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Modeling the Dispersion of Waves in a Multilayered Inhomogeneous Membrane with Fractional-Order Infusion

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Abstract: The dispersion of elastic shear waves in multilayered bodies is a topic of extensive research due to its significance in contemporary science and engineering. Anti-plane shear motion, a two-dimensional mathematical model in solid mechanics, effectively captures shear wave propagation in elastic bodies with relative mathematical simplicity. This study models the vibration of elastic waves in a multilayered inhomogeneous circular membrane using the Helmholtz equation with fractional-order infusion, effectively leveraging the anti-plane shear motion equation to avoid the computational complexity of universal plane motion equations. The method of the separation of variables and the conformable Bessel equation are utilized for the analytical examination of the model's resulting vibrational displacements, as well as the dispersion relation. Additionally, the influence of various wave phenomena, including the dependencies of the wavenumber on the frequency and the phase speed on the wavenumber, respectively, with the variational effect of the fractional order on wave dispersion is considered. Numerical simulations of prototypical cases validate the formulated model, illustrating its applicability and effectiveness. The study reveals that fractional-order infusion significantly impacts the dispersion of elastic waves in both single- and multilayer membranes. The effects vary depending on the membrane's structure and the wave propagation regime (long-wave vs. short-wave). These findings underscore the potential of fractional-order parameters in tailoring wave behavior for diverse scientific and engineering applications.

Keywords: Helmholtz equation; multilayered membrane; fractional order infusion; composite media; vibration; Bessel equation

MSC: 26A33; 35R11; 34A08; 49M41



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1. Introduction

The vibration of elastic waves is among the burning areas of great concern in elasticity, and it is repeatedly encountered in a variety of processes and applications [1,2]. In particular, the vibration of waves in circular elastic shells and membranes has been comprehensively examined in both the past and present times, including the incorporation of external forces and excitations; see [3,4] and the references therein. Further, the vibration of elastic waves in cylindrical media under the assumption of an anti-plane motion is presided over by the following equation of motion [5]

$$r^2 \frac{\partial^2 v}{\partial r^2} + r \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial \theta^2} - \frac{1}{c^2} r^2 \frac{\partial^2 v}{\partial t^2} = 0, \quad c = \sqrt{\frac{\mu}{\rho}}, \quad (1)$$

where $v = v(r, \theta, t)$ is the wave vibrational displacement and c is the transverse speed, with μ and ρ representing the material constant and density, respectively. Furthermore, with a time-harmonic solution assumption of the form

$$v(r, \theta, t) = u(r, \theta)e^{i\omega t},$$

where ω is the dimensional frequency and $i = \sqrt{-1}$. Equation (1) then becomes

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} + k^2 r^2 u = 0, \quad (2)$$

where $k = \frac{\omega}{c}$. In fact, Equation (2) is the celebrated Helmholtz equation in a cylindrical coordinate system [6] that has vast applications in the theory of thermoelasticity and electromagnetism, to mention a few.

In addition, in an attempt to generalize the classical derivative, Khalil et al. [7] conformably gave a definition of the fractional-order derivative β as follows:

$$D^\beta(v(r)) = \lim_{\varepsilon \rightarrow 0} \frac{v^{(\lceil \beta \rceil - 1)}(r + \varepsilon r^{(\lceil \beta \rceil - \beta)}) - v^{(\lceil \beta \rceil - 1)}(r)}{\varepsilon}, \quad \beta \in (w, w + 1], \quad w \in \mathbb{N} \cup \{0\}, \quad (3)$$

where $\lceil \beta \rceil$ is the smallest integer $\geq \beta$. Consequently, Khalil et al. [7] further related the above fractional-order derivative with the classical integer-order derivative using

$$D^\beta(v)(r) = r^{(\lceil \beta \rceil - \beta)} v^{(\lceil \beta \rceil)}(r), \quad \beta \in (w, w + 1], \quad w \in \mathbb{N} \cup \{0\}, \quad (4)$$

where v is $(w + 1)$ -differentiable at $r > 0$; for more on the fractional-order calculus and various submission of definitions, one may read [8–10] and the references therein. However, this study aims to make use of the Helmholtz equation, Equation (2), with a fractional infusion based on the definition put forward in Equation (3) to model the vibration of elastic waves in a multilayered inhomogeneous circular membrane. The separation of variable method [11], a simple yet efficient classical method, would be deployed for the analytical examination of the model, alongside the application of the recently devised conformable Bessel equation [12]. In essence, this study entails modeling the dispersion of waves in a multilayered body with fractional-order infusion to examine how the fractional order optimizes the long-wave and low-frequency propagation [13–15]. Certainly, when such propagation is achieved, the fractional order can then serve to effectively control and optimize wave dispersion in various engineering applications. Furthermore, the choice of the infused fractional operator is conformable [7], thereby giving Bessel equations in the reduced model, which favors its advantage in describing processes in cylindrical media. In contrast, the known Riemann–Liouville and Caputo fractional operators [2,8–10] will result in the acquisition of complicated reduced singular equations that cannot be examined analytically.

Moreover, the study of the dispersion of elastic shear waves in multilayered bodies has been comprehensively examined deeply in the literature in relation to their vast relevance in contemporary science and engineering applications. When shear waves are propagating on an elastic body, the scenario is perfectly captured using the anti-plane shear motion, which is indeed “an interesting two-dimensional mathematical model arising in solid mechanics involving a single second-order linear or quasi-linear partial differential equation. This model has the virtue of relative mathematical simplicity without loss of essential physical relevance. Anti-plane shear deformations are one of the simplest classes of deformations that solids can undergo” [16]. In view of the quoted statement, it is therefore very obvious that the equation of anti-plane shear motion can be utilized to model some sophisticated wave problems without the exploitation of the universal equations for plane motions that are computationally expensive and indirect. Certainly, the dispersion of elastic shear waves in multilayered bodies is vital in science and engineering. Previous studies, such as the one by Wang et al. [17] on complex dynamic behaviors in coupled systems and the one

by Ba et al. [18] on multilayered structures under stress, provide foundational insights. Liu et al. [19] highlighted the efficacy of fractional-order parameters in dynamic systems, aligning with our approach to wave dispersion. The theoretical models by Kai et al. [20] offer precise frameworks for analyzing wave behavior in non-homogeneous media, which is essential for our research. Advancements in material sciences, as shown by Wang et al. [21] in electromagnetic wave manipulation, demonstrate the broad applicability of wave propagation studies, while Mubarak et al. [22] portray the application of approximation method in the chlorine flow in multilayered channels via the shear equation of motion, to state but a few. In addition, the action of external forces in the system, including the action of the thermal effect, rotational frame, gravity, magnetic field presence, and viscoelasticity, to mention just a few, can be seen in various studies such as [23–32] and the references therein for up-to-date findings on wave dispersion in multilayered media. More precisely, in this study, the expected vibrational displacements in the respective layers of the structure will be determined, in addition to the determination of the consequential dispersion relation. Furthermore, we will deeply examine certain prototypical cases of the structure graphically—by numerically simulating the resulting dispersion relations—to validate the formulated model. This communication is organized as follows: Section 2 portrays the model formulation. Section 3 analytically tackles the formulated model. Section 4 gives the derivation of the resulting dispersion relation, and Section 5 gives the application and the numerical results and discussion. Section 6 outlines some concluding remarks.

2. Problem Formulation

We consider an isotropic multilayered inhomogeneous circular membrane—see Figure 1—which is modeled using the Helmholtz equations in a cylindrical coordinate system from Equation (4) as follows:

$$r^2 \frac{\partial^2 u_i}{\partial r^2} + r \frac{\partial u_i}{\partial r} + \frac{\partial^2 u_i}{\partial \theta^2} + k_i^2 r^2 u_i = 0, \quad 0 \leq \theta \leq 2\pi, \quad i = 1, 2, \dots, m, \quad (5)$$

where

$$k_i = \frac{\omega}{c_i}, \quad c_i = \sqrt{\frac{\mu_i}{\rho_i}},$$

where $u_i = u_i(r, \theta)$ are the out-of-plane vibrational fields in the respective layers of the multilayered membrane for $i = 1, 2, \dots, m$, with r as the radial variable and θ as the azimuthal variable. Moreover, the precise ranges for definitions of the individual membranes are expressed as follows:

$$u_i := \left\{ u_i(r, \theta) : r \in \begin{cases} 0 < r \leq \alpha_1, & i = 1, \\ \alpha_{(i-1)} \leq r \leq \alpha_i, & i \neq 1, \end{cases} \quad -\pi < \theta < \pi, \quad i = 1, 2, \dots, m, \right. \quad (6)$$

where α_1 is the radius of the innermost membrane and α_i , for $i = 2, 3, \dots, m$ are the thicknesses of the subsequent membranes.

Furthermore, we impose suitable boundary conditions at $\theta = -\pi$ and $\theta = \pi$ as follows:

$$\begin{cases} u_i(r, -\pi) = u_i(r, \pi), \\ \frac{\partial u_i}{\partial \theta}(r, -\pi) = \frac{\partial u_i}{\partial \theta}(r, \pi), \end{cases} \quad i = 1, 2, \dots, m, \quad (7)$$

while the following boundary data are defined along the circumferential length, precisely at $r = 0$ and $r = \alpha_m$, as follows :

$$\begin{cases} u_1(0, \theta) \text{ is bounded,} \\ u_m(\alpha_m, \theta) = f_m(\theta), \end{cases} \quad (8)$$

where $f_m(\theta)$ is a θ -dependent vibrational field, virtually, a prescribed entire function.

