An Inverse Problem for the Subdiffusion Equation with a Non-Local in Time Condition

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Abstract: An inverse problem of determining the right-hand side of the abstract subdiffusion equation with a fractional Caputo derivative is considered in a Hilbert space $H$. For the forward problem, instead of the Cauchy condition, the non-local in time condition $u(0) = u(T)$ is taken. The right-hand side of the equation has the form $g(t)f$ with a given function $g(t)$ and an unknown element $f \in H$. If the function $g(t)$ preserves its sign, then under a over-determined condition $u(t_0) = \psi, t_0 \in (0, T)$, it is proved that the solution of the inverse problem exists and is unique. An example is given showing the violation of the uniqueness of the solution for some sign-changing functions $g(t)$. For such functions $g(t)$, under certain conditions on this function, one can achieve the well-posedness of the problem by choosing properly $t_0$. Moreover, we show that for some $g(t)$, for the existence of a solution to the inverse problem, certain orthogonality conditions must be satisfied, but in this case there is no uniqueness. To the best knowledge of authors, the inverse problem with the non-local condition $u(0) = u(T)$ has been considered for the first time. Moreover, all the results obtained are new not only for the subdiffusion equation, but also for the classical diffusion equation.

Keywords: subdiffusion equation; non-local condition; inverse problem; the Caputo derivatives; the Fourier method

1. Introduction

In recent decades, it has been discovered that fractional differential equations are an effective mathematical tool for modeling various anomalous phenomena (see, e.g., [1–5]). In this regard, researchers are actively finding new methods for solving fractional-order differential equations and discovering new properties of solutions (see, e.g., [1,6–9]).

This work is devoted to the study of the inverse problem of determining the source function, the right-hand-side of the subdiffusion equation. The appearance of inverse problems is due to the fact that when mathematically modeling various processes, the properties of the medium, which in direct problems are considered given quantities, often turn out to be unknown in practice. Mathematically, this means that, along with solving the direct problem, it is also necessary to determine either the coefficients of the equations, or the initial or boundary conditions, or the properties of the domain in which the process under study occurs, or the properties of the object being studied.

Let $H$ be a separable Hilbert space with the scalar product $(\cdot, \cdot)$, the norm $|| \cdot ||$, and $A$ be a self-adjoint positive operator acting in $H$ with a domain of definition $D(A)$. Assume that the inverse of $A$ is a compact operator. Then $A$ has a complete system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues $\lambda_k$: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$.

Let $C((a, b); H)$ stand for a set of continuous vector-valued functions $h(t)$, defined on $t \in (a, b)$ with values in $H$. For $h : \mathbb{R}_+ \to H$ fractional analogs of integrals and derivatives
are defined similar to scalar functions (see, e.g., [10]) and understood in the sense of Bochner. Recall that the Caputo fractional derivative of order $\rho > 0$ of function $h(t)$ has the form (see, e.g., [9], p. 14)

$$D^\rho_t h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{h'(\xi)}{(t-\xi)^\rho} \, d\xi, \quad t > 0,$$

provided the right-hand side exists, where $\Gamma(\cdot)$ is Euler’s gamma function. If $\rho = 1$, then we assume that the fractional derivative coincides with the ordinary classical derivative: $D^1_t h(t) = \frac{dh}{dt}(t)$.

Let $\rho \in (0, 1]$ be a fixed number. Consider the following non-local boundary value problem

$$\begin{cases}
D^\rho_t u(t) + Au(t) = g(t)f, & t \in (0, T],

u(T) = u(0),
\end{cases}$$

(1)

where $g(t) \in C[0, T]$ is a given scalar function, and $f \in H$ is a known element of $H$. This case problem (1) of finding an unknown $u(t)$ is called a forward problem. The forward problem is studied in [11].

The local boundary condition is assumed to be given at one specific point. The famous example is the Cauchy condition. In problem (1) the relation between the value of the sought function at the initial and final moments of time is specified. Therefore, such problems are usually called non-local problems (see, e.g., [12]). The physical meaning of the problem under consideration can, for example, be interpreted as the following control problem: find that part of the heat source $g(t)f(x)$ that depends on the physical location of the source. That is $f(x)$, which would provide the same temperature at the final moment of time as the temperature at the initial moment.

The main purpose of our study is the inverse problem. Namely, to determine the right-hand side of equation (1), namely element $f \in H$, when it is unknown. To solve the inverse problem, one needs an extra condition. Following A.I. Prilepko and A.B. Kostin [13] (see also K.B. Sabitov [14]) we consider the additional condition in the form:

$$u(t_0) = \psi,$$  

(2)

where $t_0$ is a given fixed point of the interval $(0, T)$.

Usually, authors specify an additional condition (2) at the final time $t_0 = T$ (see, e.g., [15–17] for classical diffusion equations and [18–21] for subdiffusion equations). This choice is convenient for real models, since when the process is over it is easy to measure $u(T)$. However, in some cases, choosing $t_0 = T$ may violate the uniqueness of the solution of the inverse problem, and if $t_0 < T$ is chosen, then it is possible to achieve uniqueness in these cases too (see, e.g., [22]).

