$$

Thus, we obtain

$$\lim_{\beta\to 1}\int_0^{\frac{1}{2}}\left(\frac{\exp\left(-\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\varkappa\right)-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}\right)^p d\varkappa=\int_0^{\frac{1}{2}}(1-2\varkappa)^p d\varkappa=\frac{1}{2(p+1)}.$$

Corollary 12. *In Theorem 18, letting β tend to 1, then we obtain*

$$\begin{aligned}&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\\ &\leq\frac{\mathfrak{w}-\mathfrak{v}}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{g}'(\mathfrak{v})|^q+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+3|\mathfrak{g}'(\mathfrak{w})|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 13. *In Corollary 11, letting β tend to 1, then we obtain*

$$\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\frac{\mathfrak{w}-\mathfrak{v}}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{7|\mathfrak{g}'(\mathfrak{v})|^q+|\mathfrak{g}'(\mathfrak{w})|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{g}'(\mathfrak{v})|^q+7|\mathfrak{g}'(\mathfrak{w})|^q}{8}\right)^{\frac{1}{q}}\right].$$

Theorem 19. *Let $\mathfrak{g}:[\mathfrak{v},\mathfrak{w}]\to\mathbb{R}$ be a differentiable function on $[\mathfrak{v},\mathfrak{w}]$ such that $\mathfrak{g}'\in L[\mathfrak{v},\mathfrak{w}]$ with $0\leq\mathfrak{v}<\mathfrak{w}$. If $|\mathfrak{g}'|^q$ is convex where $q>1$, then we have*

$$\begin{aligned}&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\ &\leq(\mathfrak{w}-\mathfrak{v})\left(\frac{2-2\exp\left(-\frac{1}{2}A\right)-A\exp\left(-\frac{1}{2}A\right)}{4A\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\right)^{1-\frac{1}{q}}\\ &\times\left[\left(\frac{8(A-1)-\left(3A^2+4A-8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{v})|^q+\frac{8-\left(A^2+4A+8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{8-\left(A^2+4A+8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+\frac{8(A-1)-\left(3A^2+4A-8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{w})|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. From Lemma 2, properties of modulus, power mean integral inequality and the convexity of $|\mathfrak{g}'|^q$, we have

$$\begin{aligned}&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\ &\leq\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\left(\int_0^{\frac{1}{2}}\left|\exp\left(-\frac{1}{2}A\right)-\exp(-A\varkappa)\right|d\varkappa\right)^{1-\frac{1}{q}}\right.\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_0^{\frac{1}{2}}\left|\exp\left(-\tfrac{1}{2}A\right)-\exp(-A\varkappa)\right|\left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|^q d\varkappa\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left|\exp\left(-\tfrac{1}{2}A\right)-\exp(-A(1-\varkappa))\right|d\varkappa\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{1}{2}}^{1}\left|\exp\left(-\tfrac{1}{2}A\right)-\exp(-A(1-\varkappa))\right|\left|\mathfrak{g}'((1-\varkappa)\mathfrak{v}+\varkappa\mathfrak{w})\right|^q d\varkappa\right)^{\frac{1}{q}}\Bigg]\\
\leq&\frac{\mathfrak{w}-\mathfrak{v}}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left(\int_0^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\tfrac{1}{2}A\right)\right)d\varkappa\right)^{1-\frac{1}{q}}\\
&\times\Bigg[\left(\int_0^{\frac{1}{2}}\left(\exp(-A\varkappa)-\exp\left(-\tfrac{1}{2}A\right)\right)\left((1-\varkappa)|\mathfrak{g}'(\mathfrak{v})|^q+\varkappa\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left(\exp(-A(1-\varkappa))-\exp\left(-\tfrac{1}{2}A\right)\right)\left((1-\varkappa)\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+\varkappa|\mathfrak{g}'(\mathfrak{w})|^q\right)d\varkappa\right)^{\frac{1}{q}}\Bigg]\\
=&(\mathfrak{w}-\mathfrak{v})\left(\frac{2-2\exp\left(-\frac{1}{2}A\right)-A\exp\left(-\frac{1}{2}A\right)}{4A\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\right)^{1-\frac{1}{q}}\\
&\times\Bigg[\left(\frac{8(A-1)-\left(3A^2+4A-8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{v})|^q+\frac{8-\left(A^2+4A+8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{8-\left(A^2+4A+8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left|\mathfrak{g}'\left(\tfrac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+\frac{8(A-1)-\left(3A^2+4A-8\right)\exp\left(-\frac{1}{2}A\right)}{16A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{w})|^q\right)^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where we have used (14)–(17). The proof is achieved. □