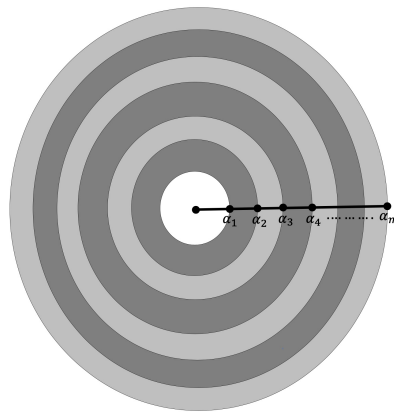


Figure 1. Multilayered inhomogeneous circular membrane.

Additionally, we further assume perfect continuity conditions in the respective interfaces of the multilayered membrane by equating the related vibrational fields u_i and the stresses $\sigma_{rz}^i = \mu_i \frac{\partial u_i}{\partial r}$ at the interfaces, $r = \alpha_i$, as follows

$$\begin{cases} u_i(\alpha_i, \theta) = u_{i+1}(\alpha_i, \theta), \\ \mu_i \frac{\partial u_i}{\partial r}(\alpha_i, \theta) = \mu_{i+1} \frac{\partial u_{i+1}}{\partial r}(\alpha_i, \theta), \end{cases} \quad i = 1, 2, \dots, m, \quad (9)$$

where μ_i values, for $i = 1, 2, \dots, m$, are the material constants in the respective layers of the multilayered membrane.

3. Problem Solution

To solve the governing problem, an analytical approach applying the method of separation of variables [11] is employed. Thus, the solution of Equation (5) is considered to admit the following solution pattern:

$$u_i(r, \theta) = R_i(r)Q(\theta), \quad i = 1, 2, \dots, m, \quad (10)$$

where $R_i(r)$ values are the respective radial solutions, while $Q(\theta)$ is the corresponding azimuthal solution. Further, upon substituting the latter equation into Equation (5), one obtains

$$-\left(r^2 \frac{R_i''}{R_i} + r \frac{R_i'}{R_i} + k_i^2 r^2\right) = \frac{Q''}{Q} = -\lambda, \quad (11)$$

which is subsequently separated into the following boundary-value problem (BVP):

$$Q'' + \lambda Q = 0, \quad Q(-\pi) = Q(\pi), \quad Q'(-\pi) = Q'(\pi), \quad (12)$$

which admits the following solution:

$$Q(\theta) = C_{1n} \cos(n\theta) + C_{2n} \sin(n\theta), \quad (13)$$

where the eigenvalues take the form $\lambda_n = n^2$, for $n = 0, 1, 2, \dots$ and further yield the following radial equation in terms of a Bessel differential equation:

$$r^2 R_i'' + r R_i' + (k_i^2 r^2 - n^2) R_i = 0, \quad i = 1, 2, \dots, m. \quad (14)$$

Most importantly, it is our aim to infuse a fractional-order derivative in the governing model; thus, we feel it is very relevant to fractionalize Equation (14) by considering the recently proposed conformable Bessel fractional differential equation by Hammad et al. [12] as follows:

$$r^{2\beta} D^\beta D^\beta R_i + \beta r^\beta D^\beta R_i + (k_i^2 r^{2\beta} - \beta^2 n^2) R_i = 0, \quad 0 < \beta \leq 1, \quad i = 1, 2, \dots, m. \quad (15)$$

Certainly, when the fractional-order β takes the integer-order, that is, when $\beta = 1$, Equation (15) is completely reduced to Equation (14). In addition, the radial solutions $R_i(r)$ are further obtained by solving Equation (15) to obtain

$$R_i(r) = \begin{cases} {}^1C_3 J_{\beta n}(k_1 r), & 0 < r \leq \alpha_1, & i = 1, \\ {}^iC_3 J_{\beta n}(k_i r) + {}^iC_4 J_{-\beta n}(k_i r), & \alpha_{(i-1)} \leq r \leq \alpha_i, & i \neq 1, \end{cases} \quad i = 1, 2, \dots, m, \quad (16)$$

where

$$J_{\pm\beta n}(k_i r) = \sum_{s=0}^{\infty} (-1)^s \frac{(k_i r)^{2\beta s \pm \beta n}}{(2\beta)^{2s \pm n} s! \Gamma(s \pm n + 1)}, \quad i = 1, 2, \dots, m,$$

is the conformable fractional Bessel function of the first kind of order βn ; ${}^iC_3$, and ${}^iC_4$ values for $i = 1, 2, \dots, m$, are constants to be determined. Furthermore, ${}^1C_4$ vanishes at the innermost membrane, that is, when $i = 1$ due to the boundedness condition at $r = 0$ that was prescribed in the first part of Equation (8). Also, when $\beta n = y$, for some integer y in Equation (16), $J_y(\cdot)$ becomes the classical Bessel function of the first kind of order y , while $\Gamma(\cdot)$ in the latter expression is the well-known gamma function.

Hence, the overall solution without the involvement of the continuity and boundary conditions, which was earlier expressed in Equation (10), is now rewritten using Equations (13) and (16) as follows:

$$u_i(r, \theta) = \begin{cases} \sum_{n=0}^{\infty} \{ {}^1a_n J_{\beta n}(k_1 r) \cos(n\theta) + {}^1b_n J_{\beta n}(k_1 r) \sin(n\theta) \}, & i = 1, \\ \sum_{n=0}^{\infty} \{ ({}^i a_n J_{\beta n}(k_i r) + {}^i \bar{a}_n J_{-\beta n}(k_i r)) \cos(n\theta) + \\ \quad ({}^i b_n J_{\beta n}(k_i r) + {}^i \bar{b}_n J_{-\beta n}(k_i r)) \sin(n\theta) \}, & i \neq 1, \end{cases} \quad (17)$$

where

$$\begin{aligned} C_{1n} {}^1C_3 &= {}^1a_n, & C_{2n} {}^1C_3 &= {}^1b_n, & C_{1n} {}^iC_3 &= {}^i a_n, \\ C_{1n} {}^iC_4 &= {}^i \bar{a}_n, & C_{2n} {}^iC_3 &= {}^i b_n, & C_{2n} {}^iC_4 &= {}^i \bar{b}_n, \end{aligned}$$

Moreover, in order to determine the explicit solution of the governing model, we now employ the prescribed second boundary condition in Equation (8) and the interfacial conditions given in Equation (5). Thus, we begin by substituting the interfacial conditions at $r = \alpha_i$ for $i = 1, 2, \dots, m$ into the acquired solution in Equation (17) to obtain the following system of algebraic equations:

$$\left\{ \begin{array}{l} \text{at } r = \alpha_1, \\ {}^1a_n J_{\beta n}(k_1 \alpha_1) = {}^2a_n J_{\beta n}(k_2 \alpha_1) + {}^2\bar{a}_n J_{-\beta n}(k_2 \alpha_1), \\ {}^1b_n J_{-\beta n}(k_1 \alpha_1) = {}^2b_n J_{\beta n}(k_2 \alpha_1) + {}^2\bar{b}_n J_{-\beta n}(k_2 \alpha_1), \\ \mu_1 {}^1a_n J'_{\beta n}(k_1 \alpha_1) = \mu_2 ({}^2a_n J'_{\beta n}(k_2 \alpha_1) + {}^2\bar{a}_n J'_{-\beta n}(k_2 \alpha_1)), \\ \mu_1 {}^1b_n J'_{-\beta n}(k_1 \alpha_1) = \mu_2 ({}^2b_n J'_{\beta n}(k_2 \alpha_1) + {}^2\bar{b}_n J'_{-\beta n}(k_2 \alpha_1)), \\ \\ \text{at } r = \alpha_j, j = 2, 3, \dots, (m-1), \\ {}^j a_n J_{\beta n}(k_j \alpha_j) + {}^j \bar{a}_n J_{-\beta n}(k_j \alpha_j) = {}^{j+1} a_n J_{\beta n}(k_{j+1} \alpha_j) + {}^{j+1} \bar{a}_n J_{-\beta n}(k_{j+1} \alpha_j), \\ {}^j b_n J_{\beta n}(k_j \alpha_j) + {}^j \bar{b}_n J_{-\beta n}(k_j \alpha_j) = {}^{j+1} b_n J_{\beta n}(k_{j+1} \alpha_j) + {}^{j+1} \bar{b}_n J_{-\beta n}(k_{j+1} \alpha_j), \\ \mu_j ({}^j a_n J'_{\beta n}(k_j \alpha_j) + {}^j \bar{a}_n J'_{-\beta n}(k_j \alpha_j)) = \mu_{j+1} ({}^{j+1} a_n J'_{\beta n}(k_{j+1} \alpha_j) + {}^{j+1} \bar{a}_n J'_{-\beta n}(k_{j+1} \alpha_j)), \\ \mu_j ({}^j b_n J'_{\beta n}(k_j \alpha_j) + {}^j \bar{b}_n J'_{-\beta n}(k_j \alpha_j)) = \mu_{j+1} ({}^{j+1} b_n J'_{\beta n}(k_{j+1} \alpha_j) + {}^{j+1} \bar{b}_n J'_{-\beta n}(k_{j+1} \alpha_j)), \\ \text{for } j = 2, 3, \dots, (m-1), \end{array} \right. \quad (18)$$

where

$$J'_{\pm\beta n}(k_i r) = \frac{1}{2} k_i (J_{\pm n \beta - 1}(k_i r) - J_{\pm n \beta + 1}(k_i r)) \Big|_{r=\alpha_i}, \quad i = 1, 2, \dots, m. \quad (19)$$

Lastly, upon deploying the second boundary condition in Equation (8) when $r = \alpha_m$, one systematically obtains

$$f_m(\theta) = \sum_{n=0}^{\infty} \{ ({}^m a_n J_{\beta n}(k_m \alpha_m) + {}^m \bar{a}_n J_{-\beta n}(k_m \alpha_m)) \cos(n\theta) + ({}^m b_n J_{\beta n}(k_m \alpha_m) + {}^m \bar{b}_n J_{-\beta n}(k_m \alpha_m)) \sin(n\theta) \}, \quad (20)$$

which, upon employing the application of Fourier's series [15], then yields the explicit expressions for the involving coefficients as follows:

$$\begin{aligned} {}^m a_n J_{\beta n}(k_m \alpha_m) + {}^m \bar{a}_n J_{-\beta n}(k_m \alpha_m) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_m(\theta) \cos(n\theta) d\theta, \\ {}^m b_n J_{\beta n}(k_m \alpha_m) + {}^m \bar{b}_n J_{-\beta n}(k_m \alpha_m) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_m(\theta) \sin(n\theta) d\theta. \end{aligned} \quad (21)$$

Furthermore, the overall general solution is plainly determined when solving the $(m+4) \times (m+4)$ coupled algebraic system of equations in Equations (18) and (21) for ${}^i a_n$, ${}^i b_n$, ${}^s \bar{a}_n$, and ${}^s \bar{b}_n$ when $i = 1, 2, \dots, m$ and $s = 2, 3, \dots, m$.

