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Measure Attractors of Stochastic Fractional Lattice Systems

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Abstract: This paper seeks to establish the measure attractors in stochastic fractional lattice systems. First, the presence of these attractor measures is proven by the uniform estimates of the solution. Subsequently, the study also looks at the upper semicontinuous dependence of the measure attractors on the noise intensity as the latter goes to zero. The given asymptotic compactness for the family of probability measures occurring with the solution probability distributions is exhibited by a uniform prior estimation of the far-field solution values.

Keywords: measure attractor; fractional order; lattice systems; asymptotic compactness

1. Introduction

In this paper, the existence of measure attractors in fractional stochastic lattice systems is discussed, which are defined on the mathematical set $\mathbb{Z}$ of integers as follows:

$$dw_m(s) + \nu(-\Delta)^p w_m(s)ds + \lambda w_m(s)ds = (f_m(w_m(s)) + g_m)ds + \varepsilon \sum_{k=1}^{\infty} (h_{m,k} + \sigma_{m,k}(w_m(s)))dW_k(s), \quad s > 0, \quad (1)$$

with the initial data

$$w_m(0) = \zeta_m, \quad (2)$$

where $w = (w_m)_{m \in \mathbb{Z}}$ is an unknown sequence and $\zeta = (\zeta_m)_{m \in \mathbb{Z}}$ is a known sequence belonging to l^2 . Additionally, the following conditions must be satisfied: for the emergence of a new $0 < \varepsilon \leq 1$, it is found that $\nu > 0$, $\lambda > 0$, $g = (g_m)_{m \in \mathbb{Z}}$, and $h = (h_{m,k})_{m \in \mathbb{Z}, k \in \mathbb{N}}$ are given in l^2 ; $\sigma_{m,k}, f_m : \mathbb{R} \rightarrow \mathbb{R}$ are nonlinear functions for every $m \in \mathbb{Z}, k \in \mathbb{N}$; $(W_k)_{k \in \mathbb{N}}$ is a sequence of standard two-sided Wiener processes on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s \in \mathbb{R}}, P)$ [1].

For lattice systems, numerous authors have described the lattice system as a mathematical model in physics, biology, and engineering. For instance, in physics, the dynamics of lattice systems have been employed in analyzing the stability of some traveling wave solutions [2,3]; in biology, the dynamics of lattice systems are often used to analyze the transmission effect of cellular excitation [4]; similarly, deterministic lattice systems are often used in stability theory in engineering [5–7]. Furthermore, stochastic lattice dynamical systems have been related to neural networks by many authors, and it has been applied to real-life problems [8–10]. In particular, one employs the dynamics of stochastic lattice systems for investigating correlations in neural networks, referring to the random increase in the course of a long period.

Firstly, Wang mentioned the existence of attractors for determining the lattice system in [11] and then proposed the attractors for stochastic lattice systems in [12,13]; Wang proved the existence of random attractors in delayed lattice systems in [14,15]. On the basis of these articles, many authors have made attractor problems such as delay systems and applied them to practical physical models [16,17].



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The concept of measure attractors seems to have been initially introduced in [18]. The existence results of measure attractors were studied in [19,20] concerning equations related to the nonlinear noise. The interconnection of measure attractors and random attractors was investigated in [21,22] for the equation with additive noise. There are several research articles that have analyzed the phenomenon of stochastic attractors in non-autonomous fractional random lattice systems, and Chen and Wang [23] have done this in detail and have proven it under certain conditions on linear multiplicative noise. Reviewing the research team's previous studies, the benefits of such systems were described as being highly valuable regarding efficient supply chain management. Furthermore, on invariant measures for stochastic delay fractional lattice dynamical systems, Li and Wang have performed some work, and it has also been validated in the existing literature [24]. One of the comprehensive objectives of the field is the attractors problem of stochastic lattice systems, which has attracted much attention in this field's literature. Other specific measures of attractors of stochastic lattice systems have been described in [25–27].

Our first goal is to prove the existence and uniqueness of solutions (1) and (2). Subsequently, with the help of the moments and tails' estimate obtained from the solutions of (1) and (2), one can prove the existence and, respectively, uniqueness of the measure attractors. Finally, we also prove that the measure attractor of the systems (1) and (2) is upper semicontinuous in the parameter ε as $\varepsilon \rightarrow 0$.

2. The Theory of Measure Attractors

In the following discussion, we designate M as a separable Banach space equipped with the norm denoted by $\|\cdot\|_M$. We define $C_b(M)$ as the collection of bounded continuous functions $\varphi : M \rightarrow \mathbb{R}$. This follows the right norm, which if put in place:

$$\|\varphi\|_\infty = \sup_{x \in M} |\varphi(x)|,$$

and let $L_b(M)$ denote the linear space of bounded Lipschitz functions on M . This is the set of all the functions $\varphi \in C_b(M)$ satisfying the condition that

$$\text{Lip}(\varphi) := \sup_{w_1, w_2 \in M, w_1 \neq w_2} \frac{|\varphi(w_1) - \varphi(w_2)|}{\|w_1 - w_2\|_M} < \infty.$$

The space $L_b(M)$ is endowed with the norm

$$\|\varphi\|_L = \|\varphi\|_\infty + \text{Lip}(\varphi).$$

Let $\mathcal{P}(M)$ represent the collection of probability measures on the space $(M, \mathcal{B}(M))$, where $\mathcal{B}(M)$ denotes the Borel σ -algebra of M . Define a metric on $\mathcal{P}(M)$ in the following manner:

$$d_{\mathcal{P}(M)}^*(w_1, w_2) = \sup_{\substack{\varphi \in L_b(M) \\ \|\varphi\|_L \leq 1}} |(\varphi, w_1) - (\varphi, w_2)|, \quad w_1, w_2 \in \mathcal{P}(M),$$

where $(w, \varphi) = \int_M \varphi(x) w(dx)$ for $\varphi \in C_b(M)$ and $w \in \mathcal{P}(M)$. As we all know, the metric space $(\mathcal{P}(M), d_{\mathcal{P}(M)}^*)$ is completed and separable. Furthermore, a sequence $\{w_n\}_{n=1}^\infty \subset \mathcal{P}(M)$ converges to a measure w in $(\mathcal{P}(M), d_{\mathcal{P}(M)}^*)$ if and only if, for any $\varphi \in L_b(M)$,

$$(\varphi, w_n) \rightarrow (\varphi, w), \text{ as } n \rightarrow \infty.$$

For $p > 0$, $\mathcal{P}_p(M)$ is denoted as a subset of $\mathcal{P}(M)$ that satisfies the following conditions:

$$\mathcal{P}_p(M) = \left\{ w \in \mathcal{P}(M) \mid \int_M \|x\|_M^p w(dx) < +\infty \right\}.$$

It is evident that $\mathcal{P}_p(M)$ constitutes a closed subspace of $\mathcal{P}(M)$. Then, $(\mathcal{P}_p(M), d_{\mathcal{P}(M)}^*)$ is also a completed and separable metric space.

Suppose $D(s)$ represents a continuous autonomous dynamical system on $\mathcal{P}_p(M)$, and for $s \geq 0$, $D(s) : \mathcal{P}_p(M) \rightarrow \mathcal{P}_p(M)$ satisfies the following conditions:

- (a) $D(0) = I_{\mathcal{P}_p(M)}$, where $I_{\mathcal{P}_p(M)}$ is the identity operating on $\mathcal{P}_p(M)$;
- (b) $D(s + v) = D(s)D(v)$, for every $s, v \geq 0$;
- (c) $D(s) : \mathcal{P}_p(M) \rightarrow \mathcal{P}_p(M)$ is continuous, for every $s \geq 0$.

