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Existence of Solutions for Caputo Sequential Fractional Differential Inclusions with Nonlocal Generalized Riemann–Liouville Boundary Conditions

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Abstract: In this study, we explore the existence and uniqueness of solutions for a boundary value problem defined by coupled sequential fractional differential inclusions. This investigation is augmented by the introduction of a novel set of generalized Riemann–Liouville boundary conditions. Utilizing Carathéodory functions and Lipschitz mappings, we establish existence results for these nonlocal boundary conditions. Utilizing fixed-point theorems designed for multi-valued maps, we obtain significant existence results for the problem, considering both convex and non-convex values. The derived results are clearly demonstrated with an illustrative example. Numerical examples are provided to validate the theoretical conclusions, contributing to a deeper understanding of fractional-order boundary value problems.

Keywords: existence of solutions; boundary conditions; fractional differential inclusions; sequential differential inclusions; fixed-point technique



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1. Introduction

FDEs have emerged as multipurpose tools for modeling phenomena in various scientific and engineering domains, offering a means to describe complex behaviors such as memory effects and long-range interactions. Over recent years, significant research attention has been directed towards developing efficient numerical and analytical methods to solve FDEs, owing to their wide-ranging applications. One notable method for numerically solving FDEs is the operational matrix approach, which was introduced to the field by researchers [1]. This method is adaptable to different boundary conditions and represents fractional derivatives as linear combinations of known functions. Numerical comparison studies, exemplified by [2], are essential for evaluating the effectiveness of different numerical techniques for solving FDEs. These studies help identify optimal methods for specific equations and boundary conditions. The utilization of fractional-order models in real-world scenarios has also attracted significant interest. For instance, researchers [3] applied the Caputo fractional-order model to analyze the dynamics of the COVID-19 pandemic, particularly in the context of implementing lockdown measures. Such studies provide valuable insights for policymakers and healthcare professionals in managing infectious diseases. Moreover, recent research has delved into VOFDEs, where the fractional order varies with time or space. References [4,5] explore numerical techniques for computing solutions to VOFDEs, highlighting the challenges and opportunities associated with this class of equations. Additionally, approximate solution methods for nonlinear FPDEs have been extensively investigated. For instance, ref. [6] presents an approach for obtaining

approximate solutions to FPDEs arising in ion-acoustic waves, showcasing the applicability of fractional calculus techniques in studying nonlinear wave phenomena.

The study of fractional calculus has surged in popularity, as it offers a powerful framework for modeling intricate phenomena characterized by memory effects, long-range interactions, and fractal behavior, especially within the domain of FBVPs. FBVPs often incorporate integral boundary conditions, with the Riemann–Liouville integral operator serving as a cornerstone for capturing nonlocal effects. Additionally, CDs are commonly utilized to depict systems exhibiting memory effects. This literature review encompasses various seminal contributions, including investigations into nonlocal FBVPs in [7] and the exploration of generalized Riemann–Liouville (R-L) integral boundary conditions. Key works include the exploration of nonlocal Hadamard fractional integral conditions for nonlinear R-L fractional differential equations [8] and the analysis of impulsive boundary value problems with R-L fractional order derivatives [9,10]. Further, stability analysis of systems governed by fractional-order differential equations with CDs is presented in [11]. Research on the existence, uniqueness, and stability of solutions for variable FBVPs is covered in [12,13]. Experimental studies highlight practical applications of fractional calculus, including fractional-order RC circuit models as discussed in [14]. Recent advancements also include multi-order fractional differential equations with integral boundary conditions in [15] and analytical solutions for nonlinear initial value problems with Caputo derivatives in [16]. Collectively, these studies enhance both theoretical insights and practical applications of FBVPs across scientific and engineering fields.

FDIs have become a key research focus because they provide a more accurate modeling approach for complex dynamic systems compared to traditional differential equations. These inclusions incorporate elements of both differential equations and set-valued mappings, making them highly suitable for describing systems with uncertainties, delays, and nonlocal interactions. The study of FDIs encompasses various types of fractional derivatives, such as the Caputo and Riemann–Liouville derivatives, which allow for the modeling of memory and hereditary properties in dynamic systems. Recent advancements in the field have highlighted the importance of nonlocal boundary conditions and integral conditions in FDIs. Research on Caputo-type sequential fractional differential inclusions with nonlocal integral boundary conditions has advanced our understanding of solution existence in complex boundary scenarios [17]. Similarly, investigations into sequential Riemann–Liouville and Caputo fractional differential inclusions, incorporating generalized fractional integral conditions, have addressed more complex system behaviors [18]. Additionally, the integration of time-varying delays and nonlocal boundary conditions has broadened the scope of FDIs in modeling real-world phenomena, as shown in studies on the controllability of nonlinear fractional differential inclusions [19–26].

The authors in [27] present novel insights into the existing results for fractional differential inclusions with boundary conditions. Their work advances the understanding of how boundary conditions influence the behavior and solution properties of these inclusions, providing valuable contributions to the field. The study contributes to the theoretical understanding of fractional calculus by investigating the existence of solutions in systems with boundary constraints, offering valuable insights into the behavior of dynamic systems under fractional order derivatives. Stochastic differential inclusions and applications are discussed in [28], where the introduction of stochastic elements adds a layer of complexity to the modeling of dynamic systems. By incorporating randomness and uncertainty, stochastic differential inclusions offer a more realistic representation of systems subject to probabilistic influences, making them particularly suitable for applications where uncertainty plays a significant role.

Ref. [29] focuses on coupled systems of fractional differential inclusions with coupled boundary conditions. The study investigates the complexities arising from the coupling of multiple differential equations and inclusionary concepts, offering insights into the be-

havior of interconnected dynamic systems. The coupled Caputo-type FDIs with boundary conditions are given as follows:

$$\begin{cases} ({}^C D^{\alpha_1})X_1(t) \in F_1(t, X_1(t), Y_1(t)), t \in Q := [0, T], \\ ({}^C D^{\alpha_2})Y_1(t) \in F_2(t, X_1(t), Y_1(t)), t \in Q := [0, T], \end{cases}$$

with involving the coupled boundary conditions:

$$\begin{aligned} X_1(0) &= \nu_1 Y_1(T), & X_1'(0) &= \nu_2 Y_1'(T), \\ X_1(0) &= \mu_1 Y_1(T), & X_1'(0) &= \mu_2 Y_1'(T). \end{aligned}$$

where ${}^C D^{\alpha_1}, {}^C D^{\alpha_2}$ denote the CFDs of order α_1, α_2 respectively, $F_1, F_2 : [0, T] \times \mathbb{R}^2 \rightarrow P(\mathbb{R})$ are given multi-valued maps, and $\nu_i, \mu_i, i = 1, 2$ are real constants.