4. Dispersion Relation

To derive the resulting dispersion relation, we revisit the governing Helmholtz equations for the vibration of waves in a multilayered inhomogeneous circular membrane expressed in Equation (5). In fact, u_i will further be conveniently represented by $u_i = R_i(r) \cos(\gamma\theta)$, where γ is the dimensional wavenumber. Moreover, the resultant infused conformable Bessel fractional differential equations for the inhomogeneous media then take the following form:

$$r^{2\beta} D^\beta D^\beta R_i + \beta r^\beta D^\beta R_i + (k_i^2 r^{2\beta} - \beta^2 \gamma^2) R_i = 0, \quad 0 < \beta \leq 1, \quad i = 1, 2, \dots, m, \quad (22)$$

which then admit the following solutions

$$R_i(r) = {}^i E_1 J_{\beta\gamma}(k_i r) + {}^i E_2 J_{-\beta\gamma}(k_i r), \quad \alpha_i \leq r \leq \alpha_{(i+1)}, \quad i = 1, 2, \dots, m, \quad (23)$$

where $J_{\pm\beta\gamma}(\cdot)$ is the conformable fractional Bessel function of the first kind of order $\beta\gamma$; ${}^i E_1$ and ${}^i E_2$ for $i = 1, 2, \dots, m$ are constants. Notably, we mention here that the solution obtained above is for a hollow multilayered inhomogeneous membrane. Certainly, this is for the sake of a practical scenario. Thus, the prescribed interfacial conditions in Equation (9) are shifted starting from $i = 2, 3, \dots, (m-1)$, while the prescribed boundary conditions at $r = \alpha_1$ and $r = \alpha_m$ are traction-free conditions, that is,

$$\sigma_{rz}^1|_{r=\alpha_1} = 0, \quad \text{and} \quad \sigma_{rz}^m|_{r=\alpha_{m+1}} = 0.$$

Further, with this development, one obtains the following dispersion equations:

$$\left\{ \begin{array}{l} \text{at } r = \alpha_1, \\ {}^1 E_1 J'_{\beta\gamma}(k_1 \alpha_1) + {}^1 E_2 J'_{-\beta\gamma}(k_1 \alpha_1) = 0, \\ \\ \text{at } r = \alpha_j, \text{ for } j = 2, 3, \dots, (m-1), \\ {}^j E_1 J_{\beta\gamma}(k_j \alpha_j) + {}^j E_2 J_{-\beta\gamma}(k_j \alpha_j) = {}^{j+1} E_1 J_{\beta\gamma}(k_{j+1} \alpha_j) + {}^{j+1} E_2 J_{-\beta\gamma}(k_{j+1} \alpha_j), \\ \mu_j ({}^j E_1 J'_{\beta\gamma}(k_j \alpha_j) + {}^j E_2 J'_{-\beta\gamma}(k_j \alpha_j)) = \mu_{j+1} ({}^{j+1} E_1 J'_{\beta\gamma}(k_{j+1} \alpha_j) + {}^{j+1} E_2 J'_{-\beta\gamma}(k_{j+1} \alpha_j)), \\ \\ \text{at } r = \alpha_m, \\ {}^m E_1 J'_{\beta\gamma}(k_m \alpha_m) + {}^1 E_2 J'_{-\beta\gamma}(k_m \alpha_m) = 0, \end{array} \right. \quad (24)$$

where $J'_{\pm\beta\gamma}(\cdot)$ follows from Equation (19). Moreover, the resulting dispersion relation will then be determined upon equating the determinant of the coefficient matrix of the above dispersion system of equations to zero.

5. Application

The present section considers two prototypical multilayered circular membranes as an application of the governing generalized multilayered inhomogeneous circular membrane that was successfully examined in the above section. More precisely, we will determine the consequential vibrational fields and the dispersion relations in two- and three-layered inhomogeneous membranes (Figure 2).

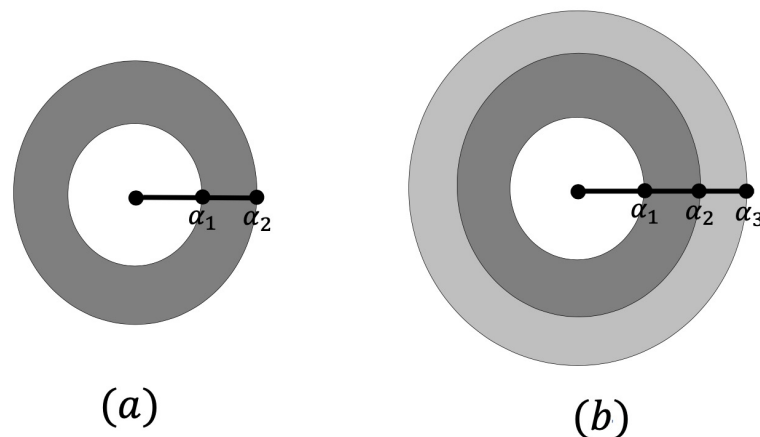


Figure 2. Prototype schema for (a) a two-layered inhomogeneous membrane, and (b) a three-layered inhomogeneous membrane.

5.1. Vibrational Displacement

This subsection undeniably presents the consequential vibrational fields for the prototype structures under consideration.

5.1.1. Two-Layered Inhomogeneous Membrane

In this regard, the explicit expressions for the resulting vibrational fields associated with the a two-layered membrane for $i = 2$ are obtained in the respective layers from Equation (17) as follows

$$\begin{aligned} u_1(r, \theta) &= \sum_{n=0}^{\infty} \{ {}^1a_n J_{\beta n}(k_1 r) \cos(n\theta) + {}^1b_n J_{\beta n}(k_1 r) \sin(n\theta) \}, \\ u_2(r, \theta) &= \sum_{n=0}^{\infty} \{ ({}^2a_n J_{\beta n}(k_2 r) + {}^2\bar{a}_n J_{-\beta n}(k_2 r)) \cos(n\theta) + \\ &\quad ({}^2b_n J_{\beta n}(k_2 r) + {}^2\bar{b}_n J_{-\beta n}(k_2 r)) \sin(n\theta) \}, \end{aligned} \quad (25)$$

where the involving coefficients ${}^1a_n, {}^1b_n, {}^2a_n, {}^2\bar{a}_n, {}^2b_n$, and ${}^2\bar{b}_n$ are determined from Equations (18) and (20); see Appendix A for their explicit expressions.

Particular case: singled-layered homogeneous membrane

Deducibly, when the second membrane is removed, thereby leaving behind a singled-layered hollow membrane, the solution found in Equation (25) reduces to the following

$$u_1(r, \theta) = \sum_{n=0}^{\infty} \{ {}^1a_n J_{\beta n}(k_1 r) \cos(n\theta) + {}^1b_n J_{\beta n}(k_1 r) \sin(n\theta) \}, \quad (26)$$

where the coefficients involved, ${}^1a_n$ and ${}^1b_n$, are determined from Equation (21) as follows:

$${}^1a_n = \frac{1}{\pi J_{\beta n}(k_1 \alpha_1)} \int_{-\pi}^{\pi} f_1(\theta) \cos(n\theta) d\theta, \quad {}^1b_n = \frac{1}{\pi J_{\beta n}(k_1 \alpha_1)} \int_{-\pi}^{\pi} f_1(\theta) \sin(n\theta) d\theta.$$

5.1.2. Three-Layered Inhomogeneous Membrane

As we proceed, the explicit expressions for the resulting vibrational fields in this regard for $i = 3$ are obtained from Equation (17) as follows:

$$\begin{aligned} u_1(r, \theta) &= \sum_{n=0}^{\infty} \{ {}^1a_n J_{\beta n}(k_1 r) \cos(n\theta) + {}^1b_n J_{\beta n}(k_1 r) \sin(n\theta) \}, \\ u_2(r, \theta) &= \sum_{n=0}^{\infty} \{ ({}^2a_n J_{\beta n}(k_2 r) + {}^2\bar{a}_n J_{-\beta n}(k_2 r)) \cos(n\theta) + \\ &\quad ({}^2b_n J_{\beta n}(k_2 r) + {}^2\bar{b}_n J_{-\beta n}(k_2 r)) \sin(n\theta) \}, \\ u_3(r, \theta) &= \sum_{n=0}^{\infty} \{ ({}^3a_n J_{\beta n}(k_3 r) + {}^3\bar{a}_n J_{-\beta n}(k_3 r)) \cos(n\theta) + \\ &\quad ({}^3b_n J_{\beta n}(k_3 r) + {}^3\bar{b}_n J_{-\beta n}(k_3 r)) \sin(n\theta) \}, \end{aligned} \quad (27)$$

where the coefficients involved, ${}^1a_n, {}^1b_n, {}^2a_n, {}^2\bar{a}_n, {}^2b_n, {}^2\bar{b}_n, {}^3a_n, {}^3\bar{a}_n, {}^3b_n$, and ${}^3\bar{b}_n$, can equally be determined from Equations (18) and (20); see Appendix B for their explicit expressions.