Denote, for $r > 0$,

$$B_M(r) = \left\{ w \in \mathcal{P}_p(M) \mid \int_M \|x\|_M^p w(dx) \leq r \right\}.$$

Definition 1. The set $B_M(r_1)$, for $r_1 > 0$, is called an absorbing set for D if, for every $r_0 > 0$, there exists $s_{r_0} > 0$ such that

$$D(s)B_M(r_0) \subseteq B_M(r_1), \quad \text{for all } s \geq s_{r_0}.$$

Definition 2. D is said to be asymptotically compact in $\mathcal{P}_p(M)$ if $\{D(s_n)w_n\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{P}_p(M)$ whenever $s_n \rightarrow \infty$ and $w_n \in B_M(r)$, for any $r > 0$.

Definition 3. A set $\star$ of $\mathcal{P}_p(M)$ is referred to as a measure attractor for D when the following conditions are met:

- (i) $\star$ is compact in $\mathcal{P}_p(M)$;
- (ii) $\star$ is invariant, that is $D(s)\star = \star$, for every $s \geq 0$;
- (iii) $\star$ attracts every set $B_M(r)$, for every $r > 0$, that is

$$\lim_{s \rightarrow \infty} d_{\mathcal{P}_p(M)}(D(s)B_M(r), \star) = 0,$$

The Hausdorff semi-metric denoted as $d_{\mathcal{P}_p(M)}$ is given by the formula: $d_{\mathcal{P}_p(M)}(A, C) = \sup_{a \in A} \inf_{c \in C} d_{\mathcal{P}(M)}^*(a, c)$ for any $A \subseteq \mathcal{P}_p(M)$ and $C \subseteq \mathcal{P}_p(M)$.

Definition 4. A mapping $\psi : \mathbb{R} \rightarrow \mathcal{P}_p(M)$ is referred to as a complete orbit of D , if for every $v \in \mathbb{R}$ and any $s \geq 0$, the following condition is met:

$$D(s)\psi(v) = \psi(s + v).$$

If there exists a $r > 0$ such that $\psi(s) \in B_M(r)$, for $s \in \mathbb{R}$, then $\psi : \mathbb{R} \rightarrow \mathcal{P}_p(M)$ is called a bounded complete orbit.

From [25] in Lemma 2.8, we have the following. This abstract theorem can be stated regarding the existence and uniqueness of measure attractors and also regards their structure.

Theorem 1. The measure attractor $\star$ in $\mathcal{P}_p(M)$ is unique for D , if and only if D has an absorbing set $B_M(r_1)$, with $r_1 > 0$ and D being asymptotically compact in $\mathcal{P}_p(M)$. The unique measure attractor $\star$ is given as follows:

$$\star = \Xi(B_M(r_1)) = \bigcap_{\tau \geq 0} \overline{\bigcup_{s \geq \tau} D(s)B_M(r_1)} = \{\psi(0) : \psi \text{ is a bounded completed orbit of } D\}. \quad (3)$$

3. Properties of Solutions

The following facts and results related to the fractional discrete Laplace operator $(-\Delta)^p$ are essentially needed. We then rewrite the stochastic systems (1) and (2) as an evolution equation in l^2 space. After that, we will show the existence, as well as the uniqueness of the solutions for the circumstances such as follows:

For $1 \leq q < \infty$, denote

$$l^q = \left\{ w = \{w_m\}_{m \in \mathbb{Z}} \mid w_m \in \mathbb{R}, \sum_{m \in \mathbb{Z}} |w_m|^q < \infty \right\}.$$

Let l^2 denote a Hilbert space, equipped with an inner product and a norm defined as follows:

$$(w, h) := \sum_{m \in \mathbb{Z}} w_m h_m, \quad \|w\|^2 := \sum_{m \in \mathbb{Z}} w_m^2, \quad w = (w_m)_{m \in \mathbb{Z}}, \quad h = (h_m)_{m \in \mathbb{Z}} \in l^2.$$

For $0 < p < 1$, define l_p by

$$l_p = \left\{ w = \{w_m\}_{m \in \mathbb{Z}} \mid w_m \in \mathbb{R}, \sum_{m \in \mathbb{Z}} \frac{|w_m|}{(1 + |m|)^{1+2p}} < \infty \right\}.$$

Obviously, $l^{q_1} \subset l^{q_2} \subset l_p$ if $1 \leq q_1 < q_2 < \infty$ and $0 < p < 1$.

The function $f_m : \mathbb{R} \rightarrow \mathbb{R}$ is globally Lipschitz continuous for all $s \in \mathbb{R}$ and uniformly so for every $m \in \mathbb{Z}$. Specifically, there exists a constant $L_f > 0$ such that for all $v_1, v_2 \in \mathbb{R}$:

$$|f_m(v_1) - f_m(v_2)| \leq L_f |v_1 - v_2|. \quad (4)$$

Moreover, the growth of $f_m(v)$ is linear in $v \in \mathbb{R}$; specifically for each $m \in \mathbb{Z}$, there exist $\alpha_m, \beta_0 > 0$, ensuring that this holds true for all $v \in \mathbb{R}$:

$$|f_m(v)| \leq \alpha_m + \beta_0 |v|, \quad (5)$$

where $\alpha = (\alpha_m)_{m \in \mathbb{Z}} \in l^2$.

In this paper, we assume that the sequences $g = (g_m)_{m \in \mathbb{Z}}$ and $h = (h_{m,k})_{m \in \mathbb{Z}, k \in \mathbb{N}}$ belong to l^2 :

$$\|g\|^2 = \sum_{m \in \mathbb{Z}} |g_m|^2 < \infty \quad \text{and} \quad \|h\|^2 = \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{N}} |h_{m,k}|^2 < \infty. \quad (6)$$

Suppose $\sigma_{m,k} : \mathbb{R} \rightarrow \mathbb{R}$ is globally Lipschitz in $v \in \mathbb{R}$ uniformly with respect to $m \in \mathbb{Z}$; more precisely, for every $k \in \mathbb{N}$, $m \in \mathbb{Z}$, there exists a constant $L_k > 0$ such that for all $v_1, v_2 \in \mathbb{R}$:

$$|\sigma_{m,k}(v_1) - \sigma_{m,k}(v_2)| \leq L_k |v_1 - v_2|, \quad (7)$$

where $L = (L_k)_{k \in \mathbb{N}} \in \ell^2$. In addition, we assume $\sigma_{m,k}(v)$ satisfies linear growth in $v \in \mathbb{R}$; that is, for all $k \in \mathbb{N}$ and $m \in \mathbb{Z}$, there exist $\delta_{m,k}, \beta_k > 0$ such that, for all $v \in \mathbb{R}$,

$$|\sigma_{m,k}(v)| \leq \delta_{m,k} + \beta_k |v|. \quad (8)$$

To simplify the notation, we use

$$\alpha = (\alpha_m)_{m \in \mathbb{Z}}, \quad L = (L_k)_{k \in \mathbb{N}}, \quad \beta = (\beta_k)_{k \in \mathbb{N}}, \quad \delta = (\delta_{m,k})_{m \in \mathbb{Z}, k \in \mathbb{N}},$$

and

$$\|\alpha\|^2 = \sum_{m \in \mathbb{Z}} |\alpha_m|^2, \quad \|L\|^2 = \sum_{k \in \mathbb{N}} |L_k|^2, \quad \|\beta\|^2 = \sum_{k \in \mathbb{N}} |\beta_k|^2, \quad \|\delta\|^2 = \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{N}} |\delta_{m,k}|^2.$$

