



Article

Antiperiodic Solutions for Impulsive ω -Weighted ρ -Hilfer Fractional Differential Inclusions in Banach Spaces

Zainab Alsheekhussain ^{1,*} , Ahmed Gamal Ibrahim ² , M. Mossa Al-Sawalha ¹ and Osama Yusuf Ababneh ³

¹ Department of Mathematics, Faculty of Science, University of Ha'il, Hail 55476, Saudi Arabia; m.alswalha@uoh.edu.sa

² Department of Mathematics, College of Science, King Faisal University, Al Hofuf 31982, Al Ahsa, Saudi Arabia; agamal@kfu.edu.sa or agamal2000@yahoo.com

³ Department of Mathematics, College of Science, Zarqa University, Zarqa 13110, Jordan; osababneh@zu.edu.jo

* Correspondence: za.hussain@uoh.edu.sa

Abstract: In this article, we construct sufficient conditions that secure the non-emptiness and compactness of the set of antiperiodic solutions of an impulsive fractional differential inclusion involving an ω -weighted ρ -Hilfer fractional derivative, $D_{0,t}^{\sigma,\rho,\omega}$, of order $\sigma \in (1,2)$, in infinite-dimensional Banach spaces. First, we deduce the formula of antiperiodic solutions for the observed problem. Then, we give two theorems regarding the existence of these solutions. In the first, by using a fixed-point theorem for condensing multivalued functions, we show the non-emptiness and compactness of the set of antiperiodic solutions; and in the second, by applying a fixed-point theorem for contraction multivalued functions, we prove the non-emptiness of this set. Because many types of famous fractional differential operators are particular cases from the operator $D_{0,t}^{\sigma,\rho,\omega}$, our results generalize several recent results. Moreover, there are no previous studies on antiperiodic solutions for this type of fractional differential inclusion, so this work is novel and interesting. We provide two examples to illustrate and support our conclusions.

Keywords: antiperiodic solutions; instantaneous impulses; ω -weighted ρ -Hilfer fractional derivative; measure of noncompactness

MSC: 26A33; 34A08; 34A60



Citation: Alsheekhussain, Z.; Ibrahim, A.G.; Al-Sawalha, M.M.; Ababneh, O.Y. Antiperiodic Solutions for Impulsive ω -Weighted ρ -Hilfer Fractional Differential Inclusions in Banach Spaces. *Fractal Fract.* **2024**, *8*, 376. <https://doi.org/10.3390/fractalfract8070376>

Academic Editor: Gani Stamov

Received: 17 May 2024

Revised: 14 June 2024

Accepted: 18 June 2024

Published: 26 June 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Fractional differential equations and fractional differential inclusions have many applications in the real world such as in medicine [1], physics [2], engineering, life and social sciences [3], and industry [4].

There are many processes and phenomena in our real life that, during their development, are exposed to external factors' effects. If their duration is negligible compared to the total duration of the studied phenomena and processes, it can be assumed that these external influences are instantaneous. If the external influences continue for a period of time, it is called noninstantaneous. An example of instantaneous impulsive motion is the motion of an elastic ball bouncing perpendicularly on a surface. The moments of the impulses are in the time when the ball touches the surface and its velocity changes rapidly. An example of movement with noninstantaneous impulses is the entry of drugs into the bloodstream and their subsequent absorption into the body, which is a gradual and continuous process. Other examples include the following [5]: the operation of a damper subjected to percussive effects; the change in a valve's shutter speed in its transition from an open to a closed state; fluctuations in a pendulum system in the case of external impulsive effects; a percussive model of a clock mechanism; percussive systems with vibrations; relaxation oscillations in electromechanical systems; electronic schemes; a remittent oscillator subjected to impulsive effects; the dynamics of a system with automatic

regulation; the passage of a solid body from a given fluid density to another fluid density; control of a satellite's orbit using the radial acceleration; a change in the speed of a chemical reaction with the addition or removal of a catalyst; disturbances in cellular neural networks; impulsive external intervention and optimization problems in the dynamics of isolated populations; death in populations as a result of impulsive effects; and impulsive external interference and the optimization problem in population dynamics of predator–prey types. For other applications in different fields in our lives, we refer the reader to Refs. [6–9]. In Refs. [10–13], recent results on different types of impulse differential equations and inclusions are presented.

The obtained results in Refs. [14–17] confirm that there are no periodic solutions for a large number of differential equations and differential inclusions with fractional order on bounded intervals. As a result, scientists have an increased interest in studying the conditions that guarantee the existence of antiperiodic solutions to fractional differential equations. Additionally, fractional differential equations and inclusions with antiperiodic solutions have applications in chemical engineering, population dynamics, and the study of blood flow and ground water [18–20]. Ahmad et al. [21] initiated the study of the existence of antiperiodic solutions for fractional differential equations by considering the following problem:

$$\begin{cases} {}^c D_{0,\zeta}^\theta x(\zeta) = \zeta(\zeta, x(\zeta)), a.e. \zeta \in \mathcal{L} = [0, T], 1 < \theta < 2 \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, \end{cases}$$

where ${}^c D_{0,\zeta}^\theta$ is the Caputo derivative of order θ with the lower limit at 0, $\zeta : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}$. Ahmad et al. [22] demonstrated the presence of antiperiodic solutions for the following problem:

$$\begin{cases} {}^c D_{0,\zeta}^\theta x(\zeta) = \zeta(\zeta, x(\zeta)), a.e. \zeta \in \mathcal{L} = [0, T], \\ x(\zeta_i^+) - x(\zeta_i^-) = I_i(x(\zeta_i^-)), x^{(1)}(\zeta_i^+) - x^{(1)}(\zeta_i^-) = J_i(x(\zeta_i^-)), i \in L_1, \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, \end{cases}$$

where $\zeta : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}$. Chai et al. [23] examined the antiperiodic solutions for the problem following:

$$\begin{cases} {}^R D_{0,\zeta}^\theta x(\zeta) = \zeta(\zeta, x(\zeta), x^{(1)}(\zeta)), \zeta \in (0, 1], \\ \lim_{\zeta \rightarrow 0} \zeta^{2-\alpha} x(\zeta) = -\lim_{\zeta \rightarrow 1} \zeta^{2-\alpha} x(\zeta), \\ \lim_{\zeta \rightarrow 0} \frac{d}{d\zeta} (\zeta^{2-\alpha} x(\zeta)) = -\lim_{\zeta \rightarrow 1} \frac{d}{d\zeta} (\zeta^{2-\alpha} x(\zeta)), \end{cases}$$

where ${}^R D_{0,\zeta}^\theta$ is the Riemann–Liouville fractional derivative of order θ with the lower limit at 0 and $\zeta : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Ibrahim [24] showed the existence of an antiperiodic solution for the following impulsive problem:

$$\begin{cases} {}^c D_{\xi_i, \zeta}^\theta x(\zeta) \in F(\zeta, x(\zeta)), \zeta \in \mathcal{L}_i, i \in L_0 \\ x(\zeta_i^+) = x(\zeta_i^-) + \mathfrak{R}_i(x(\zeta_i^-)), i \in L_1 \\ x^{(1)}(\zeta_i^+) = x^{(1)}(\zeta_i^-) + \mathfrak{R}_i(x(\zeta_i^-)), i \in L_1 \\ x(0) = -x(T), x^{(1)}(0) = -x^{(1)}(T). \end{cases} \quad (1)$$

Wang et al. [25] established various existence results of antiperiodic solutions for the following fractional differential equations and inclusions:

$$\begin{cases} {}^c D_{0,\zeta}^q x(\zeta) = \zeta(\zeta, x(\zeta)), \zeta \in \mathcal{L} = [0, T], q \in (n-1, n), \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, \dots, n-1. \end{cases}$$

and

$$\begin{cases} {}^c D_{0,\zeta}^q x(\zeta) \in F(\zeta, x(\zeta)), a.e. \zeta \in \mathcal{L}, \\ x^{(k)}(0) = -x^{(k)}(T), k = 0, 1, 2, \dots, n-1. \end{cases}$$

where ${}^c D_{0,\zeta}^q$ is the generalized Caputo derivative of order $q \in (n-1, n)$ with the lower limit at zero, and $n \in \mathbb{N}$, $\zeta: \mathcal{L} \times E \rightarrow E$ and $F: \mathcal{L} \times E \rightarrow 2^E$ is a multifunction.

Hristova et al. [26] studied the presence of antiperiodic solutions for the following problem:

$$\begin{cases} {}^R D_{a,\zeta}^{\alpha,\rho,\varrho} u(\zeta) = h(\zeta, u(\zeta)), \zeta \in [a, T], \rho, \alpha \in (0, 1) \\ I_{a,\zeta}^{1-\alpha,\rho,\varrho} u(a) = -I_{a,\zeta}^{1-\alpha,\rho,\varrho} u(T), \end{cases}$$

where ${}^R D_{a,\zeta}^{\alpha,\rho,\varrho}$ denotes the generalized proportional fractional derivatives of the Riemann–Liouville type of order α and $h: [a, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Boutiara et al. [27] presented enough assumptions for the existence and uniqueness of antiperiodic solutions for the following problem:

$$\begin{cases} {}^{CH} D_{1,\zeta}^r x(\zeta) = \zeta(\zeta, x(\zeta)), \zeta \in [1, \zeta], r \in (0, 1), \\ ax(1) + cx(\zeta) = \gamma^H I^q x(\eta) + \delta, q, \eta \in (0, 1), \end{cases}$$

where ${}^{CH} D_{1,\zeta}^r$ is the Caputo–Hadamard derivative, $a, c, \gamma \in \mathbb{R}$, and $f: [1, \zeta] \times \mathbb{R} \rightarrow \mathbb{R}$.

Redhwan et al. [28] proved the presence and uniqueness of an antiperiodic solution for the following problem:

$$\begin{cases} {}^{CK} D_{a,\zeta}^{\varsigma,\rho} x(\zeta) = \zeta(\zeta, x(\zeta), {}^{CK} D_{0,\zeta}^{\varsigma,\rho} x(\zeta)), \zeta \in [a, T], \varsigma \in (1, 2), \\ x(a) = -x(T), x^{(1)}(0) = -x^{(1)}(\zeta), \end{cases}$$

where ${}^{CK} D_{a,\zeta}^{\varsigma,\rho}$ is the Caputo–Katugampola fractional derivative of order with the lower limit at a , $\rho \in (0, 1)$ and $\zeta: [a, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

Boutiara et al. [29] proved the presence of antiperiodic solutions for the following boundary-value problem of a nonlinear Hilfer fractional differential equation:

$$\begin{cases} D_{0,\zeta}^{\alpha,\beta} x(\zeta) = \zeta(\zeta, x(\zeta)), \zeta \in (0, T], \alpha, \beta \in (0, 1), \\ \gamma I_{0,\zeta}^{1-\gamma} x(0) + \kappa x(T) = \sum_{i=1}^{i=n} c_i p_i I^{q_i} x(\eta_i), \end{cases}$$

where $D_{0,\zeta}^{\alpha,\beta}$ is the Hilfer fractional differential operator of order α and of type β with the lower limit at 0; $\gamma = \alpha + \beta - \alpha\beta$; $p_i I^{q_i} x(\eta_i)$ is the Katugampola integral; $\zeta: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$; and $a, T, d, c_i, i = 1, \dots, n$ ($n \in \mathbb{N}$) are real constants.

Yang et al. [30] investigated the presence of antiperiodic solutions for the following inclusion problem:

$$\begin{cases} {}^c D_{a,\zeta}^{\theta,\varrho} x(\zeta) \in \zeta(\varrho, x(\varrho)), \zeta \in [a, d], \\ x(a) = x(d), x^{(1)}(a) = x^{(1)}(d), \end{cases} \quad (2)$$

Alruwaily et al. [31] presented a boundary-value problem containing a different order of Caputo derivatives with coupled antiperiodic conditions.

Now, it is recognized that the ω -weighted ϱ -Hilfer fractional derivative generalizes many well-known fractional derivatives such as the Riemann–Liouville, Caputo, Hilfer, Hadamard, Caputo–Katugampola fractional, Hilfer–Hadamard, Hilfer–Katugampola, ϱ -Riemann–Liouville, ϱ -Caputo, and ϱ -Hilfer derivatives.

Motivated by the above discussion, and in order to undertake work that generalizes the majority of the above results and allows the study of the existence of antiperiodic solutions to several boundary-value problems containing one of the fractional derivatives known in the literature as Riemann–Liouville, Caputo, Hilfer, Hadamard, Katugampola, Hilfer–Hadamard, Hilfer–Katugambula, ϱ -Riemann–Liouville, ϱ -Caputo, and ϱ -Hilfer, we prove the existence of antiperiodic solutions of differential inclusions involving the ω -weight ϱ -Hilfer fractional derivative. Furthermore, we consider in our problem the

influence of instantaneous pulses in infinite-dimensional Banach spaces, and this makes our work very interesting.

Indeed, in this work, and for the first time, we prove the existence of antiperiodic solutions for the following differential inclusion involving the ω -weight ϱ -Hilfer fractional derivative in the presence of instantaneous impulses:

$$\left\{ \begin{array}{l} D_{\xi_i, \xi}^{\theta, v, \varrho, \omega} x(\xi) \in F(\xi, x(\xi)), \xi \in \mathcal{L}_i, i \in L_0, \\ \lim_{\xi \rightarrow \xi_i^+} \omega(\xi) I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \\ = \lim_{\xi \rightarrow \xi_i^-} \omega(\xi) I_{\xi_{i-1}, \xi}^{2-\lambda, \varrho, \omega} x(\xi) + \mathfrak{N}_i(x(\xi_i^-)), i \in L_1, \\ \lim_{\xi \rightarrow \xi_i^+} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\ = \lim_{\xi \rightarrow \xi_i^-} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) + \mathfrak{R}_i(x(\xi_i^-)), i \in L_1. \\ \lim_{\xi \rightarrow 0} \omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) = - \lim_{\xi \rightarrow T^-} \omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi), \\ \lim_{\xi \rightarrow 0} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) = - \lim_{\xi \rightarrow T^-} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right), \end{array} \right. \quad (3)$$

where $\mathcal{L} = [0, T]$, E is a Banach space, $T > 0$, $\mathbb{N}$ is the set of natural numbers and $m \in \mathbb{N}$ is given, $\theta \in (1, 2)$, $v \in [0, 1]$, $\lambda = \theta + 2v - \theta v$, $\xi_0 = 0 \leq \xi_1 \leq \xi_2 < \dots \leq \xi_m \leq \xi_{m+1} = T$ is a partition for $\mathcal{L}$, $L_0 = \{0, 1, \dots, m\}$, $L_1 = \{1, \dots, m\}$, $\omega : \mathcal{L} \rightarrow]0, \infty[$ and $\omega^{-1}(\theta) = \frac{1}{\omega(\theta)}$, $\varrho : \mathcal{L} \rightarrow \mathbb{R}$ is a strictly increasing continuously differentiable function with $\varrho'(\theta) \neq 0$, $\forall \theta \in \mathcal{L}$ and ϱ^{-1} is its inverse, and $D_{\xi_i, \xi}^{\theta, v, \varrho, \omega}$ is the ω -weighted ϱ -Hilfer derivative operator of order θ and of type v with the lower limit at ξ_i . Moreover, $F : \mathcal{L} \times E \rightarrow P_{ck}(E)$ (the family of non-empty convex compact subsets of E), $\mathfrak{N}_i, \mathfrak{R}_i : E \rightarrow E; i \in L_1$, $\mathcal{L}_i = (\xi_i, \xi_{i+1}]; i \in L_0$, and $I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega}$ is the ω -weighted ϱ -Riemann–Liouville derivative operator of order $2 - \lambda$ with the lower limit at ξ_i . First, we deduce the formula for antiperiodic solutions to Problem (3) (Lemma 9). Then, we prove two theorems regarding the existence of antiperiodic solutions to (3) (Theorem 1 and Theorem 2). In Theorem 1, we show that the set of antiperiodic solutions of (3) is non-empty and compact, while in Theorem 2, we assume less restrictive conditions and use the fixed-point theorem for the contraction of multivalued functions to show that the set of antiperiodic solutions of (3) is not empty.

Here, we mention some other studies on antiperiodic solutions of fractional differential equations and differential inclusions. Benyoub et al. [32] discussed the existence and uniqueness of solutions for a nonlinear antiperiodic boundary-value problem for fractional impulsive differential equations:

$$\left\{ \begin{array}{l} {}^{CF}D_{0, \xi}^{\alpha} x(\xi) = \zeta(\xi, x(\xi)), \xi \in (\xi_i, \xi_{i+1}], i = 0, 1, \dots, m, \alpha \in (0, 1). \\ \lim_{\xi \rightarrow \xi_i^+} x(\xi) = \lim_{\xi \rightarrow \xi_i^-} x(\xi) + K_i(x(\xi_i^-)), i \in 1, 2, \dots, m, \\ \lim_{\xi \rightarrow 0} x(\xi) = - \lim_{\xi \rightarrow T^-} x(\xi), \end{array} \right.$$

where ${}^{CF}D_{0, \xi}^{\alpha}$ is the Caputo–Fabrizio fractional derivative of order $\alpha \in (0, 1)$ and $\zeta : [0, \xi] \times \mathbb{R} \rightarrow \mathbb{R}$.

Yang et al. [33] studied the antiperiodic solutions for a differential equation involving the Riesz–Caputo derivative of order $\theta \in (1, 2)$.