5.2. Dispersion Relation

The subsection irrefutably derives the resulting dispersion relations with regard to the governing prototyped multilayered circular membranes.

5.2.1. Two-Layered Inhomogeneous Membrane

Upon considering the prototype for the two-layered inhomogeneous membrane, the obtained generalized equations for the dispersion relation in Equation (24) yield the consequential dispersion matrix in this regard as follows:

$$A_1 = \begin{pmatrix} J'_{\beta\gamma}(k_1 \alpha_1) & J'_{-\beta\gamma}(k_1 \alpha_1) & 0 & 0 \\ J_{\beta\gamma}(k_1 \alpha_2) & J_{-\beta\gamma}(k_1 \alpha_2) & -J_{\beta\gamma}(k_2 \alpha_2) & J_{-\beta\gamma}(k_2 \alpha_2) \\ J'_{\beta\gamma}(k_1 \alpha_2) & J'_{-\beta\gamma}(k_1 \alpha_2) & -\frac{\mu_2}{\mu_1} J'_{\beta\gamma}(k_2 \alpha_2) & -\frac{\mu_2}{\mu_1} J'_{-\beta\gamma}(k_2 \alpha_2) \\ 0 & 0 & J'_{\beta\gamma}(k_2 \alpha_3) & J'_{-\beta\gamma}(k_2 \alpha_3) \end{pmatrix}, \quad (28)$$

upon which the dispersion equation is obtained by setting the resulting determinant of the above dispersion matrix to zero.

Particular case: singled-layered homogeneous membrane

It can be deduced that, when the second membrane is removed, thereby leaving behind a singled-layered hollow membrane with traction-free boundary conditions on both surfaces, the above dispersion matrix thus reduces to the following:

$$A_2 = \begin{pmatrix} J'_{\beta\gamma}(k_1 \alpha_1) & J'_{-\beta\gamma}(k_1 \alpha_1) \\ J'_{\beta\gamma}(k_1 \alpha_2) & J'_{-\beta\gamma}(k_1 \alpha_2) \end{pmatrix}, \quad (29)$$

which reveals the following reduced and easy dispersion relation

$$J'_{-\beta\gamma}(k_1 \alpha_2) J'_{\beta\gamma}(k_1 \alpha_1) - J'_{\beta\gamma}(k_1 \alpha_1) J'_{\beta\gamma}(k_1 \alpha_2) = 0. \quad (30)$$

5.2.2. Three-Layered Inhomogeneous Membrane

Accordingly, the three-layered inhomogeneous membrane reveals from Equation (24) the following dispersion matrix:

$$A_3 = \begin{pmatrix} J'_{\beta\gamma}(k_1\alpha_1) & J'_{-\beta\gamma}(k_1\alpha_1) & 0 & 0 & 0 & 0 \\ J_{\beta\gamma}(k_1\alpha_2) & J_{-\beta\gamma}(k_1\alpha_2) & -J_{\beta\gamma}(k_2\alpha_2) & J_{-\beta\gamma}(k_2\alpha_2) & 0 & 0 \\ J'_{\beta\gamma}(k_1\alpha_2) & J'_{-\beta\gamma}(k_1\alpha_2) & -\frac{\mu_2}{\mu_1}J'_{\beta\gamma}(k_2\alpha_2) & -\frac{\mu_2}{\mu_1}J'_{-\beta\gamma}(k_2\alpha_2) & 0 & 0 \\ 0 & 0 & J_{\beta\gamma}(k_2\alpha_3) & J_{-\beta\gamma}(k_2\alpha_3) & a_{45} & a_{46} \\ 0 & 0 & J'_{\beta\gamma}(k_2\alpha_3) & J'_{-\beta\gamma}(k_2\alpha_3) & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{pmatrix}, \quad (31)$$

where

$$\begin{aligned} a_{45} &= -J_{\beta\gamma}(k_3\alpha_3), \quad a_{46} = -J_{-\beta\gamma}(k_3\alpha_3), \quad a_{55} = -\frac{\mu_3}{\mu_2}J'_{\beta\gamma}(k_3\alpha_3), \\ a_{65} &= J'_{\beta\gamma}(k_3\alpha_4), \quad a_{66} = J'_{-\beta\gamma}(k_3\alpha_4), \quad a_{56} = -\frac{\mu_3}{\mu_2}J'_{-\beta\gamma}(k_3\alpha_3). \end{aligned} \quad (32)$$

upon which the dispersion equation is obtained by setting the resulting determinant of the dispersion matrix to zero.

6. Numerical Results and Discussion

This section simulates and then discusses the obtained results numerically with the help of computational software. Indeed, in the numerical simulation, we consider the prototype two- and three-layered inhomogeneous membrane cases for brevity. Further, we deduce the result of the singled-layered homogeneous membrane by varnishing the outer layer of the two-layered membrane while imposing traction-free conditions on both sides of the hollow bodies.

6.1. Two- and Three-Layered Inhomogeneous Membranes

In this regard, the resulting dimensionless dispersion relation of the two-layered inhomogeneous membrane is obtained from Equation (28) as follows:

$$|A_1| = 0, \quad (33)$$

where the new dimensional structural parameters take the following expressions:

$$\mu = \frac{\mu_1}{\mu_2}, \quad \rho = \frac{\rho_1}{\rho_2}, \quad \alpha = \frac{\alpha_3}{\alpha_2}, \quad \alpha' = \frac{\alpha_1}{\alpha_2}, \quad (34)$$

where μ is the dimensionless material constant, ρ is the dimensionless density, and α , and α' are the respective dimensionless thicknesses. In addition, we also have the following dimensionless quantities:

$$W = \frac{\omega\alpha_2}{c_2}, \quad \Phi = \gamma\alpha_2, \quad \chi = \frac{c_2}{c_1}, \quad \tau = \frac{\beta}{\alpha_2}, \quad (35)$$

where W is the dimensionless frequency, Φ is the dimensionless wavenumber, χ is the shear speed ratio, which is also dimensionless; and τ is the scaled dimensionless fractional-order. Furthermore, the influence of the scaled fractional-order τ on the dispersion of elastic waves in the media in a bi-elastic inhomogeneous hollow membrane has been shown in Figure 3a,b, for the long-wave and short-wave propagations, respectively [33], with regard to the relationship between the phase-speed $\Theta (= W/\Phi)$ and the speed ratio $\chi (= c_2/c_1)$. In fact, we depict the harmonic curves, that is, the relationship between the dimensionless phase speed and the dimensionless speed ratio with respect to the variation in the scaled fractional-order τ in Figure 3a,b. Notably, in both Figure 3a,b, it is vividly obvious that an increase in the fractional-order τ partially/halfway stretches the harmonic curves backward, thereby implying a delay or rather a decrease in the propagation.

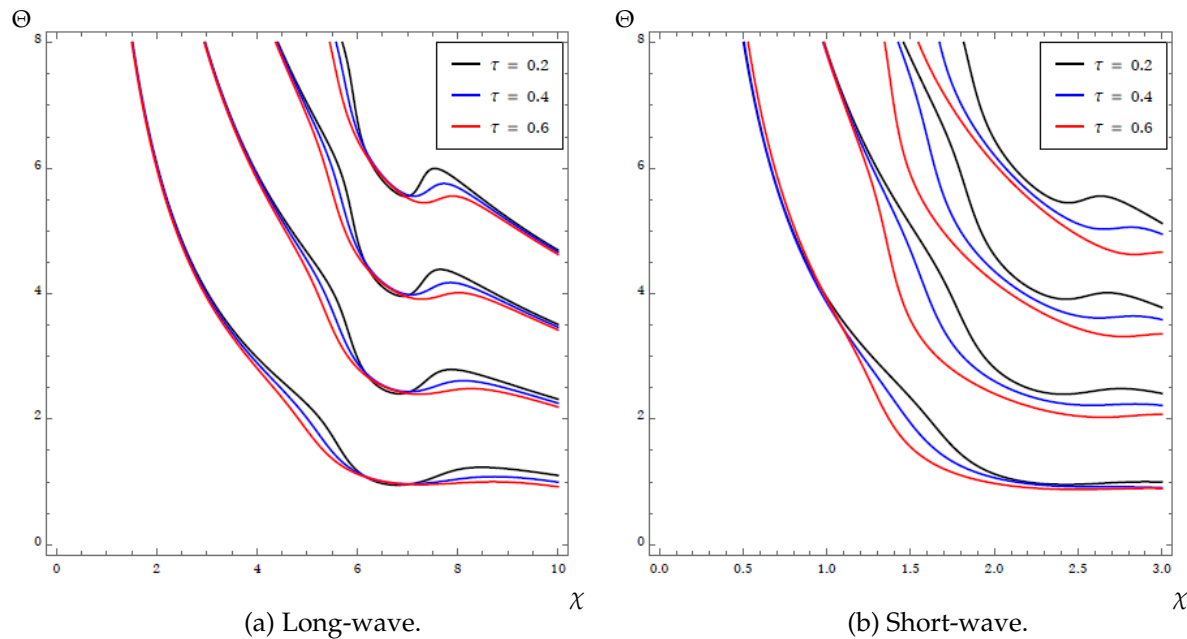


Figure 3. Harmonic curves: relationship between the dimensionless phase-speed Θ and dimensionless speed ratio χ with respect to the variation of the scaled fractional-order τ for the two-layered inhomogeneous membrane when $\alpha_1 = 0.05$, $\alpha_2 = 0.15$, and $\alpha_3 = 0.2$ while fixing the dimensionless wavenumber to be (a) $\Phi = 0.4$ and (b) $\Phi = 1.2$.

