



Article

A Fractional Magnetic System with Critical Nonlinearities

Libo Yang ^{1,*} , Shapour Heidarkhani ² and Jiabin Zuo ^{3,*} ¹ Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, China² Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah 67149, Iran; sh.heidarkhani@razi.ac.ir³ School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China

* Correspondence: yanglibo@hyit.edu.cn or yanglibo80@126.com (L.Y.); zuojiabin88@gzhu.edu.cn or zuojiabin88@163.com (J.Z.)

Abstract: In the present paper, we investigate a fractional magnetic system involving critical concave–convex nonlinearities with Laplace operators. Specifically, $(-\Delta)_A^s u_1 = \lambda_1 |u_1|^{q-2} u_1 + \frac{2\alpha_1}{\alpha_1 + \beta_1} |u_1|^{\alpha_1-2} u_1 |u_2|^{\beta_1}$ in Ω , $(-\Delta)_A^s u_2 = \lambda_2 |u_2|^{q-2} u_2 + \frac{2\beta_1}{\alpha_1 + \beta_1} |u_2|^{\beta_1-2} u_2 |u_1|^{\alpha_1}$ in Ω , $u_1 = u_2 = 0$ in $\mathbb{R}^n \setminus \Omega$, where Ω is a bounded set with Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^n$, $1 < q < 2 < \frac{n}{s}$ with $s \in (0, 1)$, λ_1, λ_2 are two real positive parameters, $\alpha_1 > 1, \beta_1 > 1, \alpha_1 + \beta_1 = 2_s^* = \frac{2n}{n-2s}$, 2_s^* is the fractional critical Sobolev exponent, and $(-\Delta)_A^s$ is a fractional magnetic Laplace operator. By using Lusternik–Schnirelmann's theory, we prove the existence result of infinitely many solutions for the magnetic fractional system.

Keywords: magnetic fractional system; infinitely many solutions; critical nonlinearity; magnetic Laplace operator

MSC: 47G20; 35J60

1. Introduction

During a short period of 20 years, a number of researchers have paid attention to nonlocal problems. The applications of nonlocal operator go far beyond the classical local operator. It has wide-ranging applications in many fields, and plays an important role in physics, probability, finance, dynamics, game theory, and so on. Naturally, as a typical nonlocal problem, the fractional elliptic equations (or fractional elliptic systems) have become a research hotspot of scholars, and many interesting results about the existence, nonexistence, and multiplicity of this class of equations are obtained. At the same time, the problems with magnetic potential function have also been considered by many scholars. The famous nonlinear magnetic Schrödinger equation

$$-(\nabla - iA)^2 u + V(x)u = f(x, |u|)u, \quad x \in \mathbb{R}^N$$

has been extensively studied, where $-(\nabla - iA)^2$ is a magnetic Schrödinger operator. In many cases, the microscopic particles are not only affected by the potential field, but also by the magnetic field; thus, the magnetic potential function $A(x) \neq 0$. The Schrödinger equation plays a very important role in quantum mechanics. It describes the changing laws of particle states in the microcosmic world and has important applications in nuclear physics, solid physics, and other fields. As research continues, a number of experts have already started studying the problems with operator like this $(-\Delta)_A^s u$. Usually, the operator $(-\Delta)_A^s u$ is called fractional magnetic Laplace operator, in a suitable sense, $(-\Delta)_A^s u$ converges to $-(\nabla - iA)^2 u$ in the limit $s \uparrow 1$. In the work of Fiscella, Pinamonti, and Vecchi [1], a fractional boundary value problem involving external magnetic field was investigated, with different assumptions about the nonlinear term, and the results of multiplicity were obtained. Wang and Xiang [2], for a Choquard equation with fractional magnetic potential,



Citation: Yang, L.; Heidarkhani, S.; Zuo, J. A Fractional Magnetic System with Critical Nonlinearities. *Fractal Fract.* **2024**, *8*, 380. <https://doi.org/10.3390/fractalfract8070380>

Academic Editor: Ivanka Stamova

Received: 12 May 2024

Revised: 23 June 2024

Accepted: 24 June 2024

Published: 27 June 2024



Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

proved the multiplicity of solutions applying variational methods. In [3], Xiang et al. studied fractional elliptic equations with an external magnetic field, combining the direct methods with the Nehari method, as well as the method of mountain pass theorem. The lowest energy solution results were obtained for this latter equation. Meanwhile, employing the symmetric mountain pass theorem, the existence of infinitely many solutions has also been obtained under some appropriate conditions. In [4], Yang et al. considered a class of magnetic fractional Schrödinger–Kirchhoff equations in $\mathbb{R}^n$, and proved the existence results for the equations under some suitable conditions. In [5], under some mild assumptions on f and V , Jin et al. obtained the existence of at least one ground state solution for a critical fractional magnetic Choquard equation. If you want to learn more works about the equations with magnetic fractional Laplace operators and magnetic Laplace operators, there are many useful results, such as [6–16] and the references contained within.

However, as far as we know, there are few works about magnetic fractional systems. Moreover, the fractional magnetic operator is a kind of non-local operator and it is defined on the complex field, resulting in complicated and difficult estimates.

Influenced by the above works, in the present paper, we investigated a system involving critical concave–convex nonlinearities driven by a magnetic fractional operator,

$$\begin{cases} (-\Delta)_A^s u_1 = \lambda_1 |u_1|^{q-2} u_1 + \frac{2\alpha_1}{\alpha_1 + \beta_1} |u_1|^{\alpha_1-2} u_1 |u_2|^{\beta_1} & \text{in } \Omega, \\ (-\Delta)_A^s u_2 = \lambda_2 |u_2|^{q-2} u_2 + \frac{2\beta_1}{\alpha_1 + \beta_1} |u_2|^{\beta_1-2} u_2 |u_1|^{\alpha_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (1)$$

where Ω is an open bounded set in $\mathbb{R}^n$, the boundary $\partial\Omega$ fulfills the Lipschitz condition, $\lambda_1 > 0, \lambda_2 > 0$ are two real parameters, $1 < q < 2 < \frac{n}{s}$ with $s \in (0, 1)$, $2_s^* = \frac{2n}{n-2s}$ is the critical exponent of fractional Sobolev space, let $\alpha_1 + \beta_1 = 2_s^*$, $\alpha_1 > 1$ and $\beta_1 > 1$, the function $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a L_{loc}^∞ -vector field, and $(-\Delta)_A^s$ is so called magnetic fractional Laplace operator. For any complex-valued function $\phi \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$, which is defined by

$$(-\Delta)_A^s \phi(x) = 2 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\phi(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} \phi(y)}{|x-y|^{n+2s}} dy, \quad x \in \mathbb{R}^n, \quad (2)$$

where $B_\epsilon(x)$ denotes a ball in $\mathbb{R}^n$, x is the center of the ball, and ϵ is the radius. Actually, the operator $(-\Delta)_A^s$ can be seen as the fractional counterpart of magnetic Laplace operator $-(\nabla - iA)^2$; for more details, the readers can see [17,18]. If $A \equiv 0$, the operator given by (2) is consistent with the usual fractional Laplace operator

$$(-\Delta)^s \phi(x) = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{\phi(x) - \phi(y)}{|x-y|^{n+2s}} dy, \quad x \in \mathbb{R}^n, \quad (3)$$

we refer the readers to reference [19] for details and properties about this operator. And when $A \equiv 0$, the problem (1) becomes

$$\begin{cases} (-\Delta)^s u_1 = \lambda_1 |u_1|^{q-2} u_1 + \frac{2\alpha_1}{\alpha_1 + \beta_1} |u_1|^{\alpha_1-2} u_1 |u_2|^{\beta_1} & \text{in } \Omega, \\ (-\Delta)^s u_2 = \lambda_2 |u_2|^{q-2} u_2 + \frac{2\beta_1}{\alpha_1 + \beta_1} |u_2|^{\beta_1-2} u_2 |u_1|^{\alpha_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \quad (4)$$

When $A \equiv 0$, $u_1 = u_2$, $\alpha_1 = \beta_1$, $\alpha_1 + \beta_1 = p$, $\lambda_1 = \lambda_2$, the problem (1) becomes

$$\begin{cases} (-\Delta)^s u_1 = \lambda_1 |u_1|^{q-2} u_1 + |u_1|^{p-2} u_1 & \text{in } \Omega, \\ u_1 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \quad (5)$$

And the more general forms of (4) and (5) are p -Laplace operators. For this class of system, in [20], Chen and Deng obtained at least two nontrivial solutions for case p with the Nehari manifold method. For the study of problems driven by fractional Laplace and p -Laplace operators, there are many valuable results, for example [21–35]. When $A \equiv 0$, $s = 1$, the

problem (1) becomes the critical elliptic equations involving concave–convex nonlinearities. For classical works, see [36,37].