In a very recent paper, Al Nuwairan et al. [34] derived the sufficient conditions for the existence of an antiperiodic solution to the following fractional differential inclusion with instantaneous impulses:

$$\begin{cases} {}^{ATC}D_{0,\xi}^\alpha x(\xi) \in F(\xi, x(\xi)), \xi \in J - \{\xi_1, \xi_2, \dots, \xi_m\}, \\ x(0) = -x(T), \frac{d}{d\xi} x(0) = -\frac{d}{d\xi} x(T), \\ x(\xi_i^+) = x(\xi_i^-) + I_i(x(\xi_i^-)), i \in L_1, \\ \frac{d}{d\xi} x(\xi_i^+) = \frac{d}{d\xi} x(\xi_i^-) + \bar{I}_i(x(\xi_i^-)), i \in L_1, \end{cases}$$

where ${}^{ATC}D_{0,\xi}^\alpha u(\xi)$ is the Atangana–Baleanu fractional derivative of order $\alpha \in (1, 2)$, and $F : \mathcal{L} \times \mathcal{L} \rightarrow P_{ck}(E)$ is a multivalued function satisfying $F(0, u_0) = \{0\}$.

It is worth noting that in a recent paper, Alsheekhussain et al. [35] proved the existence of a solution for the following fractional differential inclusion:

$$\begin{cases} D_{\kappa_i, \xi}^{\theta, v, \varrho, \omega} x(\xi) \in F(\xi, x(\xi)), a.e., \xi \in (\kappa_i, \xi_{i+1}], i \in L_0, \\ x(\xi) = g_i(\xi, x(\kappa_i^-)), \xi \in (\xi_i, \kappa_i], i \in L_1, \\ \lim_{\xi \rightarrow 0^+} I_{0, \xi}^{2-\lambda, \varrho, \omega} \omega(\xi) x(\xi) = x_0, \lim_{\xi \rightarrow 0^+} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(I_{0, \xi}^{2-\lambda, \varrho, \omega} \omega(\xi) x(\xi) \right) = x_1, \\ \lim_{\xi \rightarrow \kappa_i^-} \left(\omega(\xi) I_{0, \xi}^{2-\lambda} x(\xi) \right) = g_i(\kappa_i, x(\kappa_i^-)), i \in L_1, \\ \lim_{\xi \rightarrow \kappa_i^-} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{0, \xi}^{2-\lambda} x(\xi) \right) = g_i^*(\kappa_i, x(\kappa_i^-)), i \in L_1, \end{cases} \quad (4)$$

where $\kappa_0 = 0 < \xi_1 \leq \kappa_1 < \xi_2 \leq \kappa_2 < \dots \leq \kappa_m \leq \xi_{m+1} = T$ and $g_i, g_i^* : [0, T] \times E \rightarrow E$. For other publications on antiperiodic solutions and differential inclusions involving Hilfer fractional derivatives, we refer to [35–42].

In the following remark, we explain the importance of this study and its relationship to some of the aforementioned results.

Remark 1. 1—Following the same technique that we use to prove the existence of antiperiodic solutions to (3), we can prove the existence of antiperiodic solutions to the following problem:

$$\begin{cases} D_{0, \xi}^{\theta, v, \varrho, \omega} x(\xi) \in F(\xi, x(\xi)), \xi \in \mathcal{L}_i, i \in L_0, \\ \lim_{\xi \rightarrow \xi_i^-} \omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \\ = \lim_{\xi \rightarrow \xi_i^-} \omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) + \aleph_i(x(\xi_i^-)), i \in L_1, \\ \lim_{\xi \rightarrow \xi_i^+} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\ = \lim_{\xi \rightarrow \xi_i^+} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) + \aleph_i(x(\xi_i^-)), i \in L_1. \\ \lim_{\xi \rightarrow 0} \omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) = - \lim_{\xi \rightarrow T^-} \omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi), \\ \lim_{\xi \rightarrow 0} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) = - \lim_{\xi \rightarrow T^-} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right), \end{cases} \quad (5)$$

2—In Problem (3), the lower limit of the differential operator $D_{\xi_i, \xi}^{\theta, v, \varrho, \omega}$ is ξ_i , $i \in L_0$. That is, the lower limit changes on each $\mathcal{L}_i$. But in Problem (5), the lower limit of the differential operator $D_{0, \xi}^{\theta, v, \varrho, \omega}$ is at 0, which is fixed at 0 on each $\mathcal{L}_i$.

3—Ahmad et al. [21] studied Problem (3) in the particular cases $\omega(\xi) = 1$, $\varrho(\xi) = \xi$; $\in \mathcal{L}$, $v = 1$, $\aleph_i$ and $\aleph_i$ are the zero functions, and $\dim(E) = 1$.

4—Ahmad et al. [22] studied (3) in the particular cases $\omega(\xi) = 1$, $\varrho(\xi) = \xi$; $\in \mathcal{L}$, $v = 1$, and $\dim(E) = 1$.

5—In the special cases $\omega(\xi) = 1$, $\varrho(\xi) = \xi$; $\in \mathcal{L}$, $v = 0$, $\aleph_i$ and $\aleph_i$ are zero functions, and $\dim(E) = 1$, Problem (3) is studied in [23].

6—If we set $\omega(\xi) = 1$, $\varrho(\xi) = \xi$; $\in \mathcal{L}$, and $v = 1$, then $D_{\xi_i, \xi}^{\theta, v, \varrho, \omega} x(\xi) = D_{\xi_i, \xi}^c x(\xi)$, $\lambda = 2$, and (3) becomes Problem 1 studied in (Theorem 25, Ibrahim [24]).

7—If we set $\omega(\xi) = 1; \in \mathcal{L}$ and $v = 1$, then $D_{\xi, \xi}^{\theta, v, \rho, \omega} x(\xi) = D_{\xi, \xi}^{\theta} x(\xi)$, $\lambda = 2$, and (3) reduces to Problem (2) studied by [30].

8—In Problem (3), the impulses are instantaneous, but in Problem (4) they are noninstantaneous. Also, there are no antiperiodic boundary conditions in problem (4) like (3). Therefore, the problem we are studying is completely different from Problem (4).

Next, we illustrate the relationship between our work and relevant recent findings, along with our work's main contributions.

I—A new type of fractional differential inclusions, Problem (3), involving the ω -weighted ρ -Hilfer differential operator $D_{\xi, \xi}^{\theta, v, \rho, \omega}$ of order $\theta \in (1, 2)$ and of type v , which contains the fractional derivatives operators known in the literature by Riemann–Léouville, Caputo, Hilfer, Hadamard, Katugambula, Hilfer–Hadamard, Hilfer–Katugambula, ρ -Riemann–Liouville, ρ -Caputo, and ρ -Hilfer, is considered in the existence of instantaneous impulses and in infinite-dimensional Banach spaces.

II—The formula for the antiperiodic solutions of (3) in E is deduced (Lemma 9).

III—Two consequences of the existence of antiperiodic solutions to (3) are proven (Theorems 1 and 2).

IV—Two examples are provided to show that our results are viable (Example 1).

V—Our work generalizes the obtained results in [21–24] (Remark 1).

VI—Up to now there, has been no study about the existence of antiperiodic solutions for differential inclusions involving an ω -weighted ρ -Hilfer derivative in the presence or absence of impulses.

VII—Our method guides those interested in generalizing the majority of the above works when the nonlinear term is a multivalued function in the presence of pulses and in any Banach space.

VIII—By following our technique, one can demonstrate the existence of antiperiodic solutions for Problem (5).

IX—As a result of our work, one can study the existence of antiperiodic solutions to several boundary-value problems containing one of the fractional derivatives known in the literature by the following names: Riemann–Léouville, Caputo, Hilfer, Hadamard, Katugambula, Hilfer–Hadamard, Hilfer–Katugambula, ρ -Riemann–Liouville, ρ -Caputo, and ρ -Hilfer. Therefore, one can generalize the majority of the above works after replacing the differential operator presented in the issues discussed in these works by $D_{\xi, \xi}^{\theta, v, \rho, \omega}$. This can be considered as a proposal for future research work based on this work.

We organize our work as follows. In the second section, we remind the reader of the concepts that will be used later. In the third part, we deduce the formula for antiperiodic solutions of (3), and then, we proved two conclusions regarding the existence of these solutions. Finally, we give two examples to show that our results are viable.

2. Preliminaries and Notations

Let $AC(\mathcal{L}, E)$ be the Banach space of absolutely continuous functions from $\mathcal{L}$ to E and $L_{\omega}^{p, \rho}((0, T), E)$, $p \geq 1$ be the Banach space of all Lebesgue measurable functions u such that $u\omega(\rho')^{\frac{1}{p}} \in L^p((a, T), E)$ when $p \in [1, \infty)$, and $u\omega \in L^{\infty}((0, T), E)$ when $p = \infty$, where

$$\|u\|_{L_{\omega}^{p, \rho}((0, T), E)} := \begin{cases} (\int_0^T \|u(\tau)\omega(\tau)\|^p \rho'(\tau) d\tau)^{\frac{1}{p}}, & \text{if } p \in [1, \infty), \\ \|u\omega\|_{L^{\infty}((0, T), E)}, & \text{if } p = \infty. \end{cases}$$

We consider the Banach following spaces:

- $C_{\omega}(\mathcal{L}, E) := \{x \in C(\mathcal{L}, E) : \omega x \in C([0, T], E)\}$, where $\|x\| = \max_{s \in \mathcal{L}} \|\omega(s)x(s)\|$.
- $C_{2-\lambda, \rho, \omega}(\mathcal{L}, E) = \left\{x \in C((0, T], E) : (\rho(\cdot) - \rho(0))^{2-\lambda} x(\cdot) \in C_{\omega}([a, T], E)\right\}$, where

$$\|x\|_{C_{2-\lambda, \rho, \omega}(\mathcal{L}, E)} := \sup_{\xi \in \mathcal{L}} \|(\rho(\xi) - \rho(0))^{2-\lambda} \omega(\xi)x(\xi)\|.$$

- $PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E) := \{x : [0, T] \rightarrow E, (\varrho(\cdot) - \varrho(\xi_k))^{2-\lambda} x(\cdot) \in C_\omega(\mathcal{E}_k, E), \lim_{\theta \rightarrow \xi_k^+} (\varrho(\theta) - \varrho(\xi_k))^{2-\lambda} x(\theta) \text{ exists}, k \in L_0\}$, where

$$\|x\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} := \max \left\{ \sup_{\substack{\theta \in \mathcal{E}_k \\ k \in L_0}} (\varrho(\theta) - \varrho(\theta_k))^{2-\lambda} \|\omega(\theta)x(\theta)\|_E \right\},$$

- $AC^{1, \omega}(\mathcal{E}, E) := \{x : [a, T] \rightarrow E, x\omega \in AC(\mathcal{E}, E)\}$, where $\|x\|_{AC^{1, \omega}(\mathcal{E}, E)} = \|x\omega\|_{AC(\mathcal{E}, E)}$.
- $C_{2-\mu, \varrho, \omega}^1(\mathcal{E}, E) := \{x \in C_{2-\mu, \Phi, \omega}(\mathcal{E}, E) : D^{1, \varrho, \omega} x \in C_{2-\mu, \varrho, \omega}(\mathcal{E}, E)\}$.
- $C_{2-\lambda, \varrho, \omega}^n(\mathcal{E}, E) := \{x \in C_{2-\lambda, \Phi, \omega}^{n-1}(\mathcal{E}, E) : D^{n, \varrho, \omega} x \in C_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)\}, n \in \mathbb{N} - \{1\}$, where

$$\|x\|_{C_{2-\mu, \varrho, \omega}^n(\mathcal{E}, E)} := \sum_{k=1}^{n-1} \|D^{k, \varrho, \omega} x\|_{C_{2-\mu, \Phi, \omega}^k(\mathcal{E}, E)} + \|D^{n, \varrho, \omega} x\|_{C_{2-\mu, \varrho, \omega}(\mathcal{E}, E)}.$$

- $AC^{n, \varrho, \omega}(\mathcal{E}, E) := \{x : \mathcal{E} \rightarrow E, D^{n-1, \varrho, \omega} x \in AC^{1, \omega}(\mathcal{E}, E), D^{n, \varrho, \omega} x \in L_\omega^{1, \varrho}((0, T), E)\}; n \in \mathbb{N} - \{1\}$, where $\|x\|_{AC^{n, \varrho, \omega}(\mathcal{E}, E)} := \|x_{n-1, \varrho, \omega}\|_{AC(\mathcal{E}, E)}$.

The function $\chi_{PC_{2-\lambda, \Phi, \omega}(\mathcal{E}, E)} : P_T(PC_{2-\lambda, \Phi, \omega}(\mathcal{E}, E)) \rightarrow [0, \infty)$ given by

$$\chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(D) := \max_{k \in N_0} \chi_{C(\mathcal{E}_k, E)} \left(D_{|\mathcal{E}_k} \right),$$

is a measure of noncompactness on $PC_{2-\lambda, \Phi, \omega}(\mathcal{E}, E)$, where

$$D_{|\mathcal{E}_k} := \left\{ h^* \in C(\mathcal{E}_k, E) : h^*(\theta) = (\Phi(\theta) - \Phi(\theta_k))^{2-\lambda} \omega(\theta) h(\theta), \theta \in \mathcal{E}_k, \right. \\ \left. h^*(\theta_k) = \lim_{\theta \rightarrow \theta_k^+} h^*(\theta), h \in D \right\}.$$

Definition 1 ([43]). Let $\alpha > 0$ and $u \in L_\omega^{p, \varrho}(\mathcal{E}, E)$. The operator $I_{0, \xi}^{\alpha, \varrho, \omega} u$ is given by

$$I_{0, \xi}^{\alpha, \varrho, \omega} u(\xi) := \frac{\omega^{-1}(\xi)}{\Gamma(\alpha)} \int_0^\xi (\varrho(\xi) - \varrho(\sigma))^{\alpha-1} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma.$$

Lemma 1 ([43], Theorem 2.4). If $u \in L_\omega^{p, \varrho}(\mathcal{E}, E), 1 \leq p \leq \infty, \alpha > 0$, and $\beta > 0$, then $I_{0, \xi}^{\alpha, \varrho, \omega} I_{0, \xi}^{\beta, \varrho, \omega} u = I_{0, \xi}^{\alpha+\beta, \varrho, \omega} u$.

Definition 2 ([43]). Let $n \in \mathbb{N}, \alpha \in (n-1, n)$, and $u \in AC^{n, \varrho, \omega}(\mathcal{E}, E)$. The operator $D_{a, \xi}^{\alpha, \varrho, \omega} u(\xi)$ is defined by

$$D_{a, \xi}^{\alpha, \varrho, \omega} u(\xi) := D_\xi^{n, \varrho, \omega} \left(I_{a, \xi}^{n-\alpha, \varrho, \omega} u \right) (\xi) \\ = \frac{\omega^{-1}(\xi)}{\Gamma(n-\alpha)} D_\xi^{n, \varrho, \omega} \left(\int_a^\xi (\varrho(\xi) - \varrho(\sigma))^{n-\alpha-1} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma, \xi \geq a, \right.$$

given that the right-hand side is well defined.

Lemma 2 ([43], Proposition 1.3). i—If $\delta > 0$ and $\kappa > 0$, then

$$I_{a, \xi}^{\delta, \varrho, \omega} (\omega^{-1}(\xi) (\varrho(\xi) - \varrho(a))^{\kappa-1}) \\ = \frac{\Gamma(\kappa)}{\Gamma(\kappa + \delta)} \omega^{-1}(\xi) (\varrho(\xi) - \varrho(a))^{\kappa+\delta-1}; \forall \xi \in [a, T].$$

ii—If $n - 1 < \alpha < n$, then

$$D_{a,\xi}^{\alpha,\varrho,\omega}(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(a))^{\alpha-k} = 0, \forall k = 1, 2, \dots, n.$$

Lemma 3 ([35]). If $u \in C_{2-\lambda,\varrho,\omega}([a, T], E)$, then $\lim_{\xi \rightarrow a} \|\omega(\xi) I_{a,\xi}^{1-v(2-\theta),\varrho,\omega} u(\xi)\| = 0$.

The proof of the following properties can be derived using the same arguments as in the case $E = \mathbb{R}$ (Ref. [43], Theorems 3.3, 3.4, 3.5, and 3.6).

Lemma 4. Let $u \in L_{\omega}^{p,\varrho}([a, T], E)$. If $\alpha \in (n - 1, n)$ and $I_{a,\xi}^{n-\alpha,\varrho,\omega} u \in AC^{n,\varrho,\omega}([a, T], E)$, then for any $\xi \in [a, T]$,

$$\begin{aligned} & I_{a,\xi}^{\alpha,\varrho,\omega} \left(D_{a,\xi}^{\alpha,\varrho,\omega} u \right) (\xi) \\ &= u(\xi) - \omega^{-1}(\xi) \sum_{k=1}^n \frac{(\varrho(\xi) - \varrho(a))^{\alpha-k}}{\Gamma(\alpha - k + 1)} \lim_{\xi \rightarrow a} \left(\frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \right)^{n-k} (\omega(\xi) I^{n-\alpha} u(\xi)). \end{aligned}$$

Definition 3 ([43]). Let $n \in \mathbb{N}$, $\alpha \in]n - 1, n[$ and $u \in AC^{n,\varrho,\omega}(\mathcal{L}, E)$. The operator ${}^c D_{a,\xi}^{\alpha,\varrho,\omega} u$ is defined as

$${}^c D_{a,\xi}^{\alpha,\varrho,\omega} u(\xi) := D_{\xi}^{\alpha,\varrho,\omega} \left(u(\xi) - \sum_{k=0}^{k=n-1} \frac{(\varrho(\xi) - \varrho(a))^k}{k!} u_{k,\varrho,\omega}(0) \right).$$

Definition 4 ([35]). Let $u : [a, T] \rightarrow E$. The operator $D_{a,\theta}^{\theta,v,\varrho,\omega}$ is defined by

$$\begin{aligned} D_{a,\xi}^{\theta,v,\varrho,\omega} u(\xi) &:= I_{a,\xi}^{v(2-\theta)} D_{\xi}^{2,\varrho,\omega} I_{a,\xi}^{(2-\theta)(1-v),\varrho,\omega} u(\xi) \\ &= I_{a,\xi}^{v(2-\theta)} D_{\xi}^{2,\varrho,\omega} I_{a,\xi}^{2-\lambda,\varrho,\omega} u(\xi) \\ &= I_{a,\xi}^{v(2-\theta)} D_{a,\xi}^{\lambda,\varrho,\omega} u(\xi), \end{aligned} \quad (6)$$

as long as the right-hand side is well defined.