Figure 4a portrays the harmonic curves via the relationship between the dimensionless phase-speed Θ and wavenumber Φ with respect to the variation in the scaled fractional-order τ for the two-layered inhomogeneous membrane. It is noted that an increase in the fractional order decreases the respective harmonic modes. Furthermore, it is equally noted that the disparity with regard to the effects of the fractional order becomes clearer as the wavenumber Φ tilts towards the short-wave region; remember that short-wave propagation occurs when $\Phi \geq 1$. Additionally, Figure 4b portrays the relationship between the frequency W and the wavenumber Φ within the estimate of long-wave and low-frequency propagation, that is, when $\Phi \ll 1$ and $W \ll 1$. Remember that the vibration and dispersion of surface waves and their likes under the assumption of a long-wave and low-frequency have been expansively studied with regards to their vast applications in wave analysis and vibration control, for more on the linked findings, one may consult [23–33] and the references therein. Hence, from Figure 4b, it is noted that only the fundamental mode is able to satisfy the long-wave and low-frequency conditions. Further, it was noted that an increase in the fractional-order τ uplifts the dispersion fundamental curve; indeed, this then shrinks the range of Φ to perfectly suit the long-wave range.

Accordingly, the three-layered inhomogeneous membrane case is analyzed in Figures 5 and 6, where harmonic curves, depicting the relationship between the dimensionless phase-speed and the dimensionless speed ratio with respect to the variation of the scaled fractional-order τ is shown in Figure 5 for long and short waves, respectively. In addition, Figure 4 portrays the harmonic curves via Figure 6a, which shows the relationship between the dimensionless phase-speed Θ and wavenumber Φ , while Figure 6b portrays the relationship between the frequency W and the wavenumber Φ ; all with respect to the variation of the scaled fractional-order τ for the three-layered inhomogeneous membrane. Moreover, the same interpretation with the two-layered case is applied here. Thus, we leave it for brevity. In addition, the fixed parameter values in the case of a two-layered membrane are extended and further used for plotting the results for the three-layered scenario. Moreover, in making the plots in the three-layered membrane, a consideration of alternating layers has been made for simplicity, where $\rho_1 = \rho_3$ and $\mu_1 = \mu_3$. In addition, beyond the dimensionless quantities in the two-layered case, this case discovered new

dimensionless quantities, including $\alpha'' = \alpha_4/\alpha_2$, upon which we numerically consider $\alpha_4 = 0.2$. Notably, we can conclude from these figures that the lower the number of layers, the more modes observed concerning both the Θ vs. χ and also Θ vs. Φ scenarios; however, the fractional infusion has been noted to have the same effect in both the two- and three-layered membranes concerning the W vs. Φ relationship.

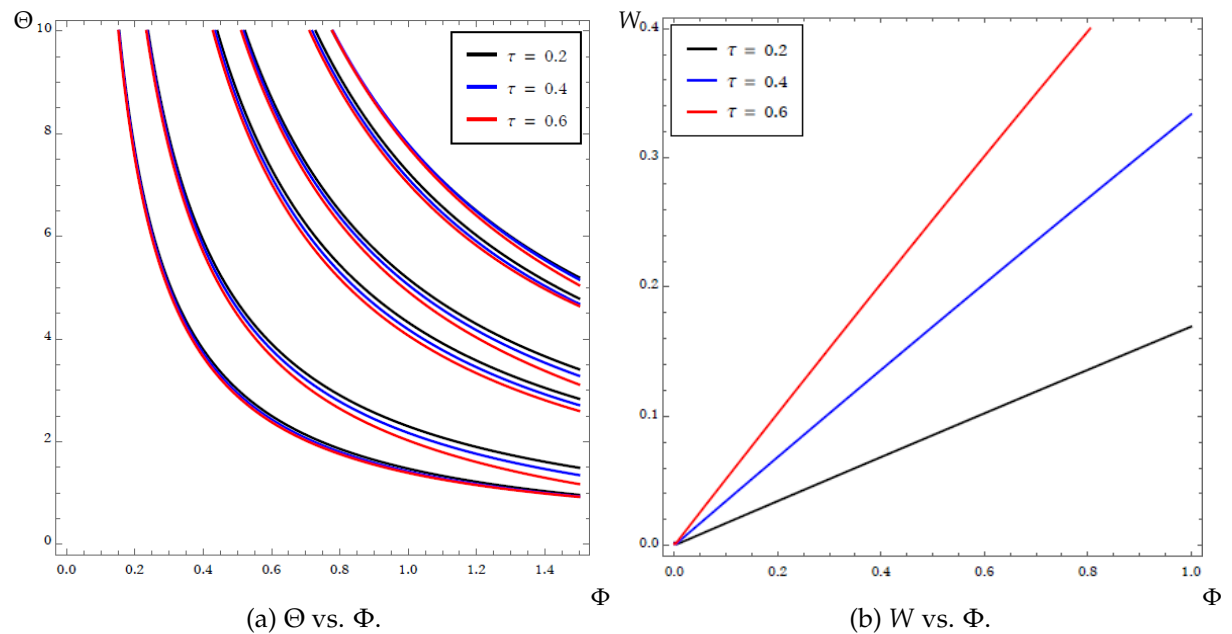


Figure 4. Harmonic curves: the relationship between the (a) dimensionless phase-speed Θ and wavenumber Φ and (b) dimensionless frequency W and wavenumber Φ with respect to the variation in the scaled fractional-order τ for the two-layered inhomogeneous membrane when $\mu = 0.3$, $\rho = 0.3$, $\alpha = 1.3333$, $\alpha' = 0.3333$.

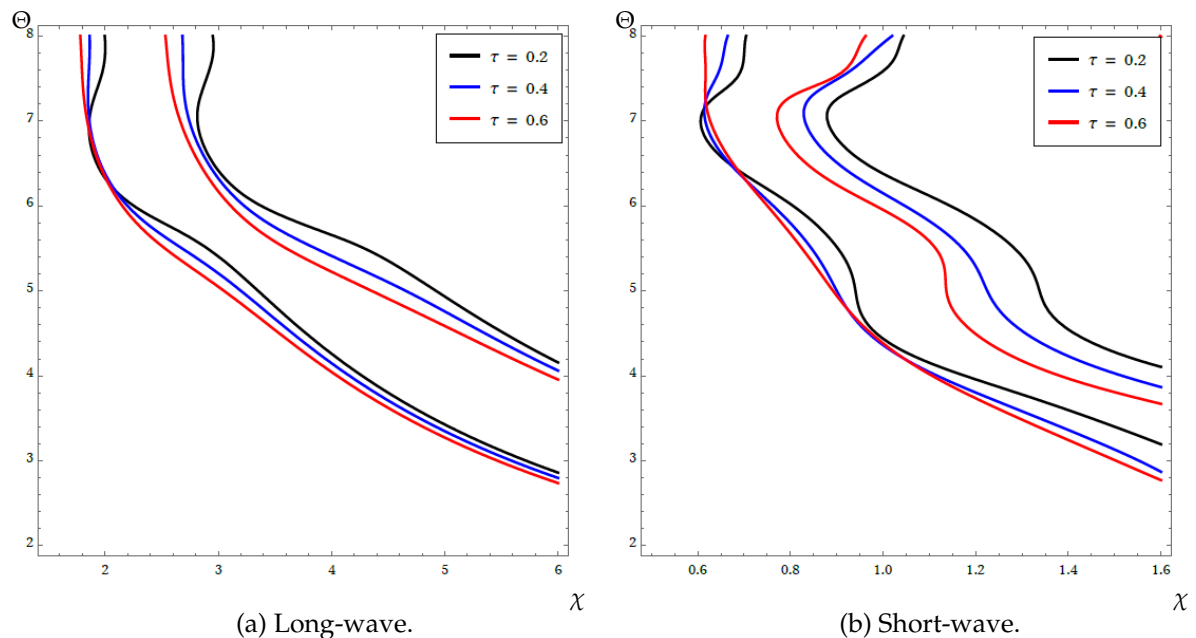


Figure 5. Harmonic curves: the relationship between the dimensionless phase-speed Θ and dimensionless speed ratio χ with respect to the variation of the scaled fractional-order τ for the three-layered inhomogeneous membrane while fixing the dimensionless wavenumber to (a) $\Phi = 0.4$ and (b) $\Phi = 1.2$.