We call the non-local boundary value problem (1) with unknown $f$ on the right and equipped with the additional condition (2) an inverse problem. The solution to the inverse problem is understood in the following sense:

**Definition 1.** A pair of functions $\{u(t), f\}$ such that $D^\rho_t u(t), Au(t) \in C((0, T]; H)$, $u(t) \in C([0, T]; H)$, $f \in H$, and satisfying conditions (1)–(2), is called a solution of the inverse problem (1), (2).

Inverse problems of determining the right-hand side of various subdiffusion equations have been studied by a number of authors due to the importance of such problems for applications. Note that the methods for studying inverse problems depend on whether $f$ or $g(t)$ is unknown. In the case when the function $g(t)$ is unknown, but $f$ is known, the inverse problem with various additional conditions is equivalently reduced to solving an integral equation (see, e.g., [23–26]). For the subdiffusion and diffusion equations, the case $g(t) \equiv 1$ and the unknown $f$ has been studied by many authors; in this case, the Fourier method.
is used to solve the inverse problem (see, e.g., [27–32] and references therein). In [33], the authors considered the inverse problem of the simultaneous determination of the order of the Riemann–Liouville fractional derivative and the source function in the subdiffusion equation. Using the Fourier method, the authors proved that the solution to this inverse problem exists and is unique. Note that in works [27,28,33], the additional condition (2) is specified at \( t_0 = T \).

Let us focus on the case where \( g(t) \neq 1 \) and \( f \) is unknown, since the present paper is devoted to this case. We note that in all the papers cited below, the Cauchy problem with the initial condition \( u(0) = \phi \) is considered instead of a non-local condition \( u(0) = u(T) \) in problem (1). For the classical diffusion equation (i.e., \( \rho = 1 \)), such an inverse problem with some additional conditions has been studied in sufficient detail in [34] (Chapter 8) and in papers [13–16,35]. Since the equation under consideration also includes the diffusion equation, we will dwell on these works in more detail. In works [15,16], abstract diffusion equations in Banach and Hilbert spaces with additional condition (2) at the final time \( t_0 = T \) are considered. In [15], in the case of a Hilbert space, the elliptic part of the equation is self-adjoint, while in [16], non-self-adjoint elliptic operator is also included.

The uniqueness criterion is found and the existence of a generalized solution is proved. The elliptic part of the diffusion equation in the work [13] is a second-order differential expression. Both non-self-adjoint and self-adjoint elliptic parts are considered. In this paper, \( g(t) \) also depends on a space variable: \( \hat{g}(t) := g(x, t) \), and as an additional condition the authors take (2) and in some cases the integral condition. In the case of a self-adjoint elliptic part, the authors managed to find a criterion for the uniqueness of the generalized solution of the inverse problem. In works [14,35], the elliptic part of the equation is \( u_{xx} \), defined on the interval and the Laplace operator on the rectangle, respectively. Considering the overdetermined condition in the form (2), a criterion for the uniqueness of the classical solution is found and the Fourier method is used to construct a classic solution.

A similar inverse problem \( (g(t) \neq 1 \) and \( f \) is unknown) for the abstract subdiffusion equation (i.e., \( 0 < \rho < 1 \)) with the Caputo derivative was studied in [36]. To define the function \( f \), the authors used the following additional condition \( \int_0^T u(t)d\mu(t) = u_T \). Authors of [18] (see also [19,20]) studied the uniqueness of the solution of the inverse problem for the subdiffusion equation, the elliptic part of which depends on time, with the additional condition \( u(T) = \phi \). It is proved that if the function \( g(t) \) does not change sign, then the solution of the inverse problem is unique. It should be especially noted that in [19] the authors constructed an example of a function \( g(t) \) that changes sign in the domain under consideration, as a result of which the uniqueness of the solution of the inverse problem is lost.

Finally, let us note paper [22], where in their Equation (1) the Laplace operator is taken as the operator \( A \) and, as noted above, the Cauchy condition is taken instead of the non-local condition. The authors have taken the additional condition (2) to find the unknowns. In terms of the function \( g(t) \), a uniqueness criterion is found, and under the condition that the criterion is satisfied, the existence of a unique solution to the inverse problem is proved.

It is well known that the inverse problem is ill-posed, i.e., the solution does not depend continuously on the given data. Therefore, in papers [37,38] (see also references therein) various regularization methods are proposed for constructing an approximate solution of the inverse problem under the condition \( u(T) = \phi \).

In the current paper we consider a new inverse problem (1), (2). To the authors’ best knowledge, the above works contain no results concerning the correct solvability of such inverse problems as we study in the present paper.

The present paper is organized as follows. In Section 2, we recall some preliminary facts related to Mittag–Leffler functions used in this paper. Section 3 is devoted to the study of the inverse problem (1), (2) of determining the right-hand side of the equation. This section shows that if the function \( g(t) \) preserves its sign, then the solution to the inverse problem (1), (2) exists and is unique. An example is given showing the viola-
tion of the uniqueness of the solution in the case of some not sign-preserving functions $g(t)$. It is also proven that under certain conditions on the function $g(t)$ it is possible to achieve well-posedness of the problem by choosing $t_0$ properly. The paper ends with the conclusion section.