Remark 2. If $v = 0$, then $D_{a,\xi}^{\theta,v,\varrho,\omega} u(\xi) = D_{a,\xi}^{\theta,\varrho,\omega} u(\xi)$, while if $v = 1$, then $D_{a,\xi}^{\theta,v,\varrho,\omega} u(\xi) = {}^c D_{a,\xi}^{\theta,\varrho,\omega} u(\xi)$.

Let us consider the following Banach spaces:

$$C_{2-\lambda,\varrho,\omega}^{\lambda}(\mathcal{L}, E) := \left\{ x \in C_{2-\lambda,\varrho,\omega}(\mathcal{L}, E), D_{0,\xi}^{\lambda,\varrho,\omega} x \in C_{2-\lambda,\varrho,\omega}(\mathcal{L}, E) \right\}.$$

and

$$C_{2-\lambda,\varrho,\omega}^{\theta,\nu}(\mathcal{L}, E) := \left\{ x \in C_{2-\lambda,\varrho,\omega}(\mathcal{L}, E), D_{0,\xi}^{\theta,\nu,\varrho,\omega} x \in C_{2-\theta,\varrho}(\mathcal{L}, E) \right\}.$$

Remark 3. If $u \in C_{2-\lambda,\varrho,\omega}^{\lambda}([a, T], E)$, then $D_{a,\xi}^{\lambda,\varrho,\omega} u(\xi)$ exists, and consequently, by (6), $D_{a,\xi}^{\theta,v,\varrho,\omega} u(\xi)$ exists. Therefore, $C_{2-\lambda,\varrho,\omega}^{\lambda}([a, T], E) \subseteq C_{2-\lambda,\varrho,\omega}^{\theta,\nu}([a, T], E)$.

Definition 5. A function $x \in PC_{2-\lambda,\varrho,\omega}(\mathcal{L}, E)$ is a solution for Problem (3) if $D_{\xi_k,\xi}^{\theta,v,\varrho,\omega} x(\xi)$ exists for $\xi \in (\xi_k, \xi_{k+1}]$; $k \in L_0$, and x verifies (3).

Lemma 5. Let $u \in C_{2-\lambda, \varrho, \omega}^{v(2-\theta)}([a, T], E)$, c, d are two fixed points in E , and $x : (a, T] \rightarrow E$ defined by

$$x(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(a))^{\lambda-2}}{\Gamma(\lambda-1)} c + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(a))^{\lambda-1}}{\Gamma(\lambda)} d \\ + I_{a, \xi}^{\theta, \varrho, \omega} u(\xi), \xi \in (a, T]. \end{cases} \quad (7)$$

Then $x \in C_{2-\lambda, \varrho, \omega}^{\lambda}([a, T], E)$, $D_{\xi, \xi}^{\theta, v, \varrho, \omega} x(\xi)$ exists for $\xi \in (a, T]$, and it is a solution for the following ω -weighted ϱ -differential equation of order θ and of type v :

$$D_{a, \xi}^{\theta, v, \varrho, \omega} x(\xi) = u(\xi), \xi \in (a, T]. \quad (8)$$

Proof. Since $1 < \theta < \lambda \leq 2$ and $v(2-\theta) < 1$, then by applying $D_{a, \xi}^{\lambda, \varrho, \omega}$ on both sides of (8) and considering (6), Lemma 1 and (ii) of Lemma 2, it follows for $\xi \in [a, T]$,

$$\begin{aligned} D_{a, \xi}^{\lambda, \varrho, \omega} x(\xi) &= D_{a, \xi}^{\lambda, \varrho, \omega} I_{a, \xi}^{\theta, \varrho, \omega} u(\xi) \\ &= D_{\xi}^{2, \varrho, \omega} I_{a, \xi}^{2-\lambda, \varrho, \omega} I_{a, \xi}^{\theta, \varrho, \omega} u(\xi) \\ &= D_{\xi}^{2, \varrho, \omega} I_{a, \xi}^{2-v(2-\theta), \varrho, \omega} u(\xi) \\ &= D_{\xi}^{1, \varrho, \omega} \left(D_{\xi}^{1, \varrho, \omega} I_{a, \xi}^{1, \varrho, \omega} I_{a, \xi}^{1-v(2-\theta), \varrho, \omega} u(\xi) \right) \\ &= D_{\xi}^{1, \varrho, \omega} I_{a, \xi}^{1-v(2-\theta), \varrho, \omega} u(\xi) \\ &= D_{a, \xi}^{v(2-\theta), \varrho, \omega} u(\xi). \end{aligned} \quad (9)$$

Since $u \in C_{2-\lambda, \varrho, \omega}^{v(2-\theta)}([a, T], E)$, $D_{a, \xi}^{v(2-\theta), \varrho, \omega} u \in C_{2-\lambda, \varrho, \omega}([a, T], E)$, and, hence, $D_{a, \xi}^{\lambda, \varrho, \omega} x \in C_{2-\lambda, \varrho, \omega}([a, T], E)$. This leads to $x \in C_{2-\lambda, \varrho, \omega}^{\lambda}([a, T], E)$. Therefore, by Remark 2, $D_{a, \xi}^{\theta, v, \varrho, \omega} x(\xi)$ exists for $\xi \in [a, T]$. Next, we have to show that $D_{a, \xi}^{\theta, v, \varrho, \omega} x(\xi) = u(\xi), \forall \xi \in [a, T]$. let $\xi \in \mathcal{L}$. Since $u \in C_{2-\lambda, \varrho, \omega}^{v(2-\theta)}([a, T], E)$, it follows that $D_{a, \xi}^{v(2-\theta), \varrho, \omega} u \in C_{2-\lambda, \varrho, \omega}([a, T], E)$, and then $D_{a, \xi}^{1, \varrho, \omega} I_{a, \xi}^{1-v(2-\theta), \varrho, \omega} u \in C_{2-\lambda, \varrho, \omega}([a, T], E)$, and hence $I_{a, \xi}^{1-v(2-\theta), \varrho, \omega} u \in C_{2-\lambda, \varrho, \omega}^1([a, T], E)$. Therefore, as a result of Lemmas 4 and 9, we obtain for $\xi \in [a, T]$,

$$\begin{aligned} D_{a, \xi}^{\theta, v, \varrho, \omega} x(\xi) &= I_{a, \xi}^{v(2-\theta)} D_{a, \xi}^{\lambda, \varrho, \omega} x(\xi) = I_{a, \xi}^{v(2-\theta)} D_{a, \xi}^{v(2-\theta), \varrho, \omega} u(\xi) \\ &= u(\xi) - \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(a))^{v(2-\theta)-1}}{\Gamma(v(2-\theta))} \lim_{\xi \rightarrow a} \left(\omega(\xi) I_{a, \xi}^{1-v(2-\theta)} u(\xi) \right). \end{aligned}$$

We use Lemma 3 to obtain $\lim_{\xi \rightarrow a} \left(\omega(\xi) I_{a, \xi}^{1-v(2-\theta)} u(\xi) \right) = 0$. Thus, $D_{a, \xi}^{\theta, v, \varrho, \omega} x(\xi) = u(\xi)$ for any $\xi \in (a, T]$. $\square$

Remark 4. The function x given in (7) is not defined at $x = a$. But if $v = 1$, then $\lambda = 2$ and x becomes defined and continuous on $[a, T]$. Moreover, $D_{0, \xi}^{\theta, v, \varrho, \omega} x(\xi) = {}^c D_{0, \xi}^{\theta, \varrho, \omega} x(\xi); \xi \in \mathcal{L}$ and, as a result of Lemma 5, the following function

$$x(\xi) = \omega^{-1}(\xi) c + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(a)) d + I_{a, \xi}^{\theta, \varrho, \omega} u(\xi), \xi \in \mathcal{L}.$$

is a solution for the ϱ -Caputo differential equation of order θ and of type v :

$${}^c D_{a, \xi}^{\theta, \varrho, \omega} x(\xi) = u(\xi), \xi \in [a, T].$$

This is consistent with what is known.

To drive our results, we need the next following lemmas. Let $P_{cc}(E) = \{A \subseteq E : A \text{ be not empty, convex, and closed}\}$.

Lemma 6 ([44], Corollary 3.3.1). *Suppose that $K \in P_{cc}(E)$ and $\Pi : K \rightarrow P_{ck}(E)$. If Π has a closed graph and χ -condensing, then Π has a fixed point.*

Lemma 7 ([44], Prop. 3.5.1). *In addition to the assumptions of Lemma 7, suppose $\Pi : U \rightarrow P_{ck}(E)$ and χ is monotone. If the set of fixed points for Π is bounded, then it is compact.*

3. Antiperiodic Solutions for Problem (3)

In this part, we show that the set of antiperiodic solutions for Problem (3) is a subset that is not empty and compact in the Banach space $PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$.

Let $p > \frac{1}{\theta-1} > 1$ be a fixed real number, and for any $z \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$, let

$$S_{F(\cdot, z)}^p := \left\{ u \in L_{\omega}^{p, \varrho}(\mathcal{L}, E) : u(\xi) \in u(\xi, z(\xi)), a.e. \text{ for } \xi \in \mathcal{L} \right\}.$$

Lemma 8. *Let $u \in C_{2-\lambda, \varrho, \omega}^{v(2-\theta)}(\mathcal{L}, E)$ and define a function $x : \mathcal{L} \rightarrow E$ by*

$$x(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{1-\lambda}}{\Gamma(\lambda-1)} c_i(x, u) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{1-\lambda}}{\Gamma(\lambda)} d_i(x, u) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u(\xi), \xi \in \mathcal{L}_i, i \in L_0, \end{cases} \quad (10)$$

where

$$d_0(x, u) = \frac{-1}{2} \left[\sum_{k=1}^{k=m+1} \varsigma_k(u) + \sum_{k=1}^{k=m} \Re_k(x(\xi_k^-)) \right], \quad (11)$$

$$d_i(x, u) = d_0(x, u) + \sum_{k=1}^i \varsigma_k(u) + \sum_{k=1}^i \Re_k(x(\xi_k^-)), i \in L_1, \quad (12)$$

$$c_0(x, u) = \frac{-1}{2} \left[\sum_{k=1}^{k=m+1} (\varrho(\xi_k) - \varrho(\xi_{k-1})) d_{k-1} + \sum_{k=1}^{k=m+1} \tau_k(u) + \sum_{k=1}^{k=m} \Re_k(x(\xi_k^-)) \right], \quad (13)$$

$$c_i(x, u) = c_0(x, u) + \sum_{k=1}^{k=i} (\varrho(\xi_k) - \varrho(\xi_{k-1})) d_{k-1} + \sum_{k=1}^{k=i} \varsigma_k(u) + \sum_{k=1}^{k=i} \Re_k(x(\xi_k^-)), i \in L_1. \quad (14)$$

where $\varsigma_k(u), \tau_k(u), k = 1, 2, \dots, m$ will be defined in relationships (23), (25), (27), and (31) which are mentioned later.

Then $x \in C_{2-\lambda, \varrho, \omega}^{\lambda}(\mathcal{L}, E)$, $D_{\xi_i, \xi}^{\theta, v, \varrho, \omega} x(\xi)$ exists for $\xi \in (0, T]$, and it is a solution for the problem

$$D_{\xi_i, \xi}^{\theta, v, \varrho, \omega} x(\xi) = u(\xi), \xi \in (\xi_i, \xi_{i+1}], i \in L_0 \quad (15)$$

subject to the impulsive conditions

$$\lim_{\xi \rightarrow \xi_i^+} \omega(\xi) I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) = \lim_{\xi \rightarrow \xi_i^-} \omega(\xi) I_{\xi_{i-1}, \xi}^{2-\lambda, \varrho, \omega} x(\xi) + \Re_i(x(\xi_i^-)), i \in L_1, \quad (16)$$

and

$$\begin{aligned} & \lim_{\xi \rightarrow \xi_i^+} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\ &= \lim_{\xi \rightarrow \xi_i^-} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) + \Re_i(x(\xi_i^-)), i \in L_1 \end{aligned} \quad (17)$$

and the antiperiodic conditions

$$\lim_{\xi \rightarrow 0} \omega(\xi) I_{0,\xi}^{2-\lambda, \varrho, \omega} x(\xi) = - \lim_{\xi \rightarrow T^-} \omega(\xi) I_{0,\xi}^{2-\lambda, \varrho, \omega} x(\xi), \quad (18)$$

and

$$\lim_{\xi \rightarrow 0} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{0,\xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) = - \lim_{\xi \rightarrow T^-} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right). \quad (19)$$

Proof. For any $\xi \in \mathcal{L}_i = (\xi_i, \xi_{i+1}]$, $i \in L_0$, define

$$\begin{aligned} x(\xi) = & \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(x, u) \\ & + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(x, u) \\ & + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u(\xi), \end{aligned} \quad (20)$$

where $c_i(x, u)$ and $d_i(x, u)$, $i = 0, 1, 2, \dots, m$ are fixed points in E . Using Lemma 5, $D_{0,\xi}^{\lambda, \varrho, \omega} x \in C_{2-\lambda, \varrho, \omega}([a, T], E)$, $x \in C_{2-\lambda, \varrho, \omega}^{\lambda}([a, T], E)$, and $D_{\xi_i, \xi}^{\lambda, \varrho, \omega} x(\xi) = u(\xi)$, $\forall \xi \in (\xi_i, \xi_{i+1}]$. Our aim is to use the boundary conditions (16)–(19) to show that $c_i(x, u)$ and d_i verifies Equations (11)–(14).

Let $i \in L_1$ be fixed. Apply $I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega}$ on both sides of (20); it results from (i) in Lemma 2 that

$$\begin{aligned} I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) = & \omega^{-1}(\xi) c_i(x, u) + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(\xi_i)) d_i(x, u) \\ & + I_{\xi_i, \xi}^{\theta+2-\lambda, \varrho, \omega} u(\xi). \end{aligned} \quad (21)$$

From (16), it yields

$$\begin{aligned} & \lim_{\xi \rightarrow \xi_i^+} \left[c_i(x, u) + (\varrho(\xi) - \varrho(\xi_i)) d_i(x, u) + \omega(\xi) I_{\xi_i, \xi}^{2-\lambda+\theta, \varrho, \omega} u(\xi) \right] \\ & = \lim_{\xi \rightarrow \xi_i^-} \left[c_{i-1}(x, u) + (\varrho(\xi) - \varrho(\xi_{i-1})) d_{i-1}(x, u) + \omega(\xi) I_{\xi_{i-1}, \xi}^{2-\lambda+\theta, \varrho, \omega} u(\xi) \right] \\ & + \aleph_i(x(\xi_i^-)), \end{aligned}$$

which leads to $c_i(x, u) = c_{i-1}(x, u) + (\varrho(\xi_i) - \varrho(\xi_{i-1})) d_{i-1}(x, u) + \omega(\xi_i) I_{\xi_{i-1}, \xi_i}^{2-\lambda+\theta, \varrho, \omega} u(\xi) + \aleph_i(x(\xi_i^-))$, and hence

$$c_{i-1}(x, u) = c_i(x, u) - (\varrho(\xi_i) - \varrho(\xi_{i-1})) d_{i-1}(x, u) - \tau_i(u) - \aleph_i(x(\xi_i^-)), \quad (22)$$

where

$$\tau_i(u) = \frac{1}{\Gamma(\theta+2-\lambda)} \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(\sigma))^{1-\lambda+\theta} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma. \quad (23)$$

Now, since $(2-\lambda) - 1 < 0$, and $\theta + 2 - \lambda - 1 = \theta + 1 - (\theta + 2v - \theta v) = 1 - v(2 - \theta) > 0$, then (22) gives us

$$\begin{aligned} & \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\ & = \omega(\xi) D_{\xi_i, \xi}^{1, \varrho, \omega} \left(I_{\xi_i, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\ & = \omega(\xi) D_{\xi_i, \xi}^{1, \varrho, \omega} \left[c_i \omega^{-1}(\xi) + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(\xi_i)) d_i \right] + D_{\xi_i, \xi}^{1, \varrho, \omega} I_{\xi_i, \xi}^{\theta+2-\lambda, \varrho, \omega} u(\xi) \\ & = d_i + \omega(\xi) I_{\xi_i, \xi}^{\theta+1-\lambda, \varrho, \omega} u(\xi), \forall \xi \in (\xi_i, \xi_{i+1}]. \end{aligned}$$

Therefore, it yields from (17) that

$$d_i(x, u) = d_{i-1}(x, u) + \omega(\xi_i) I_{\xi_{i-1}, \xi_i}^{\theta+1-\lambda, \varrho, \omega} u(\xi) + \Re_i(x(\xi_i^-)).$$

So,

$$d_{i-1}(x, u) = d_i(x, u) - \varsigma_i(u) - \Re_i(x(\xi_i^-)), \quad (24)$$

where

$$\varsigma_i(u) = \frac{1}{\Gamma(\theta+1-\lambda)} \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(\sigma))^{\theta-\lambda} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma. \quad (25)$$

Because x satisfies the antiperiodic condition (18), we obtain

$$\begin{aligned} c_0(x, u) &= - \lim_{\xi \rightarrow T^-} \omega(\xi) [\omega^{-1}(\xi) c_m(x, u) \\ &\quad + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(\xi_m)) d_m(x, u) + I_{\xi_m, \xi}^{\theta+2-\lambda, \varrho, \omega} u(\xi) d_m] \\ &= -c_m(x, u) - (\varrho(T) - \varrho(\xi_m)) d_m(x, u) - \tau_{m+1}(u), \end{aligned} \quad (26)$$

where

$$\tau_{m+1}(u) = \frac{1}{\Gamma(\theta+2-\lambda)} \int_{\xi_m}^T (\varrho(T) - \varrho(\sigma))^{\theta+1-\lambda} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma. \quad (27)$$