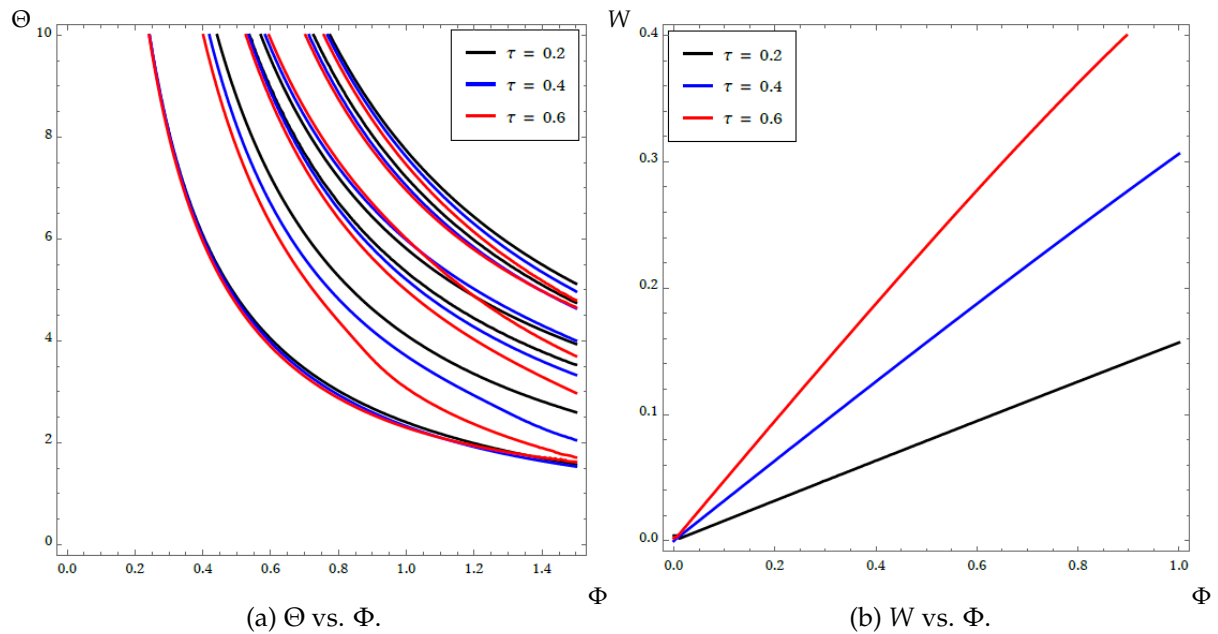


Figure 6. Harmonic curves: the relationship between the (a) dimensionless phase-speed Θ and wavenumber Φ and (b) dimensionless frequency W and wavenumber Φ with respect to the variation of the scaled fractional-order τ for the three-layered inhomogeneous membrane.

6.2. Singled-Layered Homogeneous Membrane

In the same manner, the reduced dimensionless dispersion relation for the singled-layered homogeneous membrane was obtained in Equation (30) as follows:

$$J'_{-\tau\Phi}(W)J'_{\tau\Phi}(W\alpha') - J'_{\tau\Phi}(W)J'_{-\tau\Phi}(W\alpha') = 0, \quad (36)$$

or equally, after explicitly expanding $J'_{\pm}(\cdot)$ as in Equation (19), as follows

$$(J_{-\tau\Phi-1}(W) - J_{1-\tau\Phi}(W))(J_{\tau\Phi-1}(W\alpha') - J_{\tau\Phi+1}(W\alpha')) - (J_{\tau\Phi-1}(W) - J_{\tau\Phi+1}(W))(J_{-\tau\Phi-1}(W\alpha') - J_{1-\tau\Phi}(W\alpha')) = 0, \quad (37)$$

where τ , Φ and α' are already given in Equations (34) and (35), while W now takes the expression $W = \omega\alpha_2/c_1$ since c_2 is not present in the reduced body.

Further, the dispersion of elastic waves in the reduced medium with respect to the variation in the scaled fractional-order τ is depicted in Figure 7a, portraying the harmonic curves via the relationship between the dimensionless phase-speed Θ and wavenumber Φ with respect to the variation in the scaled fractional-order τ for the single-layered homogeneous membrane. Indeed, the opposite has been noted here in comparison with the two-layered inhomogeneous membrane. Precisely, an increase in the fractional order increases the respective harmonic modes. In addition, Figure 7b portrays the relationship between the frequency and the wavenumber equally within the estimate of long-wave and low-frequency propagation. Thus, without further delay, it is noted from Figure 7b that the fundamental mode increases with an increase in the fractional order. Furthermore, the fixed parameter values in the case of a two-layered membrane are systematically used here for plotting the corresponding reduced one-layered case.

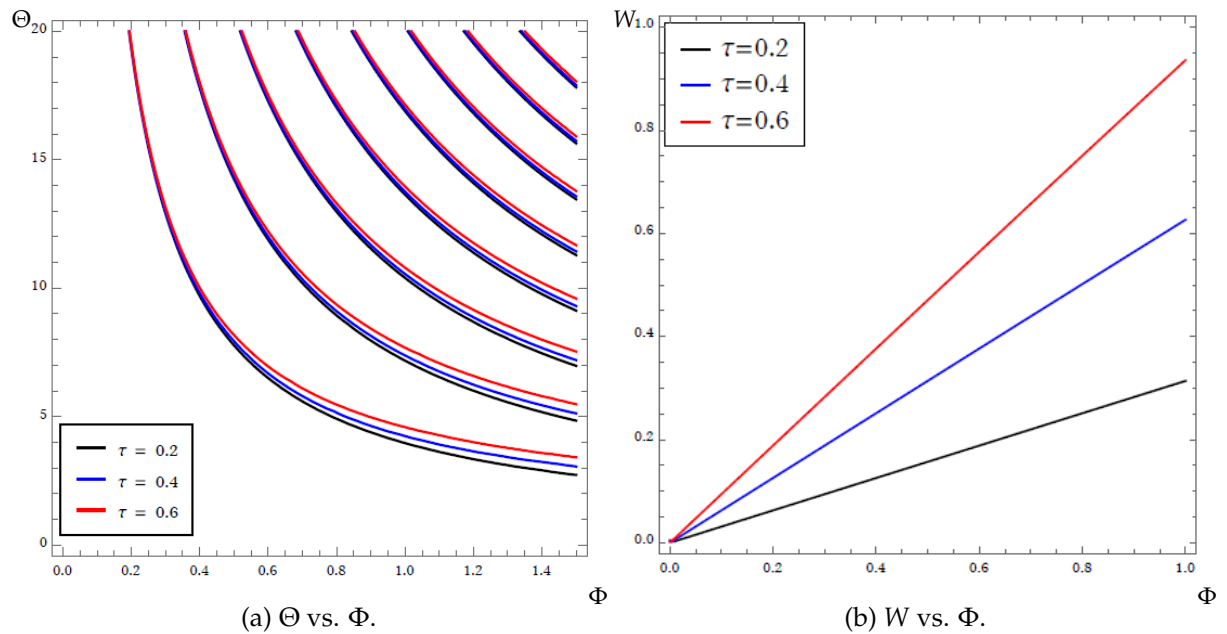


Figure 7. Harmonic curves: the relationship between the (a) dimensionless phase-speed Θ and wavenumber Φ and (b) dimensionless frequency W and wavenumber Φ with respect to the variation of the scaled fractional-order τ for the single-layered homogeneous membrane.

7. Conclusions

This research utilized the Helmholtz equation to model the dispersion of elastic waves in a multilayered inhomogeneous circular membrane with fractional infusion. Ideal continuity conditions were applied between membrane layers, with fixed conditions on the outermost layer, and traction-free conditions at the endpoints of the composite membrane. Numerical simulations of the wave expressions conclusively showed that fractional infusion significantly affected the dispersion of elastic waves in the medium. The key findings are summarized as follows:

1. The study shows that increasing the fractional-order parameter in a bi-elastic inhomogeneous hollow membrane stretches the harmonic curves backward, resulting in a delay or decrease in wave propagation. This effect is observed consistently in both long-wave and short-wave propagations.
2. For both the two- and three-layered inhomogeneous membranes, an increase in the fractional order leads to a decrease in the respective harmonic modes. This effect becomes more pronounced in the short-wave region, indicating that fractional-order parameters significantly influence short-wave propagation.
3. In the context of long-wave and low-frequency propagation, only the fundamental mode satisfies the condition; this cuts across the three prototype cases. An increase in the fractional order raises the fundamental dispersion curve, effectively shrinking the range of the wavenumber suitable for long-wave propagation. This highlights the importance of fractional-order parameters in tuning the wave behavior for specific applications.
4. In a single-layered homogeneous membrane, an increase in the fractional order results in an increase in the respective harmonic modes, contrasting with the behavior observed in the two- and three-layered inhomogeneous membranes. This indicates that the structural composition of the membrane (single vs. multilayered) plays a crucial role in determining how fractional-order parameters affect wave dispersion.
5. For both single- and multi-layered membranes, the fundamental mode increases with an increase in the fractional-order, especially within the long-wave and low-frequency propagation range. This suggests that fractional-order infusion can be effectively used to control and optimize wave dispersion in various engineering applications.

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Appendix A

The explicit form for the coefficients ${}^1a_n$, ${}^1b_n$, ${}^2a_n$, ${}^2\bar{a}_n$, ${}^2b_n$, and ${}^2\bar{b}_n$ ubvikved in Equation (25) are found from Equations (18) and (20) as follows:

$$\begin{aligned} {}^1a_n &= \mu_2 \left(J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) w_1(n), \\ {}^1b_n &= \mu_2 \left(J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) w_2(n), \\ {}^2a_n &= \left(\mu_1 J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_1\alpha_1) - \mu_2 J_{n\beta}(k_1\alpha_1) J'_{-n\beta}(k_2\alpha_1) \right) w_1(n), \\ {}^2b_n &= \left(\mu_1 J_{-n\beta}(k_2\alpha_1) J'_{-n\beta}(k_1\alpha_1) - \mu_2 J_{-n\beta}(k_1\alpha_1) J'_{-n\beta}(k_2\alpha_1) \right) w_2(n), \\ {}^2\bar{a}_n &= \left(\mu_1 J_{n\beta}(k_2\alpha_1) J'_{n\beta}(k_1\alpha_1) - \mu_2 J_{n\beta}(k_1\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) w_1(n), \\ {}^2\bar{b}_n &= \left(\mu_1 J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_1\alpha_1) - \mu_2 J_{-n\beta}(k_1\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) w_2(n), \end{aligned}$$

where $w_1(n)$ and $w_2(n)$ appearing above are expressed as

$$w_1(n) = \frac{1}{\pi\zeta_1} \int_{-\pi}^{\pi} f_2(\theta) \cos(n\theta) d\theta, \quad w_2(n) = \frac{1}{\pi\zeta_2} \int_{-\pi}^{\pi} f_2(\theta) \sin(n\theta) d\theta,$$

with ζ_1 and ζ_2 appearing above expressed as follows

$$\begin{aligned} \zeta_1 &= \mu_1 \left(J_{-n\beta}(k_2\alpha_2) J_{n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_1) J_{n\beta}(k_2\alpha_2) \right) J'_{n\beta}(k_1\alpha_1) \\ &\quad + \mu_2 J_{n\beta}(k_1\alpha_1) \left(J_{n\beta}(k_2\alpha_2) J'_{-n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) \right), \\ \zeta_2 &= \mu_1 \left(J_{-n\beta}(k_2\alpha_2) J_{n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_1) J_{n\beta}(k_2\alpha_2) \right) J'_{-n\beta}(k_1\alpha_1) \\ &\quad + \mu_2 J_{-n\beta}(k_1\alpha_1) \left(J_{n\beta}(k_2\alpha_2) J'_{-n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) \right). \end{aligned}$$