The investigation in [30] focuses on solving systems of coupled nonlinear Atangana–Baleanu-type fractional differential equations. The study introduces advanced numerical techniques that enhance the analysis of these complex systems. Additionally, ref. [31] examines nonlinear coupled differential equations and inclusions with Caputo-type sequential fractional derivatives, providing deeper insights into the behavior of such systems. Together, these studies offer practical methodologies and extend the theoretical framework for analyzing intricate dynamic systems. The study provides a comprehensive analysis of the interplay between differential equations and inclusionary concepts within the framework of fractional calculus. By incorporating Caputo-type sequential derivatives, the research extends beyond traditional differential equations to capture the memory and hereditary properties inherent in dynamic systems, offering valuable insights into the behavior of complex coupled systems with fractional order derivatives. It is given as follows:

$$\begin{cases} ({}^C D^{\alpha_1} + K_1 {}^C D^{\alpha_1-1})X_1(t) = F_1(t, X_1(t), Y_1(t)), t \in Q := [0, T], \\ ({}^C D^{\alpha_2} + K_2 {}^C D^{\alpha_2-1})Y_1(t) = F_2(t, X_1(t), Y_1(t)), t \in Q := [0, T], \\ (X_1 + Y_1)(0) = -(X_1 + Y_1)(T), \\ \int_{\eta}^{\zeta} (X_1 - Y_1)(\theta) d\theta = Q_1. \end{cases}$$

where ${}^C D^{\alpha_1} X_1(t)$ and ${}^C D^{\alpha_1} Y_1(t)$ represent the CDs and the function $F_1, F_2 : Q \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous. Subsequently, ref. [31] explores the existence of solutions for SFDEs. The study not only examines the conditions under which solutions exist but also contributes to the broader understanding of SFDEs by addressing various boundary conditions and solution methodologies.

Inspired by previous studies, our research looks into a specific type of boundary condition called generalized Riemann–Liouville. We are interested in how these boundary conditions affect complicated systems made up of different equations that are all connected. These equations involve a mathematical concept called Caputo-type sequential derivatives, which helps us understand how things change over time. By studying how these equations interact with each other and with these boundary conditions,

$$\begin{cases} ({}^C D^{\alpha_1} + K_1 {}^C D^{\alpha_1-1})X_1(t) \in F_1(t, X_1(t), Y_1(t)), t \in Q := [0, T], \\ ({}^C D^{\alpha_2} + K_2 {}^C D^{\alpha_2-1})Y_1(t) \in F_2(t, X_1(t), Y_1(t)), t \in Q := [0, T], \end{cases} \quad (1)$$

and the following boundary conditions:

$$\begin{cases} (X_1 + Y_1)(0) = -(X_1 + Y_1)(T), \\ \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^{\zeta} \frac{\theta^{\rho-1}}{(\zeta^{\rho} - \theta^{\rho})} (X_1 - Y_1)(\theta) d\theta = Q_1, \end{cases} \quad (2)$$

where ${}^C D^{\alpha_1}$ and ${}^C D^{\alpha_2}$ are the CFDs of order $1 \leq \alpha_1, \alpha_2 \leq 2$, $F_1, F_2 : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions, ${}^\rho I^q$ is the generalized Riemann–Liouville fractional integral of order $q > 0$, $\rho > 0$, and $Y_1, Y_2 : [0, T] \times \mathbb{R}^2 \rightarrow P(\mathbb{R})$ are multi-valued functions, where $P(\mathbb{R})$ denotes the family of all nonempty subsets of $\mathbb{R}$. Q_1 is a positive constant.

This study aims to determine criteria for solutions to problems (1) and (2), with particular emphasis on both convex and non-convex multi-valued maps denoted as F_1 and F_2 . The main contribution of the work as follows: (i) This study introduces new generalized R-L boundary conditions for coupled sequential fractional differential inclusions, expanding the range of applicable boundary conditions in fractional calculus. (ii) It utilizes Carathéodory functions and Lipschitz mappings to establish existence results for fractional differential inclusions, providing a robust theoretical foundation. (iii) It applies standard fixed-point theorems, including the Leray–Schauder nonlinear alternative and the Covitz–Nadler theorem, to derive significant existence results for the problem under both convex and non-convex values. (iv) It provides a specific example to demonstrate the practical application of the developed theoretical results, enhancing understanding and validation of the proposed methods.

This paper is structured as follows: Section 2 deals with basic concepts and introduces a crucial auxiliary lemma necessary to solve the problem. Section 3 investigates existence results under novel boundary conditions, while Section 4 investigates existence results using the Carathéodory function, utilizing the Leray–Schauder nonlinear alternative. Section 5 investigates the existence results under Lipschitz mapping, in particular using the Covitz–Nadler theorem. Section 6 presents a specific example to illustrate the application of these basic theorems. Finally, Section 7 summarizes the conclusion and gives an outlines future directions.

2. Preliminaries Work

This section delves into the definitions of multi-valued maps and examines essential lemmas.

Definition 1 ([32]). For an $(n - 1)$ -times absolutely continuous function $X_1 : [0, \infty) \rightarrow \mathbb{R}$, the (CD) of fractional order α is defined as:

$$I^\alpha X_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{X_1(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\alpha)$ is the gamma function, given by:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

This definition is valid provided that the right-hand side of the equation is point-wise defined over the interval $[0, \infty)$.

Definition 2 ([32]). The CD of fractional order α for an $X_1 : [0, \infty) \rightarrow \mathbb{R}$ can be written as

$${}^C D_{0+}^\alpha X_1(t) = D_{0+}^\alpha \left(X_1(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} X_1^{(k)}(0) \right), \quad X_1 > 0, n-1 < r < n.$$

Lemma 1 (Nonlinear alternative for contraction maps [33]). Let E be a Banach space:

- F_1 is contraction;
- F_2 is u.s.c and compact.

Let $F_1 : E \rightarrow \mathcal{P}_{cp,c}(X)$, $F_2 : \hat{F}_1 \rightarrow Q_{c,cp}(E)$ be two multi-valued operators satisfying the equation. Then, $G_1 = F_1 + F_2$ and either:

- G_1 has fixed point in $\hat{F}_1$; or

- there exists $u \in \partial F_1$ and $\mu \in (0, 1)$ such that $u \in \mu G_1(u)$.

Lemma 2 (Covitz–Nadler theorem [34]). Let (H, d) be a complete metric space. If $\chi : H \rightarrow Q_{cl}(H)$ is a contraction, we then adopt $\chi \neq 0$.

Lemma 3. Let $\psi_1, \psi_2 \in \mathcal{C}(Q, \mathbb{R})$, then the solution of the following system:

$$\begin{cases} ({}^C D^{\alpha_1} + K_1 {}^C D^{\alpha_1-1})X_1(t) = \psi_1(t), & t \in Q := [0, T], \\ ({}^C D^{\alpha_2} + K_1 {}^C D^{\alpha_2-1})Y_1(t) = \psi_2(t), & t \in Q := [0, T], \\ (X_1 + Y_1)(0) = -(X_1 + Y_1)(T), \\ \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} (X_1 - Y_1)(\vartheta) d\vartheta = Q_1, \end{cases} \quad (3)$$

is described by

$$\begin{aligned} X_1(t) = & \frac{e^{-K_1 t}}{2} \left[\frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(u) du \right) d\vartheta \right. \right. \\ & - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \psi_2(u) du \right) d\vartheta \\ & + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(m) dm \right) du \right) d\vartheta \right. \\ & + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \psi_2(m) dm \right) du \right) d\vartheta \left. \left. \right) \right] \\ & + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(u) du \right) d\vartheta, \end{aligned} \quad (4)$$

$$\begin{aligned} Y_1(t) = & \frac{e^{-K_1 t}}{2} \left[\frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(u) du \right) d\vartheta \right. \right. \\ & - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \psi_2(u) du \right) d\vartheta \\ & - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(m) dm \right) du \right) d\vartheta \right. \\ & + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \psi_2(m) dm \right) du \right) d\vartheta \left. \left. \right) \right] \\ & + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \psi_2(u) du \right) d\vartheta, \end{aligned} \quad (5)$$

where

$$\Delta_1 = (1 + e^{-K_1 T}), \quad \Delta_2 = \beta^\rho I^\rho e^{-K_1(\zeta)} \neq 0, \quad (6)$$

$$\begin{cases} I_1 = \left(-\int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(u) du \right) d\vartheta \right. \\ \quad \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \psi_2(u) du \right) d\vartheta \right) \\ I_2 = \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(m) dm \right) du \right) d\vartheta \right. \\ \quad \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \psi_2(m) dm \right) du \right) d\vartheta \right). \end{cases}$$