2. Preliminaries

In this section, we recall some facts about Mittag–Leffler functions and the solution of the forward problem (1), which we will use in the following section.

The action of the abstract operator $A$ under consideration on the element $h \in H$ can be written as

$$Ah = \sum_{k=1}^{\infty} \lambda_k h_k v_k,$$

where $h_k$ is the Fourier coefficient of the element $h$: $h_k = (h, v_k)$. Obviously, the domain of definition of this operator has the form

$$D(A) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^2 |h_k|^2 < \infty\}.$$

For elements of $D(A)$, we introduce the norm

$$||h||_1^2 = \sum_{k=1}^{\infty} \lambda_k^2 |h_k|^2 = ||Ah||^2,$$

and together with this norm $D(A)$ turns into a Hilbert space.

For $0 < \rho < 1$ and an arbitrary complex number $\mu$, let $E_{\rho,\mu}(z)$ denote the two-parameter Mittag–Leffler function of the complex argument $z$ (see, e.g., [5], p. 15):

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}.$$

If the parameter $\mu = 1$, then we have the classical Mittag–Leffler function: $E_{\rho}(z) = E_{\rho,1}(z)$.

Recall some properties of the Mittag–Leffler functions.

**Lemma 1** (see [39], p. 136). For any $t \geq 0$ one has

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t},$$

where constant $C$ does not depend on $t$ and $\mu$.

**Lemma 2** (see [5], p. 47). The classical Mittag–Leffler function of the negative argument $E_{\rho}(-t)$ is monotonically decreasing function for all $0 < \rho < 1$ and

$$0 < E_{\rho}(-t) < 1, \quad E_{\rho}(0) = 1.$$

**Lemma 3** (see [5], formula (4.4.17), p. 64). Let $\mu$ be an arbitrary complex number. Then, the following asymptotic formula

$$E_{\rho,\mu}(-t) = -\sum_{k=1}^{\infty} \frac{(-t)^{-k}}{\Gamma(\mu - k\rho)} + O(t^{-3}), \quad t \to \infty,$$

is valid.
Lemma 4 (see [5], formula (4.2.3), p. 57). For all $\alpha > 0$, $\mu \in \mathbb{C}$ the following recurrence relation holds:

$$E_{\rho,\mu}(-t) = \frac{1}{\Gamma(\mu)} - tE_{\rho,\mu}(t).$$

Lemma 5 (see [39], formula (1.16), p. 120). Let $\rho > 0$, $\mu > 0$ and $\lambda \in \mathbb{C}$. Then, for all positive $t$ one has

$$\frac{1}{\Gamma(\mu)} \int_0^t (t-\eta)^{\mu-1}\eta^{\rho-1} E_{\rho,\mu}(\lambda \eta^\rho) d\eta = t^{\mu+\rho-1} E_{\rho,\mu+\mu}(\lambda t^\rho).$$

To solve the inverse problem we use the following result on the forward problem: If $f \in H$ and $g \in C[0,T]$ then the unique solution of the forward problem has the form (see [11], Theorem 3 and Corollary 1)

$$u(t) = \sum_{k=1}^{\infty} \left[ \omega_k(t) + \frac{\omega_k(T)}{1 - E_{\rho}(-\lambda_k t^\rho)} E_{\rho}(-\lambda_k t^\rho) \right] v_k,$$

where

$$\omega_k(t) = f_k \int_0^t (t-\eta)^{\rho-1} E_{\rho,\mu}(\lambda_k (t-\eta)^\rho) g(\eta) d\eta,$$

and $f_k, k \geq 1$, are Fourier coefficients of the element $f$.

3. Existence and Uniqueness of the Inverse Problem

We apply the additional condition (2) to Equation (4) and denote by $\psi_k$ the Fourier coefficients of the element $\psi : \psi_k = (\psi, v_k)$. Then

$$\sum_{k=1}^{\infty} f_k [1 - E_{\rho}(-\lambda_k t^\rho)] b_{k,\rho}(t_0) + E_{\rho}(-\lambda_k t^\rho) b_{k,\rho}(T)] v_k = \sum_{k=1}^{\infty} \psi_k (1 - E_{\rho}(-\lambda_k t^\rho)) v_k,$$

where

$$b_{k,\rho}(t) = \int_0^t (t-s)^{\rho-1} E_{\rho,\mu}(\lambda_k (t-s)^\rho) g(s) ds.$$

Hence, to find $f_k$, we obtain the following equation

$$f_k \Delta_k(k, t_0, T) = \psi_k (1 - E_{\rho}(-\lambda_k t^\rho)),$$

where

$$\Delta_k(k, t_0, T) = (1 - E_{\rho}(-\lambda_k t^\rho)) b_{k,\rho}(t_0) + E_{\rho}(-\lambda_k t^\rho) b_{k,\rho}(T).$$

Of course the case $\Delta_k(k, t_0, T) = 0$ is critical. Since, according to Lemma 2

$$0 < 1 - E_{\rho}(-\lambda_k t^\rho) < 1 \quad \text{and} \quad 0 < E_{\rho}(-\lambda_k t^\rho) < 1,$$

then this can happen only when $g(t)$ changes its sign. Note that in this case, the functions $b_{k,\rho}(t_0)$ and $b_{k,\rho}(T)$ can also change their signs.