Corollary 14. *In Theorem 19, using the convexity of $|\mathfrak{g}'|^q$, we obtain*

$$
\begin{aligned}
&\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1-\beta}{2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^-}\mathfrak{g}(\mathfrak{v})+\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^+}\mathfrak{g}(\mathfrak{w})\right]\right|\\
\leq&(\mathfrak{w}-\mathfrak{v})\left(\frac{2-2\exp\left(-\frac{1}{2}A\right)-A\exp\left(-\frac{1}{2}A\right)}{4A\left(1-\exp\left(-\frac{1}{2}A\right)\right)}\right)^{1-\frac{1}{q}}\\
&\times\Bigg[\left(\frac{16A-8-\left(7A^2+12A-8\right)\exp\left(-\frac{1}{2}A\right)}{32A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{v})|^q+\frac{8-\left(A^2+4A+8\right)\exp\left(-\frac{1}{2}A\right)}{32A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{w})|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{8-\left(A^2+4A+8\right)\exp\left(-\frac{1}{2}A\right)}{32A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{v})|^q+\frac{16A-8-\left(7A^2+12A-8\right)\exp\left(-\frac{1}{2}A\right)}{32A^2\left(1-\exp\left(-\frac{1}{2}A\right)\right)}|\mathfrak{g}'(\mathfrak{w})|^q\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Remark 11. *By making β tend to 1, we find*

1. $\displaystyle\lim_{\beta\to1}\frac{2-2\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)-\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{4\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{8}.$
2. $\displaystyle\lim_{\beta\to1}\frac{8\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-1\right)-\left(3\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+4\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{16\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{5}{48}.$

3. $\lim\limits_{\beta\to1}\dfrac{8-\left(\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+4\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})+8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{16\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{48}.$
4. $\lim\limits_{\beta\to1}\dfrac{16\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-8-\left(7\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+12\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})-8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{32\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{11}{96}.$
5. $\lim\limits_{\beta\to1}\dfrac{8-\left(\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2+4\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})+8\right)\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)}{32\left(\frac{1-\beta}{\beta}(\mathfrak{w}-\mathfrak{v})\right)^2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}=\frac{1}{96}.$

Corollary 15. *In Theorem 19, letting β tend to 1, we obtain*

$$\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int\limits_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\frac{\mathfrak{w}-\mathfrak{v}}{8}\left[\left(\frac{5|\mathfrak{g}'(\mathfrak{v})|^q+\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{g}'\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)\right|^q+5|\mathfrak{g}'(\mathfrak{w})|^q}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 16. *In Corollary 14, letting β tend to 1, then we obtain*

$$\left|\frac{\mathfrak{g}(\mathfrak{v})+\mathfrak{g}(\mathfrak{w})}{2}-\frac{1}{\mathfrak{w}-\mathfrak{v}}\int\limits_{\mathfrak{v}}^{\mathfrak{w}}\mathfrak{g}(z)dz\right|\leq\frac{\mathfrak{w}-\mathfrak{v}}{8}\left[\left(\frac{11|\mathfrak{g}'(\mathfrak{v})|^q+|\mathfrak{g}'(\mathfrak{w})|^q}{12}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{g}'(\mathfrak{v})|^q+11|\mathfrak{g}'(\mathfrak{w})|^q}{12}\right)^{\frac{1}{q}}\right].$$

3. Illustrative Example

In this section, to confirm the accuracy of the established results, we provide an example with graphical representations of the three types of inequalities discussed in this work. It should be noted that the figures were generated using MATLAB 7.12.0 (R2011a) software.