2. Main Result

In order to describe our research, some notations, basic properties, and definitions about working spaces are needed, all of these will be used in the subsequent sections.

Let $L^2(\Omega, \mathbb{R})$ be the space of the measurable real function $\varphi : \Omega \rightarrow \mathbb{R}$,

$$L^2(\Omega, \mathbb{R}) := \left\{ \varphi \mid \varphi : \Omega \rightarrow \mathbb{R} \text{ is a real measurable function, with } \int_{\Omega} |\varphi(x)|^2 dx < \infty \right\},$$

equipped with norm

$$\|\varphi\|_{L^2} = \left(\int_{\Omega} |\varphi(x)|^2 dx \right)^{\frac{1}{2}} \text{ for all } \varphi \in L^2(\Omega, \mathbb{R}).$$

Denote the usual fractional Sobolev space by $H^s(\Omega)$, that is

$$H^s(\Omega) := \left\{ \varphi \mid \varphi \in L^2(\Omega, \mathbb{R}), [\varphi]_s < \infty \right\},$$

with norm

$$\|\varphi\|_{H^s} = \left(\|\varphi\|_{L^2}^2 + [\varphi]_s^2 \right)^{\frac{1}{2}},$$

where $[\varphi]_s^2 = \int_{\Omega \times \Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2s}} dx dy$, and $[\varphi]_s$ is the Gagliardo semi-norm.

Let $v : \Omega \rightarrow \mathbb{C}$ be a complex-valued function, which is measurable. Denote $L^2(\Omega, \mathbb{C})$ as a space of v , and define the real scalar product as follows:

$$\langle v, \varphi \rangle_{L^2(\Omega, \mathbb{C})} = \Re \int_{\Omega} v \bar{\varphi} dx \text{ for all } v, \varphi \in L^2(\Omega, \mathbb{C}),$$

where $\bar{\varphi}$ is the complex conjugate of φ , and $\Re \varphi$ is the real part of φ . The norm of $L^2(\Omega, \mathbb{C})$ endowed with

$$\|v\|_{L^2(\Omega, \mathbb{C})} = \left(\int_{\Omega} |v(x)|^2 dx \right)^{\frac{1}{2}} \text{ for all } v \in L^2(\Omega, \mathbb{C}).$$

Let $L^{2s*}(\Omega, \mathbb{C})$ be the space of $v : \Omega \rightarrow \mathbb{C}$, which is a Lebesgue space, and it has the following scalar product

$$\langle v, \varphi \rangle_{L^{2s*}(\Omega, \mathbb{C})} = \Re \int_{\Omega} |u(x)|^{2s*-2} v \bar{\varphi} dx,$$

for all $v, \varphi \in L^{2s*}(\Omega, \mathbb{C})$. The norm is defined by

$$\|v\|_{L^{2s*}(\Omega, \mathbb{C})} = \left(\int_{\Omega} |v(x)|^{2s*} dx \right)^{\frac{1}{2s*}} \text{ for all } v \in L^{2s*}(\Omega, \mathbb{C}).$$

Set

$$[v]_A^2 = \int_{\Omega \times \Omega} \frac{|v(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} v(y)|^2}{|x - y|^{n+2s}} dx dy,$$

where $[v]_A$ is so called magnetic Gagliardo semi-norm.

Let

$$H_A^s(\Omega, \mathbb{C}) := \{v \mid v \in L^2(\Omega, \mathbb{C}), [v]_A < \infty\}.$$

A scalar product on the space $H_A^s(\Omega, \mathbb{C})$ is defined by

$$\langle \phi, v \rangle_{s,A} = \langle \phi, v \rangle_{L^2(\Omega, \mathbb{C})} + \Re \int_{\Omega \times \Omega} \frac{[\phi(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} \phi(y)] \cdot \overline{[v(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} v(y)]}}{|x-y|^{n+2s}} dx dy,$$

and normed

$$\|v\|_{H_A^s} = \left(\|v\|_{L^2(\Omega, \mathbb{C})}^2 + [v]_A^2 \right)^{\frac{1}{2}}. \quad (6)$$

By the above definitions, we see that the space $H_A^s(\Omega, \mathbb{C})$ is consistent with the space $H^s(\Omega)$ when the function $A \equiv 0$. Moreover, according to [18], we have $C_0^\infty(\mathbb{R}^n, \mathbb{C}) \subseteq H_A^s(\mathbb{R}^n, \mathbb{C})$.

Set $Q = \mathbb{R}^{2n} \setminus (C\Omega \times C\Omega)$, and $C\Omega = \mathbb{R}^n \setminus \Omega$, then, we let

$$H := \left\{ v \mid v \in L^2(\Omega, \mathbb{C}), [v]_Q < \infty \right\},$$

where $[v]_Q^2 = \int_Q \frac{|v(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} v(y)|^2}{|x-y|^{n+2s}} dx dy$, the norm is given by

$$\|v\|_H = \left(\|v\|_{L^2(\Omega, \mathbb{C})}^2 + [v]_Q^2 \right)^{\frac{1}{2}}. \quad (7)$$

Next, we define the space H_0 as

$$H_0 := \{v \mid v \in H, v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

As the space introduced in [38], we generalize it to magnetic framework, and the real scalar product is defined as

$$\langle \phi, v \rangle_{H_0} = \Re \int_Q \frac{[\phi(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} \phi(y)] \cdot \overline{[v(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} v(y)]}}{|x-y|^{n+2s}} dx dy,$$

and normed with

$$\|v\|_{H_0} = [v]_Q = \left(\int_Q \frac{|v(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} v(y)|^2}{|x-y|^{n+2s}} dx dy \right)^{\frac{1}{2}}. \quad (8)$$

Similar to [1], we know that the norm defined by (7) is equivalent to (8), and we observe that $(H_0, \langle \cdot, \cdot \rangle_{H_0})$ is a real Hilbert space, and is also separable.

Now, we formulate the main result of our paper as follows.

Theorem 1. Let $\Omega \in \mathbb{R}^n$ be an open bounded set, $1 < q < 2 < \frac{n}{s}$ with $s \in (0, 1)$, $\alpha_1 + \beta_1 = \frac{2n}{n-2s} = 2_s^*$. Then, a positive number Λ can be founded, such that for every $0 < \lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}} < \Lambda$, the problem (1) possesses infinitely many solutions.

Remark 1. The main novelty of this result is that we provide a new method to obtain the result. As far as we know, there are not many results about the fractional system with external magnetic potential, and the conclusion of Theorem 1 is the first result for a magnetic fractional system with critical concave–convex powers.

Remark 2. When $2 < \alpha_1 + \beta_1 < 2_s^*$, the same result also can be obtained by using our method. The proof of this case is easier than the critical case, because Sobolev embedding has compactness in this subcritical case. The Palais–Smale (PS) condition holds easily for the energy functional. In this paper, we discuss the critical case, then Sobolev embedding compactness is lost, the energy functional does not satisfy the Palais–Smale (PS) condition at all levels. So in this case, we must find suitable range of c , such that the energy functional satisfies the Palais–Smale (PS) condition in this range.

Remark 3. As mentioned before, when $A \equiv 0$, the problem (1) becomes (4), for the system (4), our result of Theorem 1 also holds. Furthermore, using the present method, infinitely many solutions also can be obtained for this class p -Laplacian system. Moreover, our result is new, even for local cases.