Let us divide the set of natural numbers $\mathbb{N}$ into two groups $K_{0,\rho}$ and $K_\rho$: $\mathbb{N} = K_\rho \cup K_{0,\rho}$. Namely, the number $k$ belongs to $K_{0,\rho}$, if $\Delta_k(k, t_0, T) = 0$, and $k \in K_\rho$ if $\Delta_k(k, t_0, T) \neq 0$. Note that for some $t_0$ and $T$ the set $K_{0,\rho}$ may be empty, then $K_\rho = \mathbb{N}$. For example, if $g(t)$ is sign-preserving, then $K_\rho = \mathbb{N}$ for all $k$ regardless of the values $t_0$ and $T$.

Now we establish lower bounds for $\Delta_k(k, t_0, T)$. The lemma below states that if $g(t)$ does not change its sign, then $K_{0,\rho}$ is empty.
Lemma 6. Let \( \rho \in (0,1], g(t) \in C[0,T] \) and \( g(t) \neq 0, t \in [0,T] \). Then, there is a constant \( C > 0 \), depending on \( t_0 \) and \( T \), such that for all \( k \):

\[
|\Delta_p(k,t_0,T)| \geq \frac{C}{\lambda_k}.
\]

Proof. Since \( g(t) \neq 0 \) on the closed interval \([0,T]\), we have either \( g(t) \geq g_+ = \text{const} > 0 \) or \( g(t) \leq g_- = \text{const} < 0 \) for all \( t \in [0,T] \). It is sufficient to consider the first case. The second case can be treated in exactly the same way. For \( \tau \in (0,T] \), we have (see Lemma 5)

\[
b_{k,p}(\tau) \geq g_+ \int_0^\tau \eta^{p-1} E_{p,p}(\lambda_k \eta^p) d\eta = g_+ \tau^p E_{p,p+1}(\lambda_k \tau^p).
\]

Taking into account (see Lemma 4)

\[
E_{p,p+1}(-t) = t^{-1}(1 - E_{p}(-t))
\]

we obtain

\[
b_{k,p}(\tau) \geq \frac{1}{\lambda_k} (1 - E_p(-\lambda_k \tau^p)) g_+ \geq \frac{1}{\lambda_k} (1 - E_p(-\lambda_1 \tau^p)) g_+ \geq \frac{C_T}{\lambda_k}, \quad C_T > 0.
\]

Therefore

\[
\Delta_p(k,t_0,T) \geq (1 - E_p(-\lambda_k T^p)) \frac{C_{l_0}}{\lambda_k} + E_p(-\lambda_k t_0^p) \frac{C_T}{\lambda_k}
\]

\[
\geq (1 - E_p(-\lambda_k T^p)) \frac{C_{l_0}}{\lambda_k} + E_p(-\lambda_k t_0^p) \frac{C_T}{\lambda_k},
\]

which implies the desired assertion due to Lemma 2. \( \square \)

Theorem 1. Let \( \rho \in (0,1], g(t) \in C[0,T] \) and \( g(t) \neq 0, t \in [0,T] \). Assume \( \psi \in D(A) \). Then there exists a unique solution of the inverse problem (1) and (2):

\[
u(t) = \sum_{k=1}^{\infty} f_k \left[ b_{k,p}(t) + \frac{b_{k,p}(T)}{1 - E_p(-\lambda_k T^p)} E_p(-\lambda_k t^p) \right] v_k,
\]

\[
f = \sum_{k=1}^{\infty} f_k v_k,
\]

where

\[
f_k = \frac{\psi_k (1 - E_p(-\lambda_k T^p))}{(1 - E_p(-\lambda_k T^p)) b_{k,p}(t_0) + E_p(-\lambda_k t_0^p) b_{k,p}(T)}, \quad k \geq 1.
\]

Moreover, there is a positive constant \( C \) such that the following estimate holds:

\[
||D^p u(t)|| + ||Au(t)|| + ||f|| \leq C ||\psi||_1, \quad t > 0.
\]

Proof. The assumption of the theorem, due to Lemma 6, implies \( \Delta_p(k,t_0,T) \neq 0 \) for all \( k \in \mathbb{N} \). Hence, it follows from (5) that

\[
f_k = \frac{\psi_k (1 - E_p(-\lambda_k T^p))}{\Delta_p(k,t_0,T)},
\]

and

\[
u_k(t) = f_k \left[ b_{k,p}(t) + \frac{b_{k,p}(T)}{1 - E_p(-\lambda_k T^p)} E_p(-\lambda_k t^p) \right].
\]
With these Fourier coefficients, for \( f \) and \( u(t) \) we have the following infinite series representations:

\[
f = \sum_{k=1}^{\infty} \frac{\psi_k(1 - E_{\rho}(-\lambda_k T^\rho))}{\Delta_\rho(k, t_0, T)} v_k,
\]

and

\[
u(t) = \sum_{k=1}^{\infty} f_k \left[ b_{k,\rho}(t) + \frac{b_{k,\rho}(T)}{1 - E_{\rho}(-\lambda_k T^\rho)} E_{\rho}(-\lambda_k T^\rho) \right] v_k.
\]

Let \( F_j \) be the partial sum of series (9):

\[
F_j = \sum_{k=1}^{j} \frac{\psi_k(1 - E_{\rho}(-\lambda_k T^\rho))}{\Delta_\rho(k, t_0, T)} v_k.
\]

We show that the series \( F_j \) is convergent in \( H \).