Example 1. *Consider the function $\mathfrak{g}:[0,1]\to\mathbb{R}$ defined by $\mathfrak{g}(z)=e^z$. This function satisfies the conditions of our theorems, as its derivative, given by $\mathfrak{g}'(z)=e^z$, is convex on the interval $[\mathfrak{v},\mathfrak{w}]=[0,1]$.*

Let us note that for the function considered, for $\beta\in(0,1)$, we have :

$$\Omega(\beta)=\mathcal{I}^{\beta}_{\frac{1}{2}^-}\mathfrak{g}(0)+\mathcal{I}^{\beta}_{\frac{1}{2}^+}\mathfrak{g}(1)=\begin{cases}\frac{e^{\frac{2\beta-1}{2\beta}}-1}{2\beta-1}-e^{\frac{2\beta-1}{2\beta}}+e & \text{if} \quad \beta\neq\frac{1}{2}\\ \frac{1}{2}-e^{\frac{2\beta-1}{2\beta}}+e & \text{if} \quad \beta=\frac{1}{2}.\end{cases} \tag{18}$$

Now, we will apply the three main theorems from our study to the specified function.

1. *From Theorem 13, the Hermite–Hadamard inequality for the given function is expressed as*

$$e^{\frac{1}{2}}\leq\frac{1-\beta}{2\left(1-\exp\left(\frac{\beta-1}{2\beta}\right)\right)}\Omega(\beta)\leq\frac{1+e}{2},$$

where $\Omega(\beta)$ is defined as in (18). The above result is graphically represented in Figure 1.

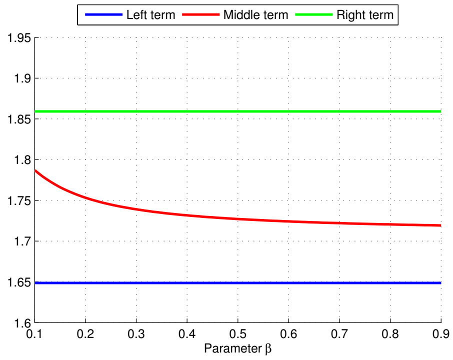

Figure 1. Hermite–Hadamard inequality.

2. *From Theorem 14, the midpoint-type inequality for the given function is given by*

$$\begin{aligned}&\left|e^{\frac{1}{2}}-\frac{1-\beta}{2\left(1-\exp\left(\frac{\beta-1}{2\beta}\right)\right)}\Omega(\beta)\right|\\ \leq&\frac{1}{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}\right)\right)}\left[(1+e)\left(\frac{19\beta^2-14\beta+3}{8(1-\beta)^2}+\frac{\beta(1-3\beta)}{2(1-\beta)^2}\exp\left(-\frac{1-\beta}{2\beta}\right)\right)\right.\\ &\left.+2e^{\frac{1}{2}}\left(\frac{(1-\beta)^2-8\beta^2}{8(1-\beta)^2}+\frac{\beta+\beta^2}{2(1-\beta)^2}\exp\left(-\frac{1-\beta}{2\beta}\right)\right)\right],\end{aligned}$$

where $\Omega(\beta)$ is defined as in (18). This result is graphically depicted in Figure 2.

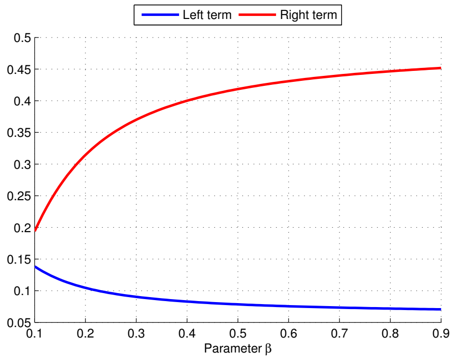

Figure 2. Midpoint-type inequality.