In addition, for $s = (s_m)_{m \in \mathbb{Z}} \in l^2$, we write $f(s) = (f_m(s_m))_{m \in \mathbb{Z}}$ and $\sigma_k(s) = (\sigma_{m,k}(s_m))_{m \in \mathbb{Z}}$. It follows from (4) and (5) that, for all $w_1, w_2 \in l^2$,

$$\|f(w_1) - f(w_2)\|^2 \leq L_f^2 \|w_1 - w_2\|^2, \quad (9)$$

and

$$\|f(w_1)\|^2 \leq 2\|\alpha\|^2 + 2\beta_0^2\|w_1\|^2. \quad (10)$$

Similarly, by (7) and (8), we have, for all $w_1, w_2 \in l^2$,

$$\sum_{k \in \mathbb{N}} \|\sigma_k(w_1) - \sigma_k(w_2)\|^2 \leq \|L\|^2 \|w_1 - w_2\|^2 \quad (11)$$

and

$$\sum_{k \in \mathbb{N}} \|\sigma_k(w_1)\|^2 \leq 2\|\delta\|^2 + 2\|\beta\|^2 \|w_1\|^2. \quad (12)$$

For $0 < p < 1$ and $u_j \in \mathbb{R}$, the fractional discrete Laplacian $(-\Delta)^p$ is defined with the semigroup method as

$$(-\Delta)^p w_m = \frac{1}{\Gamma(-p)} \int_0^\infty (e^{s\Delta} w_m - w_m) \frac{1}{s^{1+p}} ds, \quad (13)$$

where Γ denotes the Gamma function with $\Gamma(-p) = \int_0^\infty (e^{-l} - 1) \frac{1}{l^{1+p}} dl < 0$ and $v_m(s) = e^{s\Delta} w_m$ is the solution to the semidiscrete heat equation:

$$\begin{cases} \partial_s v_m = \Delta v_m, & \text{in } \mathbb{Z} \times (0, \infty), \\ v_j(0) = w_m, & \text{on } \mathbb{Z}, \end{cases} \quad (14)$$

where $\Delta w_m = -2w_m + w_{m-1} + w_{m+1}$. Based on the semidiscrete Fourier transform, it can be inferred that the solution to Equation (14) can be expressed as follows:

$$e^{s\Delta} w_m = \sum_{i \in \mathbb{Z}} G(m-i, s) w_i = \sum_{i \in \mathbb{Z}} G(i, s) w_{m-i}, \quad s \geq 0, \quad (15)$$

where the semidiscrete heat kernel $G(i, s) = e^{-2s} I_i(2s)$; in addition, I_i is the modified Bessel function of order i .

By (13) with (15), we have the pointwise formula for $(-\Delta)^p$ presented in the following statement.

Lemma 1 ([28]). *Let $0 < p < 1$ and $w = (w_m)_{m \in \mathbb{Z}} \in l_p$. Then, we have*

$$(-\Delta)^p w_m = \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} (w_m - w_i) \tilde{K}_p(m-i),$$

where the specific kernel $\tilde{K}_p$ is given by

$$\tilde{K}_p(i) = \begin{cases} \frac{4^p \Gamma(\frac{1}{2}+p)}{\sqrt{\pi} |\Gamma(-p)|} \cdot \frac{\Gamma(|i|-p)}{\Gamma(|i|+1+p)}, & i \in \mathbb{Z} \setminus \{0\}, \\ 0, & i = 0. \end{cases}$$

Furthermore, there exist positive constants $c_1^p \leq c_2^p$ such that, for any $i \in \mathbb{Z} \setminus \{0\}$,

$$\frac{c_1^p}{|i|^{1+2p}} \leq \tilde{K}_p(i) \leq \frac{c_2^p}{|i|^{1+2p}}.$$

Lemma 1 demonstrates that the fractional discrete Laplacian is a nonlocal operator on $\mathbb{Z}$ with an order of $2p$. Furthermore, it follows from Lemma 1 that $(-\Delta)^p w = ((-\Delta)^p w_m)_{m \in \mathbb{Z}}$ is a well-defined bounded function whenever $w \in l^q$ ($1 \leq q < \infty$). In particular, we also find that, for $0 < p < 1$, if $w \in l^2$, then $(-\Delta)^p w \in l^2$ and $\|(-\Delta)^p w\| \leq 4^p \|w\|$.

We also we have the following result with respect to $(-\Delta)^p$.

Lemma 2 ([23]). Let $w, h \in l^2$. Then for every $p \in (0, 1)$,

$$\begin{aligned} ((-\Delta)^p w, h) &= ((-\Delta)^{\frac{p}{2}} w, (-\Delta)^{\frac{p}{2}} h) \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} (w_m - w_i)(h_m - h_i) \tilde{K}_p(m - i). \end{aligned} \quad (16)$$

Systems (1) and (2) can be rewritten as the following stochastic equation in l^2 for $s > 0$:

$$dw(s) + \nu(-\Delta)^p w(s)ds + \lambda w(s)ds = (f(w(s)) + g)ds + \varepsilon \sum_{k=1}^{\infty} (h_k + \sigma_k(w(s)))dW_k(s), \quad (17)$$

with the initial data

$$w(0) = \zeta. \quad (18)$$

Let $L^2_{\mathcal{F}_0}(\Omega, l^2)$ denote the space of all $\mathcal{F}_0$ -measurable l^2 -valued random variables φ with $\mathbb{E}(\|\varphi\|^2) < \infty$, where $\mathbb{E}$ means the mathematical expectation.

Definition 5. Suppose $\zeta \in L^2_{\mathcal{F}_0}(\Omega, l^2)$. Then, a continuous l^2 -valued stochastic process $w(s)$ with $s \in [0, +\infty)$ is called a solution of systems (17) and (18), if $(w(s))_{s \geq 0}$ is $\mathcal{F}_s$ -adapted, $w(0) = \zeta$, $w \in L^2(\Omega, C([0, S], l^2))$ for all $S > 0$, and for each $s \geq 0$,

$$\begin{aligned} w(s) + \nu \int_0^s (-\Delta)^p w(\tau) d\tau + \lambda \int_0^s w(\tau) d\tau &= \zeta + \int_0^s (f(w(\tau)) + g) d\tau \\ &+ \varepsilon \sum_{k=1}^{\infty} \int_0^s (h_k + \sigma_k(w(\tau))) dW_k(\tau), \end{aligned} \quad (19)$$

in l^2 for almost all $\omega \in \Omega$.

Similar to [16], under conditions (4)–(8), we can obtain that, for any $\zeta \in L^2_{\mathcal{F}_0}(\Omega, l^2)$, systems (17) and (18) have a unique solution, which is written as $w(s)$. To highlight the initial values, we denote by $w(s, \zeta)$ the solution of (17) and (18) with initial conditions $w(0) = \zeta \in l^2$.

4. Uniform Estimates

This section is aimed at deriving uniform estimates for the solutions of the problems (17) and (18) as required in the proof of measure attractors. After this, we can postulate the following:

$$\lambda > \frac{4\sqrt{2}}{3}\beta_0 + \frac{8}{3}\|\beta\|^2. \quad (20)$$

Firstly, we give a Lemma 3 on the mean-squared bound.