Now, let us state the arrangement of the present paper. In Section 3, we provide some previous knowledge, recall some properties for the working space, and set up the functional framework. In the Section 4, we prove our main result. In the Section 5, we provide the conclusion.

3. Preliminary Knowledge and Functional Setting

To prove the main result, in this section, we introduce some lemmas related to the space and recall some preliminary knowledge.

Let $W = H_0 \times H_0$, we know that W is a separable Hilbert space, and the norm of space W is defined by

$$\|(u_1, u_2)\| = \left(\|u_1\|_{H_0}^2 + \|u_2\|_{H_0}^2 \right)^{\frac{1}{2}}. \quad (9)$$

To simplify the writing mode, we provide the notation

$$\langle L(u_1), v \rangle = \Re \int_Q \frac{[u_1(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u_1(y)] \cdot [\overline{v(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} v(y)}]}{|x - y|^{n+2s}} dx dy.$$

Then, we provide the following definition.

Definition 1. If $(u_1, u_2) \in W$, we have

$$\begin{aligned} \langle L(u_1), v \rangle + \langle L(u_2), v \rangle &= \Re \int_{\Omega} (\lambda_1 |u_1|^{q-2} u_1 \bar{v} + \lambda_2 |u_2|^{q-2} u_2 \bar{v}) dx \\ &\quad + \Re \left[\frac{2\alpha_1}{\alpha_1 + \beta_1} \int_{\Omega} |u_1|^{\alpha_1-2} u_1 \bar{v} |u_2|^{\beta_1} dx + \frac{2\beta_1}{\alpha_1 + \beta_1} \int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1-2} u_2 \bar{v} dx \right] \end{aligned}$$

for all $(v, v) \in W$, then (u_1, u_2) is a weak solution of the problem (1).

Now, let us define the functional $I(u_1, u_2) : W \rightarrow \mathbb{R}$ associated with the problem (1) as

$$I(u_1, u_2) = \frac{1}{2} \|(u_1, u_2)\|^2 - \frac{1}{q} \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx - \frac{2}{2_s^*} \int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1} dx. \quad (10)$$

We see that $I(u_1, u_2) \in C^1(W, \mathbb{R})$ and $I(u_1, u_2)$ is even, in addition

$$\begin{aligned} \langle I'(u_1, u_2), (v, v) \rangle &= \langle L(u_1), v \rangle + \langle L(u_2), v \rangle - \Re \int_{\Omega} (\lambda_1 |u_1|^{q-2} u_1 \bar{v} + \lambda_2 |u_2|^{q-2} u_2 \bar{v}) dx \\ &\quad + \Re \left[\frac{2\alpha_1}{2_s^*} \int_{\Omega} |u_1|^{\alpha_1-2} u_1 \bar{v} |u_2|^{\beta_1} dx + \frac{2\beta_1}{2_s^*} \int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1-2} u_2 \bar{v} dx \right]. \end{aligned}$$

It is clear that the weak solutions of the problem (1) are all the critical points of functional $I(u_1, u_2)$.

To prove the main result of Theorem 1, the following lemmas are also needed.

Lemma 1. For any $\varphi \in H_0$, we have $|\varphi| \in H^s(\Omega)$ and $\| |\varphi| \|_{H^s} \leq \|\varphi\|_{H_0}$.

Proof. Using the same process as in [3], we can obtain the result of Lemma 1 easily; so, the proving details are omitted here. $\square$

Lemma 2 ([1]). If $\Omega \subset \mathbb{R}^n$ is an open bounded set, then we have that $H_0 \hookrightarrow L^p(\Omega, \mathbb{C})$ is a continuous embedding for any $p \in [1, 2_s^*]$; If $\Omega \subset \mathbb{R}^n$ is an open bounded set and the boundary $\partial\Omega$ is Lipschitz, then we have that $H_0 \hookrightarrow L^p(\Omega, \mathbb{C})$ is a compact embedding for any $p \in [1, 2_s^*)$.

Next, we recall the definition of the Palais–Smale condition. For convenience, from then on, we write it as the (PS) condition for short.

Definition 2. ((PS) sequence) Let E be a Banach space, the functional $I \in C^1(E, \mathbb{R})$, $c \in \mathbb{R}$, if $\lim_{n \rightarrow \infty} I(u_{1n}, u_{2n}) = c$ in E and $\lim_{n \rightarrow \infty} I'(u_{1n}, u_{2n}) = 0$ in E^* , then we have $\{(u_{1n}, u_{2n})\} \subset E$ is a (PS) sequence of I , where E^* is the dual space of E .

Definition 3. ((PS) condition) If any (PS) sequence $\{(u_{1n}, u_{2n})\} \subset E$ has a convergent subsequence, the functional I satisfies the (PS) condition.

Remark 4. By Lemma 2, we know that the embedding $H_0 \hookrightarrow L^{2^*}(\Omega, \mathbb{C})$ is not compact for the critical case, the functional $I(u_1, u_2)$ does not satisfy the (PS) condition at all levels, then more strategies are needed to overcome this difficulty. Hence, we need to prove that $I(u_1, u_2)$ satisfies the (PS) condition at a certain level. The following definition is needed,

$$S = \inf_{v \in H_0 \setminus \{0\}} \frac{\|v\|_{H_0}^2}{\|v\|_{2_s^*}^2}. \quad (11)$$

It is easy to see that $S > 0$ is well defined.

In two steps, we show that the function I satisfies the (PS) condition. Firstly, we prove the boundedness of the (PS) sequence.

Lemma 3. Let $\{(u_{1n}, u_{2n})\} \subset W$ be a (PS) sequence of $I(u_1, u_2)$, we have that $\{(u_{1n}, u_{2n})\}$ is bounded.

Proof. Let us prove it by contradiction; assume that $\|(u_{1n}, u_{2n})\| \rightarrow \infty$ as $n \rightarrow \infty$. Since (u_{1n}, u_{2n}) is a (PS) sequence of $I(u_1, u_2)$, there exists a constant $C > 0$, such that $|I(u_{1n}, u_{2n})| \leq C$ and $|\langle I'(u_{1n}, u_{2n}), (u_{1n}, u_{2n}) \rangle| \leq C\|(u_{1n}, u_{2n})\|$, we get

$$\begin{aligned} C(1 + \|(u_{1n}, u_{2n})\|) &\geq I(u_{1n}, u_{2n}) - \frac{1}{2_s^*} \langle I'(u_{1n}, u_{2n}), (u_{1n}, u_{2n}) \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \|(u_{1n}, u_{2n})\|^2 - \left(\frac{1}{q} - \frac{1}{2_s^*}\right) \int_{\Omega} \lambda_1 |u_{1n}|^q + \lambda_2 |u_{2n}|^q dx. \end{aligned} \quad (12)$$

According to (11) and the Hölder inequality, we obtain

$$\begin{aligned} \int_{\Omega} \lambda_1 |u_{1n}|^q + \lambda_2 |u_{2n}|^q dx &\leq |\Omega|^{\frac{2_s^*-q}{2_s^*}} (\lambda_1 \|u_{1n}\|_{L^{2_s^*}(\Omega, \mathbb{C})}^q + \lambda_2 \|u_{2n}\|_{L^{2_s^*}(\Omega, \mathbb{C})}^q) \\ &\leq |\Omega|^{\frac{2_s^*-q}{2_s^*}} S^{-\frac{q}{2}} (\lambda_1 \|u_{1n}\|_{H_0}^q + \lambda_2 \|u_{2n}\|_{H_0}^q) \\ &\leq |\Omega|^{\frac{2_s^*-q}{2_s^*}} S^{-\frac{q}{2}} (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u_1, u_2)\|^q. \end{aligned} \quad (13)$$

Placing (13) into (12), we have

$$\begin{aligned} C(1 + \|(u_{1n}, u_{2n})\|) &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \|(u_{1n}, u_{2n})\|^2 \\ &\quad - \left(\frac{1}{q} - \frac{1}{2_s^*}\right) |\Omega|^{\frac{2_s^*-q}{2_s^*}} S^{-\frac{q}{2}} (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u_{1n}, u_{2n})\|^q. \end{aligned} \quad (14)$$