Next, for $\xi \in (0, \xi_1]$, we have from (i) of Lemma 2,

$$\begin{aligned} &I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \\ &= I_{0, \xi}^{2-\lambda, \varrho, \omega} [\omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(0))^{\lambda-2}}{\Gamma(\lambda-1)} c_0(x, u) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(0))^{\lambda-1}}{\Gamma(\lambda)} d_0(x, u) \\ &\quad + I_{0, \xi}^{\theta, \varrho, \omega} u(\xi)] \\ &= \omega^{-1}(\xi) c_0(x, u) + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(0)) d_0(x, u) + I_{0, \xi}^{\theta+2-\lambda, \varrho, \omega} u(\xi). \end{aligned}$$

Then,

$$\begin{aligned} &\lim_{\xi \rightarrow 0^+} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\ &= \lim_{\xi \rightarrow 0^+} \omega(\xi) D_{\xi}^{1, \varrho, \omega} \left(I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) = \omega(0) \lim_{\xi \rightarrow 0} D_{\xi}^{1, \varrho, \omega} \left(I_{0, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\ &= \omega(0) \lim_{\xi \rightarrow 0^+} D_{\xi}^{1, \varrho, \omega} \left[\omega^{-1}(\xi) c_0(x, u) + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(0)) d_0(x, u) + I_{0, \xi}^{\theta+2-\lambda, \varrho, \omega} u(\xi) \right] \\ &= \omega(0) \lim_{\xi \rightarrow 0^+} \left[\omega^{-1}(\xi) d_0(x, u) + I_{0, \xi}^{\theta+1-\lambda, \varrho, \omega} u(\xi) \right] \\ &= d_m(x, u) + \frac{1}{\Gamma(\theta+1-\lambda)} \int_{\xi_m}^T (\varrho(T) - \varrho(\sigma))^{\theta-\lambda} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma. \end{aligned} \quad (28)$$

Similarly, for $\xi \in (\xi_m, T]$, we have

$$\begin{aligned} &I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \\ &= I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} [\omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_m(x, u) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_m(x, u) \\ &\quad + I_{\xi_m, \xi}^{\theta, \varrho, \omega} u(\xi)] \\ &= \omega^{-1}(\xi) c_m(x, u) + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(\xi_i)) d_m(x, u) + I_{\xi_m, \xi}^{\theta+2-\lambda, \varrho, \omega} u(\xi). \end{aligned}$$

Then,

$$\begin{aligned}
 & \lim_{\xi \rightarrow T^-} \frac{1}{\varrho'(\xi)} \frac{d}{d\xi} \left(\omega(\xi) I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\
 &= \lim_{\xi \rightarrow T^-} \omega(\xi) D_{\xi}^{1, \varrho, \omega} \left(I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) = \omega(T) \lim_{\xi \rightarrow T^-} D_{\xi}^{1, \varrho, \omega} \left(I_{\xi_m, \xi}^{2-\lambda, \varrho, \omega} x(\xi) \right) \\
 &= \omega(T) \lim_{\xi \rightarrow T^-} D_{\xi}^{1, \varrho, \omega} \left[\omega^{-1}(\xi) c_m(x, u) + \omega^{-1}(\xi) (\varrho(\xi) - \varrho(\xi_i)) d_m(x, u) + I_{\xi_m, \xi}^{\theta+2-\lambda, \varrho, \omega} u(\xi) \right] \\
 &= \omega(T) \lim_{\xi \rightarrow T^-} \left[\omega^{-1}(\xi) d_m(x, u) + I_{\xi_m, \xi}^{\theta+1-\lambda, \varrho, \omega} u(\xi) \right] \\
 &= d_m(x, u) + \frac{1}{\Gamma(\theta+1-\lambda)} \int_{\xi_m}^T (\varrho(T) - \varrho(\sigma))^{\theta-\lambda} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma.
 \end{aligned} \tag{29}$$

Therefore, from (28), (29), and the antiperiodic condition (19), we obtain

$$-d_0(x, u) = d_m(x, u) + \varsigma_{m+1}(u), \tag{30}$$

where

$$\varsigma_{m+1}(u) = \frac{1}{\Gamma(\theta+1-\lambda)} \int_{\xi_m}^T (\varrho(T) - \varrho(\sigma))^{\theta-\lambda} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma. \tag{31}$$

Now, (24) leads to

$$d_0(x, u) = d_m(x, u) - \sum_{k=1}^{k=m} \varsigma_k(u) - \sum_{k=1}^{k=m} \Re_k(x(\xi_k^-)).$$

This equation and (30) imply

$$d_0(x, u) = \frac{-1}{2} \left[\sum_{k=1}^{k=m+1} \varsigma_k(u) + \sum_{k=1}^{k=m} \Re_k(x(\xi_k^-)) \right].$$

So, (11) is verified, and, consequently, we obtain from (24)

$$d_i(x, u) = d_0(x, u) + \sum_{k=1}^i \varsigma_k(u) + \sum_{k=1}^i \Re_k(x(\xi_k^-)), i \in L_1,$$

and this confirms the validity of the relationship (12). Likewise, it yields from (22)

$$c_0(x, u) = c_m(x, u) - \sum_{k=i}^{k=m} (\varrho(\xi_k) - \varrho(\xi_{k-1})) d_{k-1}(x, u) - \sum_{k=i}^{k=m} \tau_k(u) - \sum_{k=i}^{k=m} \aleph_k(x(\xi_k^-)).$$

By substituting into this equation the value of $c_m(x, u)$ from (26), it results in

$$c_0(x, u) = \frac{-1}{2} \left[\sum_{k=1}^{k=m+1} (\varrho(\xi_k) - \varrho(\xi_{k-1})) d_{i-1}(x, u) + \sum_{k=1}^{k=m+1} \tau_k(u) + \sum_{k=1}^{k=m} \aleph_k(x(\xi_k^-)) \right].$$

So, (13) is true, and consequently, from (22), we have

$$c_i(x, u) = c_0(x, u) + \sum_{k=1}^{k=i} (\varrho(\xi_i) - \varrho(\xi_{i-1})) d_{i-1}(x, u) + \sum_{k=1}^{k=i} \tau_i(u) + \sum_{k=1}^{k=i} \aleph_i(x(\xi_i^-)), i \in L_1.$$

Then, (14) is verified. $\square$

As a consequence of Lemma 8, we give the definition of solutions for (3).

Definition 6. By an antiperiodic solution for (3), it means a function x given by

$$x(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(x, u) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(x, u) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u(\xi), \xi \in \mathcal{L}_i, i \in L_0, \end{cases} \quad (32)$$

where $u \in S_{F(\cdot, x)}^{v(2-\theta), 1}$ and $c_i(x, u), d_i(x, u)$ satisfy the identities (11)–(14).

We mention some properties for $L_{\omega}^{p, \varrho}((0, T), \mathbb{R}), p \geq 1$.

Remark 5. For $p \geq 1$, $\varphi \in L_{\omega}^{p, \varrho}((0, T), \mathbb{R}) \iff \omega \varphi(\varrho')^{\frac{1}{p}} \in L^p((0, T), \mathbb{R})$.

Remark 6. $p > q \rightarrow L_{\omega}^{p, \varrho}((0, T), \mathbb{R}) \subseteq L_{\omega}^{q, \varrho}((0, T), \mathbb{R})$. In fact, let $u \in L_{\omega}^{p, \varrho}((0, T), \mathbb{R})$ and set $A = \{\sigma \in [0, T] : |\omega(\sigma)u(\sigma)| < 1\}$. Then

$$\begin{aligned} \int_0^T |\omega(\sigma)u(\sigma)|^q \varrho'(\sigma) d\sigma &\leq \int_T |\omega(\sigma)u(\sigma)|^q \varrho'(\sigma) d\sigma + \int_{[0, T] - T} |\omega(\sigma)u(\sigma)|^q \varrho'(\sigma) d\sigma \\ &\leq \int_A \varrho'(\sigma) d\sigma + \int_{[0, T] - A} |\omega(\sigma)u(\sigma)|^p \varrho'(\sigma) d\sigma < \infty. \end{aligned}$$

Lemma 9. If $p > \frac{1}{\theta-1}$ and $\varphi \in L_{\omega}^{p, \varrho}((0, T), \mathbb{R})$, then

$$\int_0^T (\varrho(T) - \varrho(\sigma))^{\theta-\lambda} \omega(\sigma) \varphi(\sigma) \varrho'(\sigma) d\sigma \leq \eta_1 \|\varphi\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})}, \quad (33)$$

and

$$\int_0^T (\varrho(T) - \varrho(\sigma))^{\theta-1} \omega(\sigma) \varphi(\sigma) \varrho'(\sigma) d\sigma \leq \eta_2 \|\varphi\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})}, \quad (34)$$

where $\eta_1 = \left(\frac{p}{p-1}(\theta-\lambda) + 1\right)^{\frac{p}{p-1}} \varrho(T)^{(\theta+1-\lambda)-\frac{1}{p}}$ and $\eta_2 = \left(\frac{p\theta-1}{p-1}\right)^{\frac{p}{p-1}} \varrho(T)^{\theta-\frac{1}{p}}$.

Proof. Let $q = \frac{p}{p-1}$ and $h(\sigma) = (\varrho(T) - \varrho(\sigma))^{\theta-\lambda} \omega^{-1}(\sigma)$. Since $\frac{1}{\theta+1-\lambda} < \frac{1}{\theta-1}$, then $\frac{1}{\theta+1-\lambda} < p$, and consequently, $q(\theta-\lambda) + 1 > 0$. Then

$$\begin{aligned} \left(\int_0^T |h(\sigma) \omega(\sigma)|^q \varrho'(\sigma) d\sigma\right)^{\frac{1}{q}} &= \left(\int_0^T (\varrho(T) - \varrho(\sigma))^{q(\theta-\lambda)} \varrho'(\sigma) d\sigma\right)^{\frac{1}{q}} \\ &= \left(\frac{\varrho(T)^{q(\theta-\lambda)+1}}{q(\theta-\lambda) + 1}\right)^{\frac{1}{q}}. \end{aligned}$$

Then $h \in L_{\omega}^{q, \varrho}((0, T), E)$, and hence, $h\omega(\varrho')^{\frac{1}{q}} \in L^q((a, T), E)$. Moreover, the assumption $\varphi \in L_{\omega}^{p, \varrho}((0, T), E)$ leads to $\varphi\omega(\varrho')^{\frac{1}{p}} \in L^p((0, T), E)$. Since $\frac{1}{p} + \frac{1}{q} = 1$, Holder's inequality implies

$$\begin{aligned}
& \int_0^T \left[h(\sigma) \omega(\sigma) (\varrho'(\sigma))^{\frac{1}{q}} \right] \left[\varphi(\sigma) \omega(\sigma) (\varrho'(\sigma))^{\frac{1}{p}} \right] d\sigma \\
& \leq \left(\int_0^T |h(\sigma) \omega(\sigma)|^q \varrho'(\sigma) d\sigma \right)^{\frac{1}{q}} \left(\int_0^T |\varphi(\sigma) \omega(\sigma)|^p \varrho'(\sigma) d\sigma \right)^{\frac{1}{p}} \\
& = \left(\frac{\varrho(T)^{q(\theta-\lambda)+1}}{q(\theta-\lambda)+1} \right)^{\frac{1}{q}} \|\varphi\|_{L_{\omega}^{p,q}((0,T),E)} \\
& = \left(\frac{\varrho(T)^{\frac{p}{p-1}(\theta-\lambda)+1}}{\frac{p}{p-1}(\theta-\lambda)+1} \right)^{\frac{p-1}{p}} \|\varphi\|_{L_{\omega}^{p,q}((0,T),E)} \\
& = \eta_1 \varrho(T)^{(\theta+1-\lambda)-\frac{1}{p}}.
\end{aligned}$$

Since $\frac{1}{p} + \frac{1}{p} = 1$, we obtain

$$\begin{aligned}
& \int_0^T (\varrho(T) - \varrho(\sigma))^{\theta-\lambda} \omega(\sigma) \varphi(\sigma) \varrho'(\sigma) d\sigma \\
& = \int_0^T \left[(\varrho(T) - \varrho(\sigma))^{\theta-\lambda} \omega^{-1}(\sigma) (\varrho'(\sigma))^{\frac{1}{q}} \omega(\sigma) \right] \left[\varphi(\sigma) \omega(\sigma) (\varrho'(\sigma))^{\frac{1}{p}} \right] d\sigma \\
& = \int_0^T \left[h(\sigma) (\varrho'(\sigma))^{\frac{1}{q}} \omega(\sigma) \right] \left[\varphi(\sigma) \omega(\sigma) (\varrho'(\sigma))^{\frac{1}{p}} \right] d\sigma \\
& \leq \eta_1 \|\varphi\|_{L^p((0,T),E)}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
& \left\| \int_0^T (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \varphi(\sigma) \omega(\sigma) d\sigma \right\| \\
& = \eta_2 \|\varphi\|_{L^p((0,T),E)}, i \in L_1.
\end{aligned}$$

□

Theorem 1. Suppose that $F : \mathcal{E} \times E \rightarrow P_{ck}(E)$ and $\aleph_i, \Re_i : E \rightarrow E, i \in L_1$. We assume the following conditions:

(F₁) For any $z \in PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, the multivalued function $\xi \rightarrow F(\xi, z(\xi))$ is measurable and the set $S_{F(\cdot, z)}^{v(2-\theta), p}$ is not empty.

(F₂) There is a $\varphi \in L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)$, $p > \frac{1}{\theta-1}$ such that for any $z \in PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, and for almost $\xi \in \mathcal{E}$

$$\|F(\xi, z(\xi))\| = \sup_{y \in F(\xi, z(\xi))} \|y\| \leq \varphi(\xi)(1 + \|z\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}).$$

(F₃) If $z_n \in PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, $z_n \rightarrow z \in PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, $u_n \in S_{F(\cdot, z_n)}^{v(2-\theta), p}$, and $u_n \rightarrow u \in L_{\omega}^{p, \varrho}(\mathcal{E}, E)$, then $u \in S_{F(\cdot, z)}^{v(2-\theta), p}$.

(F₄) There is a $\beta \in L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)$, $p > \frac{1}{\theta-1}$ such that for any bounded subset $D \subseteq PC_{2-\lambda, \varrho, \omega}$ and any $i \in L_0$, we have for a.e. $\xi \in \mathcal{E}_i$

$$\begin{aligned}
& \chi_E(F(\xi, z(\xi)) : z \in D) \\
& \leq (\varrho(\xi) - \varrho(\xi_i))^{2-\lambda} \beta(\xi) \chi_E\{(z(\xi) : z \in D)\},
\end{aligned}$$

where χ is the Hausdorff measure of noncompactness on E .

(H) For any $i \in L_1$, $\aleph_i, \Re_i : E \rightarrow E$ are continuous and map any bounded set to a relatively compact subset, and there is $\delta_i, \delta_i^* > 0$ such that for any $u \in PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, one has

$$\|\mathfrak{N}_i(u(\xi))\| \leq \delta_i \|u\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L},$$

and

$$\|\mathfrak{R}_i(u(\xi))\| \leq \delta_i^* \|u\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L}.$$

Then, the set of solutions of Problem (3) is a non-empty and compact subset in $PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$ provided that

$$\begin{aligned} \varepsilon_1 = & \|\varphi\|_{L_{\omega}^{p, \varrho}(\mathcal{L}, \mathbb{R})} \left[\frac{m\eta_1}{\Gamma(\lambda-1)\Gamma(\theta-\lambda+1)} \left(\frac{\varrho(T)}{2} \left(\frac{9}{2} + \frac{1}{\theta+1-\lambda} \right) + 1 \right) \right. \\ & \left. + \frac{3\varrho(T)m\eta_1}{2\Gamma(\theta+1-\lambda)\Gamma(\lambda)} + \frac{\eta_2}{\Gamma(\theta)} \right] + \kappa m \left[\frac{1}{\Gamma(\lambda-1)} \left(\frac{9\varrho(T)}{4} + \frac{3}{2} \right) + \frac{3\varrho(T)}{2\Gamma(\lambda)} \right] \\ & < 1, \end{aligned} \quad (35)$$

and

$$\begin{aligned} \varepsilon_2 = & \|\beta\|_{L^p((0, T), E)} \left[\frac{3m\eta_1}{\Gamma(\lambda-1)2\Gamma(\theta+1-\lambda)} + \frac{m\eta_1}{\Gamma(\lambda)} \left(\frac{9\varrho(T)+4}{4\Gamma(\theta+1-\lambda)} \right. \right. \\ & \left. \left. + \frac{\varrho(T)}{2\Gamma(\theta+2-\lambda)} \right) + \frac{\eta_2\varrho(T)}{\Gamma(\theta)} \right], \end{aligned} \quad (36)$$

where $\kappa = \max\left(\sum_{i=0}^{i=n} \delta_i, \sum_{i=0}^{i=n} \delta_i^*\right)$.