Lemma 3. Suppose (4)–(8) and (20) hold. Then, for every $R > 0$, $\zeta \in L^2_{\mathcal{F}_0}(\Omega, l^2)$ and $0 < \varepsilon \leq 1$, there exists $S = S(R) \geq 0$; additionally, $\gamma > 0$ such that, for all $s \geq S$, the solution w of systems (17) and (18) satisfies

$$\mathbb{E}(\|w(s)\|^2) \leq M, \quad (21)$$

where $\mathbb{E}(\|\zeta\|^2) \leq R$ and M is a positive constant independent of ε , ζ , and R .

Proof. Using Ito's formula and by (19), we obtain the result for $s \geq 0$:

$$\begin{aligned} & \|w(s)\|^2 + 2\nu \int_0^s \left\| (-\Delta)^{\frac{p}{2}} w(\tau) \right\|^2 d\tau + 2\lambda \int_0^s \|w(\tau)\|^2 d\tau \\ &= \|\zeta\|^2 + 2 \int_0^s (w(\tau), f(w(\tau))) d\tau + 2 \int_0^s (w(\tau), g) d\tau + \varepsilon^2 \sum_{k=1}^{\infty} \int_0^s \|h_k + \sigma_k(w(\tau))\|^2 d\tau \\ &+ 2\varepsilon \sum_{k=1}^{\infty} \int_0^s (w(\tau), h_k + \sigma_k(w(\tau))) dW_k(\tau), \end{aligned} \quad (22)$$

which implies that, for $s \geq 0$ and $\Delta s \geq 0$,

$$\begin{aligned} & \mathbb{E}(\|w(s + \Delta s)\|^2) - \mathbb{E}(\|w(s)\|^2) \\ &= -2\nu \int_s^{s+\Delta s} \mathbb{E} \left(\left\| (-\Delta)^{\frac{p}{2}} w(\tau) \right\|^2 \right) d\tau - 2\lambda \int_s^{s+\Delta s} \mathbb{E}(\|w(\tau)\|^2) d\tau \\ &+ 2 \int_s^{s+\Delta s} \mathbb{E}(w(\tau), f(w(\tau)) + g) d\tau + \varepsilon^2 \sum_{k=1}^{\infty} \int_s^{s+\Delta s} \mathbb{E}(\|h_k + \sigma_k(w(\tau))\|^2) d\tau. \end{aligned} \quad (23)$$

By utilizing (10) and Young's inequality, we obtain the following result for the third term on the right-hand side of (23):

$$\begin{aligned} & 2 \int_s^{s+\Delta s} \mathbb{E}(w(\tau), f(w(\tau)) + g) d\tau \\ & \leq (2\sqrt{2}\beta_0 + \frac{\lambda}{2}) \int_s^{s+\Delta s} \mathbb{E}(\|w(\tau)\|^2) d\tau \\ & + \frac{\sqrt{2}}{\beta_0} \Delta s \|\alpha\|^2 + \frac{2}{\lambda} \Delta s \|g\|^2. \end{aligned} \quad (24)$$

We can obtain the last term on the right-hand side of Equation (23) using (12):

$$\begin{aligned} & \varepsilon^2 \sum_{k=1}^{\infty} \int_s^{s+\Delta s} \mathbb{E}(\|h_k + \sigma_k(w(\tau))\|^2) d\tau \\ & \leq \sum_{k=1}^{\infty} \int_s^{s+\Delta s} \mathbb{E}(\|h_k + \sigma_k(w(\tau))\|^2) d\tau \\ & \leq 4\Delta s \sum_{m \in \mathbb{Z}} \sum_{k=1}^{\infty} (h_{m,k}^2 + \delta_{m,k}^2) + 4\|\beta\|^2 \int_s^{s+\Delta s} \mathbb{E}(\|w(\tau)\|^2) d\tau. \end{aligned} \quad (25)$$

From (23)–(25), we can obtain that, for $s > 0$,

$$\begin{aligned} D^+ \mathbb{E}(\|w(s)\|^2) & \leq - \left(\frac{3}{2} \lambda - 2\sqrt{2}\beta_0 - 4\|\beta\|^2 \right) \mathbb{E}(\|w(s)\|^2) \\ & - 2\nu \mathbb{E} \left(\left\| (-\Delta)^{\frac{p}{2}} w(s) \right\|^2 \right) \\ & + \frac{\sqrt{2}}{\beta_0} \|\alpha\|^2 + \frac{2}{\lambda} \|g\|^2 + 4 \left(\sum_{k=1}^{\infty} \|h_k\|^2 + \|\delta\|^2 \right). \end{aligned} \quad (26)$$

Obviously, (20) indicates the existence of a sufficiently small $\gamma > 0$ such that

$$\lambda > \frac{4\sqrt{2}}{3} \beta_0 + \frac{8}{3} \|\beta\|^2 + \frac{4}{3} \gamma. \quad (27)$$

By (26), (27), and the Gronwall inequality, for $s \geq 0$,

$$\begin{aligned} \mathbb{E}(\|w(s)\|^2) &\leq \left[\mathbb{E}(\|\zeta\|^2)\right] e^{-\gamma s} - \mu \int_0^s e^{-\gamma(s-\tau)} \mathbb{E}\left(\|w(\tau)\|^2 + \left\|(-\Delta)^{\frac{p}{2}} w(\tau)\right\|^2\right) d\tau \\ &\quad + \frac{1}{\gamma} \left(\frac{\sqrt{2}}{\beta_0} \|\alpha\|^2 + \frac{2}{\lambda} \|g\|^2 + 4\left(\sum_{k=1}^{\infty} \|h_k\|^2 + \|\delta\|^2\right)\right) \\ &\leq \left[\mathbb{E}(\|\zeta\|^2)\right] e^{-\gamma s} + \frac{1}{\gamma} \left(\frac{\sqrt{2}}{\beta_0} \|\alpha\|^2 + \frac{2}{\lambda} \|g\|^2 + 4\left(\sum_{k=1}^{\infty} \|h_k\|^2 + \|\delta\|^2\right)\right) \\ &\leq M. \end{aligned} \quad (28)$$

where $\mu = \min\{2\gamma, 2\nu\}$. This concludes the proof. $\square$

The lemma below provides the uniform estimates for the tails of solutions to the problem (17) and (18), the proof of the asymptotic compactness of the probability distribution of the solutions. Before that, let us a smooth function χ on $\mathbb{R}$ such that $0 \leq \chi(s) \leq 1$ for all $s \in \mathbb{R}$, $\chi(s) = 0$ for $|s| \leq 1$, and $\chi(s) = 1$ for $|s| \geq 2$. We remind that χ' is bounded on $\mathbb{R}$, that is there exists such a constant c_0 such that $|\chi'(s)| \leq c_0$ for all $s \in \mathbb{R}$. Also, the function χ has the following:

Lemma 4 ([23]). *Let χ be the smooth function defined above and $0 < p < 1$. Then, for every $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have*

$$\sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} \left| \chi\left(\frac{|m|}{n}\right) - \chi\left(\frac{|i|}{n}\right) \right|^2 \tilde{K}_p(m-i) \leq \frac{L_p^2}{n^{2p}},$$

where L_p is a positive constant that is independent of m and n .

Following this, we give the uniform estimates on the tails of the solutions of (17) and (18).

Lemma 5. *Suppose (4)–(8) and (20) hold. Then, for every $R > 0$, $0 < \varepsilon \leq 1$ and $\epsilon > 0$, there exist $S = S(R, \epsilon) > 0$ with $N = N(R, \epsilon) \in \mathbb{N}$ such that, for all $s \geq S$ and $n \geq N$, the solution w of systems (17) and (18) satisfies*

$$\sum_{|m| \geq n} \mathbb{E}(|w_m(s, \zeta)|^2) \leq \epsilon.$$

where $\mathbb{E}(\|\zeta\|^2) \leq R$, $\zeta \in L_{\mathcal{F}_0}^2(\Omega, l^2)$.