Obviously,

$$\begin{aligned} C\left(\frac{1}{\|(u_{1n}, u_{2n})\|^2} + \frac{1}{\|(u_{1n}, u_{2n})\|}\right) &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \\ &\quad - \left(\frac{1}{q} - \frac{1}{2_s^*}\right) |\Omega|^{\frac{2_s^*-q}{2_s^*}} S^{-\frac{q}{2}} (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \frac{1}{\|(u_{1n}, u_{2n})\|^{2-q}}. \end{aligned} \quad (15)$$

According to the assumptions, we obtain that $\|(u_{1n}, u_{2n})\| \rightarrow \infty$ as $n \rightarrow \infty$. In (15), we take the limit as $n \rightarrow \infty$, there must be

$$0 < \frac{1}{2} - \frac{1}{2_s^*} \leq 0 \quad (16)$$

as $1 < q < 2 < 2_s^*$. This is contradictory; the sequence $\{(u_{1n}, u_{2n})\}$ must be bounded in W . The proof is completed. $\square$

Lemma 4. If $\{(u_{1n}, u_{2n})\}$ is a (PS) sequence of I , and $(u_{1n}, u_{2n}) \rightharpoonup (u_1, u_2)$ in W , then $I'(u_1, u_2) = 0$, and $I(u_1, u_2) \geq -C^*(\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})$, where constant C^* is positive and

$$C^* = \left(\frac{1}{q} - \frac{1}{2_s^*}\right) \frac{2-q}{2} \left(\frac{2_s^* - q}{2_s^* - 2}\right)^{\frac{2}{2-q}} \left\| \frac{2(2_s^* - q)}{2_s^*(2-q)} S^{-\frac{q}{2-q}} \right\|.$$

Proof. Considering that $\{(u_{1n}, u_{2n})\}$ is a (PS) sequence of I in W , Let $(v, v) \in W$, we have

$$\begin{aligned} \langle I'(u_{1n}, u_{2n}) - I'(u_1, u_2), (v, v) \rangle &= \langle L(u_{1n}), v \rangle - \langle L(u_1), v \rangle + \langle L(u_{2n}), v \rangle - \langle L(u_2), v \rangle \\ &\quad - \lambda_1 \Re \int_{\Omega} (|u_{1n}|^{q-2} u_{1n} - |u_1|^{q-2} u_1) \bar{v} dx \\ &\quad - \lambda_2 \Re \int_{\Omega} (|u_{2n}|^{q-2} u_{2n} - |u_2|^{q-2} u_2) \bar{v} dx \\ &\quad - \frac{2\alpha_1}{2_s^*} \Re \int_{\Omega} (|u_{1n}|^{\alpha_1-2} u_{1n} |u_{2n}|^{\beta_1} - |u_1|^{\alpha_1-2} u_1 |u_2|^{\beta_1}) \bar{v} dx \\ &\quad - \frac{2\beta_1}{2_s^*} \Re \int_{\Omega} (|u_{1n}|^{\alpha_1} |u_{2n}|^{\beta_1-2} u_{2n} - |u_1|^{\alpha_1} |u_2|^{\beta_1-2} u_2) \bar{v} dx, \end{aligned} \quad (17)$$

where $\langle L(\cdot), \cdot \rangle$ is given as before. Meanwhile, by Lemma 3, we know that (u_{1n}, u_{2n}) is bounded, and by Lemma 2, we have that there exists a subsequence (we still denote it by $\{(u_{1n}, u_{2n})\}$), such that

$$\begin{aligned} (u_{1n}, u_{2n}) &\rightharpoonup (u_1, u_2) \text{ in } W, \quad u_{1n} \rightharpoonup u_1, \quad u_{2n} \rightharpoonup u_2 \text{ in } L^{2_s^*}(\Omega, \mathbb{C}), \\ u_{1n} &\rightarrow u_1, \quad u_{2n} \rightarrow u_2 \text{ in } L^r(\Omega, \mathbb{C}), \quad 1 \leq r < 2_s^*, \\ u_{1n} &\rightarrow u_1, \quad u_{2n} \rightarrow u_2 \text{ a.e. in } \mathbb{R}^n, \end{aligned} \quad (18)$$

we have

$$\frac{u_{1n}(x) - e^{i(x-y)A(\frac{x+y}{2})} u_{1n}(y)}{|x-y|^{\frac{n+2s}{2}}} \rightarrow \frac{u_1(x) - e^{i(x-y)A(\frac{x+y}{2})} u_1(y)}{|x-y|^{\frac{n+2s}{2}}} \text{ in } L^2(\Omega, \mathbb{C})$$

and

$$\frac{u_{2n}(x) - e^{i(x-y)A(\frac{x+y}{2})} u_{2n}(y)}{|x-y|^{\frac{n+2s}{2}}} \rightarrow \frac{u_2(x) - e^{i(x-y)A(\frac{x+y}{2})} u_2(y)}{|x-y|^{\frac{n+2s}{2}}} \text{ in } L^2(\Omega, \mathbb{C}).$$

Since

$$\begin{aligned} \frac{v(x) - e^{i(x-y)A(\frac{x+y}{2})} v(y)}{|x-y|^{\frac{n+2s}{2}}} &\in L^2(\Omega, \mathbb{C}), \\ \frac{v(x) - e^{i(x-y)A(\frac{x+y}{2})} v(y)}{|x-y|^{\frac{n+2s}{2}}} &\in L^2(\Omega, \mathbb{C}) \end{aligned}$$

for all $v, v \in H_0$, we claim that

$$\lim_{n \rightarrow \infty} \langle L(u_{1n}), v \rangle = \langle L(u_1), v \rangle, \quad \lim_{n \rightarrow \infty} \langle L(u_{2n}), v \rangle = \langle L(u_2), v \rangle.$$

In addition, we know

$$|u_{1n}|^{q-2} u_{1n} \rightarrow |u_1|^{q-2} u_1, \quad |u_{2n}|^{q-2} u_{2n} \rightarrow |u_2|^{q-2} u_2 \text{ in } L^{\frac{q}{q-1}}(\Omega, \mathbb{C})$$

and

$$\begin{aligned} |u_{1n}|^{\alpha_1-2} u_{1n} |u_{2n}|^{\beta_1} &\rightarrow |u_1|^{\alpha_1-2} u_1 |u_2|^{\beta_1}, \\ |u_{1n}|^{\alpha_1} |u_{2n}|^{\beta_1-2} u_{2n} &\rightarrow |u_1|^{\alpha_1} |u_2|^{\beta_1-2} u_2 \text{ in } L^{\frac{2_s^*}{2_s^*-1}}(\Omega, \mathbb{C}) \end{aligned}$$

for all $v, v \in L^q(\Omega, \mathbb{C}) \cap L^{2_s^*}(\Omega, \mathbb{C})$, as $n \rightarrow \infty$. Therefore, we can obtain

$$\begin{aligned} \Re \int_{\Omega} (|u_{1n}|^{\alpha_1-2} u_{1n} |u_{2n}|^{\beta_1} - |u_1|^{\alpha_1-2} u_1 |u_2|^{\beta_1}) \bar{v} dx &\rightarrow 0, \\ \Re \int_{\Omega} (|u_{2n}|^{q-2} u_{2n} - |u_2|^{q-2} u_2) \bar{v} dx &\rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \Re \int_{\Omega} (|u_{1n}|^{q-2} u_{1n} - |u_1|^{q-2} u_1) \bar{v} dx &\rightarrow 0, \\ \Re \int_{\Omega} (|u_{1n}|^{\alpha_1} |u_{2n}|^{\beta_1-2} u_{2n} - |u_1|^{\alpha_1} |u_2|^{\beta_1-2} u_2) \bar{v} dx &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence,

$$\langle I'(u_{1n}, u_{2n}) - I'(u_1, u_2), (v, v) \rangle \rightarrow 0, \text{ as } n \rightarrow \infty, \text{ for all } (v, v) \in W. \quad (19)$$