Proof. Equation (3) can be equivalently written as

$$\begin{cases} ({}^C D^{\alpha_1} + K_1 {}^C D^{\alpha_1-1}) X_1(t) = \psi_1(t), \\ ({}^C D^{\alpha_2} + K_1 {}^C D^{\alpha_2-1}) Y_1(t) = \psi_2(t). \end{cases} \quad (7)$$

Rewriting the equation in (7) as ${}^C D^{\alpha_1}(1 + K_1 {}^C D^{-1}) X_1(t) = \psi_1(t)$, and ${}^C D^{\alpha_2}(1 + K_2 {}^C D^{-1}) Y_1(t) = \psi_2(t)$, then applying the integral operators $I_{0+}^{\alpha_1}$ and $I_{0+}^{\alpha_2}$ to these equations, we obtain

$$\begin{cases} X_1(t) = c_0 e^{-K_1 t} + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(s-a)^{(\alpha_1-2)}}{\Gamma(\alpha_1-1)} \psi_1(q) dq \right) d\vartheta, \\ Y_1(t) = d_0 e^{-K_1 t} + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(s-a)^{(\alpha_2-2)}}{\Gamma(\alpha_2-1)} \psi_2(q) dq \right) d\vartheta, \end{cases} \quad (8)$$

where c_0, d_0 are arbitrary constants. Using BCs (2) in (8), we obtain

$$c_0 + d_0 = I_1 \quad (9)$$

$$c_0 - d_0 = I_2. \quad (10)$$

Solving (9) and (10) for c_0 and d_0 yields

$$\begin{aligned} c_0 = \frac{1}{2} \Bigg\{ & \frac{1}{\Delta_1} \left(-\int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(u) du \right) d\vartheta \right. \\ & \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \psi_2(u) du \right) d\vartheta \right) \\ & + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(m) dm \right) du \right) d\vartheta \right. \\ & \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \psi_2(m) dm \right) du \right) d\vartheta \right) \Bigg\}, \end{aligned}$$

and

$$\begin{aligned} d_0 = \frac{1}{2} \Bigg\{ & \frac{1}{\Delta_1} \left(-\int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(u) du \right) d\vartheta \right. \\ & \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \psi_2(u) du \right) d\vartheta \right) \\ & - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \psi_1(m) dm \right) du \right) d\vartheta \right. \\ & \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \psi_2(m) dm \right) du \right) d\vartheta \right) \Bigg\}. \end{aligned}$$

The solutions presented in (4) and (5) are derived by incorporating the values of c_0 and d_0 into Equation (8). $\square$

3. Existence Results under New Boundary Conditions

A function $(X_1, Y_1) \in \mathcal{C}^2(Q, \mathbb{R}) \times \mathcal{C}^2(Q, \mathbb{R})$ satisfies the new kind of BCs

$$\beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} (X_1 - Y_1)(\vartheta) d\vartheta = Q_1,$$

and there exist functions

$$\Lambda(X_1, Y_1)(T) = (\Lambda_1(X_1, Y_1)(T), \Lambda_2(X_1, Y_1)(T)),$$

where

$$\begin{aligned} \Lambda_1(X_1, Y_1)(T) &= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} F_1(u, X_1(u), Y_1(u)) du \right) d\vartheta \right. \right. \right. \\ &\quad - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} F_2(u, X_1(u), Y_1(u)) du \right) d\vartheta \Bigg) \\ &\quad + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} F_1(m, X_1(m), Y_1(m)) dm \right) du \right) d\vartheta \right. \\ &\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} F_2(m, X_1(m), Y_1(m)) dm \right) du \right) d\vartheta \right) \right\} \right] \\ &\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} F_1(u, X_1(u), Y_1(u)) du \right) d\vartheta, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \Lambda_2(X_1, Y_1)(T) &= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} F_1(u, X_1(u), Y_1(u)) du \right) d\vartheta \right. \right. \right. \\ &\quad - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} F_2(u, X_1(u), Y_1(u)) du \right) d\vartheta \Bigg) \\ &\quad - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} F_1(m, X_1(m), Y_1(m)) dm \right) du \right) d\vartheta \right. \\ &\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} F_2(m, X_1(m), Y_1(m)) dm \right) du \right) d\vartheta \right) \right\} \right] \\ &\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} F_2(u, X_1(u), Y_1(u)) du \right) d\vartheta. \end{aligned} \quad (12)$$

This is referred to as a coupled solution for the system in (2).

To simplify the calculations, we introduce the following notation:

$$Y_1 = \frac{e^{-K_1 t}}{2} \left[\frac{1}{\Delta_1} \left\{ \frac{T^{\alpha_1-1}}{K_1 \Gamma(\alpha_1)} (1 - e^{K_1 \vartheta}) \right\} \right]$$

$$+ \frac{1}{\Delta_2} \left\{ \frac{\beta}{K_1^2 \Gamma(\alpha_1)} \frac{\zeta^{\alpha_1-1+\rho q}}{\rho^q} \frac{\Gamma\left(\frac{(\alpha_1-1+\rho)}{\rho}\right)}{\Gamma\left(\frac{\alpha_1-1+\rho q+\rho}{\rho}\right)} (\zeta K_1 + e^{-K_1 \zeta} - 1) \right\} \Bigg], \quad (13)$$

$$Y_2 = \frac{e^{-K_1 t}}{2} \left[\frac{1}{\Delta_1} \left\{ \frac{T^{\alpha_2-1}}{K_1 \Gamma(\alpha_2)} (1 - e^{K_1 \theta}) \right\} + \frac{1}{\Delta_2} \left\{ \frac{\beta}{K_1^2 \Gamma(\alpha_2)} \frac{\zeta^{\alpha_2-1+\rho q}}{\rho^q} \frac{\Gamma\left(\frac{(\alpha_2-1+\rho)}{\rho}\right)}{\Gamma\left(\frac{\alpha_2-1+\rho q+\rho}{\rho}\right)} (\zeta K_1 + e^{-K_1 \zeta} - 1) \right\} \right], \quad (14)$$

and

$$\Phi = \min \left\{ 1 - \left[\|\gamma_2\| \left(2Y_1 + \frac{t^{\alpha_1-1}}{K_1 \Gamma(\alpha_1)} (1 - e^{-K_1 t}) \right) + \|k_2\| \left(2Y_2 + \frac{t^{\alpha_2-1}}{K_2 \Gamma(\alpha_2)} (1 - e^{-K_2 t}) \right) \right], \right. \\ \left. 1 - \left[\|\gamma_3\| \left(2Y_1 + \frac{t^{\alpha_1-1}}{K_1 \Gamma(\alpha_1)} (1 - e^{-K_1 t}) \right) + \|k_3\| \left(2Y_2 + \frac{t^{\alpha_2-1}}{K_2 \Gamma(\alpha_2)} (1 - e^{-K_2 t}) \right) \right] \right\}.$$

Let

$$V_{F_1(X_1, Y_1)} = \{H_1 \in L^1(Q, \mathbb{R}) : H_1(t) \in F_1(t, X_1(t), Y_1(t)), \text{ for a.e } t \in Q\},$$

and

$$V_{F_2(X_1, Y_1)} = \{H_2 \in L^1(Q, \mathbb{R}) : H_2(t) \in F_2(t, X_1(t), Y_1(t)), \text{ for a.e } t \in Q\},$$

describe the sets of F_1, F_2 selections for each $(X_1, Y_1) \in U \times U$.