For this, applying Parseval’s identity, we obtain

\[
||F_j||^2 = \sum_{k=1}^{j} \left| \frac{\psi_k(1 - E_{\rho}(-\lambda_k T^\rho))}{\Delta_\rho(k, t_0, T)} \right|^2.
\]

The latter, by virtue of Lemmas 2 and 6, implies the following estimate:

\[
||F_j||^2 \leq C \sum_{k=1}^{j} \lambda_k^2 |\psi_k|^2 \leq C \||\psi||^2.
\]

Now, letting \( j \to \infty \) and taking into account that \( \psi \in D(A) \), we obtain \( f \in H \).

Now, consider series (10) for \( u(t) \). Using the representation (7) for \( f_k \), this series can be written in the form

\[
u(t) = \sum_{k=1}^{\infty} \frac{\psi_k(1 - E_{\rho}(-\lambda_k T^\rho))}{\Delta_\rho(k, t_0, T)} \left[ b_{k,\rho}(t) + \frac{b_{k,\rho}(T)}{1 - E_{\rho}(-\lambda_k T^\rho)} E_{\rho}(-\lambda_k T^\rho) \right] v_k.
\]

Let \( S_j(t) \) be the \( j \)-th partial sum of series (12). Then, by virtue of Parseval’s identity, we have

\[
||AS_j(t)||^2 = \sum_{k=1}^{j} \lambda_k^2 |\psi_k|^2 \left[ |b_{k,\rho}(t)| + |b_{k,\rho}(T)| \right]^2.
\]

Applying Lemmas 2 and 6, we obtain

\[
||AS_j(t)||^2 \leq C \sum_{k=1}^{j} \lambda_k^2 |\psi_k|^2 \left[ |b_{k,\rho}(t)| + |b_{k,\rho}(T)| \right]^2,
\]

with a constant \( C > 0 \). Due to Lemmas 1 and 5, we obtain

\[
||AS_j(t)||^2 \leq C \sum_{k=1}^{j} \lambda_k^2 |\psi_k|^2 \max_{0 \leq t \leq T} |g(t)|^2 \left[ |t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) + T^\rho E_{\rho,\rho+1}(-\lambda_k T^\rho)|^2 \right]
\]

\[
\leq C \sum_{k=1}^{j} \lambda_k^2 |\psi_k|^2 \max_{0 \leq t \leq T} |g(t)|^2 \left[ \frac{t^\rho}{1 + \lambda_k t^\rho} + \frac{T^\rho}{1 + \lambda_k T^\rho} \right]^2
\]

\[
\leq C \sum_{k=1}^{j} \lambda_k^2 |\psi_k|^2 \max_{0 \leq t \leq T} |g(t)|^2 \leq C_1 ||\psi||^2, \quad t \geq 0.
\]
Now, letting $j \to \infty$, we obtain $Au(t) \in C([0,T];H)$ and in particular $u(t) \in C([0,T];H)$. Further, it follows from Equation (1) that

$$D^q_j S_j(t) = -AS_j(t) + \sum_{k=1}^{j} f_k g(t) v_k, \quad t > 0. \quad (14)$$

Therefore, it follows from estimates (11) and (13), that

$$||D^q_j S_j(t)||^2 \leq C||\psi||^2_{1}, \quad t > 0,$$

and

$$D^q_j u(t) \in C((0,T];H).$$

To prove the uniqueness of the solution, assume that there exist two distinct pairs $\{u_1, f_1\}$ and $\{u_2, f_2\}$, satisfying the inverse problem (1) and (2). We need to show that $u(t) \equiv u_1(t) - u_2(t) \equiv 0$, and $f \equiv f_1 - f_2 = 0$. For the pair $\{u(t), f\}$, we have the infinitely many non-local boundary value problems:

$$\begin{cases}
D^q_j u(t) + Au(t) = f(t), & t \in (0,T], \\
u(T) = u(0), \\
u(t_0) = 0, & t_0 \in (0,T).
\end{cases} \quad (15)$$

Let $u_k(t) = (u(t), v_k)$ and $f_k = (f, v_k)$ be Fourier coefficients of $u(t)$ and $f$, respectively. Then, due to the self-adjointness of operator $A$, we obtain

$$D^q_j u_k(t) = (D^q_j u, v_k) = -(Au, v_k) + f_k g(t) = -(u, Av_k) + f_k g(t) = -\lambda_k u_k(t) + f_k g(t).$$