Proof. We define a multivalued function Ψ on $PC_{2-\lambda, \varrho, \omega}$ in the following manner: let $z \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$. Condition (F_1) assures the existence of a function $u \in S_{F(.,z)}^{v(2-\theta), p}$. We consider $x \in (z)$ if and only if

$$x(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z, u) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z, u) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u(\xi), \xi \in \mathcal{L}_i, i \in L_0, \end{cases} \quad (37)$$

where $c_i(z, u)$ and $d_i(z, u)$ are given by (8)–(11). As a result of Lemma 5 and Remark 2, this function belonging to $PC_{2-\lambda, \varrho, \omega}^\lambda(\mathcal{L}, E) \subseteq PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$, and $D_{\xi_i, \xi}^{\theta, v, \varrho, \omega} x(\xi)$ exists for $\xi \in \mathcal{L}_i, \forall i \in L_0$. Furthermore, x is a solution for Problem (3). We show, by making use of Lemma 6, that Ψ has a fixed point. Since the values of F are convex, the values of Ψ are also convex. The proof will be performed through the following steps:

Step 1. We show in this step the existence of a natural number n_0 with $(T_{n_0}) \subseteq T_{n_0}$, where for any positive real number δ ,

$$T_\delta = \left\{ x \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E) : \|x\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)} \leq \delta \right\}.$$

If this is not true, there are two sequences $z_n, x_n \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$ such that $x_n \in (z_n)$, $\|z_n\|_{PC_{2-\lambda, \varrho, \omega}} \leq n$ and $\|x_n\|_{PC_{2-\lambda, \varrho, \omega}} > n$. Using the definition of Ψ , we can find $u_n \in S_{F(.,z_n)}^{v(2-\theta), p}; n \in \mathbb{N}$ such that

$$x_n(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z_n, u_n) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z_n, u_n) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u_n(\xi), \xi \in \mathcal{L}_i, i \in L_0. \end{cases} \quad (38)$$

It results from (F_2) that

$$\|u_n(\xi)\| \leq \varphi(\xi)(1+n), a.e. \quad (39)$$

Therefore, from (33), (34), and (39), it holds for any $i \in L_1$

$$\begin{aligned}
& \left\| \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(s))^{\theta-\lambda} \varrho'(s) \omega(s) u_n(s) ds \right\| \\
& \leq (1+n) \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(s))^{\theta-\lambda} \varrho'(s) \omega(s) \varphi(s) ds \\
& \leq (1+n) \left(\frac{p(\theta+1-\lambda)-1}{p-1} \right)^{\frac{p}{p-1}} \varrho(\xi_i)^{\frac{p(\theta+1-\lambda)-1}{p}} \|\varphi\|_{L_{\omega}^{p,\varrho}((0,T),\mathbb{R})} \\
& \leq (1+n) \eta_1 \|\varphi\|_{L_{\omega}^{p,\varrho}((0,T),\mathbb{R})},
\end{aligned} \tag{40}$$

$$\begin{aligned}
& \left\| \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(s))^{\theta-\lambda+1} \varrho'(s) \omega(s) u_n(s) ds \right\| \\
& \leq (1+n) \varrho(T) \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(s))^{\theta-\lambda} \varrho'(s) \omega(s) \varphi(s) ds \\
& \leq (1+n) \varrho(T) \eta_1 \|\varphi\|_{L_{\omega}^{p,\varrho}((0,T),\mathbb{R})},
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
& \left\| \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(s))^{\theta-1} \varrho'(s) u_n(s) \omega(s) ds \right\| \\
& \leq \varrho(T) (1+n) \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(s))^{\theta-2} \varrho'(s) \varphi(s) \omega(s) ds \\
& \leq \varrho(T) \left(\frac{\varrho(\xi)^{\frac{p}{p-1}(\theta-2)+1}}{\frac{p}{p-1}(\theta-2)+1} \right)^{\frac{p-1}{p}} \|\varphi\|_{L^p((0,T),E)} \\
& = \varrho(T) \varrho(\xi)^{\theta-1-\frac{1}{p}} \left(\frac{p\theta-p-1}{p-1} \right)^{\frac{p}{p-1}} \|\varphi\|_{L^p((0,T),E)} \\
& \leq \varrho(T) \varrho(T)^{\theta-1-\frac{1}{p}} \left(\frac{p\theta-p-1}{p-1} \right)^{\frac{p}{p-1}} \|\varphi\|_{L^p((0,T),E)} \\
& = \varrho(T)^{\theta-\frac{1}{p}} \left(\frac{p\theta-p-1}{p-1} \right)^{\frac{p}{p-1}} \|\varphi\|_{L^p((0,T),E)} \\
& = \eta_2 \|\varphi\|_{L^p((0,T),E)}, i \in L_1.
\end{aligned} \tag{42}$$

Considering (23), (25), (27), (31), and (39), it yields from (40)–(43) that

$$\begin{aligned}
& \sum_{i=1}^{i=m+1} \|\tau_i(u_n)\| \\
& \leq \frac{(1+n)}{\Gamma(2-\lambda+\theta)} \sum_{i=1}^{i=m+1} \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(s))^{\theta-\lambda+1} \varrho'(s) \varphi(s) \omega(s) ds \\
& = \frac{(1+n)m\varrho(T)}{\Gamma(2-\lambda+\theta)} \eta_1 \|\varphi\|_{L_{\omega}^{p,\varrho}((0,T),\mathbb{R})}
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
& \sum_{i=1}^{i=m+1} \|\varsigma_i(u_n)\| \leq \frac{(1+n)}{\Gamma(\theta-\lambda+1)} \sum_{i=1}^{i=m+1} \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(s))^{\theta-\lambda} \varrho'(s) \varphi(s) \omega(s) ds \\
& \leq \frac{(1+n)m\eta_1}{\Gamma(\theta-\lambda+1)} \|\varphi\|_{L_{\omega}^{p,\varrho}((0,T),\mathbb{R})}.
\end{aligned} \tag{45}$$

As a result of (11)–(14), (44), and (45), we obtain

$$\begin{aligned} \|d_0(z_n, u_n)\| &= \frac{1}{2} \left\| \sum_{k=1}^{k=m+1} \varsigma_k(u) + \sum_{k=1}^{k=m} \Re_k(z_n(\xi_k^-)) \right\| \\ &\leq \frac{(1+n)m\eta_1}{2\Gamma(\theta-\lambda+1)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \frac{\kappa m}{2} n, \end{aligned} \quad (46)$$

$$\begin{aligned} \|d_i(z_n, u_n)\| &= \|d_0(z_n, u_n) + \sum_{k=1}^i \varsigma_k(u) + \sum_{k=1}^i \Re_k(z_n(\xi_k^-))\| \\ &\leq \frac{3(1+n)m\eta_1}{2\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \frac{3\kappa mn}{2}, \forall i \in L_1, \end{aligned} \quad (47)$$

$$\begin{aligned} \|c_0(z_n, u_n)\| &= \frac{1}{2} \left\| \sum_{k=1}^{k=m+1} (\varrho(\xi_k) - \varrho(\xi_{k-1})) d_{i-1}(z_n, u) + \sum_{k=1}^{k=m+1} \tau_k(u) + \sum_{k=1}^{k=m} \Re_k(z_n(\xi_k^-)) \right\| \\ &\leq \frac{1}{2} \left[\frac{3(1+n)m\eta_1\varrho(T)}{2\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \frac{3\kappa mn\varrho(T)}{2} \right. \\ &\quad \left. + \frac{m(1+n)\varrho(T)\eta_1}{\Gamma(\theta-\lambda+2)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \kappa mn \right] \\ &= \frac{(1+n)m\eta_1\varrho(T)}{2\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} \left[\frac{3}{2} + \frac{1}{\theta+1-\lambda} \right] + \kappa mn \left[\frac{3\varrho(T)}{4} + \frac{1}{2} \right]. \end{aligned} \quad (48)$$

and for $i \in L_1$,

$$\begin{aligned} \|c_i(z_n, u_n)\| &= \|c_0(z_n, u) + \sum_{k=1}^{k=i} (\varrho(\xi_k) - \varrho(\xi_{k-1})) d_{k-1} + \sum_{k=1}^{k=i} \varsigma_k(u) + \sum_{k=1}^{k=i} \Re_k(z_n(\xi_k^-))\| \\ &\leq \frac{(1+n)m\eta_1\varrho(T)}{2\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} \left[\frac{3}{2} + \frac{1}{\theta+1-\lambda} \right] + \kappa mn \left[\frac{3\varrho(T)}{4} + \frac{1}{2} \right] \\ &\quad + \varrho(T) \left[\frac{3(1+n)m\eta_1}{2\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \frac{3\kappa mn}{2} \right] \\ &\quad + \frac{(1+n)m\eta_1}{\Gamma(\theta-\lambda+1)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \kappa mn \\ &\leq \frac{(1+n)m\eta_1}{\Gamma(\theta-\lambda+1)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} \left[\frac{\varrho(T)}{2} \left(\frac{9}{2} + \frac{1}{\theta+1-\lambda} \right) + 1 \right] \\ &\quad + \kappa mn \left(\frac{9\varrho(T)}{4} + \frac{3}{2} \right). \end{aligned} \quad (49)$$

Next, for any $\xi \in \mathcal{E}_0$, we obtain from (38) and (43)–(46) that

$$\begin{aligned} &(\varrho(\xi) - \varrho(0))^{2-\lambda} \omega(\xi) \|x_n(\xi)\| \\ &\leq \frac{1}{\Gamma(\lambda-1)} c_0(x_n, u_n) + \frac{\varrho(\xi) - \varrho(0)}{\Gamma(\lambda)} d_0(x_n, u_n) \\ &\quad + \frac{(1+n)}{\Gamma(\theta)} \int_0^\xi (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \varphi(\sigma) \omega(\sigma) d\sigma \\ &\leq \frac{(1+n)m\eta_1\varrho(T)}{2\Gamma(\lambda-1)\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} \left[\frac{3}{2} + \frac{1}{\theta+1-\lambda} \right] + \kappa mn \left[\frac{3\varrho(T)}{4} + 1 \right] \\ &\quad + \frac{\varrho(T)}{\Gamma(\lambda)} \left[\frac{(1+n)m\eta_1}{2\Gamma(\theta-\lambda+1)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \frac{\kappa m}{2} n \right] \\ &\quad + \frac{\eta_2(1+n)}{\Gamma(\theta)} \|\varphi\|_{L_\omega^{p,q}}, \end{aligned} \quad (50)$$

and for any $\xi \in \mathcal{L}_i, i \in L_1$,

$$\begin{aligned}
 & (\varrho(\xi) - \varrho(\xi))^{2-\lambda} \omega(\xi) \|x_n(\xi)\| \\
 & \leq \frac{1}{\Gamma(\lambda-1)} c_i(z_n, u_n) + \frac{\varrho(T)}{\Gamma(\lambda)} d_i(z_n, u_n) \\
 & + \frac{(1+n)}{\Gamma(\theta)} \int_0^\xi (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \varphi(\sigma) \omega(\sigma) d\sigma \\
 & \leq \frac{(1+n)m\eta_1}{\Gamma(\lambda-1)\Gamma(\theta-\lambda+1)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} \left[\frac{\varrho(T)}{2} \left(\frac{9}{2} + \frac{1}{\theta+1-\lambda} \right) + 1 \right] \\
 & + \frac{\kappa mn}{\Gamma(\lambda-1)} \left(\frac{9\varrho(T)}{4} + 2 \right) \\
 & + \frac{\varrho(T)}{\Gamma(\lambda)} \left[\frac{3(1+n)m\eta_1}{2\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \frac{3\kappa mn}{2} \right] \\
 & + \frac{\eta_2(1+n)}{\Gamma(\theta)} \|\varphi\|_{L_\omega^{p,q}}.
 \end{aligned} \tag{51}$$

It yields from (50) and (51) that

$$\begin{aligned}
 n & < \|x_n\|_{PC_{2-\lambda,\varrho,\omega}(\mathcal{L},E)} \\
 & \leq \frac{1}{\Gamma(\lambda-1)} \max_{i \in L_0} \|c_i(z_n, u_n)\| + \frac{1}{\Gamma(\lambda)} \max_{i \in L_0} \left(\frac{\varrho(\xi) - \varrho(\xi_i)}{\Gamma(\lambda)} \|d_i(z_n, u_n)\| \right) \\
 & + \frac{\eta_2(1+n)}{\Gamma(\theta)} \|\varphi\|_{L_\omega^{p,q}}.
 \end{aligned}$$

We divide both sides of this inequality by n and then, letting $n \rightarrow \infty$, we obtain

$$\begin{aligned}
 1 & < \frac{m\eta_1}{\Gamma(\lambda-1)\Gamma(\theta-\lambda+1)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} \left[\frac{\varrho(T)}{2} \left(\frac{9}{2} + \frac{1}{\theta+1-\lambda} \right) + 1 \right] \\
 & + \frac{\kappa m}{\Gamma(\lambda-1)} \left(\frac{9\varrho(T)}{4} + 2 \right) \\
 & + \frac{\varrho(T)}{\Gamma(\lambda)} \left[\frac{3m\eta_1}{2\Gamma(\theta+1-\lambda)} \|\varphi\|_{L_\omega^{p,q}((0,T),\mathbb{R})} + \frac{3\kappa m}{2} \right] \\
 & + \frac{\eta_2}{\Gamma(\theta)} \|\varphi\|_{L_\omega^{p,q}} \\
 & = \|\varphi\|_{L_\omega^{p,q}(\mathcal{L},\mathbb{R})} \left[\frac{m\eta_1}{\Gamma(\lambda-1)\Gamma(\theta-\lambda+1)} \left(\frac{\varrho(T)}{2} \left(\frac{9}{2} + \frac{1}{\theta+1-\lambda} \right) + 1 \right) \right. \\
 & + \frac{3\varrho(T)m\eta_1}{2\Gamma(\theta+1-\lambda)\Gamma(\lambda)} + \frac{\eta_2}{\Gamma(\theta)} \left. \right] \\
 & + \kappa m \left[\frac{1}{\Gamma(\lambda-1)} \left(\frac{9\varrho(T)}{4} + 2 \right) + \frac{3\varrho(T)}{2\Gamma(\lambda)} \right] \\
 & = \varepsilon_1,
 \end{aligned}$$

where ε_1 is given in (35). Since $\varepsilon_1 < 1$, the last inequality leads to a contradiction, and our claim in this step is proved.

Step 2. Our claim in this step is to show that the graph of is closed on T_{n_0} .

Suppose that $(z_n)_{n \geq 1}, (x_n)_{n \geq 1}$ are two sequences in T_{n_0} with $z_n \rightarrow z, x_n \rightarrow x$ in T_{n_0} , and $z_n \in (x_n); n \in \mathbb{N}$. Then, there is $u_n \in S_{F(\cdot, z_n)}^{v(2-\theta),p}$, and x_n is given by (38). Note that by $(F_2), \|u_n(\xi)\| \leq \varphi(\xi)(1+n_0)$, a.e. and hence, $\{u_n\}_{n \geq 1}$ is weakly compact in $L_\omega^{p,q}(E, \mathbb{R}^+), P > 2$. Making use of Mazur's lemma, it can be found as a subsequence that

$\{u_{n_k}\}_{k \geq 1}$ and $u_{n_k} \rightarrow u \in L_{\omega}^{1,q}(E, \mathbb{R}^+)$. Then, there is a subsequence $\{u_n^*\}_{n \geq 1}$ of $\{u_n\}_{n \geq 1}$ such that u_n^* tends to u almost everywhere. Now, for any $n \in \mathbb{N}$, set

$$x_n^*(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z_n, u_n^*) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z_n, u_n^*) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u_n^*(\xi), \xi \in \mathcal{E}_i, i \in L_0. \end{cases}$$

Obviously, (x_n^*) is a subsequence of (x_n^*) , and hence $x_n^* \rightarrow x$ in $PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$. We have to show that

$$x(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z, u) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z, u) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u(\xi), \xi \in \mathcal{E}_i, i \in L_0, \end{cases} \quad (52)$$

and $u \in S_{F(\cdot, z)}^{v(2-\theta), p}$. Using Lebesgue's convergence theorem, we obtain $\tau_i(u_{n_k}) \rightarrow \tau_i(u)$, $\varsigma_i(u_{n_k}) \rightarrow \varsigma_i(u)$ for any $i \in L_1 \cup \{m+1\}$, and

$$\lim_{n \rightarrow \infty} I_{\xi_i, \xi}^{\theta, \varrho, \omega} u_{n_k}(\xi) = I_{\xi_i, \xi}^{\theta, \varrho, \omega} u(\xi), \forall \xi \in \mathcal{E}_i, i \in L_0,$$

Moreover, assumption (H) gives us

$$\lim_{n \rightarrow \infty} \aleph_i(z_n(\xi_n^-)) = \aleph_i(z(\xi_n^-)) \forall i \in L_1,$$

and

$$\lim_{n \rightarrow \infty} \Re_i(z_n(\xi_n^-)) = \Re_i(z(\xi_n^-)), \forall i \in L_1.$$

As a result, $c_i(z_n, u_{n_k}) \rightarrow \tau_i(z, u)$ and $d_i(z_n, u_{n_k}) \rightarrow d_i(z, u)$, $\forall i \in L_0$. It yields from this discussion the validity of (52). Furthermore, assumption (F₃) leads to $u \in S_{F(\cdot, z)}^{v(2-\theta), p}$, and this proves our aim in this step.