Appendix B

The explicit expressions for ${}^1a_n$, ${}^1b_n$, ${}^2a_n$, ${}^2\bar{a}_n$, ${}^2b_n$, ${}^2\bar{b}_n$, ${}^3a_n$, ${}^3\bar{a}_n$, ${}^3b_n$, and ${}^3\bar{b}_n$ with regard to the three-layered membrane appearing in Equation (27) are determined accordingly as follows:

$$\begin{aligned} {}^1a_n &= \mu_2 \mu_3 \left(J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) \times \\ &\quad \left(J_{n\beta}(k_3\alpha_2) J'_{-n\beta}(k_3\alpha_2) - J_{-n\beta}(k_3\alpha_2) J'_{n\beta}(k_3\alpha_2) \right) w_1(n), \\ {}^1b_n &= \mu_2 \mu_3 \left(J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) \times \\ &\quad \left(J_{n\beta}(k_3\alpha_2) J'_{-n\beta}(k_3\alpha_2) - J_{-n\beta}(k_3\alpha_2) J'_{n\beta}(k_3\alpha_2) \right) w_2(n), \end{aligned}$$

$$\begin{aligned}
{}^2a_n &= -\mu_3 \left(J_{n\beta}(k_3\alpha_2) J'_{-n\beta}(k_3\alpha_2) - J_{-n\beta}(k_3\alpha_2) J'_{n\beta}(k_3\alpha_2) \right) \times \\
&\quad \left(\mu_1 J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_1\alpha_1) - \mu_2 J_{n\beta}(k_1\alpha_1) J'_{-n\beta}(k_2\alpha_1) \right) w_1(n), \\
{}^2b_n &= -\mu_3 \left(J_{n\beta}(k_3\alpha_2) J'_{-n\beta}(k_3\alpha_2) - J_{-n\beta}(k_3\alpha_2) J'_{n\beta}(k_3\alpha_2) \right) \times \\
&\quad \left(\mu_1 J_{-n\beta}(k_2\alpha_1) J'_{-n\beta}(k_1\alpha_1) - \mu_2 J_{-n\beta}(k_1\alpha_1) J'_{-n\beta}(k_2\alpha_1) \right) w_2(n), \\
{}^2\bar{a}_n &= \mu_3 \left(J_{n\beta}(k_3\alpha_2) J'_{-n\beta}(k_3\alpha_2) - J_{-n\beta}(k_3\alpha_2) J'_{n\beta}(k_3\alpha_2) \right) \times \\
&\quad \left(\mu_1 J_{n\beta}(k_2\alpha_1) J'_{n\beta}(k_1\alpha_1) - \mu_2 J_{n\beta}(k_1\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) w_1(n), \\
{}^2\bar{b}_n &= \mu_3 \left(J_{n\beta}(k_3\alpha_2) J'_{-n\beta}(k_3\alpha_2) - J_{-n\beta}(k_3\alpha_2) J'_{n\beta}(k_3\alpha_2) \right) \times \\
&\quad \left(\mu_1 J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_1\alpha_1) - \mu_2 J_{-n\beta}(k_1\alpha_1) J'_{n\beta}(k_2\alpha_1) \right) w_2(n), \\
{}^3a_n &= (\mu_1 X_1 J'_{n\beta}(k_1\alpha_1) + \mu_2 Z_1 J_{n\beta}(k_1\alpha_1)) w_1(n), \\
{}^3b_n &= -(\mu_1 X_2 J'_{-n\beta}(k_1\alpha_1) + \mu_2 Z_2 J_{-n\beta}(k_1\alpha_1)) w_2(n), \\
{}^3\bar{a}_n &= -(\mu_1 X_3 J'_{-n\beta}(k_1\alpha_1) + \mu_2 Z_3 J_{-n\beta}(k_1\alpha_1)) w_1(n), \\
{}^3\bar{b}_n &= (\mu_1 X_4 J'_{-n\beta}(k_1\alpha_1) + \mu_2 Z_4 J_{-n\beta}(k_1\alpha_1)) w_2(n),
\end{aligned}$$

where X_j , and Z_j , for $j = 1, 2, 3, 4$ appearing in the later expressions are expressed as follows:

$$\begin{aligned}
X_1 &= \mu_2 J_{-n\beta}(k_3\alpha_2) \left(J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) - J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_2) \right) \\
&\quad + \mu_3 \left(J_{-n\beta}(k_2\alpha_2) J_{n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_1) J_{n\beta}(k_2\alpha_2) \right) J'_{-n\beta}(k_3\alpha_2), \\
Z_1 &= \mu_2 J_{-n\beta}(k_3\alpha_2) \left(J'_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) - J'_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) \right) \\
&\quad + \mu_3 J'_{-n\beta}(k_3\alpha_2) \left(J_{n\beta}(k_2\alpha_2) J'_{-n\beta}(k_2\alpha_1) - J_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) \right), \\
X_2 &= \mu_2 J_{-n\beta}(k_3\alpha_2) \left(J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_2) - J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) \right) \\
&\quad + \mu_3 \left(J_{-n\beta}(k_2\alpha_1) J_{n\beta}(k_2\alpha_2) - J_{-n\beta}(k_2\alpha_2) J_{n\beta}(k_2\alpha_1) \right) J'_{-n\beta}(k_3\alpha_2), \\
Z_2 &= \mu_2 J_{-n\beta}(k_3\alpha_2) \left(J'_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) - J'_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) \right) \\
&\quad + \mu_3 J'_{-n\beta}(k_3\alpha_2) \left(J_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) - J_{n\beta}(k_2\alpha_2) J'_{-n\beta}(k_2\alpha_1) \right), \\
X_3 &= \mu_2 J_{-n\beta}(k_3\alpha_2) \left(J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_2) - J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) \right) \\
&\quad + \mu_3 \left(J_{-n\beta}(k_2\alpha_1) J_{n\beta}(k_2\alpha_2) - J_{-n\beta}(k_2\alpha_2) J_{n\beta}(k_2\alpha_1) \right) J'_{-n\beta}(k_3\alpha_2), \\
Z_3 &= \mu_2 J_{-n\beta}(k_3\alpha_2) \left(J'_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) - J'_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) \right) \\
&\quad + \mu_3 J'_{-n\beta}(k_3\alpha_2) \left(J_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) - J_{n\beta}(k_2\alpha_2) J'_{-n\beta}(k_2\alpha_1) \right), \\
X_4 &= \mu_2 J_{n\beta}(k_3\alpha_2) \left(J_{n\beta}(k_2\alpha_1) J'_{-n\beta}(k_2\alpha_2) - J_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) \right) \\
&\quad + \mu_3 \left(J_{-n\beta}(k_2\alpha_1) J_{n\beta}(k_2\alpha_2) - J_{-n\beta}(k_2\alpha_2) J_{n\beta}(k_2\alpha_1) \right) J'_{n\beta}(k_3\alpha_2), \\
Z_4 &= \mu_2 J_{n\beta}(k_3\alpha_2) \left(J'_{-n\beta}(k_2\alpha_1) J'_{n\beta}(k_2\alpha_2) - J'_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) \right) \\
&\quad + \mu_3 J'_{n\beta}(k_3\alpha_2) \left(J_{-n\beta}(k_2\alpha_2) J'_{n\beta}(k_2\alpha_1) - J_{n\beta}(k_2\alpha_2) J'_{-n\beta}(k_2\alpha_1) \right),
\end{aligned}$$

while $w_1(n)$ and $w_2(n)$ take the following expressions:

$$w_1(n) = \frac{1}{\pi\chi_1} \int_{-\pi}^{\pi} f_3(\theta) \cos(n\theta) d\theta, \quad w_2(n) = \frac{1}{\pi\chi_2} \int_{-\pi}^{\pi} f_3(\theta) \sin(n\theta) d\theta,$$

with χ_1 and χ_2 in the above integrals expressed as follows:

$$\begin{aligned}\chi_1 &= \mu_1 X_5 J'_{n\beta}(k_1 \alpha_1) + \mu_2 Z_5 J_{n\beta}(k_1 \alpha_1), \\ \chi_2 &= \mu_1 X_5 J'_{-n\beta}(k_1 \alpha_1) + \mu_2 Z_5 J_{-n\beta}(k_1 \alpha_1),\end{aligned}$$

along with X_5 , and Z_5 in the latter expression found as follows:

$$\begin{aligned}X_5 &= \mu_2 (J_{-n\beta}(k_3 \alpha_3) J_{n\beta}(k_3 \alpha_2) - J_{-n\beta}(k_3 \alpha_2) J_{n\beta}(k_3 \alpha_3)) \times \\ &\quad (J_{n\beta}(k_2 \alpha_1) J'_{-n\beta}(k_2 \alpha_2) - J_{-n\beta}(k_2 \alpha_1) J'_{n\beta}(k_2 \alpha_2)) \\ &\quad + \mu_3 (J_{-n\beta}(k_2 \alpha_2) J_{n\beta}(k_2 \alpha_1) - J_{-n\beta}(k_2 \alpha_1) J_{n\beta}(k_2 \alpha_2)) \times \\ &\quad (J_{n\beta}(k_3 \alpha_3) J'_{-n\beta}(k_3 \alpha_2) - J_{-n\beta}(k_3 \alpha_3) J'_{n\beta}(k_3 \alpha_2)), \\ Z_5 &= \mu_3 (J_{n\beta}(k_2 \alpha_2) J'_{-n\beta}(k_2 \alpha_1) - J_{-n\beta}(k_2 \alpha_2) J'_{n\beta}(k_2 \alpha_1)) \times \\ &\quad (J_{n\beta}(k_3 \alpha_3) J'_{-n\beta}(k_3 \alpha_2) - J_{-n\beta}(k_3 \alpha_3) J'_{n\beta}(k_3 \alpha_2)) \\ &\quad - \mu_2 (J_{-n\beta}(k_3 \alpha_3) J_{n\beta}(k_3 \alpha_2) - J_{-n\beta}(k_3 \alpha_2) J_{n\beta}(k_3 \alpha_3)) \times \\ &\quad (J'_{-n\beta}(k_2 \alpha_2) J'_{n\beta}(k_2 \alpha_1) - J'_{-n\beta}(k_2 \alpha_1) J'_{n\beta}(k_2 \alpha_2)).\end{aligned}$$

References

- Ewing, W.M.; Jardetzky, W.S.; Press, F.; Beiser, A. Elastic Waves in Layered Media. *Phys. Today* **1957**, *10*, 27–28. [\[CrossRef\]](#)
- Asghar, S.; Zaman, F.; Masud, A. Dispersion of Love-type waves in a vertically inhomogeneous intermediate layer. *J. Phys. Earth* **1990**, *38*, 213–221. [\[CrossRef\]](#)
- Mubarak, A.M.; Althobaiti, S.; Nuruddeen, R.I. Heat and wave interactions in a thermoelastic coaxial solid cylinder driven by laser heating sources. *Case Stud. Therm. Eng.* **2022**, *38*, 102338. [\[CrossRef\]](#)
- Althobaiti, S.; Mubarak, A.; Nuruddeen, R.I.; Gomez-Aguilar, J.F. Wave propagation in an elastic coaxial hollow cylinder when exposed to thermal heating and external load. *Results Phys.* **2022**, *38*, 105582. [\[CrossRef\]](#)
- Nuruddeen, R.I.; Nawaz, R.; Zaigham Zia, Q.M. Asymptotic approach to anti-plane dynamic problem of asymmetric three-layered composite plate. *Math. Methods Appl. Sci.* **2021**, *44*, 10933–10947. [\[CrossRef\]](#)
- Wolfram MathWorld Helmholtz Differential Equation-Circular Cylindrical Coordinates. Available online: <https://mathworld.wolfram.com/HelmholtzDifferentialEquationCircularCylindricalCoordinates.html> (accessed on 24 September 2023).
- Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–701. [\[CrossRef\]](#)
- Kukla, S.; Siedlecka, U. Time-fractional heat conduction in a finite composite cylinder with heat source. *J. Appl. Math. Comput. Mech.* **2020**, *19*, 85–94. [\[CrossRef\]](#)
- Iyiola, O.S.; Zaman, F.D. A note on analytical solutions of nonlinear fractional 2-D heat equation with non-local integral terms. *Pramana* **2016**, *87*, 1–6. [\[CrossRef\]](#)
- Anwar, A.M.O.; Jarad, F.; Baleanu, D.; Ayaz, F. Fractional Caputo heat equation within the double Laplace transform. *Rom. J. Phys.* **2013**, *58*, 15–22.
- Haberman, R. *Elementary Applied Differential Equations*; A Paramaout Communication Company: Englewood Cliffs, NJ, USA, 1987.
- Hammad, M.A.; AlSharif, S.; Khalil, R.; Shmasneh, A. Fractional Bessel differential equation and fractional Bessel functions. *Ital. J. Pure Appl. Math.-N* **2022**, *47*, 521–531.
- Dai, H.-H.; Kaplunov, J.; Prikachikov, D.A. Long-wave model for the surface wave in a coated half-space. *Proc. R. Soc. A Math. Phys. Eng. Sci.* **2010**, *466*, 3097–3116. [\[CrossRef\]](#)
- Kaplunov, J.; Prikachikova, L.; Alkinidri, M. Antiplane shear of an asymmetric sandwich plate. *Contin. Mech. Thermodyn.* **2021**, *33*, 1247–1262. [\[CrossRef\]](#)
- Mubarak, A.; Prikachikov, D.; Kudaibergenov, A. Explicit model for surface waves on an elastic half-space coated by a thin vertically inhomogeneous layer. *Dyn. Syst. Theo. Appl.* **2019**, 267–275. [77306-9_23](#). [\[CrossRef\]](#)
- Horgan, C.O. Anti-plane shear deformations in linear and nonlinear solid mechanics. *SIAM Rev.* **1995**, *37*, 53–81. [\[CrossRef\]](#)
- Wang, Z.; Chen, M.; Xi, X.; Tian, H.; Yang, R. Multi-chimera states in a higher-order network of FitzHugh–Nagumo oscillators. *Eur. Phys. J. Spec. Top.* **2024**, *233*, 779–786. [\[CrossRef\]](#)
- Ba, Z.; Wu, M.; Liang, J. 3D dynamic responses of a multi-layered transversely isotropic saturated half-space under concentrated forces and pore pressure. *Appl. Math. Model.* **2020**, *80*, 859–878. [\[CrossRef\]](#)
- Liu, L.; Zhang, S.; Zhang, L.; Pan, G.; Yu, J. Multi-UUV maneuvering counter-game for dynamic target scenario based on fractional-order recurrent neural network. *IEEE Trans. Cybern.* **2023**, *53*, 4015–4028. [\[CrossRef\]](#) [\[PubMed\]](#)

20. Kai, Y.; Chen, S.; Zhang, K.; Yin, Z. Exact solutions and dynamic properties of a nonlinear fourth-order time-fractional partial differential equation. *Waves Random Complex Media* **2022**, 1–12. [\[CrossRef\]](#)
21. Wang, C.; Wen, P.; Huang, Y.; Chen, K.; Xu, K. Terahertz dual-band bandpass filter based on spoof surface plasmon polaritons with wide upper stopband suppression. *Opt. Express* **2024**, 32, 22748–22758. [\[CrossRef\]](#)
22. Mubarak, A.M.; Nuruddeen, R.I.; Gomez-Aguilar, J.F. Closed-form asymptotic solution for the transport of chlorine concentration in composite pipes. *Phys. Scr.* **2024**, 99, 075201. [\[CrossRef\]](#)
23. Sahin, O.; Erbas, B.; Kaplunov, J.; Savsek, T. The lowest vibration modes of an elastic beam composed of alternating stiff and soft components. *Arch. Appl. Mech.* **2020**, 90, 339–352. [\[CrossRef\]](#)
24. Abd-Alla, A.M.; Abo-Dahab, S.M.; Khan, A. Rotational effects on magneto-thermoelastic Stoneley, Love, and Rayleigh waves in fibre-reinforced anisotropic general viscoelastic media of higher order. *Comput. Mater. Contin.* **2017**, 53, 49–72. [\[CrossRef\]](#)
25. Ebrahimi, F.; Seyfi, A. A wave propagation study for porous metal foam beams resting on an elastic foundation. *Waves Random Complex Media* **2021**, 31, 1–15. [\[CrossRef\]](#)
26. Liu, Z.; Qi, H. Dynamic anti-plane behavior of rare earth giant magnetostrictive medium with a circular cavity defect in semi-space. *Sci. Rep.* **2021**, 11, 13442. [\[CrossRef\]](#) [\[PubMed\]](#)
27. Mandi, A.; Kundu, S.; Chandra, P.; Pati, P. An analytic study on the dispersion of Love wave propagation in double layers lying over inhomogeneous half-space. *J. Solid Mech.* **2019**, 11, 570–580. [\[CrossRef\]](#)
28. Zenkour, A.M.; Radwan, A.F. Analysis of multilayered composite plates resting on elastic foundations in thermal environment using a hyperbolic model. *J. Braz. Soc. Mech. Sci.* **2017**, 39, 2801–2816. [\[CrossRef\]](#)
29. Afsar, H.; Peiwei, G.; Alshamrani, A.; Aldandani, M.; Alam, M.M.; Aljohani, A.F. Dimensionless dynamics: Multi-peak and envelope solitons in perturbed nonlinear Schrödinger equation with Kerr law nonlinearity. *AIP Phys. Fluids* **2024**, 36, 067126. [\[CrossRef\]](#)
30. Erbas, B.; Kaplunov, J.; Nobili, A.; Kilic, G. Dispersion of elastic waves in a layer interacting with a Winkler foundation. *J. Acoust. Soc. Am.* **2018**, 144, 2918–2925. [\[CrossRef\]](#) [\[PubMed\]](#)
31. Ahmad, F.; Zaman, F.D. Exact and asymptotic solutions of the elastic wave propagation problem in a rod. *Int. J. Pure Appl. Math.* **2006**, 27, 123–127.
32. Afsar, H.; Akbar, Y. Energy distribution of bifurcated waveguide structure with planar and non-planar surfaces. *Math. Mech. Solids* **2022**, 28, 1155–1169. [\[CrossRef\]](#)
33. Kaplunov, J.D.; Kossovich, L.Y.; Nolde, E.V. *Dynamics of Thin Walled Elastic Bodies*; Academic Press: San Diego, CA, USA, 1998.

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