By implementing Lemma 3, the following operators $\Lambda_1, \Lambda_2 : U \times U \rightarrow W(U \times U)$ by:

$$\Lambda_1(X_1, Y_1)(t) = \{M_1 \in U \times U : \text{exists } H_1 \in V_{F_1(X_1, Y_1)}, H_2 \in V_{F_2(X_1, Y_1)} \ni M_1(X_1, Y_1)(t) \\ = P_1(X_1, Y_1)(t), \forall t \in Q\}, \quad (15)$$

and

$$\Lambda_2(X_1, Y_1)(t) = \{M_2 \in U \times U : \text{exists } H_1 \in V_{F_1(X_1, Y_1)}, H_2 \in V_{F_2(X_1, Y_1)} \ni M_2(X_1, Y_1)(t) \\ = P_2(X_1, Y_1)(t), \forall t \in Q\}, \quad (16)$$

where

$$P_1(X_1, Y_1)(t) \\ = \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\theta)} \left(\int_0^\theta \frac{(\theta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\theta \right. \right. \right. \\ \left. \left. - \int_0^T e^{-K_1(T-\theta)} \left(\int_0^\theta \frac{(\theta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\theta \right) \right. \\ \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \right.$$

$$\begin{aligned}
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
& P_2(X_1, Y_1)(t) \\
& = \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& \quad - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \Bigg) \\
& \quad - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& \quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right\} \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta.
\end{aligned} \tag{18}$$

Operator $\Lambda : U \times U \rightarrow W(U \times U)$ is defined as follows:

$$\Lambda(X_1, Y_1)(t) = \begin{pmatrix} \Lambda_1(X_1, Y_1)(t) \\ \Lambda_2(X_1, Y_1)(t) \end{pmatrix},$$

where Λ_1 and Λ_2 are defined in (15) and (16), respectively.

4. Existence Results via the Carathéodory Function

We begin by demonstrating the existence of solutions for the boundary value problem (BVP) (2) using the nonlinear alternative of Leray–Schauder. Following this, we present the assumptions that underpin the main results of our study:

- (A1) $F_1, F_2 : Q \times \mathbb{R}^2 \rightarrow W(\mathbb{R})$ are L^1 -Carathéodory and have convex values.
- (A2) There exists continuous increasing functions $\gamma_1, \gamma_2, k_1, k_2 : [0, \infty) \rightarrow [0, \infty)$ and functions $l_1, l_2 \in C(Q, \mathbb{R}^+)$, such that

$$\begin{aligned}
\|F_1(t, X_1, Y_1)\|_W &:= \sup\{|H_1| : H_1 \in F_1(t, X_1, Y_1)\} \leq l_1(t)[\gamma_1(\|X_1\|) + k_1(\|Y_1\|)] \\
&\quad \text{for each } (t, X_1, Y_1) \in Q \times \mathbb{R}^2, \\
\|F_2(t, X_1, Y_1)\|_W &:= \sup\{|H_2| : H_2 \in F_2(t, X_1, Y_1)\} \leq l_2(t)[\gamma_2(\|X_1\|) + k_2(\|Y_1\|)] \\
&\quad \text{for each } (t, X_1, Y_1) \in Q \times \mathbb{R}^2.
\end{aligned}$$

- (A3) There exists a non-negative $Z > 0 \ni$

$$\frac{Z}{(2Y_1)\|l_1\|(\gamma_1(Z) + k_1(Z)) + (2Y_2)\|l_2\|(\gamma_2(Z) + k_2(Z))} > 1,$$

where Y_1, Y_2 are defined by (13) and (14).

- (A4) $F_1, F_2 : Q \times \mathbb{R}^2 \rightarrow W_{cp}(\mathbb{R})$ are such that $F_1(\cdot, X_1, Y_1) : Q \rightarrow W_{cp}(\mathbb{R}^2)$ and $F_2(\cdot, X_1, Y_1) : Q \rightarrow W_{cp}(\mathbb{R}^2)$ are measurable for each $X_1, Y_1 \in \mathbb{R}$.
- (A5)

$$\Pi_d(F_1(t, X_1, Y_1), F_1(t, \hat{X}_1, \hat{Y}_1)) \leq \Theta_1(t)(|X_1 - \hat{X}_1| + |Y_1 - \hat{Y}_1|),$$

and

$$\Pi_d(F_2(t, X_1, Y_1), F_2(t, \hat{X}_1, \hat{Y}_1)) \leq \Theta_2(t)(|X_1 - \hat{X}_1| + |Y_1 - \hat{Y}_1|),$$

$$\forall t \in Q \text{ and } X_1, Y_1, \hat{X}_1, \hat{Y}_1 \in \mathbb{R} \text{ with } \Theta_1, \Theta_2 \in C(Q, \mathbb{R}^+) \text{ and } d(0, F_1(t, 0, 0)) \leq \Theta_1(t), \\ d(0, F_2(t, 0, 0)) \leq \Theta_2(t) \forall t \in Q.$$

Theorem 1. Assuming that conditions (A1)–(A3) are satisfied, it can be established that there is at least one solution to the system defined by Equations (1) and (2) within the interval Q .

Proof. Consider the operators Λ_1 and Λ_2 defined by $\Lambda_1, \Lambda_2 : U \times U \rightarrow W(U \times U)$, as given by Equations (15) and (16), respectively. Using assumption (A3), the sets $V_{F_1(X_1, Y_1)}$ and $V_{F_2(X_1, Y_1)}$ are nonempty for each $(X_1, Y_1) \in U \times U$. Then, for $H_1 \in V_{F_1(X_1, Y_1)}$, $H_2 \in V_{F_2(X_1, Y_1)}$ for $(X_1, Y_1) \in U \times U$, we have

$$\begin{aligned} M_1(X_1, Y_1)(t) &= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\ &\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right) \right. \\ &\quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \\ &\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\ &\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta, \end{aligned} \quad (19)$$

and

$$\begin{aligned} M_2(X_1, Y_1)(t) &= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\ &\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right) \right. \\ &\quad \left. - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \\ &\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\ &\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta, \end{aligned} \quad (20)$$

where $M_1 \in \Lambda_1(X_1, Y_1)$, $M_2 \in \Lambda_2(X_1, Y_1)$, and so $(M_1, M_2) \in \Lambda(X_1, Y_1)$. We will demonstrate that the operator Λ satisfies the criteria of the Leray–Schauder nonlinear alternative by following a structured approach. First, we will establish that $\Lambda(X_1, Y_1)$ possesses a convex-valued property. Specifically, for each pair $(M_i, \hat{M}_i) \in (\Lambda_1, \Lambda_2)$, where $i = 1, 2$, we need to show that there exist functions $H_{1i} \in V_{F_1(X_1, Y_1)}$ and $H_{2i} \in V_{F_2(X_1, Y_1)}$, for $i = 1, 2$, such that for every $t \in Q$, the following holds:

$$\begin{aligned}
M_i(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \Bigg) \\
& + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \left. \left. \left. \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right) \right\} \right] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta,
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\hat{M}_i(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \Bigg) \\
& - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \left. \left. \left. \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right) \right\} \right] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta.
\end{aligned} \tag{22}$$