3. *From Theorem 17, the trapezoid-type inequality for the given function is stated as*

$$\begin{aligned}&\left|\frac{1+e}{2}-\frac{1-\beta}{2\left(1-\exp\left(\frac{\beta-1}{2\beta}\right)\right)}\Omega(\beta)\right|\\ \leq&\frac{1}{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}\right)\right)}\left[(1+e)\left(\frac{\beta-2\beta^2}{(1-\beta)^2}-\frac{3-2\beta-9\beta^2}{8(1-\beta)^2}\exp\left(-\frac{1-\beta}{2\beta}\right)\right)\right.\\ &\left.+2e^{\frac{1}{2}}\left(\frac{\beta^2}{(1-\beta)^2}-\frac{(1-\beta)^2+4\beta(1-\beta)+8\beta^2}{8(1-\beta)^2}\exp\left(-\frac{1-\beta}{2\beta}\right)\right)\right],\end{aligned}$$

where $\Omega(\beta)$ is defined as in (18). This result is illustrated graphically in Figure 3.

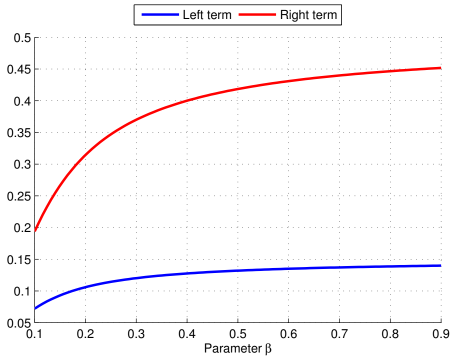

Figure 3. Trapezoid-type inequality.

The three representations shown in Figures 1–3 confirm the accuracy and precision of our results.

4. Some Applications

For arbitrary real numbers $\mathfrak{v}, \mathfrak{w}$ we have the following:

The arithmetic mean: $\mathcal{A}(\mathfrak{v}, \mathfrak{w}) = \frac{\mathfrak{v}+\mathfrak{w}}{2}$.

The logarithmic mean: $\mathcal{L}(\mathfrak{v}, \mathfrak{w}) = \begin{cases} \mathfrak{v} & \text{if} \quad \mathfrak{v} = \mathfrak{w} \\ \frac{\mathfrak{w}-\mathfrak{v}}{\ln \mathfrak{w} - \ln \mathfrak{v}} & \text{if} \quad \mathfrak{v} \neq \mathfrak{w} \end{cases}, 0 < \mathfrak{v} \leq \mathfrak{w}$.

The harmonic mean: $\mathcal{H}(\mathfrak{v}, \mathfrak{w}) = \frac{2\mathfrak{v}\mathfrak{w}}{\mathfrak{v}+\mathfrak{w}}$, $0 < \mathfrak{v} < \mathfrak{w}$.

Proposition 1. *Let* $0 < \mathfrak{v} < \mathfrak{w}$, *then we have*

$$\mathcal{A}^{-1}(\mathfrak{v}, \mathfrak{w}) \leq \frac{1-\beta}{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)} \left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}} \mathfrak{g}(\mathfrak{v}) + \mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}} \mathfrak{g}(\mathfrak{w})\right] \leq \mathcal{H}^{-1}(\mathfrak{v}, \mathfrak{w}).$$

Proof. Taking $\mathfrak{g}(z) = \frac{1}{z}$ for $z > 0$ in Theorem 17. □

Proposition 2. *Let* $0 < \mathfrak{v} < \mathfrak{w}$, *then we have*

$$\exp\{\mathcal{A}(\mathfrak{v}, \mathfrak{w})\} \leq \frac{1-\beta}{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)} \left[\mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{-}} \mathfrak{g}(\mathfrak{v}) + \mathcal{I}^{\beta}_{\left(\frac{\mathfrak{v}+\mathfrak{w}}{2}\right)^{+}} \mathfrak{g}(\mathfrak{w})\right] \leq \mathcal{A}(\exp(\mathfrak{v}), \exp(\mathfrak{w})).$$