Therefore, for $u_k(t)$, $t > 0$, we have the non-local boundary value problem

$$D^q_j u_k(t) + \lambda_k u_k(t) = f_k g(t), \quad t > 0, \quad u_k(T) = u_k(0), \quad \forall k \geq 1, \quad (15)$$

and the additional condition

$$u_k(t_0) = 0. \quad (16)$$

For fixed $f_k$, the unique solution of non-local boundary value problem (15) has the form (see Formula (14))

$$u_k(t) = f_k \left[ b_{k,p}(t) + \frac{b_{k,p}(T)}{1 - E_p(-\lambda_k T^p)} E_p(-\lambda_k t^p) \right], \quad \forall k \geq 1. \quad (17)$$

Now, applying the additional condition in (16), we have

$$u_k(t_0) = f_k \left[ (1 - E_p(-\lambda_k T^p)) b_{k,p}(t_0) + \frac{b_{k,p}(T) E_p(-\lambda_k t_0^p)}{1 - E_p(-\lambda_k T^p)} \right]$$

$$= \frac{f_k \Delta_p(k, t_0, T)}{1 - E_p(-\lambda_k T^p)} = 0. \quad (18)$$

In accordance with Lemma 6, we have $\Delta_p(k, t_0, T) \neq 0$ for all $k \in \mathbb{N}$. Moreover, Lemma 2 implies $1 - E_p(-\lambda_k T^p) \neq 0$ for all $k \in \mathbb{N}$. Therefore, Equation (18) is valid only if $f_k = 0$ for all $k \in \mathbb{N}$ and due to the completeness of the set of eigenvectors $\{v_k\}$ in $H$, we obtain $f = 0$. In turn, Equation (17) implies $u_k(t) \equiv 0$ for all $k \in \mathbb{N}$, and consequently, $u(t) \equiv 0$. □
Now, let us consider the case when \( g(t) \) changes its sign. In this case, the function \( \Delta_\rho(k, t_0, T) \) may become zero, and as a result, the set \( K_{0, \rho} \) may be non-empty. As a consequence, in this case the solution of the inverse problem may not be unique. Consider the following example.

**Example 1.** Let the inverse problem have the form

\[
\begin{aligned}
D_t^\rho u(x, t) - u_{xx}(x, t) &= f(x)g(t), \\ u(0, t) &= u(\pi, t) = 0, \\ u(x, 0) &= u(x, T), \\ u(x, T/2) &= 0, 
\end{aligned}
\]

where \( 0 < \rho \leq 1, f \in L^2(0, \pi) \) and \( g \in C[0, T] \). First, note that this problem has a trivial solution \((u, f) = (0, 0)\).

Assume \( T = 1 \). One can easily verify that the pair of functions

\[
u(x, t) = \omega(t) \sin(x), \quad f(x) = \sin(x),
\]

also satisfy problem (19) with

\[
g(t) = D_t^\rho \omega(t) + \omega(t),
\]

where \( \omega(t) = (t - \frac{1}{2})^2 \). The function \( g(t) \) is continuous on the interval \([0, 1]\) and can be written in the form

\[
g(t) = \frac{t^2 - \rho}{\Gamma(3 - \rho)} - \frac{t^{1-\rho}}{\Gamma(2 - \rho)} + \left(t - \frac{1}{2}\right)^2 - 2 - (t + \rho) t^{1-\rho}.
\]

The reason for existing of a non-unique solution in this example is the function \( g(t) \) is not sign-preserving. Indeed, one has \( g(0) = \frac{1}{4} > 0 \), and

\[
g\left(\frac{1}{2}\right) = -\frac{3 - \rho}{2^{1-\rho}\Gamma(3 - \rho)} < 0,
\]

for all \( 0 < \rho \leq 1 \).

An important feature of the non-local problem being considered, which comes out from Theorem 1, is the relation of the eigenvalues of the operator \( A \) to the set of zeros of the function \( \Delta_\rho(k, t_0, T) \). In order to further analyze such a connection, we separately consider the cases of diffusion \((\rho = 1)\) and subdiffusion \((0 < \rho < 1)\) equations.

**Lemma 7.** Let \( \rho = 1, g(t) \in C^1[0, T] \) and \( g(t_0) \neq 0 \). Then there exists a number \( k_0 \) such that for all \( k \geq k_0 \), the following estimate holds:

\[
|\Delta_1(k, t_0, T)| \geq \frac{C}{\lambda_k},
\]

where constant \( C \) depends on \( k_0, t_0, \) and \( T \).
Proof. By integrating by parts and utilizing the mean value theorem, we have

$$b_{k,1}(t_0) = \int_0^{t_0} e^{-\lambda_k s} g(t_0 - s) ds$$

$$= \frac{1}{\lambda_k} g(t_0 - s) e^{-\lambda_k s} \bigg|_0^{t_0} - \frac{1}{\lambda_k} \int_0^{t_0} e^{-\lambda_k s} g'(t_0 - s) ds$$

$$= \frac{1}{\lambda_k} [g(t_0) - g(0)e^{-\lambda_k t_0}] + \frac{g'(\xi_k)}{\lambda_k^2} [e^{-\lambda_k t_0} - 1], \quad \xi_k \in [0, t_0].$$