Step 3. For every $i \in L_0$, let

$$\Omega_{|\bar{\mathcal{E}}_i} = \left\{ x^* : \bar{\mathcal{E}}_i \rightarrow E, x^*(\xi) = (\varrho(\xi) - \varrho(\sigma_i))^{2-\lambda} \omega(\xi) x(\xi), \xi \in \mathcal{E}_i, \right. \\ \left. x^*(\xi_i) = \lim_{\xi \rightarrow \xi_i^+} x^*(\xi), x \in \Psi(z), z \in T_{n_0} \right\}.$$

In this step, we demonstrate that $\Omega_{|\bar{\mathcal{E}}_i}$ is equicontinuous in $PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$. Let $i \in L_0$ and $x^* \in \Omega_{|\bar{\mathcal{E}}_i}$. Then, there exist $z \in T_{n_0}$ and $x \in \Psi(z)$ such that $x^*(\xi) = (\varrho(\xi) - \varrho(\xi_i))^{2-\lambda} \omega(\xi) x(\xi)$, $\xi \in \mathcal{E}_0$ and $x^*(0) = \lim_{\xi \rightarrow 0^+} x^*(\xi)$. According to the definition of Ψ , there is $u \in S_{F(\cdot, z)}^{v(2-\theta)}$ with

$$x(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z, u) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z, u) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u(\xi), \xi \in \mathcal{E}_i. \end{cases}$$

Then,

$$x^*(\xi) = \frac{c_0(z, u)}{\Gamma(\lambda-1)} + \frac{(\varrho(\xi) - \varrho(\xi_i))}{\Gamma(\lambda)} d_i(z, u) \\ + \frac{(\varrho(\xi) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\theta)} \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(s))^{\theta-1} \varrho'(\sigma) u(\sigma) \omega(\sigma) d\sigma.$$

Now, assume that $\xi \in \mathcal{E}_i$ and $\delta > 0$ with $\xi + \delta \in \mathcal{E}_i$. We have

$$\begin{aligned}
 & \|x^*(\xi + \delta) - x^*(\xi)\| \\
 & \leq \frac{|\varrho(\xi + \delta) - \varrho(\xi)|}{\Gamma(\lambda)} d_i(z, u) \\
 & + \left| \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\theta)} \int_{\xi_i}^{\xi+\delta} (\varrho(\xi + \delta) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right. \\
 & \left. - \frac{(\varrho(\xi) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\theta)} \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right| \\
 & \leq \frac{|\varrho(\xi + \delta) - \varrho(\xi)|}{\Gamma(\lambda)} d_i(x, u) \\
 & + \left| \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\lambda)} \int_{\xi}^{\xi+\delta} (\varrho(\xi + \delta) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right| \\
 & + \left| \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\lambda)} \int_{\xi_i}^{\xi} (\varrho(\xi + \delta) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right. \\
 & \left. - \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\lambda)} \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right| \\
 & + \left| \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\lambda)} \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right. \\
 & \left. - \frac{(\varrho(\xi) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\theta)} \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right|.
 \end{aligned}$$

Then,

$$\begin{aligned}
 & \|x^*(\xi + \delta) - x^*(\xi)\| \\
 & \leq \frac{|\varrho(\xi + \delta) - \varrho(\xi)|}{\Gamma(\lambda)} x_1 \\
 & + \left| \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\lambda)} \int_{\xi}^{\xi+\delta} (\varrho(\xi + \delta) - \varrho(s))^{\theta-1} \varrho'(\sigma) \omega(\sigma) f(\sigma) d\sigma \right| \\
 & + \left| \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\lambda)} \right| \times \\
 & \left| \int_{\xi_i}^{\xi} (\varrho(\xi + \delta) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right. \\
 & \left. - \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right| \\
 & + \left| \frac{(\varrho(\xi + \delta) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\lambda)} - \frac{(\varrho(\xi) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\theta)} \right| \times \\
 & \left| \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) u(\sigma) d\sigma \right|.
 \end{aligned} \tag{53}$$

Since $\theta > 1$, ϱ is continuous, and $\|u(\sigma)\| \leq \varphi(\sigma)(1 + n_0)$, it results from (52) that $\lim_{\delta \rightarrow 0} \|x^*(\xi + \delta) - x^*(\xi)\| = 0$, independent of the choice of x and z .

Step 4. In this step, we demonstrate that the set $\mathfrak{S} = \bigcap_{n \in \mathbb{N}} T_n$ is not empty and compact in $PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, where $T_0 = T_{n_0}$, $T_n = \Psi(T_{n-1})$; $n \geq 1$. Since each T_n is closed, then from the Cantor intersection property [44], it is sufficient to prove the following relation:

$$\lim_{n \rightarrow \infty} \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_n) = 0, \tag{54}$$

where $\chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_n)$ is the measure of noncompactness on $PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, which is presented in the Introduction. Let $n \in \mathbb{N}$ and $\epsilon > 0$. Utilizing Lemma 5 in [45], there exists a sequence $(u_r)_{r \geq 1}$ in $T_n = \Psi(T_{n-1})$ such that

$$\begin{aligned} \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_n) &\leq 2 \left\{ \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} u_r : r \geq 1 \right\} + \epsilon \\ &= 2 \max_{i \in L_0} \chi_{C_{\omega}(\overline{\mathcal{E}_i}, E)} \left(Y_{|\overline{\mathcal{E}_i}} \right) + \epsilon, \end{aligned} \quad (55)$$

where $\Omega_{|\overline{\mathcal{E}_i}}$ is given in (5). Now, from step (3), the sets $(Y_{|\overline{\mathcal{E}_i}}), i \in L_0$ are equicontinuous, and consequently, (55) becomes

$$\chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_n) \leq \max_{i \in L_0} \max_{\xi \in \overline{\mathcal{E}_i}} \chi \left\{ (\varrho(\xi) - \varrho(\xi_i))^{2-\lambda} \omega(\xi) u_r(\xi) : r \geq 1 \right\} + \epsilon, \quad (56)$$

where χ is the measure of compactness in E .

Now, since $u_r \in T_n = \Psi(T_{n-1})$, there is $z_r \in T_{n-1}$ with $u_r \in \Psi(z_r)$. By the definition of Ψ , there is $u_r \in S_{F(\cdot, z_r)}^{v(2-\theta), p}$, such that

$$u_r(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z_r, u_r) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z_r, u_r) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} u_r(\xi), \xi \in \mathcal{E}_i, i \in L_0. \end{cases} \quad (57)$$

Then, for any $i \in L_0$, we have

$$\begin{aligned} \Lambda_i &= \max_{\xi \in \mathcal{E}_i} \chi \left\{ (\varrho(\xi) - \varrho(\xi_i))^{2-\lambda} \omega(\xi) u_r(\xi) : r \geq 1 \right\} \\ &\leq \frac{\mathcal{U}_i}{\Gamma(\lambda-1)} + \frac{\Omega_i}{\Gamma(\lambda)} + \Delta_i, \end{aligned} \quad (58)$$

where

$$\mathcal{U}_i = \chi \{ d_i(u_r, u_r) : r \geq 1 \}; i \in L_1, \quad (59)$$

$$\Omega_i = \chi \{ c_i(u_r, u_r) : r \geq 1 \}; i \in L_1, \quad (60)$$

and

$$\Delta_i = \chi \left\{ (\varrho(\xi) - \varrho(\xi_i))^{2-\lambda} \omega(\xi) I_{\xi_i, \xi}^{\theta, \varrho, \omega} u_r(\xi) : r \geq 1 \right\}, i \in L_0. \quad (61)$$

Also,

$$Y_k = \chi \{ \varsigma_k(u_r) : r \geq 1 \}, k \in \{1, 2, \dots, m+1\}, \quad (62)$$

$$\Pi_k = \chi \{ \tau_k(u_r) : r \geq 1 \}, k \in \{1, 2, \dots, m+1\}. \quad (63)$$

Due to (F_4) , it results for almost everywhere that $\sigma \in \mathcal{E}_i; i \in L_0$,

$$\begin{aligned} \chi \{ u_r(\sigma) : r \geq 1 \} &\leq \chi_E(\cup_{r \in \mathbb{N}} F(\sigma, z_r(\sigma))) \\ &\leq \varsigma(\sigma) \left((\varrho(\sigma) - \varrho(\xi_i))^{2-\lambda} \omega(\sigma) \chi \{ z_r(\sigma) : r \in \mathbb{N} \} \right. \\ &\quad \left. \leq \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1}) \beta(\sigma) \right). \end{aligned} \quad (64)$$

Next, from (40)–(43), (62)–(64), it results for $k \in \{1, 2, \dots, m+1\}$ that

$$\begin{aligned}
Y_k &\leq \frac{1}{\Gamma(\theta+1-\lambda)} \int_{\xi_{k-1}}^{\xi_k} (\varrho(\xi_k) - \varrho(\sigma))^{\theta-\lambda} \varrho'(\sigma) \chi\{u_r(\sigma) : r \geq 1\} \omega(\sigma) d\sigma. \\
&\leq \frac{\chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{\Gamma(\theta+1-\lambda)} \int_{\xi_{k-1}}^{\xi_k} (\varrho(\xi_k) - \varrho(\sigma))^{\theta-\lambda} \beta(\sigma) \varrho'(\sigma) \omega(\sigma) d\sigma \\
&\leq \frac{\chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{\Gamma(\theta+1-\lambda)} \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})},
\end{aligned} \tag{65}$$

and

$$\Pi_k \leq \frac{\chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{\Gamma(\theta+2-\lambda)} \varrho(T) \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})}. \tag{66}$$

Since $\mathfrak{R}_k$ and $\mathfrak{N}_k; k \geq 1$ map bounded sets to relatively compact sets, it yields from (65) and (66) that

$$\mathfrak{U}_0 \leq \frac{m \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{2\Gamma(\theta+1-\lambda)} \varrho(T) \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})}. \tag{67}$$

and

$$\mathfrak{U}_i \leq \frac{3m \eta_1 \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{2\Gamma(\theta+1-\lambda)} \|\beta\|_{L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)}. \tag{68}$$

Similarly,

$$\begin{aligned}
\Omega_0 &\leq \frac{1}{2} \left[\sum_{k=1}^{k=m+1} (\varrho(\xi_k) - \varrho(\xi_{k-1})) \mathfrak{U}_{k-1} + \sum_{k=1}^{k=m+1} \Pi_k \right] \\
&\leq \frac{3m \eta_1 \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{4\Gamma(\theta+1-\lambda)} \|\beta\|_{L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)} \sum_{k=1}^{k=m+1} \varrho(\xi_k) - \varrho(\xi_{k-1}) \\
&\quad + \frac{m \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{2\Gamma(\theta+2-\lambda)} \varrho(T) \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})} \\
&\leq \frac{3m \eta_1 \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1}) \varrho(T)}{4\Gamma(\theta+1-\lambda)} \|\beta\|_{L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)} \\
&\quad + \frac{m \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{2\Gamma(\theta+2-\lambda)} \varrho(T) \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})},
\end{aligned} \tag{69}$$

and for $i \in L_0$,

$$\begin{aligned}
\Omega_i &= \Omega_0 + \sum_{k=1}^{k=i} (\varrho(\xi_k) - \varrho(\xi_{k-1})) \mathfrak{U}_{k-1} + \sum_{k=1}^{k=i} Y_k \\
&\leq \frac{3m \eta_1 \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1}) \varrho(T)}{4\Gamma(\theta+1-\lambda)} \|\beta\|_{L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)} \\
&\quad + \frac{m \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{2\Gamma(\theta+2-\lambda)} \varrho(T) \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})} \\
&\quad + \frac{3m \eta_1 \varrho(T) \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{2\Gamma(\theta+1-\lambda)} \|\beta\|_{L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)} \\
&\quad + \frac{m \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1})}{\Gamma(\theta+1-\lambda)} \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})} \\
&= \chi_{PC_{2-\lambda, \varrho, \omega}}(\mathcal{E}, E)(T_{n-1}) m \eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})} \left[\frac{9\varrho(T) + 4}{4\Gamma(\theta+1-\lambda)} \right. \\
&\quad \left. + \frac{\varrho(T)}{2\Gamma(\theta+2-\lambda)} \varrho(T) \right].
\end{aligned} \tag{70}$$

Also, it results from (43) and (63) that

$$\begin{aligned}\Delta_i &\leq \frac{(\varrho(\xi) - \varrho(\xi_i))^{2-\lambda}}{\Gamma(\theta)} \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \omega(\sigma) \chi\{u_r(\sigma) d\sigma : r \in \mathbb{N}\} \\ &\leq \frac{\varrho(T) \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1})}{\Gamma(\theta)} \int_{\xi_i}^{\xi} (\varrho(\sigma) - \varrho(\xi_i))^{\theta-1} \varrho'(\sigma) \omega(\sigma) \beta(\sigma) d\sigma \\ &= \frac{\varrho(T)}{\Gamma(\theta)} \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1}) \eta_2 \|\varphi\|_{L^p((0, T), E)}, i \in L_0.\end{aligned}\quad (71)$$

Next, from (57) and (67)–(71), we obtain for $i \in L_0$,

$$\begin{aligned}\Lambda_i &\leq \frac{3m\eta_1 \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1})}{\Gamma(\lambda - 1)2\Gamma(\theta + 1 - \lambda)} \|\beta\|_{L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)} \\ &\quad + \frac{\chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1}) m\eta_1 \|\beta\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})}}{\Gamma(\lambda)} \left(\frac{9\varrho(T) + 4}{4\Gamma(\theta + 1 - \lambda)} + \frac{\varrho(T)}{2\Gamma(\theta + 2 - \lambda)} \right) \\ &\quad + \frac{\varrho(T)}{\Gamma(\theta)} \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1}) \eta_2 \|\varphi\|_{L^p((0, T), E)} \\ &= \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1}) \|\varphi\|_{L^p((0, T), E)} \times \\ &\quad \left[\frac{3m\eta_1}{\Gamma(\lambda - 1)2\Gamma(\theta + 1 - \lambda)} + \frac{m\eta_1}{\Gamma(\lambda)} \left(\frac{9\varrho(T) + 4}{4\Gamma(\theta + 1 - \lambda)} + \frac{\varrho(T)}{2\Gamma(\theta + 2 - \lambda)} \right) \right. \\ &\quad \left. + \frac{\eta_2 \varrho(T)}{\Gamma(\theta)} \right].\end{aligned}\quad (72)$$

Then, relations (55)–(57) and (73) lead to

$$\chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_n) \leq \varepsilon_2 \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_{n-1}), \forall n \geq 1, \quad (74)$$

where ε_2 is given by (36). It follows from (74) that

$$\chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_n) \leq \varepsilon_2^n \chi_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}(T_0), \forall n \geq 1,$$

and consequently, it yields, from (36), the validity of (53). This shows that T is non-empty and compact in $PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$.

Step 5. As a result of steps 1 through 4, $(x); x \in \mathfrak{S}$ is not empty, convex, and compact in $PC_{2-\lambda, \varrho, \omega}$, so the multivalued function $: \mathfrak{S} \rightarrow P_{ck}(PC_{2-\lambda, \varrho, \omega})$ satisfies the assumptions of Lemma 6, and hence, there exists $x \in \mathfrak{S}$ with $x \in (x)$. Then, x is an antiperiodic solution to problem (3). Due to the first step, we conclude that the set of fixed points of Ψ is bounded, and by using Lemma 7, we deduce that the set of antiperiodic solutions to problem (3) is compact. $\square$

Now, we give another existence result of antiperiodic solutions for Problem (3).

Theorem 2. Suppose that $F : \mathcal{E} \times E \rightarrow P_{ck}(E)$ and $\mathfrak{N}_i, \mathfrak{R}_i : E \rightarrow E; i \in L_1$. We assume the following conditions:

- (F₁)^{*} For any $z \in PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, the set $S_{F(\cdot, z)}^{v(2-\theta), p}$ is not empty.
- (F₂)^{*} There is a $\varphi \in L_{\omega}^{p, \varrho}(\mathcal{E}, \mathbb{R}^+)$, $p > \frac{1}{\theta-1}$ such that
- (i) For every $z_1, z_2 \in PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)$, and for almost $\xi \in \mathcal{E}$

$$h(F(\xi, z_1(\xi)), F(\xi, z_2(\xi))) \leq \varphi(\xi) \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)},$$

where h is the Hausdorff distance.

(ii) For almost $\xi \in \mathcal{L}$,

$$h(F(\xi, 0), \{0\}) \leq \varphi(\xi).$$

(H)* There is $\kappa^* > 0$ such that for any $i \in L_1$ and any $u, v \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$, one has

$$\|\Re_i(u(\xi)) - \Re_i(v(\xi))\| \leq \kappa^* \|u - v\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L}$$

and

$$\|\Re_i(u(\xi)) - \Re_i(v(\xi))\| \leq \kappa^* \|u - v\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L}.$$

Then, Problem (3) has a solution provided that

$$\begin{aligned} \varepsilon_3 = & \|\varphi\|_{L_{\omega}^{p, \varrho}(\mathcal{L}, \mathbb{R})} \left[\frac{m\eta_1}{\Gamma(\lambda-1)\Gamma(\theta-\lambda+1)} \left(\frac{\varrho(T)}{2} \left(\frac{9}{2} + \frac{1}{\theta+1-\lambda} \right) + 1 \right) \right. \\ & \left. + \frac{3\varrho(T)m\eta_1}{2\Gamma(\theta+1-\lambda)\Gamma(\lambda)} + \frac{\eta_2}{\Gamma(\theta)} \right] + \kappa^* m \left[\frac{1}{\Gamma(\lambda-1)} \left(\frac{9\varrho(T)}{4} + \frac{3}{2} \right) + \frac{3\varrho(T)}{2\Gamma(\lambda)} \right] \\ & < 1. \end{aligned} \quad (75)$$

Proof. Let Ψ be a multivalued function defined on $PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$ as in Theorem 1. Since by $(F_2)^*$,

$$\begin{aligned} \|F(\xi, z)\| &= h(F(\xi, z), F(\xi, 0)) + h(F(\xi, 0), \{0\}) \\ &\leq \varphi(\xi) \|z\| + \varphi(\xi) = \varphi(\xi) (1 + \|z\|), a.e. \end{aligned}$$

So, the assumption (F_2) of Theorem 1 is satisfied. Moreover, the assumption (i) in $(F_2)^*$ implies (F_3) of Theorem 1. Then, by the same argument in step 2 of Theorem 1, we conclude that the set of values of Ψ is not empty and is closed as subsets in $PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$. Next, we demonstrate that Ψ is a contraction. For this purpose, let $z_1, z_2 \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$ and $x_1 \in \Psi(z_1)$. According to the definition of Ψ , there is a function $f \in S_{F(\cdot, z_1)}^{v(2-\theta), p}$ such that

$$x_1(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z_1, f) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z_1, f) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} f(\xi), \xi \in \mathcal{L}_i, i \in L_0. \end{cases} \quad (76)$$

Define a multivalued function $\Xi : \mathcal{L} \rightarrow 2^E$ by

$$\Xi(\xi) = \left\{ u \in F(\xi, z_2(\xi)) : \|u - f(\xi)\| \leq \varphi(\xi) \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, a.e. \right\}.$$