Proof. Taking $\mathfrak{g}(z) = \exp z$ for $z > 0$ in Theorem 17. □

Proposition 3. *Let* $0 < \mathfrak{v} < \mathfrak{w}$, *then we have*

$$\begin{aligned} &\Bigg| \frac{\mathfrak{w}-\mathfrak{v}}{2\beta} \left(\mathcal{L}\left(\exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right), \exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right)\right) + 4\mathcal{L}\left(\exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right), \exp\left(\frac{1-\beta}{\beta}\mathfrak{w}\right)\right)\right) \\ &- \frac{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}{1-\beta} - \exp\left(\frac{1-\beta}{\beta}\mathcal{A}(\mathfrak{v}, \mathfrak{w})\right) \Bigg| \\ \leq &\frac{2\beta}{\beta(1-\beta)} \left(\frac{(1-\beta)(\mathfrak{w}-\mathfrak{v})-2\beta}{\beta} + 2\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right) \mathcal{A}\left(\exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right), \exp\left(\frac{1-\beta}{\beta}\mathfrak{w}\right)\right). \end{aligned}$$

Proof. Taking $\mathfrak{g}(z) = \exp\left(\frac{1-\beta}{\beta}z\right)$ for $z > 0$ in Corollary 1. □

Proposition 4. *Let* $0 < \mathfrak{v} < \mathfrak{w}$, *then we have*

$$\begin{aligned} &\Bigg| \frac{\mathfrak{w}-\mathfrak{v}}{2\beta} \mathcal{L}\left(\exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right), \exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right)\right) + 4\mathcal{L}\left(\exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right), \exp\left(\frac{1-\beta}{\beta}\mathfrak{w}\right)\right) \\ &- \frac{2\left(1-\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}{(1-\beta)} \mathcal{A}\left(\exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right), \exp\left(\frac{1-\beta}{\beta}\mathfrak{w}\right)\right) \Bigg| \end{aligned}$$

$$\leq \frac{2\left(2\beta - [(1-\beta)(\mathfrak{w}-\mathfrak{v})+2\beta]\exp\left(-\frac{1-\beta}{2\beta}(\mathfrak{w}-\mathfrak{v})\right)\right)}{\beta(1-\beta)}\mathcal{A}\left(\exp\left(\frac{1-\beta}{\beta}\mathfrak{v}\right), \exp\left(\frac{1-\beta}{\beta}\mathfrak{w}\right)\right).$$

Proof. Taking $\mathfrak{g}(z) = \exp\left(\frac{1-\beta}{\beta}z\right)$ for $z > 0$ in Corollary 9. □

5. Conclusions

This research successfully extends the realm of Hermite–Hadamard inequalities through the innovative use of fractional integral operators with exponential kernels, as initially proposed by Ahmad et al. [16]. The newly formulated inequalities enrich the existing literature on fractional calculus by linking them with classical integral inequalities and expanding their potential applications. The detailed numerical examples presented not only verify the theoretical results but also highlight the sensitivity of these inequalities to changes in the parameter β. This study offers new insights into both the fractional integrals that we investigated and classical integrals as the fractional order β tends to 1. These findings pave the way for further explorations into more complex functions and different types of fractional operators, potentially opening up new avenues for research in applied mathematics and engineering.

Author Contributions: Conceptualization: H.L., H.X., A.K., and A.L.; Formal analysis: H.L., B.M., W.S., and H.X.; Investigation: H.L., W.S., and A.L.; Methodology: B.M., H.X., and A.K.; Project administration: H.X.; Software: A.L.; Supervision: B.M., and A.K; Validation: B.M., and A.L.; Visualization: W.S., and H.X.; Writing—original draft: B.M., H.X., and A.L.; Writing—review and editing: H.L., B.M., W.S., H.X., A.K., and A.L. All authors have read and agreed to the published version of the manuscript.

Funding: The first author is supported by the Science and Technology Research Project of Jiangxi Provincial Department of Education (GJJ2201238), the Project of Jiangxi Province Education Science Planning Project in China (23YB176).

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their gratitude to the reviewers for their valuable comments and suggestions, which have significantly improved the quality of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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