Obviously, there exists a number $k_0$ such that for all $k \geq k_0$ one has

$$\left| \frac{1}{2} g(t_0) \right| \geq \left| e^{-\lambda_k t_0} g(0) \right| + \left| \frac{g'(\xi_k)}{\lambda_k} [e^{-\lambda_k t_0} - 1] \right|.$$

Thus

$$|b_{k,1}(t_0)| \geq \frac{|g(t_0)|}{2\lambda_k}. \quad (21)$$

Since function $g(t)$ is bounded, then for any $\tau \in [0, T]$ we have

$$|b_{k,1}(\tau)| \leq \frac{C_0}{\lambda_k}. \quad (22)$$

Therefore

$$|\Delta_1(k, t_0, T)| = |(1 - e^{-\lambda_k T})b_{k,1}(t_0) + e^{-\lambda_k t_0}b_{k,1}(T)|$$

$$\geq |b_{k,1}(t_0)| - |e^{-\lambda_k T}b_{k,1}(t_0)| - |e^{-\lambda_k t_0}b_{k,1}(T)|$$

$$\geq \left| \frac{g(t_0)}{2\lambda_k} \right| - \frac{C_0}{e^{\lambda_k T} \lambda_k} - \frac{C_0}{e^{-\lambda_k t_0} \lambda_k}.$$

Thus, for sufficiently large $k$, we obtain the statement of the lemma. \(\square\)

Corollary 1. Under conditions of Lemma 7, estimate (20) holds for all $k \in K_1$.

Corollary 2. Under conditions of Lemma 7, the set $K_{0,1}$ contains a finite number of elements.

In the case of subdiffusion equation ($\rho \in (0, 1)$) the following assertion holds:

Lemma 8. Let $\rho \in (0, 1)$, $g(t) \in C^1[0, T]$ and $g(0) \neq 0$. Then, there exist numbers $T_0 > 0$ and $k_0$ such that, for all $T \leq T_0$ and $k \geq k_0$, the following estimates holds:

$$|\Delta_\rho(k, t_0, T)| \geq \frac{C}{\lambda_k}, \quad (23)$$

where constant $C$ depends on $k_0, t_0,$ and $T$. 
**Theorem 2.** Let \( g \) and \( \psi \) satisfy the conditions of Lemma 8 are satisfied and \( T \) is sufficiently small. Then, for \( \tau \in (0, T] \) and sufficiently large \( \rho \), we have

\[
\sum_{k=1}^{\infty} f_k \varphi_k(t) = \sum_{k=1}^{\infty} f_k v_k \leq \sum_{k=1}^{\infty} b_{k, \rho}(\tau) v_k,
\]

where

\[
b_{k, \rho}(\tau) = \int_0^\tau g(\tau - s) s^{\rho-1} E_{\rho, \rho+1}(-\lambda_k s^\rho) ds
\]

Using Formula (3), the integral in (24) can be represented as

\[
\int_0^\tau s^\rho E_{\rho, \rho+1}(-\lambda_k s^\rho) ds = \tau^{\rho+1} E_{\rho, \rho+2}(-\lambda_k \tau^\rho).
\]

The latter due to the asymptotic estimate of the Mittag–Leffler functions (Lemma 3) implies

\[
b_{k, \rho}(\tau) = \frac{g(0)}{\lambda_k} + \frac{g'(\xi_k)}{\lambda_k} \tau + O\left(\frac{1}{(\lambda_k t_0^\rho)^2}\right).
\]

Now, without loss of generality, we assume that \( g(0) > 0 \). Then, for sufficiently small \( \tau \) and sufficiently large \( k \) we obtain the lower estimate

\[
b_{k, \rho}(\tau) \geq \frac{g(0)}{2\lambda_k}.
\]

Therefore

\[
\Delta_\rho(k, t_0, T) \geq \left(1 - E_\rho(-\lambda_k T^\rho)\right) \frac{g(0)}{2\lambda_k} + E_\rho(-\lambda_k t_0^\rho) \frac{g(0)}{2\lambda_k}.
\]

and since the classical Mittag–Leffler function of the negative argument is a monotonically decreasing function (see Lemma 2), then the desired result follows. \( \square \)

**Corollary 3.** Under conditions of Lemma 8, estimate (23) holds for all \( T \leq T_0 \) and \( k \in K_\rho \).

**Corollary 4.** Under conditions of Lemma 8, if \( T \) is sufficiently small, then the set \( K_{0, \rho} \) has a finite number elements.

**Theorem 2.** Let \( g(t) \in C^1[0, T] \), \( \psi \in D(A) \). Further, we assume that for \( \rho = 1 \) the conditions of Lemma 7 are satisfied, \( t_0 \in (0, T) \) is chosen correspondingly and for \( \rho \in (0, 1) \), the conditions of Lemma 8 are satisfied and \( T \) is sufficiently small.