Proof. Given $n \in \mathbb{N}$, denote by $\chi_n = \left(\chi\left(\frac{|m|}{n}\right)\right)_{m \in \mathbb{Z}}$ and $\chi_n w = \left(\chi\left(\frac{|m|}{n}\right) w_m\right)_{m \in \mathbb{Z}}$ for $w = (w_m)_{m \in \mathbb{Z}}$. By (19), we obtain

$$\begin{aligned} d(\chi_n w(s)) + \nu \chi_n (-\Delta)^p w(s) ds + \lambda \chi_n w(s) ds &= (\chi_n f(w(s))) + \chi_n g ds \\ &\quad + \varepsilon \sum_{k=1}^{\infty} (\chi_n h_k + \chi_n \sigma_k(w(s))) dW_k(s). \end{aligned} \quad (29)$$

By (29), Ito's formula, and taking the expectation, we obtain that, for all $s \geq 0$,

$$\begin{aligned} \mathbb{E}(\|\chi_n w(s)\|^2) &= \|\chi_n \zeta\|^2 - 2\nu \int_0^s \mathbb{E}((-\Delta)^p w(\tau), \chi_n^2 w(\tau)) d\tau \\ &\quad - 2\lambda \int_0^s \mathbb{E}(\|\chi_n w(\tau)\|^2) d\tau + 2 \int_0^s \mathbb{E}(\chi_n w(\tau), \chi_n f(w(\tau)) + \chi_n g) d\tau \\ &\quad + \varepsilon^2 \sum_{k=1}^{\infty} \int_0^s \mathbb{E}(\|\chi_n h_k + \chi_n \sigma_k(w(\tau))\|^2) d\tau, \end{aligned} \quad (30)$$

which implies that, for $s \geq 0$ and $\Delta s > 0$,

$$\begin{aligned}
 & \mathbb{E} \left(\|\chi_n w(s + \Delta s)\|^2 \right) - \mathbb{E} \left(\|\chi_n w(s)\|^2 \right) \\
 &= -2\nu \int_s^{s+\Delta s} \mathbb{E} \left((-\Delta)^p w(\tau), \chi_n^2 w(\tau) \right) d\tau \\
 & \quad - 2\lambda \int_s^{s+\Delta s} \mathbb{E} \left(\|\chi_n w(\tau)\|^2 \right) d\tau \\
 & \quad + 2 \int_s^{s+\Delta s} \mathbb{E} (\chi_n w(\tau), \chi_n f(w(\tau)) + \chi_n g) d\tau \\
 & \quad + \varepsilon^2 \sum_{k=1}^{\infty} \int_s^{s+\Delta s} \mathbb{E} \left(\|\chi_n h_k + \chi_n \sigma_k(w(\tau))\|^2 \right) d\tau.
 \end{aligned} \tag{31}$$

We first estimate the first term on the right-hand side of (31). We have

$$\begin{aligned}
 & -2\nu \left((-\Delta)^p w, \chi_n^2 w \right) = -2\nu \left((-\Delta)^{\frac{p}{2}} w, (-\Delta)^{\frac{p}{2}} \chi_n^2 w \right) \\
 &= -\nu \sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} (w_m - w_i) \left(\chi^2 \left(\frac{|m|}{n} \right) w_m - \chi^2 \left(\frac{|i|}{n} \right) w_i \right) \tilde{K}_p(m-i) \\
 &= -\nu \sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} \left(\chi^2 \left(\frac{|m|}{n} \right) - \chi^2 \left(\frac{|i|}{n} \right) \right) (w_m - w_i) w_m \tilde{K}_p(m-i) \\
 & \quad - \nu \sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} \chi^2 \left(\frac{|i|}{n} \right) (w_m - w_i)^2 \tilde{K}_p(m-i).
 \end{aligned} \tag{32}$$

By Lemma 4, one has

$$\begin{aligned}
 & \sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} \left| \left(\chi^2 \left(\frac{|m|}{n} \right) - \chi^2 \left(\frac{|i|}{n} \right) \right) (w_m - w_i) w_m \tilde{K}_p(m-i) \right| \\
 &= \sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} \left| \left(\chi \left(\frac{|m|}{n} \right) + \chi \left(\frac{|i|}{n} \right) \right) \left(\chi \left(\frac{|m|}{n} \right) - \chi \left(\frac{|i|}{n} \right) \right) (w_m - w_i) w_m \tilde{K}_p(m-i) \right| \\
 &\leq 2 \sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} \left| \left(\chi \left(\frac{|m|}{n} \right) - \chi \left(\frac{|i|}{n} \right) \right) (w_m - w_i) w_m \tilde{K}_p(m-i) \right| \\
 &\leq 2 \|w\| \left[\sum_{m \in \mathbb{Z}} \left(\sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} \left(\chi \left(\frac{|m|}{n} \right) - \chi \left(\frac{|i|}{n} \right) \right)^2 \tilde{K}_p(m-i) \right) \times \left(\sum_{\substack{i \in \mathbb{Z} \\ i \neq m}} (w_m - w_i)^2 \tilde{K}_p(m-i) \right) \right]^{\frac{1}{2}} \\
 &\leq 2\sqrt{2} \frac{L_p}{n^p} \|w\| \left(\sum_{m \in \mathbb{Z}} \sum_{\substack{i \in \mathbb{Z} \\ i \neq j}} (w_m - w_i)^2 \tilde{K}_p(m-i) \right)^{\frac{1}{2}} \\
 &\leq 2\sqrt{2} \frac{L_p}{n^p} \|w\| \left\| (-\Delta)^{\frac{p}{2}} w \right\| \\
 &\leq \sqrt{2} \frac{L_p}{n^p} \left(\|w\|^2 + \left\| (-\Delta)^{\frac{p}{2}} w \right\|^2 \right).
 \end{aligned} \tag{33}$$

Then, it can be inferred from (32) and (33) that

$$\begin{aligned}
 & -2\nu \int_s^{s+\Delta s} \mathbb{E} \left((-\Delta)^p w(\tau), \chi_n^2 w(\tau) \right) d\tau \\
 &\leq \sqrt{2} \frac{L_p}{n^p} \nu \int_s^{s+\Delta s} \mathbb{E} \left(\|w(\tau)\|^2 + \left\| (-\Delta)^{\frac{p}{2}} w(\tau) \right\|^2 \right) d\tau.
 \end{aligned} \tag{34}$$

For any positive value $\epsilon > 0$, there exists a value of N_1 denoted as $N_1(\epsilon) > 0$, which ensures that, for all $n > N_1$,

$$\begin{aligned} & \sqrt{2} \frac{L^p}{n^p} \nu \int_s^{s+\Delta s} \mathbb{E} \left(\|w(\tau)\|^2 + \left\| (-\Delta)^{\frac{s}{2}} w(\tau) \right\|^2 \right) d\tau \\ & \leq \epsilon \int_s^{s+\Delta s} \mathbb{E} \left(\|w(\tau)\|^2 + \left\| (-\Delta)^{\frac{p}{2}} w(\tau) \right\|^2 \right) d\tau. \end{aligned} \quad (35)$$