As $I'(u_{1n}, u_{2n}) = o(1)$, as $n \rightarrow \infty$, we have the conclusion

$$I'(u_1, u_2) = 0. \quad (20)$$

Specially, $\langle I'(u_1, u_2)(u_1, u_2) \rangle = 0$, we have

$$2 \int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1} dx = \|(u_1, u_2)\|^2 - \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx. \quad (21)$$

So

$$\begin{aligned} I(u_1, u_2) &= I(u_1, u_2) - \frac{1}{2^*} \langle I'(u_1, u_2)(u_1, u_2) \rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \|(u_1, u_2)\|^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right) \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx. \end{aligned} \quad (22)$$

By Hölder's inequality, Young's inequality, and the definition of S , we have

$$\begin{aligned} \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx &\leq |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} (\lambda_1 \|u_1\|_{H_0}^q + \lambda_2 \|u_2\|_{H_0}^q) \\ &= \left(\left[\frac{2}{q} \frac{s}{n} \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1}\right]^{\frac{q}{2}} \|u_1\|_{H_0}^q\right) \left(\left[\frac{2}{q} \frac{s}{n} \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1}\right]^{-\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \lambda_1\right. \\ &\quad \left.+ \left(\left[\frac{2}{q} \frac{s}{n} \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1}\right]^{\frac{q}{2}} \|u_2\|_{H_0}^q\right) \left(\left[\frac{2}{q} \frac{s}{n} \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1}\right]^{-\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \lambda_2\right) \\ &\leq \frac{s}{n} \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1} \|(u_1, u_2)\|^2 + C_1 (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}}), \end{aligned} \quad (23)$$

where

$$\begin{aligned} C_1 &= \frac{2-q}{2} \left(\left[\frac{2}{q} \frac{s}{n} \left(\frac{1}{q} - \frac{1}{2^*}\right)^{-1}\right]^{-\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \right)^{\frac{2}{2-q}} \\ &= \frac{2-q}{2} \left(\frac{2^*-q}{2^*-2} \right)^{\frac{q}{2-q}} |\Omega|^{\frac{2(2^*-q)}{2^*(2-q)}} S^{-\frac{q}{2-q}}. \end{aligned}$$

Putting (23) into (22), we have

$$I(u_1, u_2) \geq -C^* (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}}), \quad (24)$$

where

$$C^* = \left(\frac{1}{q} - \frac{1}{2^*}\right) C_1 = \left(\frac{1}{q} - \frac{1}{2^*}\right) \frac{2-q}{2} \left(\frac{2^*-q}{2^*-2}\right)^{\frac{q}{2-q}} |\Omega|^{\frac{2(2^*-q)}{2^*(2-q)}} S^{-\frac{q}{2-q}}.$$

As $1 < q < 2 < 2^*$, the constant $C^* > 0$. The result of Lemma 4 is obtained. $\square$

Next, we need to prove that the (PS) sequence of I in W admits a convergent subsequence. Let

$$S_{2^*}^* := \inf_{(u_1, u_2) \in W \setminus \{(0,0)\}} \frac{\|(u_1, u_2)\|^2}{\left(\int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1} dx\right)^{\frac{2}{2^*}}}. \quad (25)$$

Lemma 5. The function I satisfies the (PS) condition in W for $-\infty < c < \delta$, where $\delta = \frac{2s}{n} \left(\frac{S_{2^*}^*}{2}\right)^{\frac{n}{2^*}} - C^* (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})$, C^* is provided by Lemma 4 and $S_{2^*}^*$ is provided by (25).

Proof. We know that $\{(u_{1n}, u_{2n})\} \subset W$ is a (PS) sequence; there must be

$$\frac{1}{2} \|(u_{1n}, u_{2n})\|^2 - \frac{1}{q} \int_{\Omega} (\lambda_1 |u_{1n}|^q + \lambda_2 |u_{2n}|^q) dx - \frac{2}{2^*} \int_{\Omega} |u_{1n}|^{\alpha_1} |u_{2n}|^{\beta_1} dx = c + o(1), \quad (26)$$

and

$$\|(u_{1n}, u_{2n})\|^2 - \int_{\Omega} (\lambda_1 |u_{1n}|^q + \lambda_2 |u_{2n}|^q) dx - 2 \int_{\Omega} |u_{1n}|^{\alpha_1} |u_{2n}|^{\beta_1} dx = o(\|(u_{1n}, u_{2n})\|). \quad (27)$$

According to Lemma 3, the boundedness of $\{(u_{1n}, u_{2n})\}$ is obtained. Hence, we know that there exists a subsequence of $\{(u_{1n}, u_{2n})\}$ (still denoted by $\{(u_{1n}, u_{2n})\}$), $(u_{1n}, u_{2n}) \rightharpoonup$

(u_1, u_2) in W . By Lemma 4, we obtain that $(u_1, u_2) \in W$ is a critical point of $I(u_1, u_2)$. We also know that $u_{1n} \rightarrow u_1$ and $u_{2n} \rightarrow u_2$ in $L^r(\Omega, \mathbb{C})$ with $1 \leq r < 2_s^*$, then

$$\int_{\Omega} (\lambda_1 |u_{1n}|^q + \lambda_2 |u_{2n}|^q) dx \rightarrow \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx, \quad (28)$$

as $n \rightarrow \infty$. Considering the famous Brézis-Lieb lemma, we have

$$\|(u_{1n}, u_{2n})\| = \|(u_{1n} - u_1, u_{2n} - u_2)\| + \|(u_1, u_2)\| + o(1) \quad (29)$$

and

$$\int_{\Omega} |u_{1n}|^{\alpha_1} |u_{2n}|^{\beta_1} dx = \int_{\Omega} |u_{1n} - u_1|^{\alpha_1} |u_{2n} - u_2|^{\beta_1} dx + \int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1} dx + o(1). \quad (30)$$

Taking (29) and (30) into (26), we have

$$\frac{1}{2} \|(u_{1n} - u_1, u_{2n} - u_2)\| - \frac{2}{2_s^*} \int_{\Omega} |u_{1n} - u_1|^{\alpha_1} |u_{2n} - u_2|^{\beta_1} dx = c - I(u_1, u_2) + o(1), \quad (31)$$

furthermore, we place (29) and (30) into (27), and we find

$$\|(u_{1n} - u_1, u_{2n} - u_2)\|^2 = 2 \int_{\Omega} |u_{1n} - u_1|^{\alpha_1} |u_{2n} - u_2|^{\beta_1} dx + o(1). \quad (32)$$

Let us assume

$$\|(u_{1n} - u_1), (u_{2n} - u_2)\|^2 \rightarrow a, \quad 2 \int_{\Omega} |u_{1n} - u_1|^{\alpha_1} |u_{2n} - u_2|^{\beta_1} dx \rightarrow b \quad (33)$$

as $n \rightarrow \infty$. If $a = 0$, we complete the proof. Next, assuming $a > 0$ and combining (32) with (33), we find $a \leq 2S_{2_s^*}^{-\frac{2_s^*}{2}} \cdot a^{\frac{2_s^*}{2}}$; hence,

$$a \geq 2 \left(\frac{S_{2_s^*}}{2} \right)^{-\frac{n}{2s}}. \quad (34)$$

Take the limit in (31) as $n \rightarrow \infty$, by (33), (34), and Lemma 4, we have

$$\begin{aligned} c &= \frac{1}{2}a - \frac{m}{2_s^*} + I(u_1, u_2) = \left(\frac{1}{2} - \frac{1}{2_s^*} \right)a + I(u_1, u_2) \\ &\geq \frac{s}{n}a + I(u_1, u_2) \geq \frac{2s}{n} \left(\frac{S_{2_s^*}}{2} \right)^{-\frac{n}{2s}} - C^*(\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}}) = \delta. \end{aligned} \quad (35)$$

As $-\infty < c < \delta$, we have the contradiction. Hence, the proof is completed. $\square$