Let $0 \leq \nu \leq 1$. Then, for each $t \in Q$, we arrive at

$$\begin{aligned}
& [\nu M_1 + (1-\nu)M_2](t) \\
= & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} [\nu H_{11}(u) + (1-\nu)H_{11}(u)] du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} [\nu H_{21}(u) + (1-\nu)H_{21}(u)] du \right) d\vartheta \Bigg) \\
& + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} [\nu H_{11}(m) + (1-\nu)H_{11}(m)] dm \right) du \right) d\vartheta \right. \\
& + \left. \left. \left. \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} [\nu H_{21}(m) + (1-\nu)H_{21}(m)] dm \right) du \right) d\vartheta \right) \right) \right\} \right] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} [\nu H_{11}(u) + (1-\nu)H_{11}(u)] du \right) d\vartheta,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
& [\nu \hat{M}_1 + (1-\nu)\hat{M}_2](t) \\
= & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} [\nu \hat{H}_{11}(u) + (1-\nu)\hat{H}_{11}(u)] du \right) d\vartheta \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} [v\hat{H}_{21}(u) + (1-v)\hat{H}_{21}(u)] du \right) d\vartheta \\
& + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} [v\hat{H}_{11}(m) + (1-v)\hat{H}_{11}(m)] dm \right) du \right) d\vartheta \right. \\
& \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} [v\hat{H}_{21}(m) + (1-v)\hat{H}_{21}(m)] dm \right) du \right) d\vartheta \right) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} [v\hat{H}_{21}(u) + (1-v)\hat{H}_{21}(u)] du \right) d\vartheta.
\end{aligned} \tag{24}$$

This will lead us to verify the required conditions for Λ according to the Leray–Schauder principle.

Given the convexity of the values associated with F_1 and F_2 , it follows that the sets $V_{F_1(X_1, Y_1)}$ and $V_{F_2(X_1, Y_1)}$ are also convex. Consequently, for any $v \in [0, 1]$, we have $vM_1 + (1-v)M_2 \in \Lambda_1$ and $v\hat{M}_1 + (1-v)\hat{M}_2 \in \Lambda_2$. This implies that $v(M_1, \hat{M}_1) + (1-v)(M_2, \hat{M}_2) \in \Lambda$. Consider a non-negative scalar r , and let $B_r = \{(X_1, Y_1) \in U \times U \mid \|(X_1, Y_1)\| \leq r\}$ denote a bounded subset of $U \times U$. For this bounded set, there exist functions $H_1 \in V_{F_1(X_1, Y_1)}$ and $H_2 \in V_{F_2(X_1, Y_1)}$ such that:

$$\begin{aligned}
M_1(X_1, Y_1)(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \\
& + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Bigg) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta,
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
M_2(X_1, Y_1)(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \\
& - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Bigg) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta.
\end{aligned} \tag{26}$$

Then, we obtain

$$\begin{aligned}
& |M_1(X_1, Y_1)(t)| \\
& \leq \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \|l_1\| (\gamma_1(r) + k_1(r)) du \right) d\vartheta \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \|l_2\|(\gamma_1(r) + k_1(r))(u) du \right) d\vartheta \\
& + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \|l_1\|(\gamma_1(r) + k_1(r))(m) dm \right) du \right) d\vartheta \right. \\
& \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \|l_2\|(\gamma_1(r) + k_1(r))(m) dm \right) du \right) d\vartheta \right) \Bigg\} \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \|l_1\|(\gamma_1(r) + k_1(r))(u) du \right) d\vartheta,
\end{aligned}$$

and

$$|M_2(X_1, Y_1)(t)| \leq Y_1 \|l_1\|(\gamma_1(r) + k_1(r)) + Y_2 \|l_2\|(\gamma_2(r) + k_2(r)).$$

Thus, we obtain

$$\begin{aligned}
\|M_1, M_2\| &= \|M_1(X_1, Y_1)\| + \|M_2(X_1, Y_1)\| \\
&\leq 2Y_1 \|l_1\|(\gamma_1(r) + k_1(r)) + 2Y_2 \|l_2\|(\gamma_2(r) + k_2(r)).
\end{aligned}$$

Subsequently, we proceed to establish the equicontinuous of the operator Λ . Let $t_1, t_2 \in Q$ with $t_1 < t_2$. Then there exists $H_1 \in V_{F_1(X_1, Y_1)}$, $H_2 \in V_{F_2(X_1, Y_1)}$ such that

$$\begin{aligned}
M_1(X_1, Y_1)(t) &= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \\
& + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Bigg\} \\
& \left. + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta, \right.
\end{aligned}$$

and

$$\begin{aligned}
M_2(X_1, Y_1)(t) &= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \\
& - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Bigg\} \\
& \left. + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta, \right.
\end{aligned}$$

$$|M_1(X_1, Y_1)(t_2) - M_1(X_1, Y_1)(t_1)|$$

$$\begin{aligned}
&\leq \left| \frac{e^{-K_1 t_2} - e^{-K_1 t_1}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right) \right. \right. \\
&\quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right] \right| \\
&\quad + \left| \int_0^{t_1} (e^{-K_1(t_2-\vartheta)} - e^{-K_1(t_1-\vartheta)}) \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \\
&\quad \left. + \int_{t_1}^{t_2} e^{-K_1(t_2-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right|.
\end{aligned}$$

Likewise, one can construct

$$\begin{aligned}
&|M_2(X_1, Y_1)(t_2) - M_2(X_1, Y_1)(t_1)| \\
&\leq \left| \frac{e^{-K_1 t_2} - e^{-K_1 t_1}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right) \right. \right. \\
&\quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right] \right| \\
&\quad + \left| \int_0^{t_1} (e^{-K_1(t_2-\vartheta)} - e^{-K_1(t_1-\vartheta)}) \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right. \\
&\quad \left. + \int_{t_1}^{t_2} e^{-K_1(t_2-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right|.
\end{aligned}$$