From the third term on the right-hand side of Equation (31), by applying Formula (10), we obtain

$$\begin{aligned} & 2 \int_s^{s+\Delta s} \mathbb{E}(\chi_n w(\tau), \chi_n(f(w(\tau) + g))) d\tau \\ & \leq \left(2\sqrt{2}\beta_0 + \frac{\lambda}{2} \right) \int_s^{s+\Delta s} \mathbb{E} \left(\|\chi_n w(\tau)\|^2 \right) d\tau + \frac{\sqrt{2}}{\beta_0} \Delta s \sum_{|m| \geq n} \alpha_m^2 + \frac{2}{\lambda} \Delta s \sum_{|m| \geq n} g_m^2. \end{aligned} \quad (36)$$

The final term on the right-hand side of Equation (31) can be obtained through reference to Formula (11), and we obtain

$$\begin{aligned} & \epsilon^2 \sum_{k=1}^{\infty} \int_s^{s+\Delta s} \mathbb{E} \left(\|\chi_n h_k + \chi_n \sigma_k(w(\tau))\|^2 \right) d\tau \\ & \leq 4\Delta s \sum_{|m| \geq n} \sum_{k=1}^{\infty} \left(h_{m,k}^2 + \delta_{m,k}^2 \right) + 4\|\beta\|^2 \int_s^{s+\Delta s} \mathbb{E} \left(\|\theta_n w(\tau)\|^2 \right) d\tau. \end{aligned} \quad (37)$$

Since $\alpha = (\alpha_m)_{m \in \mathbb{Z}}$, $g = (g_m)_{m \in \mathbb{Z}}$, $h = (h_{m,k})_{m \in \mathbb{Z}, k \in \mathbb{N}}$ and $\delta = (\delta_{m,k})_{m \in \mathbb{Z}, k \in \mathbb{N}} \in l^2$, we find that there exists $N_2 = N_2(\epsilon) \geq N_1$ such that, for all $n \geq N_2$,

$$\sum_{|m| \geq n} \alpha_m^2 + \sum_{|m| \geq n} g_m^2 + \sum_{|m| \geq n} \sum_{k=1}^{\infty} \left(h_{m,k}^2 + \delta_{m,k}^2 \right) < \epsilon. \quad (38)$$

We obtain from (34)–(38) that, for $s > 0$ and $n \geq N_2$,

$$\begin{aligned} D^+ \mathbb{E} \left(\|\chi_n w(s)\|^2 \right) & \leq - \left(\frac{3}{2} \lambda - 2\sqrt{2}\beta_0 - 4\|\beta\|^2 \right) \mathbb{E} \left(\|\chi_n w(s)\|^2 \right) \\ & \quad + \epsilon \mathbb{E} \left(\|w(s)\|^2 + \left\| (-\Delta)^{\frac{p}{2}} w(s) \right\|^2 \right) \\ & \quad + \epsilon \left(5 + \frac{\sqrt{2}}{\beta_0} + \frac{2}{\lambda} \right). \end{aligned} \quad (39)$$

By (39), (27), and the Gronwall inequality, there exists $\gamma > 0$ such that, for $s \geq 0$ and $n \geq N_2$,

$$\begin{aligned} \mathbb{E} \left(\|\chi_n w(s)\|^2 \right) & \leq \left[\mathbb{E} \left(\|\chi_n \zeta\|^2 \right) \right] e^{-\gamma s} \\ & \quad + \epsilon \int_0^s e^{-\gamma(s-\tau)} \mathbb{E} \left(\|w(\tau)\|^2 + \left\| (-\Delta)^{\frac{p}{2}} w(\tau) \right\|^2 \right) d\tau + \frac{1}{\gamma} \epsilon \left(5 + \frac{\sqrt{2}}{\beta_0} + \frac{2}{\lambda} \right). \end{aligned} \quad (40)$$

Since $\mathbb{E}(\|\zeta\|^2) \leq R$, we have $\mathbb{E}(\|\chi_n \zeta\|^2) e^{-\gamma s} \leq \mathbb{E}(\|\zeta\|^2) e^{-\gamma s} \leq R e^{-\gamma s}$; moreover, $R e^{-\gamma s} \rightarrow 0$, as $s \rightarrow \infty$. It follows from (40) and Lemma 3 that there exists $S = S(R, \epsilon) > 0$ such that, for all $s \geq S$ and $n \geq N_2$,

$$\mathbb{E} \left(\|\chi_n u(s)\|^2 \right) \leq \epsilon + \epsilon M + \frac{1}{\gamma} \epsilon \left(5 + \frac{\sqrt{2}}{\beta_0} + \frac{2}{\lambda} \right).$$

This completes the proof. $\square$

5. Main Results

This section focuses on the existence and the uniqueness of measure attractors for Equations (17) and (18) in l^2 . Before proceeding with the analysis of systems (17) and (18), it is pertinent to mention that, for any initial time $s_0 \geq 0$ and any $\mathcal{F}_{s_0}$ -measurable $\zeta \in L^2(\Omega, l^2)$, systems (17) and (18) possesses a unique solution, defined for all times s extending from s_0 to infinity, denoted as $w(s, s_0, \zeta)$ from this point onward.

As usual, if $\varphi : l^2 \rightarrow \mathbb{R}$ is a bounded Borel function, then for $0 \leq j \leq s$ and $\zeta \in l^2$, we set

$$(p_{j,s}\varphi)(\zeta) = \mathbb{E}(\varphi(w(s, j, \zeta))),$$

and

$$p(j, \zeta; s, \Gamma) = (p_{j,s}1_\Gamma)(\zeta),$$

where $\Gamma \in \mathcal{B}(l^2)$ and 1_Γ is the characteristic function of Γ . We also write the operator $p_{0,s}$ as p_s . The following properties of $\{p_{j,s}\}_{0 \leq j \leq s}$ are standard (see, e.g., [13]), and the proof is omitted.

Lemma 6. Suppose (4)–(8) and (20) hold. Then, we have the following:

- (i) The family $\{p_{j,s}\}_{0 \leq j \leq s}$ is Feller; that is, for any $0 \leq j \leq s$, the function $p_{j,s}\varphi \in C_b(l^2)$ if also φ is.
- (ii) The family $\{p_{j,s}\}_{0 \leq j \leq s}$ is homogeneous: for any $0 \leq j \leq s$,

$$p(j, \zeta; s, \cdot) = p(0, \zeta; s - j, \cdot), \quad \forall \zeta \in l^2.$$

- (iii) For every $j \geq 0$ and $\zeta \in l^2$, the process $\{w(s, j, \zeta)\}_{s \geq j}$ is an l^2 -valued Markov process.

For $s \geq 0$ and $w \in \mathcal{P}(l^2)$, define

$$p_*(s)w(\cdot) = \int_{l^2} p(0, \zeta; s, \cdot)w(d\zeta). \quad (41)$$

In fact, $p_*(s) : \mathcal{P}(l^2) \rightarrow \mathcal{P}(l^2)$ is the duality operator of $p(s)$.

Lemma 7. Suppose (4)–(8) and (20) hold. Then, the $p_*(s)$ is a continuous autonomous dynamical system with respect to the Markov semigroup on $\mathcal{P}_2(l^2)$ generated by (17) and (18). For $s \geq 0$, $p_*(s) : \mathcal{P}_2(l^2) \rightarrow \mathcal{P}_2(l^2)$ satisfies the following conditions:

- (a) $p_*(0) = I_{\mathcal{P}_2(l^2)}$;
- (b) $p_*(s+r) = p_*(s)p_*(r)$, for any $s, r \geq 0$;
- (c) $p_*(s) : \mathcal{P}_2(l^2) \rightarrow \mathcal{P}_2(l^2)$ is continuous for every $s \geq 0$.