4. Proof of Main Result

Now, we provide the proof of Theorem 1 in this section. Firstly, we also consider the functional $I(u_1, u_2)$, by the Hölder inequality and $S_{2_s^*}$ given by (25), we have

$$\begin{aligned} I(u_1, u_2) &\geq \frac{1}{2} \|(u_1, u_2)\|^2 - \frac{2}{2_s^*} S_{2_s^*}^{-\frac{2_s^*}{2}} \|(u_1, u_2)\|^{2_s^*} - \frac{1}{q} |\Omega|^{\frac{2_s^*-q}{2s}} (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u_1, u_2)\|^q \\ &= \frac{1}{2} \|(u_1, u_2)\|^2 - C_1 \|(u_1, u_2)\|^{2_s^*} - C_2 (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u_1, u_2)\|^q, \end{aligned} \quad (36)$$

where $C_1 = \frac{2}{2_s^*} S_{2_s^*}^{-\frac{2_s^*}{2}}$ and $C_2 = \frac{1}{q} |\Omega|^{\frac{2_s^*-q}{2s}}$. Assume that

$$f(t) = \frac{1}{2}t^2 - C_1 t^{2_s^*} - C_2 (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} t^q, \quad t \geq 0, \quad (37)$$

where C_1 and C_2 as above. Obviously,

$$I(u_1, u_2) \geq h(\|(u_1, u_2)\|). \quad (38)$$

As $1 < q < 2 < 2_s^*$, we can have that $f(t) < 0$ for t near 0, and $f(t) \rightarrow -\infty$ as $t \rightarrow \infty$. From the structure of $f(t)$, we know that there exist a small $R_0 > 0$ and

$$\Lambda = C_2^{-\frac{2}{2-q}} \left(\frac{1}{2} R_0^{2-q} - C_1 R_0^{2_s^*-q} \right)^{\frac{2}{2-q}} \quad (39)$$

such that $f_\Lambda(R_0) > 0$ for $0 < \lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}} < \Lambda$. Let

$$R_1 = \max\{t \in (0, R_0) : f_\Lambda(t) \leq 0\}, \quad R_2 = \min\{t > R_0 : f_\Lambda(t) \leq 0\}.$$

Hence, we have $0 < R_1 < R_2$ and $f(R_1) = f(R_2) = 0$. It can be seen that there must be a $\Lambda > 0$, for $0 < \lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}} < \Lambda$, we obtain

- (i) $f(t)$ has a positive maximum;
- (ii) $\delta \geq 0$, where $\delta = \frac{2s}{n} \left(\frac{S_{2_s^*}}{2} \right)^{\frac{n}{2s}} - C^* (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})$ is given in Lemma 5, C^* is given by Lemma 4 and $S_{2_s^*}$ is given by (25).

Clearly, if $0 \leq t \leq R_1$, $f(t) \leq 0$; if $R_1 < t < R_2$, $f(t) > 0$; if $t \geq R_2$, $f(t) \leq 0$. Let $e(t) \in C^\infty$ be a non-increasing function such that $0 \leq e(t) \leq 1$ for all $t \in \mathbb{R}_0^+$. Define $e(t) = 1$, if $0 \leq t \leq R_1$; $e(t) = 0$, if $t \geq R_2$. Set

$$g(u_1, u_2) = e(\|(u_1, u_2)\|).$$

Considering the same idea as in [39,40], we define the truncation functional

$$J(u_1, u_2) = \frac{1}{2} \|(u_1, u_2)\|^2 - \frac{1}{q} \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx - \frac{2}{2_s^*} \int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1} g(u_1, u_2) dx. \quad (40)$$

Let

$$\tilde{f}(t) = \frac{1}{2} t^2 - c_1 t^{2_s^*} e(t) - c_2 (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} t^q, \quad t \geq 0. \quad (41)$$

As $0 \leq e(t) \leq 1$, we can see that $\tilde{f}(t) \geq f(t)$ for all $t \geq 0$. In the same way as (36), we have

$$J(u_1, u_2) \geq \frac{1}{2} \|(u_1, u_2)\|^2 - c_1 e(\|(u_1, u_2)\|) \|(u_1, u_2)\|^{2_s^*} - c_2 (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u_1, u_2)\|^q. \quad (42)$$

It is easy to see that

$$J(u_1, u_2) \geq \tilde{f}(\|(u_1, u_2)\|). \quad (43)$$

By the definitions of $f(t)$, $\tilde{f}(t)$, and $e(t)$, we find

$$\tilde{f}(t) = f(t), \quad \text{if } 0 \leq t \leq R_1,$$

$$\tilde{f}(t) \geq f(t) \geq 0, \quad \text{if } R_1 \leq t \leq R_2,$$

and if $t > R_2$, we have

$$\begin{aligned} \tilde{f}(t) &= \frac{1}{2} t^2 - c_2 (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} t^q \\ &= t^q \left(\frac{1}{2} t^{2-q} - c_2 (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \right). \end{aligned}$$

When $0 < \lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}} < \Lambda$, we have $\tilde{f}(t) \geq 0$. Hence,

$$\tilde{f}(t) \geq 0 \quad \text{for } t > R_1. \quad (44)$$

To show our main result, we give two lemmas, which will be used in the next proof.

Lemma 6. *There exists a constant $\Lambda > 0$, for any $0 < \lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}} < \Lambda$, we have*

- (i) $J(u_1, u_2) \in C^1(W, \mathbb{R})$;

- (ii) If $J(u_1, u_2) < 0$, then $\|(u_1, u_2)\| < R_1$ and $J(\phi, \varphi) = I(\phi, \varphi)$ for all (ϕ, φ) is in a small neighborhood of $(u_1, u_2) \in W$.
- (iii) $J(u_1, u_2)$ satisfies the (PS) condition on W .

Proof.

- (i) According to the definition of $e(t)$, we can see that $e(t) \equiv 1$ for (u_1, u_2) near $(0, 0)$; hence, $J(u_1, u_2) \in C^1(W, \mathbb{R})$ holds;
- (ii) In contradiction, we suppose that $J(u_1, u_2) < 0$ and $\|(u_1, u_2)\| \geq R_1$. From (43) and (44), we know that $J(u_1, u_2) \geq \tilde{f}(\|(u_1, u_2)\|)$ and $\tilde{f}(t) \geq 0$, for $t \geq R_1$. We have $J(u_1, u_2) \geq 0$, which contradicts $J(u_1, u_2) < 0$, so $\|(u_1, u_2)\| < R_1$.
- (iii) Let $\{(u_{1n}, u_{2n})\} \subset W$ be a (PS) sequence of $J(u_1, u_2)$. If $c < 0$, we can assume that $J(u_{1n}, u_{2n}) < 0$, $J'(u_{1n}, u_{2n}) \rightarrow 0$ as $n \rightarrow \infty$. With the result of (ii), that is $J(u_1, u_2) < 0$ and $\|(u_1, u_2)\| < R_1$, we obtain $I(u_{1n}, u_{2n}) = J(u_{1n}, u_{2n})$ and $I'(u_{1n}, u_{2n}) = J'(u_{1n}, u_{2n})$. Thus, by Lemma 5, we deduce that $J(u_1, u_2)$ satisfies the (PS) condition for $c < 0$.

□

Lemma 7. Given $n \in \mathbb{N}$, there exists a $\xi = \xi(n) > 0$, such that

$$\gamma(\{(u_1, u_2) \in W : J(u_1, u_2) \leq -\xi\}) \geq n,$$

γ denotes the genus of the set.