Since $f(\xi) \in F(\xi, z_1(\xi))$, a.e., the assumption (i) in $(F_2)^*$ confirms that the set of values of Ξ is not empty. In view of Proposition 3.4 in [46], the multifunction Ξ is measurable. Because the set of values of Ξ is non-empty and compact, then by Ref. [47] (Theorem 1.3.1), there is an $Y \in S_{F(\cdot, z_2)}^{v(2-\theta), p}$ with

$$\|Y(\xi) - f(\xi)\| \leq \varphi(\xi) \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, a.e. \quad (77)$$

Next, we define

$$x_2(\xi) = \begin{cases} \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-2}}{\Gamma(\lambda-1)} c_i(z_2, Y) + \omega^{-1}(\xi) \frac{(\varrho(\xi) - \varrho(\xi_i))^{\lambda-1}}{\Gamma(\lambda)} d_i(z_2, Y) \\ + I_{\xi_i, \xi}^{\theta, \varrho, \omega} Y(\xi), \xi \in \mathcal{L}_i, i \in L_0. \end{cases} \quad (78)$$

Note that $x_2 \in (z_2)$. In order to demonstrate that Ψ is a contraction, it is enough to prove $h((z_1), (z_2)) < \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}$, and this will be satisfied if the following inequality

$$\|x_1 - x_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)} < \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}. \quad (79)$$

holds. By (11)–(14), (23), (25), (27), (31), (77), and $(H)^*$, it results in

$$\begin{aligned} & \|d_0(z_1, f) - d_0(z_2, Y)\| \\ & \leq \frac{1}{2} \sum_{k=1}^{k=m+1} \|\varsigma_k(f) - \varsigma_k(Y)\| + \frac{1}{2} \sum_{k=1}^{k=m} \|\Re_k(z_1(\xi_k^-)) - \Re_k(z_2(\xi_k^-))\| \\ & = \frac{1}{2\Gamma(\theta - \lambda + 1)} \sum_{i=1}^{i=m+1} \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(\sigma))^{\theta-\lambda} \varrho'(\sigma) \|f(\sigma) - Y(\sigma)\| \omega(\sigma) d\sigma \\ & \quad + \kappa^* m \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \\ & \leq \frac{\|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}}{2\Gamma(\theta - \lambda + 1)} \sum_{i=1}^{i=m+1} \int_{\xi_{i-1}}^{\xi_i} (\varrho(\xi_i) - \varrho(\sigma))^{\theta-\lambda} \varrho'(\sigma) \varphi(\xi) \omega(\sigma) d\sigma \end{aligned} \quad (80)$$

$$\begin{aligned} & + \frac{\kappa^* m}{2} \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \\ & \leq \frac{\|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} m \eta_1}{2\Gamma(\theta - \lambda + 1)} \|\varphi\|_{L^p((0, T), E)} + \kappa m \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \\ & = \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \left[\frac{m \eta_1}{2\Gamma(\theta - \lambda + 1)} \|\varphi\|_{L^p((0, T), E)} + \frac{\kappa^* m}{2} \right], \end{aligned} \quad (81)$$

$$\begin{aligned} & \|d_i(z_1, f) - d_i(z_2, Y)\| \\ & \leq \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \left[\frac{3m \eta_1}{2\Gamma(\theta - \lambda + 1)} \|\varphi\|_{L^p((0, T), E)} + \frac{3\kappa^* m}{2} \right], \forall i \in L_1, \end{aligned} \quad (82)$$

$$\begin{aligned} & \|c_0(z_1, f) - c_0(z_2, Y)\| \\ & \leq \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \left[\frac{m \eta_1 \varrho(T)}{2\Gamma(\theta + 1 - \lambda)} \|\varphi\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})} \left(\frac{3}{2} + \frac{1}{\theta + 1 - \lambda} \right) \right. \\ & \quad \left. + \kappa^* m \left(\frac{3\varrho(T)}{4} + \frac{1}{2} \right) \right], \end{aligned} \quad (83)$$

and for $i \in L_1$,

$$\begin{aligned} & \|c_i(z_1, f) - c_i(z_2, Y)\| \\ & \leq \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \left[\frac{m \eta_1}{\Gamma(\theta - \lambda + 1)} \|\varphi\|_{L_{\omega}^{p, \varrho}((0, T), \mathbb{R})} \left(\frac{\varrho(T)}{2} \left(\frac{9}{2} + \frac{1}{\theta + 1 - \lambda} \right) + 1 \right) \right. \\ & \quad \left. + \kappa^* m \left(\frac{9\varrho(T)}{4} + \frac{3}{2} \right) \right]. \end{aligned} \quad (84)$$

From (76), (78), and (81)–(84), it yields

$$\begin{aligned} & \|x_1 - x_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)} \\ & \leq \frac{1}{\Gamma(\lambda - 1)} \max_{i \in L_0} \|c_i(z_1, f) - c_i(z_2, Y)\| \\ & \quad + \frac{1}{\Gamma(\lambda)} \max_{i \in L_0} (\varrho(\xi) - \varrho(\xi_i) \|d_i(z_1, f) - d_i(z_2, Y)\|) \\ & \quad + \frac{1}{\Gamma(\theta)} \max_{i \in L_0} \int_{\xi_i}^{\xi} (\varrho(\xi) - \varrho(\sigma))^{\theta-1} \varrho'(\sigma) \|f(\sigma) - Y(\sigma)\| \omega(\sigma) d\sigma \\ & \leq \varepsilon_3 \|x_1 - x_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{E}, E)}, \end{aligned}$$

where ε_3 is given by (75). Since $\varepsilon_3 < 1$, (75) is verified. Then, Ψ is a contraction, and hence, it has a fixed point [48], which is a solution for Problem (3). $\square$

Remark 7. If there is $h \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$ such that $f(\xi) = I_{\xi_i, \xi}^{v(2-\theta), \varrho} h(\xi), \forall \xi \in \mathcal{L}_k, k \in L_0$, then,

$$D_{\xi_i, \xi}^{v(2-\theta), \varrho, \omega} f(\xi) = D_{\xi_i, \xi}^{v(2-\theta), \varrho, \omega} I_{\xi_i, \xi}^{v(2-\theta), \varrho, \omega} h(\xi) = h(\xi), \forall \xi \in \mathcal{L}_k, k \in L_0,$$

and this implies that $f \in PC_{2-\lambda, \varrho, \omega}^{v(2-\theta)}(\mathcal{L}, E)$.

As a result, the assumption (F_1) will be fulfilled if the following assumption is satisfied:

$(F_1)^*$ For any $z \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$, the multivalued function $\xi \rightarrow F(\xi, z(\xi))$ is measurable and there is $f \in L_{\omega}^{p, \varrho}(I, E)$ and $h \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$ with $f(\xi) \in F(\xi, z(\xi)), a.e.$ and $f(\xi) = I_{\xi_k, \xi}^{v(2-\theta), \varrho, \omega} h(\xi), \forall \xi \in \mathcal{L}_k, k \in L_0$.

Indeed,

$$D_{\xi_k, \xi}^{v(2-\theta), \varrho, \omega} f(\xi) = D_{\xi_k, \xi}^{v(2-\theta), \varrho, \omega} I_{\xi_k, \xi}^{v(2-\theta), \varrho, \omega} h(\xi) = h(\xi).$$

Then, $f \in S_{F(\cdot, z)}^{v(2-\theta), p}$.

4. Examples

In this section, we give two examples to show that our results are applicable.

Example 1. Let E be a Hilbert space, Ω a non-empty, convex, and compact subset of E , $T = 3$, $\mathcal{L} = [0, 3]$, $m = 2$, $\xi_0 = 0$, $\xi_1 = 1$, $\xi_2 = 2$, $\xi_3 = 3$, $\theta = \frac{4}{3}$, $v = \frac{1}{2}$, and $P = 4$. Then, $v(2-\theta) = \frac{1}{6}$, $\frac{1}{\theta-1} = 3 < p, \lambda = \theta + 2v - \theta v = \frac{3}{2}$ and $2-\lambda = \frac{1}{2}$, $\omega : \mathcal{L} \rightarrow]0, \infty[$, $\varrho : \mathcal{L} \rightarrow \mathbb{R}$ be a strictly increasing continuously differentiable function with $\varrho'(s) \neq 0$, for any $s \in \mathcal{L}$ and $\rho = \sup\{\|x\| : x \in \Omega\}$. We define a multifunction $F : \mathcal{L} \times E \rightarrow P_{ck}(E)$ such that for any $z \in PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)$,

$$F(\xi, z(\xi)) = \begin{cases} \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \|z(0)\| \Omega, \xi \in [0, 1], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(1))^{\frac{1}{2}} \|z(1)\| \Omega, \xi \in (1, 2], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{2}} \|z(2)\| \Omega, \xi \in (2, 3], \end{cases} \quad (85)$$

where $\eta > 0$. Let $z \in PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)$. Obviously, the multivalued function $\xi \rightarrow F(\cdot, z(\cdot))$ is measurable. Let $f_z, h_z : \mathcal{L} \rightarrow E$ be defined by

$$f_z(\xi) = \begin{cases} \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \|z(0)\| v, \xi \in [0, 1], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(1))^{\frac{1}{2}} \|z(1)\| v, \xi \in (1, 2], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{2}} \|z(2)\| v, \xi \in (2, 3], \end{cases}$$

and

$$h_z(\xi) = \begin{cases} \frac{\eta \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})} I_{\xi_1, \xi}^{\frac{1}{6}, \varrho, \omega} \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(1))^{\frac{1}{3}} \right) \|z(0)\| v, \xi \in [0, 1], \\ \frac{\eta \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})} \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(1))^{\frac{1}{3}} \right) \|z(1)\| v, \xi \in (1, 2], \\ \frac{\eta \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})} I_{\xi_1, \xi}^{\frac{1}{6}, \varrho, \omega} \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(1))^{\frac{1}{3}} \right) \|z(2)\| v, \xi \in (2, 3], \end{cases}$$

where $v \in \Omega$ is a fixed point. Obviously, $f(\xi) \in F(\xi, z(\xi)), a.e.$ and, by the first assertion of Lemma 2, we obtain

$$\begin{aligned} I_{0, \xi}^{v(2-\theta), \varrho, \omega} h_z(\xi) &= I_{0, \xi}^{\frac{1}{6}, \varrho, \omega} h_u(\xi) \\ &= \frac{\eta \|z(0)\| v \Gamma(\frac{3}{2})}{\Gamma(\frac{4}{3})} I_{0, \xi}^{\frac{1}{6}, \varrho, \omega} \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{3}} \right) \\ &= \eta \|z(0)\| v \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{3}} \right); \xi \in [0, 1], \end{aligned}$$

$$\begin{aligned}
I_{\xi_1, \xi}^{v(2-\theta), \varrho, \omega} h_z(\xi) &= I_{\xi_1, \xi}^{\frac{1}{6}, \varrho, \omega} h_u(\xi) \\
&= \frac{\eta \|z(1)\| v \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{4}{3}\right)} I_{\xi_1, \xi}^{\frac{1}{6}, \varrho, \omega} \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(1))^{\frac{1}{3}}\right) \\
&= \eta \|z(1)\| v \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{3}}\right); \xi \in [1, 2],
\end{aligned}$$

and

$$\begin{aligned}
I_{\xi_2, \xi}^{v(2-\theta), \varrho, \omega} h_z(\xi) &= I_{\xi_2, \xi}^{\frac{1}{6}, \varrho, \omega} h_u(\xi) \\
&= \frac{\eta \|z(2)\| v \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{4}{3}\right)} I_{\xi_2, \xi}^{\frac{1}{6}, \varrho, \omega} \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{3}}\right) \\
&= \eta \|z(2)\| v \left(\omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{3}}\right); \xi \in [2, 3].
\end{aligned}$$

Therefore, $f_z(\xi) = I_{\xi_k, \xi}^{v(2-\theta), \varrho, \omega} h_u(\xi)$, $\forall \xi \in \mathcal{L}_k, k \in \{0, 1, 2\}$, and hence by Remark 7, (F_1) is satisfied. In addition, for any $y \in F(\xi, z(\xi))$, one has

$$\begin{aligned}
\|y(\xi)\| &\leq \begin{cases} \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \rho \|z(0)\|, & \xi \in [0, 1], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \rho \|z(1)\|, & \xi \in (1, 2], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{2}} \rho \|z(2)\|, & \xi \in (2, 3]. \end{cases} \\
&\leq \eta \omega^{-2}(\xi) \lambda \|z\|_{PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)}.
\end{aligned}$$

Then, (F_2) is verified with $\varphi(\xi) = \eta \omega^{-2}(\xi) \rho$, $\forall \xi \in [0, 3]$. Note that

$$\begin{aligned}
\|\varphi\|_{L_{\omega}^{p, \varrho}(\mathcal{L}, R^+)} &= \left(\int_0^3 |\varphi(s) \omega(s)|^p \varrho'(s) ds \right)^{\frac{1}{p}} \\
&= \left(\int_0^3 |\varphi(s) \omega(s)|^4 \varrho'(s) ds \right)^{\frac{1}{4}} \\
&= \eta \rho \int_0^3 |\omega(s)|^2 \varrho'(s) ds.
\end{aligned} \tag{86}$$

Now, let D be a bounded subset of $PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E) = PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)$, $z_1, z_2 \in D$, $\xi \in (0, 1]$, and $x \in F(\xi, z_1(\xi))$. Then,

$$x = \begin{cases} \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \|z_1(0)\| e^*, & \xi \in [0, 1], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \|z_1(1)\| e^*, & \xi \in (1, 2], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{2}} \|z_1(2)\| e^*, & \xi \in (2, 3], \end{cases}$$

where $e^* \in \Omega$. Set

$$y = \begin{cases} \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \|z_2(0)\| e^*, & \xi \in [0, 1], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \|z_2(1)\| e^*, & \xi \in (1, 2], \\ \eta \omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{2}} \|z_2(2)\| e^*, & \xi \in (2, 3]. \end{cases}$$

From (32) and (33), it yields $y \in F(\xi, z_2(\xi))$, and if $\xi \in [0, 1]$, then

$$\begin{aligned}
\|x - y\| &\leq \eta \rho \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} [|\|z_1(\xi)\| - \|z_2(\xi)\||] \\
&\leq \eta \rho \omega^{-1}(\xi)(\varrho(\xi) - \varrho(0))^{\frac{1}{2}} \|(z_1 - z_2)(\xi)\|.
\end{aligned}$$

Similarly,

$$\|x - y\| \leq \eta \rho \omega^{-1}(\xi)(\varrho(\xi) - \varrho(1))^{\frac{1}{2}} \|(z_1 - z_2)(\xi)\|, \forall \xi \in (1, 2].$$

and

$$\|x - y\| \leq \eta \rho \omega^{-1}(\xi)(\varrho(\xi) - \varrho(2))^{\frac{1}{2}} \|(z_1 - z_2)(\xi)\|, \forall \xi \in (2, 3].$$

As a result, $\|x - y\| \leq \eta \rho \omega^{-1}(\xi) \|z_1 - z_2\|_{PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)}$, and consequently,

$$h(F(\xi, z_1(\xi)), F(\xi, z_1(\xi))) \leq \eta \rho \omega^{-2}(\xi) \|z_1 - z_2\|_{PC_{2-\lambda, \varrho, \omega}}. \quad (87)$$

and for almost $\xi \in \mathcal{L}_k, k = 0, 1, 2$,

$$\begin{aligned} \chi_E(F(\xi, z(\xi)) : z \in D) \\ \leq (\varrho(\xi) - \varrho(s_k))^{2-\lambda} \beta(\xi) \chi_E\{z(\xi) : z \in D\}. \end{aligned}$$

Therefore, both (F_3) and (F_4) hold, where $\beta(\xi) = \eta \rho \omega^{-2}(\xi); \xi \in \mathcal{L}$. Note that

$$\begin{aligned} \|\beta\|_{L_{\omega}^{p, \varrho}(\mathcal{L}, R^+)} &= \left(\int_0^3 |\varphi(s) \omega(s)|^p \varrho'(s) ds \right)^{\frac{1}{p}} \\ &= \left(\int_0^3 |\varphi(s) \omega(s)|^4 \varrho'(s) ds \right)^{\frac{1}{4}} \\ &= \eta \rho \int_0^3 |\omega(s)|^2 \varrho'(s) ds. \end{aligned} \quad (88)$$

Next, let Y and Y^* be two linear bounded compact operators from E to E and define $\aleph_i, \Re_i : E \rightarrow E, i = 1, 2$, as follows:

$$\aleph_i(x) = \delta_i Y \left((\varrho(\xi) - \varrho(s_k))^{2-\lambda} \omega(\xi) x \right), \forall \xi \in \mathcal{L}_i, \forall x \in E, \quad (89)$$

and

$$\Re_i(x) = \delta_i^* Y^* \left((\varrho(\xi) - \varrho(s_k))^{2-\lambda} \omega(\xi) x \right), \forall \xi \in \mathcal{L}_i, \forall x \in E, \quad (90)$$

where $\delta_i, \delta_i^* > 0$. Then, $\aleph_i$ and $\Re_i$ are continuous, and for any $u \in PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)$, we have

$$\|\aleph_i(u(\xi))\| \leq \delta_i \|Y\| \|u\|_{PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L},$$

and

$$\|\Re_i(u(\xi))\| \leq \delta_i^* \|Y^*\| \|u\|_{PC_{\frac{1}{2}, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L},$$

Moreover, by the compactness of both Y and Y^* , $\aleph_i$ and $\Re_i$ map any bounded set to a relatively compact subset. So, condition (H) is satisfied and $\kappa = \max\{\delta_i, \delta_i^*, i = 1, 2\}$.