Consequently, the operator $\Lambda(X_1, Y_1)$ is equicontinuous. According to the Ascoli–Arzelà theorem, $\Lambda(X_1, Y_1)$ is a completely continuous operator. As stated in [35], a completely continuous operator has a closed graph when it is also upper semicontinuous. Therefore, we need to show that Λ has a closed graph. Let $(X_{1n}, Y_{1n}) \rightarrow (X_{1*}, Y_{1*})$, $(M_n, \hat{M}_n) \in \Lambda(X_{1n}, Y_{1n})$, and $(M_n, \hat{M}_n) \rightarrow (M_*, \hat{M}_*)$. We must demonstrate that $(M_*, \hat{M}_*) \in \Lambda(X_{1*}, Y_{1*})$. Remember that $(M_n, \hat{M}_n) \in \Lambda(X_{1n}, Y_{1n})$ implies that there exist $H_{1n} \in V_{F_1(X_1, Y_1)}$ and $H_{2n} \in V_{F_2(X_1, Y_1)}$ such that

$$\begin{aligned}
M_n(X_{1n}, Y_{1n})(t) &= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{2n}(u) du \right) d\vartheta \right) \right. \\
&\quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(m) dm \right) du \right) d\vartheta \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{2n}(m) dm \right) du \right) d\vartheta \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(u) du \right) d\vartheta,
\end{aligned}$$

and

$$\begin{aligned}
\hat{M}_n(X_{1n}, Y_{1n})(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{2n}(u) du \right) d\vartheta \Bigg) \\
& - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{2n}(m) dm \right) du \right) d\vartheta \Bigg) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{2n}(u) du \right) d\vartheta.
\end{aligned}$$

Consider the $\Pi_1, \Pi_2 : L^1(Q, U \times U) \rightarrow C(Q, U \times U)$ continuous linear operators given by

$$\begin{aligned}
\Pi_1(X_1, Y_1)(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \Bigg) \\
& + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Bigg) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
\Pi_2(X_1, Y_1)(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \Bigg) \\
& - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Bigg) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta.
\end{aligned} \tag{28}$$

It can be inferred from [36] that $(X_1, Y_1) \circ (V_{F_1}, V_{F_2})$ constitutes a closed graph operator. Furthermore, we have $(M_n, \hat{M}_n) \in (X_1, Y_1) \circ (V_{F_1(X_{1n}, Y_{1n})}, V_{F_2(X_{1n}, Y_{1n})})$ for all n . Since $(X_{1n}, Y_{1n}) \rightarrow (X_{1*}, Y_{1*})$, $(M_n, \hat{M}_n) \rightarrow (M_*, \hat{M}_*)$, it follows that $H_{1n} \in V_{F_1(X_1, Y_1)}$, $H_{2n} \in V_{F_2(X_1, Y_1)}$ such that

$$\begin{aligned} M_*(X_{1*}, Y_{1*})(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} (H_{1*})(u) du \right) d\vartheta \right. \right. \right. \\ & - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} (H_{2*})(u) du \right) d\vartheta \Big) \\ & + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} (H_{1*})(m) dm \right) du \right) d\vartheta \right. \\ & + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} (H_{2*})(m) dm \right) du \right) d\vartheta \Big) \Big\} \Big] \\ & + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} (H_{1*})(u) du \right) d\vartheta, \end{aligned}$$

and

$$\begin{aligned} \hat{M}_*(X_{1*}, Y_{1*})(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} (H_{1*})(u) du \right) d\vartheta \right. \right. \right. \\ & - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} (H_{2*})(u) du \right) d\vartheta \Big) \\ & - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} (H_{1*})(m) dm \right) du \right) d\vartheta \right. \\ & + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} (H_{2*})(m) dm \right) du \right) d\vartheta \Big) \Big\} \Big] \\ & + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} (H_{2*})(u) du \right) d\vartheta, \end{aligned}$$

i.e., $(M_n, \hat{M}_n) \in \Lambda(X_{1*}, Y_{1*})$. Finally, we discuss a priori on solutions. Let $(X_1, Y_1) \in \nu\Lambda(X_1, Y_1)$. Then exists $H_1 \in V_{F_1(X_1, Y_1)}$, $H_2 \in V_{F_2(X_1, Y_1)}$ such that

$$\begin{aligned} X_1(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\ & - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \Big) \\ & + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\ & + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Big) \Big\} \Big] \\ & + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta, \end{aligned} \tag{29}$$

and

$$\begin{aligned}
Y_1(t) = & \frac{e^{-K_1 t}}{2} \left[\frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \\
& - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \Big) \\
& - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \\
& + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \Big) \Big] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta.
\end{aligned} \tag{30}$$

For each $t \in Q$, we achieve

$$\begin{aligned}
\|X_1, Y_1\| &= \|X_1\| + \|Y_1\| \\
&\leq 2Y_1 l_1(\gamma_1(\|X_1\|) + k_1(\|Y_1\|)) + 2Y_2 l_2(\gamma_2(\|X_1\|) + k_2(\|Y_1\|)),
\end{aligned}$$

which signifies that

$$\frac{\|(X_1, Y_1)\|}{2Y_1 l_1(\gamma_1(\|X_1\|) + k_1(\|Y_1\|)) + 2Y_2 l_2(\gamma_2(\|X_1\|) + k_2(\|Y_1\|))} \leq 1.$$

According to (A3), Z exists such that $\|(X_1, Y_1)\| \neq Z$. Let us adopt

$$E = \{(X_1, Y_1) \in U \times U : \|(X_1, Y_1)\| < Z\}.$$

The operator $\Lambda : \bar{E} \rightarrow W_{cv,cp}(U) \times W_{cv,cp}(U)$ is continuous and compact (c.c.) and upper semicontinuous (u.s.c.). There is no $(X_1, Y_1) \in \partial E$ such that $(X_1, Y_1) \in v\Lambda(X_1, Y_1)$ for some $v \in (0, 1)$ by E selection. Consequently, based on the Leray–Schauder nonlinear alternative, we can infer that Λ possesses a fixed point $(X_1, Y_1) \in \bar{E}$, thus serving as a solution to the system (1.2). $\square$

5. Existence Results for Lipschitz Mappings

The subsequent outcome leverages Covitz and Nadler's theorem for multi-valued maps as presented in [34].

Let (U, d) represent a metric space generated by the normed space $(U, \|\cdot\|)$. For further definitions and details, see [28,34].

Theorem 2. *Assumptions (A4) and (A5) hold. Then, the System (1) and (2) possesses at least one solution within the interval Q , provided that:*

$$(2Y_1)\|\Theta_1\| + (2Y_2)\|\Theta_2\| < 1. \tag{31}$$

Proof. Assuming (A4) that the sets $V_{F_1(X_1, Y_1)}$ and $V_{F_2(X_1, Y_1)}$ are nonempty for each $(X_1, Y_1) \in U \times U$, H_1 and H_2 have measurable selections (see Theorem III.6 in Castaing and Valadier, 1977). Subsequently, we proceed to establish that the operator Λ satisfies the theorem of Covitz and Nadler [34].