Proof. It follows from Lemma 3 that, for $s \geq 0$, $p_*(s) : \mathcal{P}_2(l^2) \rightarrow \mathcal{P}_2(l^2)$. It is obvious that the operator p_* satisfies condition (a). By the properties (ii) and (iii) in Lemma 6, we obtain the property (b). We now prove (c). We need to claim that, if $w_n \rightarrow w$ in $\mathcal{P}_2(l^2)$, then $p_*(s)w_n \rightarrow p_*(s)w$ in $\mathcal{P}_2(l^2)$, for every $s \geq 0$. By the property (i) in Lemma 6, we obtain, for any $\varphi \in C_b(l^2)$ and $s \geq 0$,

$$\lim_{n \rightarrow \infty} (p_*(s)w_n, \varphi) = \lim_{n \rightarrow \infty} (w_n, p(s)\varphi) = (w, p(s)\varphi) = (p_*(s)w, \varphi). \quad (42)$$

Equation (42) implies that $p_*(s)w_n$ weakly converges to $p_*(s)w$, which means that $p_*(s)w_n \rightarrow p_*(s)w$ in $(\mathcal{P}_2(l^2), d_{\mathcal{P}_2(l^2)}^*)$. This concludes the proof. $\square$

By Lemma 3, we obtain an absorbing set of p_* associated with (17) and (18).

Lemma 8. Suppose (4)–(8) and (21) hold. Then, for every $R > 0$, there exists $S = S(R) > 0$ such that, for all $s \geq S$, p_* associated with (17) and (18), it is satisfied that

$$p_*(s)B_{l^2}(R) \subseteq B_{l^2}(M),$$

where $M > 0$ is a constant independent of R .

We now present the asymptotical compactness of p_* associated with (17) and (18).

Lemma 9. If (4)–(8) and (20) hold, then p_* is asymptotically compact in $\mathcal{P}_2(l^2)$, that is $\{p_*(s_n)w_n\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{P}_2(l^2)$, whenever $s_n \rightarrow \infty$ and $w_n \in B_{l^2}(R)$, for any $R > 0$.

Proof. To complete the proof, by Prokhorov theorem, it is to be proven that the sequence $\{\mathcal{L}(w(s_n, \zeta_n))\}_{n=1}^\infty$ is tight. It follows from Lemma 3 that, for $R > 0$, there exists a $N_1 = N_1(R) \in \mathbb{N}$ such that, for all $\mathbb{E}(\|\zeta_n\|^2) \leq R$ with $\zeta_n \in L^2_{\mathcal{F}_0}(\Omega, l^2)$ and $n > N_1$,

$$\mathbb{E}(\|w(s_n, \zeta_n)\|^2) \leq H, \quad (43)$$

where $H > 0$ is a constant independent of R . By Chebyshev's inequality, we obtain from (43) that, for all $\mathbb{E}(\|\zeta_n\|^2) \leq R$ with $\zeta_n \in L^2_{\mathcal{F}_0}(\Omega, l^2)$ and $n > N_1$,

$$P(\|w(s_n, \zeta_n)\| > R_1) \leq \frac{H}{R_1^2} \rightarrow 0 \quad \text{as } R_1 \rightarrow \infty.$$

Hence, for every $\eta > 0$ and $k \in \mathbb{N}$, there exists $R_2 = R_2(\eta, k) > 0$ such that, for all $\mathbb{E}(\|\zeta_n\|^2) \leq R$ with $\zeta_n \in L^2_{\mathcal{F}_0}(\Omega, l^2)$ and $n > N_1$,

$$P\{\|w(s_n, \zeta_n)\| > R_2\} < \frac{\eta}{2^{k+1}}. \quad (44)$$

By Lemma 5, we infer that, for each $R > 0$, $\eta > 0$ and $k \in \mathbb{N}$, there exists an integer $n_k = n_k(R, \eta, k)$ and $A_k = A_k(R, \eta, k) > N_1$ such that, for all $\mathbb{E}(\|\zeta_n\|^2) \leq R$ with $\zeta_n \in L^2_{\mathcal{F}_0}(\Omega, l^2)$ and $n \geq A_k$,

$$\mathbb{E}\left(\sum_{|m|>n_k} |w_m(s_n, \zeta_n)|^2\right) < \frac{\eta}{2^{2k+1}},$$

and hence, for all $\mathbb{E}(\|\zeta_n\|^2) \leq R$ with $\zeta_n \in L^2_{\mathcal{F}_0}(\Omega, l^2)$ and $n \geq A_k$,

$$P\left(\left\{\sum_{|m|>n_k} |w_m(s_n, \zeta_n)| > \frac{1}{2^k}\right\}\right) \leq 2^k \mathbb{E}\left(\sum_{|m|>n_k} |w_m(s_n, \zeta_n)|^2\right) < \frac{\eta}{2^{k+1}}. \quad (45)$$

Given $k > 0$, set

$$Y_{1,k} = \{v \in l^2 : \|v\| \leq R_2\}, \quad (46)$$

$$Y_{2,k} = \left\{v \in l^2 : \sum_{|m|>n_k} |v_m| \leq \frac{1}{2^k}\right\}, \quad (47)$$

and

$$Y_k = Y_{1,k} \cap Y_{2,k}. \quad (48)$$

By (46), we see that the set $\{(v_m)_{|m|\leq n_k} : v \in Y_k\}$ is bounded in the finite-dimensional space $\mathbb{R}^{2n_k+1}$ and, hence, precompact. Consequently, $\{(v_m)_{|m|\leq n_k} : v \in Y_k\}$ has a finite

open cover of balls with radius $\frac{1}{\sqrt{2^k}}$, which along with (46) implies that the set $\{v : v \in Y_k\}$ has a finite open cover of balls with radius $\frac{1}{\sqrt{2^{k-1}}}$ in l^2 . For each $k \in \mathbb{N}$, there exists a compact set K_k such that, for all $n \leq A_k$, $P(\{w(s_n, \zeta_n) \in K_k\}) > 1 - \frac{\eta}{2^k}$. Then, by (44) and (45), there exists a set $\mathcal{Y}_k = Y_k \cup K_k$, which has a finite open cover of balls with radius $\frac{1}{\sqrt{2^{k-1}}}$ in l^2 , such that, for all $n \in \mathbb{N}$, $P(\{w(s_n, \zeta_n) \in \mathcal{Y}_k\}) > 1 - \frac{\eta}{2^k}$. Set $\mathcal{Y} = \bigcap_{k=1}^{\infty} \mathcal{Y}_k$. Then, $\mathcal{Y}$ is a closed and totally bounded subset of l^2 and, hence, is compact. For all $n \in \mathbb{N}$,

$$P(\{w(s_n, \zeta_n) \in \mathcal{Y}\}) > 1 - \sum_{m=1}^{\infty} \frac{\eta}{2^k} = 1 - \eta, \quad (49)$$

as desired. $\square$

Subsequently, we firmly establish the existence and uniqueness of measure attractors pertaining to Equations (17) and (18) on $\mathcal{P}_2(l^2)$.