Proof. Fix $n \in \mathbb{N}$, set $W_n \subset W$ is a n -dimensional subspace, we know that W_n is a finite space. Let $(u_1, u_2) \in W_n$ with $\|(u_1, u_2)\| = 1$, for $0 < t < R_1$, we have

$$I(tu_1, tu_2) = J(tu_1, tu_2) = \frac{1}{2}t^2 - \frac{t^q}{q} \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx - \frac{2t^{2_s^*}}{2_s^*} \int_{\Omega} |u_1|^{\alpha_1} |u_2|^{\beta_1} g(u_1, u_2) dx.$$

Define

$$\beta_{1n} = \inf\left\{\int_{\Omega} |u_1|^q dx : (u_1, u_2) \in W_n, \|(u_1, u_2)\| = 1\right\} > 0.$$

It is clear that all the norms are equivalent in the finite space, we find

$$J(tu_1, tu_2) \leq \frac{1}{2}t^2 - \frac{1}{q}\beta_{1n} \min(\lambda_1, \lambda_2)t^q.$$

For any $\xi > 0$ and $0 < t < R_1$ such that $J(tu_1, tu_2) \leq -\xi$ for $(u_1, u_2) \in W_n$ with $\|(u_1, u_2)\| = 1$. Denote

$$S_t = \{(u_1, u_2) \in W \text{ and } \|(u_1, u_2)\| = t\},$$

then $\gamma(S_t \cap W_n) = n$ and

$$S_t \cap W_n \subset \{(u_1, u_2) \in W : J(u_1, u_2) \leq -\xi\}.$$

Using the monotonicity property of genus, the result

$$\gamma(\{(u_1, u_2) \in W : J(u_1, u_2) \leq -\xi\}) \geq \gamma(S_t \cap W_n) = n$$

holds. □

Now, let us give the proof of Theorem 1.

Proof of Theorem 1. Let

$$\omega_n = \{B | B \in W \setminus \{(0, 0)\}, B \text{ is closed and symmetric}, \gamma(B) \geq n\},$$

where $\gamma(B)$ denotes the genus of set B . We also define

$$c_n = \inf_{B \in \omega_n, (u_1, u_2) \in B} \sup J(u_1, u_2),$$

and

$$\Gamma_c = \{(u_1, u_2) \in W \mid J(u_1, u_2) = c, J'(u_1, u_2) = 0\}.$$

Assume that $0 < \lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}} < \Lambda$, where Λ is given as above, we show that if $n, k \in \mathbb{N}$ such that $c = c_n = c_{n+1} = c_{n+2} = \dots = c_{n+k}$, then $\gamma(\Gamma_c) \geq k + 1$.

Next, we provide the details. Let

$$J^{-\xi} = \{(u_1, u_2) \in W : J(u_1, u_2) \leq -\xi\}.$$

By Lemma 7, we obtain that there exists a $\xi(n) > 0$, such that $\gamma(J^{-\xi}) \geq n$. As $J(u_1, u_2)$ is continuous and even, $J^{-\xi} \in \omega_n$ and $c_n \leq \xi(n) < 0$.

In addition, for $\|(u_1, u_2)\| \geq R_2$, by the Hölder inequality, we have

$$\begin{aligned} J(u_1, u_2) &= \frac{1}{2} \|(u_1, u_2)\|^2 - \frac{1}{q} \int_{\Omega} (\lambda_1 |u_1|^q + \lambda_2 |u_2|^q) dx \\ &\geq \frac{1}{2} \|(u_1, u_2)\|^2 - \frac{1}{q} |\Omega|^{\frac{2^*_q - q}{2^*_q}} (\lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u_1, u_2)\|^q. \end{aligned}$$

As $q < 2$, it is easy to see $J(u_1, u_2)$ is coercive and bounded from below. We know that $c_n > -\infty$ for all $n \in \mathbb{N}$.

We may assume that $c = c_n = c_{n+1} = \dots = c_{n+k} < 0$. According to Lemma 6, function $J(u_1, u_2)$ satisfies the (PS) condition, and from the definition of Γ_c , we see that Γ_c is a compact set.

Through contradiction, if $\gamma(\Gamma_c) \leq k$; then, there is a set U with $\Gamma_c \subset U$, where U is a closed and symmetric set. As the genus has a continuity property (see the details in [41]), we have $\gamma(U) \leq k$. Considering that $c < 0$, we can let

$$U \subset J^0 = \{(u_1, u_2) \in W : J(u_1, u_2) \leq 0\}.$$

According to [42], for some $0 < \tau < -c$, there is an odd homeomorphism $h : W \rightarrow W$, such that $h(J^{c+\tau} - U) \subset J^{c-\tau}$. In fact $c = c_{n+k} = \inf_{B \in \omega_n} \sup_{(u_1, u_2) \in B} J(u_1, u_2)$, there exists a set

$B \in \omega_n$, such that $\sup_{(u_1, u_2) \in B} J(u_1, u_2) < c + \tau$, that is

$$h(B - U) \subset h(J^{c+\tau} - U) \subset J^{c-\tau},$$

we have

$$\sup_{(u_1, u_2) \in h(B-U)} J(u_1, u_2) \leq c - \tau. \quad (45)$$

By $\gamma(U) \leq k$, we have

$$\gamma(h(\overline{(B-U)})) \geq \gamma(\overline{(B-U)}) \geq \gamma(B) - \gamma(U) \geq n. \quad (46)$$

From this, we can obtain

$$h(\overline{(B-U)}) \in \omega_n \text{ and } \sup_{(u_1, u_2) \in h(B-U)} J(u_1, u_2) \geq c_n = c.$$

That is a contradiction of (45); we have showed that $\gamma(\Gamma_c) \geq k + 1$. Naturally, if for all $n \in \mathbb{N}$, we have $\omega_{n+1} \subset \omega_n$ and $c_n \leq c_{n+1} \leq 0$. If all c_n are distinct, there must be a result that $\gamma(\Gamma_c) \geq 1$, and each element of the sequence $\{c_n\}$ is a negative critical value of $J(u_1, u_2)$.

If for some n_0 , there exists a $k \geq 1$, such that

$$c = c_{n_0} = c_{n_0+1} = \cdots = c_{n_0+k},$$

and as above, we know $\gamma(\Gamma_c) \geq k + 1$, this tells us that infinitely many distinct elements are contained in Γ_c . This means that J has infinitely many distinct critical values.

From Lemma 6, we know that $I(u_1, u_2) = J(u_1, u_2)$ if $J(u_1, u_2) < 0$. It is easy to see that the above critical values of J are also the critical values of I . We have proven that problem (1) has infinitely distinct weak solutions. We finish the proof. $\square$

5. Conclusions

By using Lusternik–Schnirelmann's theory, let $\Omega \in \mathbb{R}^n$ be an open bounded set, under the conditions $1 < q < 2 < \frac{n}{s}$, $s \in (0, 1)$, $\alpha_1 + \beta_1 = \frac{2n}{n-2s} = 2_s^*$, we find a positive number Λ , when $0 < \lambda_1^{\frac{2}{2-q}} + \lambda_2^{\frac{2}{2-q}} < \Lambda$, for the fractional magnetic system involving critical concave–convex nonlinearities with Laplace operators, the existence result of infinitely many solutions is obtained.

Author Contributions: The contribution of all authors is the same. All authors have read and agreed to the published version of the manuscript

Funding: The research of Jiabin Zuo were supported by the Guangdong Basic and Applied Basic Research Foundation (2022A1515110907) and the Guangdong Basic and Applied Basic Research Foundation (2024A1515012389).