Next, we illustrate that $\Lambda(X_1, Y_1) \in W_{cl}(U) \times W_{cl}(U)$ for each $(X_1, Y_1) \in U \times U$. Let $(M_n, \hat{M}_n) \in \Lambda(X_{1n}, Y_{1n})$ be such that $(M_n, \hat{M}_n) \rightarrow (M, \hat{M})$ in $U \times U$. Then, $(M, \hat{M}) \in U \times U$ and there exist $H_{1n} \in V_{F_1(X_{1n}, Y_{1n})}$ and $H_{2n} \in V_{F_2(X_{1n}, Y_{1n})}$ such that

$$M_n(X_{1n}, Y_{1n})(t)$$

$$\begin{aligned}
&= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{2n}(u) du \right) d\vartheta \right) \right. \\
&\quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{2n}(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\
&\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{1n}(u) du \right) d\vartheta,
\end{aligned}$$

and

$$\begin{aligned}
&\hat{M}_n(X_{1n}, Y_{1n})(t) \\
&= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right) \right. \\
&\quad \left. - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\
&\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta.
\end{aligned}$$

Due to the compact values of F_1 and F_2 , we select subsequences to ensure the convergence of H_{1n} and H_{2n} to H_1 and H_2 , respectively, in the space $L^1(Q, \mathbb{R})$. Therefore, $H_1 \in V_{F_1}(X_1, Y_1)$ and $H_2 \in V_{F_2}(X_1, Y_1)$ for each $t \in Q$.

$$\begin{aligned}
&M_n(X_{1n}, Y_{1n})(t) \rightarrow M(X_1, Y_1)(t) \\
&= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right) \right. \\
&\quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\
&\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta,
\end{aligned}$$

and

$$\hat{M}_n(X_{1n}, Y_{1n})(t) \rightarrow \hat{M}(X_1, Y_1)(t)$$

$$\begin{aligned}
&= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta \right) \right. \\
&\quad \left. - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_1(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\
&\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_2(u) du \right) d\vartheta.
\end{aligned}$$

As a consequence, $(M, \hat{M}) \in \Lambda$, indicating that Λ is closed. Following that, we demonstrate that the equation exists (defined by (31)) such that

$$\Pi_d(\Lambda(X_1, Y_1), \Lambda(\hat{X}_1, \hat{Y}_1)) \leq \hat{\rho}(\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|) \text{ for each } X_1, \hat{X}_1, Y_1, \hat{Y}_1 \in U.$$

Let $(X_1, \hat{X}_1), (Y_1, \hat{Y}_1) \in U \times U$ and $(M_1, \hat{M}_1) \in \Lambda(X_1, Y_1)$. Then, there exists $H_{11} \in V_{F_1(X_1, Y_1)}$ and $H_{21} \in V_{F_2(X_1, Y_1)} \ni$; for each $t \in Q$, we have

$$\begin{aligned}
&M_1(X_{1n}, Y_{1n})(t) \\
&= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(u) du \right) d\vartheta \right) \right. \\
&\quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\
&\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(u) du \right) d\vartheta,
\end{aligned}$$

and

$$\begin{aligned}
&\hat{M}_1(X_{1n}, Y_{1n})(t) \\
&= \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(u) du \right) d\vartheta \right. \right. \right. \\
&\quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(u) du \right) d\vartheta \right) \right. \\
&\quad \left. - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(m) dm \right) du \right) d\vartheta \right. \right. \\
&\quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(m) dm \right) du \right) d\vartheta \right) \right\} \right] \\
&\quad + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(u) du \right) d\vartheta.
\end{aligned}$$

Utilizing (A5), we acquire

$$\Pi_d(F_1(t, X_1, Y_1), F_1(t, \hat{X}_1, \hat{Y}_1)) \leq \Theta_1(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|)$$

and

$$\Pi_d(F_2(t, X_1, Y_1), F_2(t, \hat{X}_1, \hat{Y}_1)) \leq \Theta_2(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|).$$

So, there exists $H_1 \in F_1(t, X_1(t), Y_1(t))$ and $H_2 \in F_2(t, X_1(t), Y_1(t))$ such that

$$|H_{11}(t) - u| \leq \Theta_1(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|)$$

and

$$|H_{21}(t) - v| \leq \Theta_2(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|).$$

Define $Y_{11}, Y_{12} : Q \rightarrow W(\mathbb{R})$ by

$$Y_{11}(t) = \{H_1 \in L^1(Q, \mathbb{R}) : \Theta_1(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|)\}$$

and

$$Y_{12}(t) = \{H_2 \in L^1(Q, \mathbb{R}) : \Theta_2(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|)\}.$$

There are functions $H_{12}(t), H_{22}(t)$ that are an observable selection for Y_{11}, Y_{12} because the multi-valued operators $Y_{11} \cap F_1(t, X_1(t), Y_1(t))$ and $Y_{12} \cap F_2(t, X_1(t), Y_1(t))$ are measurable (Proposition III.4 in [37]). Also, $H_{12}(t) \in F_1(t, X_1(t), Y_1(t))$, $H_{22}(t) \in F_2(t, X_1(t), Y_1(t))$ such that $\forall t \in Q$, we arrive

$$|H_{11}(t) - H_{12}(t)| \leq \Theta_1(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|)$$

and

$$|H_{21}(t) - H_{22}(t)| \leq \Theta_2(t)(|X_1(t) - \hat{X}_1(t)| + |Y_1(t) - \hat{Y}_1(t)|).$$

Let

$$\begin{aligned} \mathcal{M}_2(X_{1n}, Y_{1n})(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(u) du \right) d\vartheta \right. \right. \right. \\ & - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(u) du \right) d\vartheta \Bigg) \\ & + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(m) dm \right) du \right) d\vartheta \right. \\ & + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(m) dm \right) du \right) d\vartheta \Bigg) \Bigg] \\ & + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(u) du \right) d\vartheta, \end{aligned}$$

and

$$\begin{aligned} \hat{\mathcal{M}}_2(X_{1n}, Y_{1n})(t) = & \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(u) du \right) d\vartheta \right. \right. \right. \\ & - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(u) du \right) d\vartheta \Bigg) \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} H_{11}(m) dm \right) du \right) d\vartheta \right. \\
& \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(m) dm \right) du \right) d\vartheta \right) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} H_{21}(u) du \right) d\vartheta,
\end{aligned}$$

Hence,

$$\begin{aligned}
& |M_1(X_1, Y_1)(t) - M_2(X_1, Y_1)(t)| \\
& \leq \frac{e^{-K_1 t}}{2} \left[\left\{ \frac{1}{\Delta_1} \left(- \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \Theta_1(u) (|X_1(u) - \hat{X}_1(u)| + |Y_1(u) - \hat{Y}_1(u)|)(u) du \right) d\vartheta \right. \right. \right. \\
& \quad \left. \left. - \int_0^T e^{-K_1(T-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \Theta_2(u) (|X_1(u) - \hat{X}_1(u)| + |Y_1(u) - \hat{Y}_1(u)|)(u) du \right) d\vartheta \right) \right. \\
& \quad \left. + \frac{1}{\Delta_2} \left(Q_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \right. \right. \right. \\
& \quad \left. \left. \times \left(\int_0^u \frac{(u-m)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \Theta_1(u) (|X_1(m) - \hat{X}_1(m)| + |Y_1(m) - \hat{Y}_1(m)|)(m) dm \right) du \right) d\vartheta \right. \right. \\
& \quad \left. \left. + \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} \left(\int_0^\vartheta e^{-K_1(\vartheta-u)} \right. \right. \right. \\
& \quad \left. \left. \times \left(\int_0^u \frac{(u-m)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \Theta_2(u) (|X_1(m) - \hat{X}_1(m)| + |Y_1(m) - \hat{Y}_1(m)|)(m) dm \right) du \right) d\vartheta \right) \Bigg] \\
& + \int_0^t e^{-K_1(t-\vartheta)} \left(\int_0^\vartheta \frac{(\vartheta-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \Theta_1(u) (|X_1(u) - \hat{X}_1(u)| + |Y_1(u) - \hat{Y}_1(u)|)(u) du \right) d\vartheta, \\
& \leq Y_1 \|\Theta_1\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|) + Y_2 \|\Theta_2\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|).
\end{aligned}$$