1. If the set \( K_{0, \rho} \) is empty, i.e., \( \Delta_\rho(k, t_0, T) \neq 0 \), for all \( k \), then there exists a unique solution of the inverse problem (1) and (2):

\[
f = \sum_{k=1}^{\infty} f_k \varphi_k,
\]

\[
u(t) = \sum_{k=1}^{\infty} f_k \left[b_{k, \rho}(T) + \frac{b_{k, \rho}(T)}{1 - E_\rho(-\lambda_k T^\rho)} E_\rho(-\lambda_k T^\rho)\right] \varphi_k.
\]
where
\[ f_k = \frac{\psi_k(1 - E_p(-\lambda_k T^p))}{\Delta_p(k, t_0, T)}. \]

(2) If the set \( K_{0,\rho} \) is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions
\[ \psi_k = (\psi, v_k) = 0, \quad k \in K_{0,\rho}, \]
be satisfied. In this case, the solution to the problem (1) and (2) exists, but is not unique. In this case the solution has the representation
\[ f = \sum_{k \in K_{\rho}} \frac{\psi_k(1 - E_p(-\lambda_k T^p))}{\Delta_p(k, t_0, T)} v_k + \sum_{k \in K_{0,\rho}} f_k v_k, \]
\[ u(t) = \sum_{k=1}^{\infty} f_k \left[ b_{k,\rho}(t) + \frac{b_{k,\rho}(T)}{1 - E_p(-\lambda_k T^p)} E_p(-\lambda_k T^p) \right] v_k, \]
where \( f_k, k \in K_{0,\rho}, \) are arbitrary real numbers.

**Proof.** The proof of the first part of the theorem is completely analogous to the proof of Theorem 1.

In what concerns the proof of the second part of the theorem, we note the following. If \( k \in K_{\rho} \), then again it follows from (5) that the relations in (7) and (8) hold. Then the convergence of the corresponding series can be proved in the same way as in Theorem 1.

If \( k \in K_{0,\rho} \), i.e., \( \Delta_p(k, t_0, T) = 0 \), then the solution of Equation (5) with respect to \( f_k \) exists if and only if condition (25) is satisfied. In this case, the solution of the equation contains arbitrary numbers \( f_k \). As shown above (see Corollaries 2 and 4), under the conditions of the theorem, the set \( K_{0,\rho}, \rho \in (0,1] \), contains only a finite number of elements. \( \Box \)

Note that we considered an abstract subdiffusion equation with a self-adjoint operator \( A \) in a Hilbert space \( H \). The choice of an abstract operator allows us to generalize various known models. Since the operator \( A \) is only required to have a complete orthonormal system of eigenfunctions, then as \( A \) one can consider any of the elliptic operators given in the work by Ruzhansky et al. [40]. For example, let us take \( L_2(\Omega), \Omega \subset \mathbb{R}^N \), as the Hilbert space \( H \) and let \( A \) be the Laplace operator \((-\Delta)\) with the Dirichlet condition. Assume that the boundary \( \partial \Omega \) of \( \Omega \) is sufficiently smooth. Then, problem (1) takes the form
\[ \begin{cases} D^\rho_t u(x,t) - \Delta u(x,t) = f(x)g(t), & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, & x \in \partial \Omega, \quad t \in [0,T], \\ u(x,0) = u(x,T), & x \in \Omega, \\ u(x,t_0) = \psi(x), & x \in \Omega, \quad t_0 \in (0,T), \end{cases} \]
where \( g \in C[0,T] \) and \( \psi \in L_2(\Omega) \) are given functions and \( f \) is an unknown function in \( L_2(\Omega) \).

Let \( v_k(x) \) and \( \lambda_k \) be eigenfunctions and eigenvalues of the spectral problem:
\[ -\Delta v(x) = \lambda v(x), \quad v(x) = 0, \quad x \in \partial \Omega. \]

Assume that all the conditions of Theorem 2 (or Theorem 1) are satisfied and \( \psi \) belongs to the Sobolev space \( W^2_2(\Omega) \) and \( \psi(x) = 0, x \in \partial \Omega \). Then, \( \psi \in D(A) \) and the statements of Theorem 2 (or, respectively, Theorem 1) are valid.

4. Conclusions

The work is devoted to one of the important applications of the inverse problem—the problem of determining the source function in subdiffusion equations. The unknown
right-hand side is given in the form \( g(t)f \), where \( f \in H \) is unknown. In contrast to the well-known works, the non-local time condition is taken instead of the Cauchy condition. If \( g(t) \) is sign-preserving, then the existence and uniqueness of the solution of the inverse problem with the over-determination condition \( u(t_0) = \psi \) is proved with \( t_0 \in (0, T) \). An example is constructed showing the lack of uniqueness of the solution if \( g(t) \) is not sign-preserving. The meaning of taking \( t_0 \in (0, T) \) instead of \( T \) in the over-determined condition is that for some sign-changing functions \( g(t) \), the choice of \( t_0 \) (as it is shown in Theorem 2) can ensure the uniqueness of the solution. Note that if \( t_0 = T \), then in Theorem 2 there will be no unique solution to the inverse problem. In the absence of uniqueness of the solution, the conditions for the orthogonality of the element \( \psi \) to some eigenfunctions of the operator \( A \) are found, the fulfillment of which ensures the existence of a solution to the inverse problem. It should be noted that in this case, the solution is not unique.

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