Theorem 2. *If (4)–(8) and (20) hold, then p_* associated with (17) and (18) has a unique measure attractor $\star$ on $\mathcal{P}_2(l^2)$ whose structure is characterized by (3).*

Proof. We further note that, by Lemma 8, p_* has an absorbing state and is asymptotically compact on $\mathcal{P}_2(l^2)$ according to Lemma 9. Thus, regarding the claims stated above and related to Theorem 1, and following the above discussion, the existence of a unique measure attractor for p_* and the structure of this attractor can be obtained easily. $\square$

6. Upper Semicontinuity of Measure Attractors

We next show that, for stochastic fractional lattice systems, the measure attractors are upper semicontinuous with respect to ε going to zero. First, we provide a sufficient condition for the upper semicontinuity of measure attractors of a family of dynamical systems in $\mathcal{P}_2(M)$. Suppose Λ is an interval of $\mathbb{R}$, and for each $\lambda \in \Lambda$, φ_λ is a autonomous dynamical system on $\mathcal{P}_2(M)$. Suppose that, for each $\lambda \in \Lambda$, φ_λ has a measure attractor $\star_\lambda$. Assume that there exists $\lambda_0 \in \Lambda$ such that, for $s \in \mathbb{R}^+$,

$$\lim_{n \rightarrow \infty} \sup_{w \in \star_{\lambda_n}} d_{\mathcal{P}_2(M)}^*(\varphi_{\lambda_n}(s)w, \varphi_{\lambda_0}(s)w) = 0, \quad (50)$$

for any $\lambda_n \rightarrow \lambda_0$.

We further assume the existence of a constant $r > 0$, with the condition that

$$K = \bigcup_{\lambda \in \Lambda} \star_\lambda \subseteq B_M(r). \quad (51)$$

We hereby present the upper semicontinuity of $\star_\lambda$ as $\lambda \rightarrow \lambda_0$.

Theorem 3 ([25]). *Suppose (50) and (51) hold. Then,*

$$\lim_{\lambda \rightarrow \lambda_0} d_{\mathcal{P}_2(M)}(\star_\lambda, \star_{\lambda_0}) = 0. \quad (52)$$

Subsequently, we utilize Theorem 3 in relation to the stochastic fractional lattice systems denoted by (17) and (18), specifically with $\varepsilon \in [0, 1]$. It is worth mentioning that all outcomes discussed in previous sections remain applicable when $\varepsilon = 0$, where the actual proof becomes simpler. Moving forward, we will denote the solution for systems (17) and (18) as $w^\varepsilon(s, \zeta)$ with an initial condition of $\zeta \in L^2_{\mathcal{F}_0}(\Omega, l^2)$ to underscore the solutions' dependence on the parameter ε . For values of $\varepsilon \in [0, 1]$, let $p^\varepsilon(s)$ represent the transition probability of $w^\varepsilon(s, \zeta)$, while $p_*^\varepsilon(s)$ denotes the duality operator corresponding to p^ε . Let $\star_\varepsilon$ be the measure attractor of p_*^ε .

Next, we proceed to demonstrate the convergence of solutions for problem (17) and (18) as $\varepsilon \rightarrow 0$.

Lemma 10. Suppose (4)–(8) hold. Then, for every $r > 0$ and $s \geq 0$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{w \in B_{l^2}(r)} d_{\mathcal{P}_2(l^2)}^* \left(p_*^\varepsilon(s)w, p_*^0(s)w \right) = 0.$$

Proof. Using a similar argument to that presented in Lemma 6.2 of [24], we can deduce that, for $s \geq 0$,

$$\mathbb{E} \left(\left\| w^\varepsilon(s, \zeta) - w^0(s, \zeta) \right\|^2 \right) \leq c_1 \varepsilon^2 S e^{c_1 s}, \quad (53)$$

where c_1 is independent of ε ; hence, for all $r > 0$, $S > 0$ and $s \in [0, S]$,

$$\sup_{\mathbb{E}(\|\zeta\|^2) \leq r} \mathbb{E} \left(\left\| w^\varepsilon(s, \zeta) - w^0(s, \zeta) \right\|^2 \right) \leq c_1 \varepsilon^2 S e^{c_1 s}. \quad (54)$$

For any $\phi \in L_b(l^2)$ with $\|\phi\|_L \leq 1$, we have, for every $\eta > 0$ and $s \geq 0$,

$$\begin{aligned} & \sup_{\mathbb{E}(\|\zeta\|^2) \leq r} \left| \mathbb{E}(\phi(w^\varepsilon(s, \zeta))) - \mathbb{E}(\phi(w^0(s, \zeta))) \right| \\ & \leq \sup_{\mathbb{E}(\|\zeta\|^2) \leq r} \mathbb{E} \left(\left| \phi(w^\varepsilon(s, \zeta)) - \phi(w^0(s, \zeta)) \right| \right) \\ & \leq \sup_{\mathbb{E}(\|\zeta\|^2) \leq r} \mathbb{E} \left(2 \wedge \left\| w^\varepsilon(s, \zeta) - w^0(s, \zeta) \right\| \right) \\ & \leq 2 \sup_{\mathbb{E}(\|\zeta\|^2) \leq r} P \left\{ \left\| w^\varepsilon(s, \zeta) - w^0(s, \zeta) \right\| \geq \frac{\eta}{3} \right\} \\ & \quad + \frac{\eta}{3} \sup_{\mathbb{E}(\|\zeta\|^2) \leq r} P \left\{ \left\| w^\varepsilon(s, \zeta) - w^0(s, \zeta) \right\| < \frac{\eta}{3} \right\}. \end{aligned} \quad (55)$$

By Chebyshev's inequality and (54), there exists $0 < \varepsilon_1 = \varepsilon_1(\eta) \leq 1$, so that, for any value of ε , where $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned} & \sup_{\mathbb{E}(\|\zeta\|^2) \leq r} P \left\{ \left\| w^\varepsilon(s, \zeta) - w^0(s, \zeta) \right\| \geq \frac{\eta}{3} \right\} \\ & \leq \sup_{\mathbb{E}(\|\zeta\|^2) \leq r} \frac{\mathbb{E}(\left\| w^\varepsilon(s, \zeta) - w^0(s, \zeta) \right\|^2)}{\left(\frac{\eta}{3}\right)^2} \leq \frac{\eta}{3}. \end{aligned} \quad (56)$$

By (55) and (56), we have

$$\sup_{\mathbb{E}(\|\zeta\|^2) \leq r} \left| \mathbb{E}(\phi(w^\varepsilon(s, \zeta))) - \mathbb{E}(\phi(w^0(s, \zeta))) \right| \leq \eta, \quad \forall \varepsilon \in (0, \varepsilon_1).$$

Since ϕ is arbitrary, we can obtain that

$$\sup_{\mathbb{E}(\|\zeta\|^2) \leq r} \sup_{\substack{\phi \in L_b(l^2) \\ \|\phi\|_L \leq 1}} \left| \mathbb{E}(\phi(w^\varepsilon(s, \zeta))) - \mathbb{E}(\phi(w^0(s, \zeta))) \right| \leq \eta, \quad \forall \varepsilon \in (0, \varepsilon_1).$$

That is,

$$\sup_{w^\varepsilon \in B_{l^2}(r)} d_{\mathcal{P}(l^2)}^* \left(p_*^\varepsilon(t)w^\varepsilon, p_*^0(s)w^\varepsilon \right) \leq \eta, \quad \forall \varepsilon \in (0, \varepsilon_1),$$

as desired. $\square$

Using Lemma 8, it can be confirmed that there exists a constant r greater than zero, such that

$$\bigcup_{\varepsilon \in [0,1]} \star_{\varepsilon} \subseteq B_{l^2}(r). \quad (57)$$

The primary findings of this section are presented below.

Theorem 4. Suppose (4)–(8) and (20) hold. Then,

$$\lim_{\varepsilon \rightarrow 0} d_{\mathcal{P}(l^2)}(\star_{\varepsilon}, \star_0) = 0. \quad (58)$$

Proof. According to (50) and (51), we can obtain Theorem 3. So, based on (57) and Lemma 10, we obtain Theorem 4 immediately from Theorem 3. $\square$

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