Thus,

$$\|H_1(X_1, Y_1) - H_2(X_1, Y_1)\| \leq Y_1 \|\Theta_1\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|) + Y_2 \|\Theta_2\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|).$$

Similarly, we can define that

$$\|\hat{H}_1(X_1, Y_1) - \hat{H}_2(X_1, Y_1)\| \leq Y_1 \|\Theta_1\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|) + Y_2 \|\Theta_2\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|).$$

Therefore

$$\|(H_1, \hat{H}_1), (H_2, \hat{H}_2)\| \leq 2Y_1 \|\Theta_1\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|) + 2Y_2 \|\Theta_2\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|).$$

Similarly, by interchanging the positions of (X_1, Y_1) and $(\hat{X}_1, \hat{Y}_1)$, we get

$$\|\Pi_d(P(X_1, Y_1), P(\hat{X}_1, \hat{Y}_1))\| \leq 2Y_1 \|\Theta_1\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|) + 2Y_2 \|\Theta_2\| (\|X_1 - \hat{X}_1\| + \|Y_1 - \hat{Y}_1\|).$$

Given the assumption, Λ satisfies the contraction condition in (31). Therefore, according to the Covitz and Nadler fixed point theorem, Λ has a fixed point (X_1, Y_1) , which solves the system in (2). $\square$

6. Example

Example 1. Consider the following system with boundary conditions:

$$\begin{cases} ({}^C D^{\alpha_1} + K_1 {}^C D^{\alpha_1-1})X_1(t) \in F_1(t, X_1(t), Y_1(t)), t \in Q := [0, T], \\ ({}^C D^{\alpha_2} + K_2 {}^C D^{\alpha_2-1})Y_1(t) \in F_2(t, X_1(t), Y_1(t)), t \in Q := [0, T], \\ (X_1 + Y_1)(0) = -(X_1 + Y_1)(T), \\ \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^\zeta \frac{\vartheta^{\rho-1}}{(\zeta^\rho - \vartheta^\rho)} (X_1 - Y_1)(\vartheta) d\vartheta = Q_1. \end{cases} \quad (32)$$

where $\alpha_1 = 3/2, \alpha_2 = 4/3, \beta = 3/4, \rho = 2/3, \zeta = 3/2, T = 2, Q_1 = 1, H_1(t, X_1, Y_1) = \left[\frac{-1}{16} \frac{|X_1|}{1+|X_1|}, 0 \right] \cup \left[0, \frac{1}{16} \frac{|\sin(Y_1)|}{1+|\sin(Y_1)|} \right]$ and $H_2(t, X_1, Y_1) = \left[\frac{-1}{16} \frac{|Y_1|}{1+|Y_1|}, 0 \right] \cup \left[0, \frac{1}{16} \frac{|\cos(X_1)|}{1+|\cos(X_1)|} \right]$, and on the other hand,

$$\begin{aligned} \Pi_d(H_1(t, X_1, Y_1), H_1(t, \widehat{X}_1, \widehat{Y}_1)) &\leq \frac{1}{16}|X_1 - \widehat{X}_1| + \frac{1}{16}|Y_1 - \widehat{Y}_1|, \forall X_1, \widehat{X}_1, Y_1, \widehat{Y}_1 \in \mathbb{R}, \\ \Pi_d(H_2(t, X_1, Y_1), H_2(t, \widehat{X}_1, \widehat{Y}_1)) &\leq \frac{1}{16}|X_1 - \widehat{X}_1| + \frac{1}{16}|Y_1 - \widehat{Y}_1|, \forall X_1, \widehat{X}_1, Y_1, \widehat{Y}_1 \in \mathbb{R}. \end{aligned}$$

Implementing the aforementioned data, we calculate $Y_1 = 0.23460$ and $Y_2 = 0.15674$. Consequently, we find that $(2Y_1)\Theta_1 + (2Y_2)\Theta_2 \approx 0.0489175 < 1$. Given that all the conditions specified in Theorem 2 are met, we can conclude that a solution to the system (32) exists.

7. Conclusions

In this study, we present a novel class of coupled generalized R-L boundary conditions for addressing coupled nonlinear SFDIs with CDs. By employing multi-valued maps, we demonstrate the existence of solutions. Utilizing fixed-point theorems suited for multi-valued maps, we establish robust solutions for the problem, considering both convex and non-convex scenarios within these maps. Our results offer valuable insights and open new pathways for theoretical advancements in the field. The implications of our results are significant for the scientific community, providing a deeper understanding of complex dynamic systems and enhancing the accuracy of modeling techniques. We believe these discoveries will be of substantial interest to researchers working on fractional differential equations and their applications.

As a possible avenue for future research, we propose exploring the controllability and sensitivity of solutions in systems of coupled fractional differential equations that incorporate a combination of CDs. Investigating how these factors influence system behavior could lead to new methods for managing and predicting the dynamics of such systems, with potential applications across various scientific and engineering disciplines. The system,

$$\begin{cases} ({}^C D^{\alpha_1} + K_1 {}^C D^{\alpha_1-1})X_1(t) \in F_1(t, X_1(t), Y_1(t)), & t \in Q := [0, T], \\ ({}^C D^{\alpha_2} + K_2 {}^C D^{\alpha_2-1})Y_1(t) \in F_2(t, X_1(t), Y_1(t)), & t \in Q := [0, T], \end{cases}$$

has the coupled boundary conditions:

$$\begin{aligned} (X_1 + Y_1)(0) &= -(X_1 + Y_1)(T), \\ \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1} f(V_1 - Y_1)(s)}{(t^\eta - s^\eta)^{1-\delta}} ds &= Q_1, \quad t \in (0, T). \end{aligned}$$

Such efforts have the potential to enrich our understanding of fractional differential equations and their applicability in various fields. Furthermore, future investigations should focus on expanding the realm of novel coupled Erdélyi–Kober fractional boundary conditions, thereby broadening the scope of solutions and insights within this field.

Remark 1. The key differences and advantages between our obtained results using fractional-order differential inclusions and those in the integer-order sense are significant. Fractional-order models inherently capture the memory and hereditary properties of various processes, which integer-order models cannot. This results in a more accurate and realistic representation of complex dynamical systems, particularly in fields such as anomalous diffusion and biological processes. The nonlocal nature of fractional derivatives provides a more comprehensive description of system dynamics, allowing for better prediction and control. Additionally, our use of generalized R-L boundary conditions and fixed-point theorems for multi-valued maps offers a novel approach to establishing existence results, which can lead to new insights and more robust solutions compared to traditional integer-order methods. These advantages underscore the enhanced applicability and effectiveness of fractional-order models in various practical scenarios.

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Abbreviations

The following abbreviations are used in this manuscript:

BVP	Boundary Value Problems
CDs	Caputo Derivatives
CFDs	Caputo Fractional Derivatives
FDEs	Fractional Differential Equations
FPDEs	Fractional Partial Differential Equations
FDIs	Fractional Differential Inclusions
FO	Fractional Order
FBVPs	Fractional-Order Boundary Value Problems
VOFDEs	Variable-Order Fractional Differential Equations
SFDEs	Sequential Fractional Differential Equations.

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