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Fiscella, A.; Pinamonti, A.; Vecchi, E. Multiplicity results for magnetic fractional problems. *J. Differ. Equ.* **2017**, *263*, 4617–4633. [\[CrossRef\]](#)
2. Wang, F.; Xiang, M. Multiplicity of solutions to a nonlocal Choquard equation involving fractional magnetic operators and critical exponent. *SIAM Electron. J. Differ. Equ.* **2016**, *2016*, 1–11.
3. Xiang, M.; Pucci, P.; Squassina, M.; Zhang, B. Nonlocal Schrödinger–Kirchhoff equations with external magnetic field. *Discret. Contin. Dyn. Syst.* **2017**, *37*, 1631–1649. [\[CrossRef\]](#)
4. Yang, L.; Zuo, J.; An, T. Existence of entire solutions for critical Sobolev–Hardy problems involving magnetic fractional operator. *Complex Var. Elliptic Equ.* **2021**, *66*, 1864–1880. [\[CrossRef\]](#)
5. Jin, Z.; Sun, H.; Zhang, J. Existence of ground state solutions for critical fractional Choquard equations involving periodic magnetic field. *Adv. Nonlinear Stud.* **2022**, *22*, 372–389. [\[CrossRef\]](#)
6. Ambrosio, V. Existence and concentration results for some fractional Schrödinger equations in $\mathbb{R}^N$ with magnetic fields. *Commun. Partial Differ. Equ.* **2019**, *44*, 637–680. [\[CrossRef\]](#)
7. Di Cosmo, J.; Van Schaftingen, J. Semiclassical stationary states for nonlinear Schrödinger equations under a strong external magnetic field. *J. Differ. Equ.* **2015**, *259*, 596–627. [\[CrossRef\]](#)
8. Pinamonti, A.; Squassina, M.; Vecchi, E. Magnetic BV-functions and the Bourgain–Brezis–Mironescu formula. *Adv. Calc. Var.* **2019**, *12*, 225–252. [\[CrossRef\]](#)
9. Zhang, B.; Squassina, M.; Xia, Z. Fractional NLS equations with magnetic field, critical frequency and critical growth. *Manuscripta Math.* **2018**, *155*, 115–140.
10. Yang, L.; An, T. Infinitely many solutions for magnetic fractional problems with critical Sobolev–Hardy nonlinearities. *Math. Methods Appl. Sci.* **2018**, *41*, 9607–9617. [\[CrossRef\]](#)
11. Sun, M.; Shi, S.; Repovš, D. Degenerate fractional Kirchhoff-type system with magnetic fields and upper critical growth. *Mediterr. J. Math.* **2022**, *19*, 170. [\[CrossRef\]](#)
12. Liang, S.; Repovš, D.; Zhang, B. Fractional magnetic Schrödinger–Kirchhoff problems with convolution and critical nonlinearities. *Math. Methods Appl. Sci.* **2020**, *43*, 2473–2490. [\[CrossRef\]](#)
13. Liang, S.; Repovš, D.; Zhang, B. On the fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity. *Comput. Math. Appl.* **2018**, *75*, 1778–1794. [\[CrossRef\]](#)
14. Ji, C.; Rădulescu, V. Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger system. *Calc. Var. Partial Differ. Equ.* **2020**, *59*, 115. [\[CrossRef\]](#)
15. Ambrosio, V. On a fractional magnetic Schrödinger equation in R with exponential critical growth. *Nonlinear Anal.* **2019**, *183*, 117–148. [\[CrossRef\]](#)

16. Ambrosio, V. Multiple concentrating solutions for a fractional Kirchhoff equation with magnetic fields. *Discret. Contin. Dyn. Syst.* **2006**, *40*, 781–815. [\[CrossRef\]](#)
17. Squassina, M.; Volzone, B. Bourgain-Brézis-Mironescu formula for magnetic operators. *C. R. Math. Acad. Sci. Paris* **2016**, *354*, 825–831. [\[CrossRef\]](#)
18. D’Avenia, P.; Squassina, M. Ground states for fractional magnetic operators. *ESAIM Control Optim. Calc. Var.* **2018**, *24*, 1–24. [\[CrossRef\]](#)
19. Di Nezza, E.; Palatucci, G.; Valdinoci, E. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [\[CrossRef\]](#)
20. Chen, W.; Deng, S. The Nehari manifold for p -Laplacian system involving concave-convex nonlinearities. *Nonlinear Anal.* **2016**, *27*, 80–92. [\[CrossRef\]](#)
21. Chen, W.; Deng, S. The Nehari manifold for non-local elliptic operators involving concave-convex nonlinearities. *Z. Angew. Math. Phys.* **2015**, *66*, 1387–1400. [\[CrossRef\]](#)
22. Chen, W.; Squassina, M. Critical nonlocal systems with concave-convex towers. *Adv. Nonlinear Stud.* **2016**, *16*, 821–842. [\[CrossRef\]](#)
23. Chang, X.; Wang, Z. Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity. *Nonlinearity* **2006**, *26*, 479–494. [\[CrossRef\]](#)
24. Hou, G.; Ge, B.; Lu, J. Infinitely many solutions for sublinear fractional Schrödinger-type equations with general potentials. *Electron. J. Diff. Equ.* **2018**, *97*, 1–13.
25. Molica Bisci, G.; Rădulescu, V. Ground state solutions of scalar field fractional Schrödinger equations. *Calc. Var. Partial Diff. Equ.* **2015**, *54*, 2985–3008. [\[CrossRef\]](#)
26. Felmer, P.; Quaas, A.; Tan, J. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proc. R. Soc. Edinb. Sect. A* **2012**, *142*, 1237–1262. [\[CrossRef\]](#)
27. Pucci, P.; Xiang, M.; Zhang, B. Existence and multiplicity of entire solutions for fractional p -Kirchhoff equations. *Adv. Nonlinear Anal.* **2016**, *5*, 27–55. [\[CrossRef\]](#)
28. Chen, W.; Gui, Y. Multiplicity of solutions for fractional p - q -Laplacian system involving critical concave-convex nonlinearities. *Appl. Math. Lett.* **2018**, *96*, 81–88.
29. Chu, C.; Sun, J.; Suo, H. Multiplicity of positive solutions for critical fractional equation involving concave-convex nonlinearities and sign-changing weight functions. *Mediterr. J. Math.* **2016**, *13*, 4437–4446. [\[CrossRef\]](#)
30. Zhang, J.; Hsu, T. Multiplicity of positive solutions for a nonlocal elliptic problem involving critical Sobolev-Hardy exponents and concave-convex nonlinearities. *Acta Math. Sci.* **2020**, *40*, 679–699. [\[CrossRef\]](#)
31. Liu, Z.; Ding, Y.; Liu, C.; Zhao, C. Existence and uniqueness of solutions for singular fractional differential equation boundary value problem with p -Laplacian. *Adv. Differ. Equ.* **2020**, *2020*, 83. [\[CrossRef\]](#)
32. Guo, L.; Wang, Y.; Liu, H.; Li, C.; Zhao, J.; Chu, H. On iterative positive solutions for a class of singular infinite-point p -Laplacian fractional differential equations with singular source terms. *J. Appl. Anal. Comput.* **2023**, *13*, 2827–2842. [\[CrossRef\]](#)
33. Jia, Z. Global boundedness of weak solutions for an attraction-repulsion chemotaxis system with p -Laplacian diffusion and nonlinear production. *Discret. Contin. Dyn. Syst. B* **2023**, *28*, 4847–4863. [\[CrossRef\]](#)
34. Wang, X.; Wang, Z.; Jia, Z. Global weak solutions for an attraction-repulsion chemotaxis system with p -Laplacian diffusion and logistic source. *Acta Math. Sci.* **2024**, *44*, 909–924. [\[CrossRef\]](#)
35. Shi, S.; Zhai, Z.; Zhang, L. Characterizations of the viscosity solution of a nonlocal and nonlinear equation induced by the fractional p -Laplace and the fractional p -convexity. *Adv. Calc. Var.* **2006**, *17*, 195–207. [\[CrossRef\]](#)
36. Ambrosetti, A.; Brezis, H.; Cerami, G. Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* **1994**, *122*, 519–543. [\[CrossRef\]](#)
37. Brown, K.; Zhang, Y. The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function. *J. Diff. Equ.* **1994**, *122*, 481–499. [\[CrossRef\]](#)
38. Servadei, R.; Valdinoci, E. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.* **2012**, *389*, 887–898. [\[CrossRef\]](#)
39. Garcia, A.; Peral, A. Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Am. Math. Soc.* **1991**, *323*, 877–895. [\[CrossRef\]](#)
40. Fiscella, A.; Valdinoci, E. A critical Kirchhoff type problem involving a nonlocal operator. *Nonlinear Anal.* **2014**, *94*, 156–170. [\[CrossRef\]](#)
41. Rabinowitz, P.H. Minmax methods in critical points theory with application to differential equations. In *CBMS Regional Conference Series in Mathematics*; American Mathematical Society: Providence, RI, USA, 1986; Volume 65.
42. Ambrosetti, A.; Rabinowitz, P.H. Dual variational methods in critical point theory and application. *J. Funct. Anal.* **1973**, *14*, 349–381. [\[CrossRef\]](#)

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.