By considering (86), (87) and applying Theorem (1), Problem (3) has a solution, where $\theta = \frac{4}{3}, \nu = \frac{1}{4}, \lambda = \frac{3}{2}, F, \aleph_i$, and $\Re_i$ are given by (85), (89), and (90), provided that

$$\begin{aligned} \varepsilon_1 &= \eta \rho \int_0^3 |\omega(s)|^2 \varrho'(s) ds \left[\frac{2\eta_1}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{6}\right)} \left(\frac{57\varrho(3)}{20} + 1 \right) + \frac{6\varrho(3)\eta_1}{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{6}\right)} + \frac{\eta_2}{\Gamma\left(\frac{4}{3}\right)} \right] \\ &\quad + 2\kappa \left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{9\varrho(3)}{4} + \frac{3}{2} \right) + \frac{3\varrho(3)}{2\Gamma\left(\frac{3}{2}\right)} \right] \\ &< 1, \end{aligned} \quad (91)$$

and

$$\epsilon_2 = \eta \lambda \int_0^3 |\omega(s)|^2 \varrho'(s) ds \left[\frac{6\eta_1}{2\Gamma(\frac{1}{2})\Gamma(\frac{5}{6})} + \frac{2\eta_1}{\Gamma(\frac{3}{2})} \left(\frac{9\varrho(3) + 4}{4\Gamma(\frac{5}{6})} + \frac{\varrho(3)}{2\Gamma(\frac{11}{6})} \right) + \frac{\eta_2 \varrho(3)}{\Gamma(\frac{4}{3})} \right] < 1, \quad (92)$$

where $\eta_1 = \left(\frac{14}{18}\right)^{\frac{4}{3}} \varrho(3)^{\frac{7}{12}}$ and $\eta_2 = \left(\frac{9}{7}\right)^{\frac{4}{3}} \varrho(3)^{\frac{13}{12}}$. By choosing $\eta, \delta_i, \delta_i^*, i = 1, 2$, and ρ small enough, the relationships (91) and (92) are satisfied.

Example 2. Let $E, \Omega, T, \mathcal{L}, m, \xi_i, i = 0, 1, 2, 3, \theta, \nu, P$, and $F : \mathcal{L} \times E \rightarrow P_{ck}(E)$ and λ be as in Example (1). Assume that $\omega : \mathcal{L} \rightarrow]0, \infty[$, and let $\varrho : \mathcal{L} \rightarrow \mathbb{R}$ be a strictly increasing continuously differentiable function with $\varrho'(s) \neq 0$, for any $s \in \mathcal{L}$. It follows from (87) that F satisfies $(F_2)^*$. Next, we define $\aleph_i, \Re_i : E \rightarrow E, i = 1, 2$, as follows:

$$\aleph_i(x) = \kappa^* \left((\varrho(\xi) - \varrho(s_k))^{2-\lambda} \omega(\xi) x \right), \forall \xi \in \mathcal{L}_i, \forall x \in E, \quad (93)$$

and

$$\Re_i(x) = \kappa^* \left((\varrho(\xi) - \varrho(s_k))^{2-\lambda} \omega(\xi) x \right), \forall \xi \in \mathcal{L}_i, \forall x \in E, \quad (94)$$

where $\kappa^* > 0$. Then, for any $i \in L_1$ and any $u_1, u_2 \in PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)$, one has

$$\begin{aligned} & \| \aleph_i(u_1(\xi)) - \aleph_i(u_2(\xi)) \| \\ &= \kappa^* ((\varrho(\xi) - \varrho(s_k))^{2-\lambda} \omega(\xi) \|u_1(\xi) - u_2(\xi)\|) \leq \kappa^* \|u_1 - u_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L}. \end{aligned}$$

Likewise,

$$\begin{aligned} & \| \Re_i(u_1(\xi)) - \Re_i(u_2(\xi)) \| \\ &= \kappa^* ((\varrho(\xi) - \varrho(s_k))^{2-\lambda} \omega(\xi) \|u_1(\xi) - u_2(\xi)\|) \leq \kappa^* \|u_1 - u_2\|_{PC_{2-\lambda, \varrho, \omega}(\mathcal{L}, E)}, \forall \xi \in \mathcal{L}. \end{aligned}$$

Therefore, condition (H^*) is verified. By using Theorem (2), Problem (3) has a solution, where $\theta = \frac{4}{3}, \nu = \frac{1}{4}, \lambda = \frac{3}{2}, F, \aleph_i$, and $\Re_i$ are given by (85), (93), and (94), provided that (75) holds. As above, by choosing η, κ^* , and ρ small enough, relation (75) becomes verified.

5. Discussion and Conclusions

Kaslik et al. [14] showed that unlike the integer-order derivative, the fractional-order derivative of a periodic function cannot be a function with the same period. Consequently, there is no periodic solution for a large number of fractional-order differential systems on bounded intervals. As a result, much attention has been devoted to the study of antiperiodic solutions for differential equations and differential inclusions containing different fractional differential operators. Additionally, it is well-known that the ω -weighted ϱ -Hilfer fractional differential operator $D_{\xi_i, \xi}^{\theta, \nu, \varrho, \omega}$ is a generalization for several fractional differential operators. Therefore, in this work, we investigated two theorems concerning the existence of antiperiodic solutions for an impulsive differential inclusion containing $D_{\xi_i, \xi}^{\theta, \nu, \varrho, \omega}$ of order $\theta \in (1, 2)$ in Banach spaces. In the first theorem, it was shown that the set of antiperiodic solutions for the objective problem is not empty and compact. To achieve this target, the formula of antiperiodic solutions was first concluded; then, by making use of adequate fixed-point theorems for multivalued functions, two existence theorems of antiperiodic solutions for the considered problem were proven. Our work generalizes the obtained results in [21–24] (see Remark 1 and Remark 2). In addition, our procedure can be applied to generalize all the works mentioned in the Introduction when the considered fractional

differential operator in them is replaced by $D_{\xi_i, \xi}^{\theta, v, \varrho, \omega}$ and the space dimension is made infinite.

We propose the following next directions for future research:

- I. Search the presence of S -asymptotically ω -periodic solutions for Problem (3).
- II. Extend the recent work performed in [26,28,32,33] when the considered fractional differential operator is replaced by $D_{\xi_i, \xi}^{\theta, v, \varrho, \omega}$ and the space dimension is made infinite.
- III. Generalize the recent work performed by Al Nuwairan et al. [34], when the Atangana–Baleanu fractional derivative is replaced with a ϱ -Atangana–Baleanu fractional derivative.

Author Contributions: Methodology, Z.A., A.G.I., M.M.A.-S. and O.Y.A.; Investigation, Z.A., A.G.I., M.M.A.-S. and O.Y.A.; Resources, Z.A., A.G.I., M.M.A.-S. and O.Y.A.; Writing—original draft, Z.A., A.G.I., M.M.A.-S. and O.Y.A.; Writing—review & editing, Z.A., A.G.I., M.M.A.-S. and O.Y.A.; Funding acquisition, Z.A., A.G.I., M.M.A.-S. and O.Y.A.. All authors have read and agreed to the published version of the manuscript.

Funding: This paper is funded by the Deanship of Scientific Research, University of Ha'il, Kingdom of Saudi Arabia, project number RG-23 130.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Etemad, S.; Avci, I.; Kumar, P.; Baleanu, D.; Rezapour, S. Some novel mathematical analysis on the fractal-fractional model of the AH1N1/09 virus and its generalized Caputo-type version. *Chaos Solit. Fractals* **2022**, *162*, 112511. [CrossRef]
2. Tarasov, V.E. (Ed.) *Handbook of Fractional Calculus with Applications: Applications in Physics, Part A*; De Gruyter: Berlin, Germany, 2019; Volume 4.
3. Baleanu, D.; Lopes, A.M. *Handbook of Fractional Calculus with Applications: Applications in Engineering, Life and Social Sciences, Part A*; De Gruyter: Berlin, Germany, 2019.
4. Benito, F.; Salgado, M.; Sampayo, R.; Torres, A.; Fuentes, C. *Application of Fractional Calculus to Oil Industry*; INTECH: Rijeka, Croatia, 2017; Available online: <https://www.researchgate.net/publication/317636690> (accessed on 16 May 2024).
5. Dishlieva, K. Impulsive Differential Equations and Applications. *J. Appl. Comput. Math.* **2012**, *1*, e117. [CrossRef]
6. Agur, Z.; Cojocaru, L.; G. Mazaur, G.; Anderson, R.M.; Danon, Y.L. Pulse mass measles vaccination across age shorts. *Proc. Natl. Acad. Sci. USA* **1993**, *90*, 11698–11702. [CrossRef]
7. Church, K. Applications of Impulsive Differential Equations to the Control of Malaria Outbreaks and Introduction to Impulse Extension Equations: A General Framework to Study the Validity of Ordinary Differential Equation Models with Discontinuities in State. Master's Thesis, University of Ottawa, Ottawa, ON, Canada, 2014. [CrossRef]
8. Ballinger, G.; Liu, X. Boundedness for impulsive delay differential equations and applications in populations growth models. *Nonlinear Anal. Theory Methods Appl.* **2003**, *53*, 1041–1062.
9. Benchohra, M.; Karapinar, E.; Lazreg, J.E.; Salim, A. Fractional Differential Equations with instantaneous impulses. In *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*; Springer Nature: Cham, Switzerland, 2023; pp. 77–111.
10. Alsheekhussain, Z.; Ibrahim, A.G.; Jawarneh, Y. Properties of solution sets for ϱ -Caputo fractional non-instantaneous impulsive semi-linear differential inclusions with infinite delay. *Fractal Fract.* **2023**, *7*, 545. [CrossRef]
11. Agarwal, R.P.; Hristova, S.; O'Regan, D. *Non-Instantaneous Impulses in Differential Equations*; Springer: New York, NY, USA, 2017.
12. Terzieva, R. Some phenomena for non-instantaneous impulsive differential equations. *Int. J. Pure Appl. Math.* **2018**, *119*, 483–490.
13. Xu, H.F.; Zhu, Q.X.; Zheng, W.X. Exponential stability of stochastic nonlinear delay systems subject to multiple periodic impulses. *IEEE Trans. Autom. Control* **2023**, *69*, 2621–2628. [CrossRef]
14. Kaslik, E.; Sivasundaram, S. Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions. *Nonlinear Anal. Real World Appl.* **2012**, *13*, 1489–1497. [CrossRef]
15. Feckan, M. Note on periodic solutions of fractional differential equations. *Math. Methods Appl. Sci.* **2018**, *41*, 5065–5073. [CrossRef]
16. Kang, Y.M.; Xie, Y.; Lu, J.C.; Jiang, J. On the nonexistence of non-constant exact periodic solutions in a class of the Caputo fractional-order dynamical systems. *Nonlinear Dyn.* **2015**, *82*, 1259–1267. [CrossRef]
17. Abdurahman, A.; Anton, F.; Bordes, J. Half-string oscillator approach to string field theory (Ghost sector. I). *Nuclear Phys. B* **1993**, *397*, 260–282. [CrossRef]
18. Pinsky, S.; Tritmann, U. Antiperiodic boundary conditions in super symmetric discrete light cone quantization. *Phys. Rev. D* **2000**, *62*, 87701. [CrossRef]

19. Delvos, F.J.; Knochf, L. Lacunary interpolation by anti-periodic trigonometric polynomials. *BIT Numer. Math.* **1999**, *39*, 439–450. [[CrossRef](#)]
20. Wang, J.R.; Feckan, M.; Zhou, Y. Nonexistence of periodic solutions and asymptotically periodic solutions for fractional differential equations. *Commun. Nonlinear Sci. Num. Simul.* **2013**, *18*, 246–256. [[CrossRef](#)]
21. Ahmad, B.; Nieto, J.J. Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory. *Topol. Methods Nonlinear Anal.* **2010**, *35*, 295–304.
22. Ahmad, B.; Nieto, J.J. Existence of solutions for impulsive anti-periodic boundary value problems of fractional order. *Taiwan. J. Math.* **2011**, *15*, 981–993. [[CrossRef](#)]
23. Chai, G. Anti-periodic boundary value problems of fractional differential equations with the Riemann-Liouville fractional derivative. *Adv. Differ. Equ.* **2013**, *2013*, 306. [[CrossRef](#)]
24. Ibrahim, A.G. Differential Equations and inclusions of fractional order with impulse effect in Banach spaces. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 69–109. [[CrossRef](#)]
25. Wang, J.R.; Ibrahim, A.G.; Feckan, M. Differential inclusions of arbitrary fractional order with anti-periodic conditions in Banach spaces. *Electron. J. Qual. Theory Differ. Equ.* **2016**, *34*, 1–22. [[CrossRef](#)]
26. Hristova, S.; Abbas, M.A. Fractional differential equations with anti-periodic fractional integral boundary conditions via the generalized proportional fractional derivatives. *AIP Conf. Proc.* **2022**, *2459*, 030014. [[CrossRef](#)]
27. Boutiara, A.; Benbachir, M.; Guerbati, K. Boundary value problem for non-linear Caputo–Hadamard fractional differential equation with Hadamard fractional integral and anti-periodic conditions. *Facta Univ. Nis. Ser. Math. Inf.* **2021**, *36*, 735–748. [[CrossRef](#)]
28. Redhwan, R.S.; Shaikh, S.L.; Abdo, M.S. Implicit fractional differential equation with anti-periodic boundary condition involving Caputo–Katugampola type. *AIMS Math.* **2020**, *5*, 3714–3730. [[CrossRef](#)]
29. Boutiara, A.; Benbachir, M.; Guerbati, K. Boundary value problems for Hilfer fractional differential equations with Katugampola fractional integral anti-periodic conditions. *Mathematica* **2021**, *63*, 208–221. [[CrossRef](#)]
30. Yang, D.; Bai, C. Existence of solutions for anti-periodic fractional differential inclusions with q -Caputo fractional derivative. *Discrete Dyn. Nat. Soc.* **2019**, *2019*, 9824623. [[CrossRef](#)]
31. Alruwaily, Y.; Aljoudi, S.; Almaghamisi, L.; Ben Makhlouf, A.; Alghamdi, N. Existence and uniqueness results for different orders coupled system of fractional integro-differential equations with anti-periodic nonlocal integral boundary conditions. *Symmetry* **2023**, *15*, 182. [[CrossRef](#)]
32. Benyoub, M.; Kacem, B. Anti-periodic boundary value problems for Caputo–Fabrizio fractional impulsive differential equations. *Math. Moravica* **2022**, *26*, 49–62. [[CrossRef](#)]
33. Yang, D.; Bai, C. Existence of solutions for anti-periodic fractional differential inclusions involving q -Riesz–Caputo fractional derivative. *Mathematics* **2019**, *7*, 630. [[CrossRef](#)]
34. Al Nuwairan, M.; Ibrahim, A.G. Solutions and anti-periodic solutions for impulsive differential equations and inclusions containing Atangana–Baleanu fractional derivative of order $\mu \in (1,2)$ in infinite dimensional Banach spaces. *AIMS Math.* **2024**, *9*, 10386–10415. [[CrossRef](#)]
35. Alsheekhussain, Z.; Ibrahim, A.G.; Al-Sawalha, M.M.; Jawarneh, Y. The existence of solutions for ω -weighted q -Hilfer fractional differential inclusions of order $\mu \in (1,2)$ with non-instantaneous impulses in Banach Spaces. *Fractal Fract.* **2024**, *8*, 144. [[CrossRef](#)]
36. Agarwal, R.P.; Ahmad, B.; Alsaedi, A. Fractional-order differential equations with anti-periodic boundary conditions: A survey. *Bound. Value Probl.* **2017**, *2017*, 173. [[CrossRef](#)]
37. Ahmad, B.; Alruwaily, Y.; Alsaedi, A.; Nieto, J.J. Fractional integro-differential equations with dual anti-periodic boundary conditions. *Differ. Integral Equ.* **2020**, *33*, 181–206. [[CrossRef](#)]
38. Benhamida, W.; Graef, J.R.; Hamani, S. Boundary value problems for fractional differential equations with integral and anti-periodic conditions in a Banach space. *Prog. Fract. Differ. Appl.* **2018**, *4*, 65–70. [[CrossRef](#)]
39. Mshary, N.; Hamedy, A.; Ghanem, S.; Sayed, A.M. Hilfer–Katugampola fractional stochastic differential inclusions with Clarke sub-differential. *Heliyon* **2024**, *10*, e29667. [[CrossRef](#)]
40. Sivasankar, S.; Udhayakumar, R.; Muthukumaran, V. A new conversation on the existence of Hilfer fractional stochastic Volterra–Fredholm integro-differential inclusions via almost sectorial operators. *Nonlinear Anal. Model. Control* **2023**, *28*, 288–307. [[CrossRef](#)]
41. Sayed Ahmed, A.M.; Hamdy, A. Hilfer–Katugampola fractional stochastic differential equations with nonlocal conditions. *Int. J. Nonlinear Anal.* **2023**, *14*, 1205–1214. [[CrossRef](#)]
42. Khalid, K.H.; Zada, A.; Popa, I.L.; Samei, M.E. Existence and stability of a q -Caputo fractional jerk differential equation having anti-periodic boundary conditions. *Bound. Value Probl.* **2024**, *2024*, 28. [[CrossRef](#)]
43. Jarad, F.; Abdeljawad, T.; Shah, K. On the weighted fractional operators of a function with respect to another function. *Fractals* **2020**, *28*, 2040011. [[CrossRef](#)]
44. Kamenskii, M.; Obukhowskii, V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*; Walter de Gruyter: Berlin, Germany, 2001.
45. Bothe, D. Multivalued perturbation of m -accretive differential inclusions. *Israel J. Math.* **1998**, *108*, 109–138. [[CrossRef](#)]
46. Castaing, C.; Valadier, M. *Convex Analysis and Measurable Multifunctions*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1977; Lecture Notes in Mathematics Volume 580.

47. Hiai, F.; Umegaki, H. Integrals, conditional expectations, and martingales of multivalued functions. *J. Multivar. Anal.* **1977**, *7*, 149–182. [[CrossRef](#)]
48. Covitz, H.; Nadler, S.B. Multivalued contraction mapping in generalized metric space. *Israel J. Math.* **1970**, *8*, 5–11. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.