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An Efficient Numerical Scheme for a Time-Fractional Black–Scholes Partial Differential Equation Derived from the Fractal Market Hypothesis

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Abstract: Since the early 1970s, the study of Black–Scholes (BS) partial differential equations (PDEs) under the Efficient Market Hypothesis (EMH) has been a subject of active research in financial engineering. It has now become obvious, even to casual observers, that the classical BS models derived under the EMH framework fail to account for a number of realistic price evolutions in real-time market data. An alternative approach to the EMH framework is the Fractal Market Hypothesis (FMH), which proposes better and clearer explanations of market behaviours during unfavourable market conditions. The FMH involves non-local derivatives and integral operators, as well as fractional stochastic processes, which provide better tools for explaining the dynamics of evolving market anomalies, something that classical BS models may fail to explain. In this work, using the FMH, we derive a time-fractional Black–Scholes partial differential equation (tfBS-PDE) and then transform it into a heat equation, which allows for ease of implementing a high-order numerical scheme for solving it. Furthermore, the stability and convergence properties of the numerical scheme are discussed, and overall techniques are applied to pricing European put option problems.

Keywords: fractal market hypothesis; option pricing; fractional Black–Scholes PDEs; numerical methods; convergence; stability analysis

MSC: 60G22; 65L20; 65M12; 91G50



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1. Introduction

The concept of option pricing is both a theoretical and practical problem in computational finance. At the centre of option pricing theory is the Black–Scholes equation which is fundamental in general asset-pricing theory. The original derivation of the Black–Scholes (BS) equation is attributed to the work of Black and Scholes in 1973 (see [1]). The derivation of this model is based on the concept of the Efficient Market Hypothesis (EMH). A key primary concern with the EMH and the classical BS assumptions, in general, is the assumption that asset price returns follow a normal distribution. Of course, in the early days of trading, the Black–Scholes equation in its classical form was a very efficient asset-pricing tool and as such, it is widely used in the derivative pricing spectrum. Lately, however, scientific evidence has shown that the evolution of big data and high-frequency trading in finance, presumably informed by classical contemporary theory, and somewhat accomplished with little to no knowledge of the evolution of market dynamics, has led to the formation of a number of speculative market bubbles, which, in turn, have led to a lot of financial troubles. For a comprehensive account of speculative bubbles and their role in financial crises, among other market anomalies, see, for example, Refs. [2–4] and the references therein.

Among other factors, the recurrent occurrence of unexpected fluctuations, i.e., a sudden unexpected rise/fall in stock prices, renders the use of the classical Black–Scholes approach inappropriate. In [5], Taleb referred to these rare occurrences as “black swan events”, after his popular book entitled *The Black Swan: The Impact of the Highly Improbable*. According to Taleb [5], a black swan event is “A highly improbable event with three principal characteristics: it is unpredictable; it carries a massive impact; and, after the fact, we concoct an explanation that makes it appear less random, and more predictable than it was”. According to Kleinert [4], the existence of black swan events in financial markets is a severe obstacle to all hedging attempts. Kleinert [4] further argued that based on the central limit theorem, one would normally consider price changes as a result of random steps of a given finite size and demonstrated that these series of random walks build up to a complete Gaussian process. However, in contrast, our observations are that in the event that a black swan event occurs, some incremental steps can become very irregular, resulting in a combined process that exhibits power-law properties. It is, therefore, of vital importance that both numerical and analytical (where possible) models beyond the classical BS setup that have the capacity to explain the effects of “black swan events”, as well as their solutions, are developed.

The birth of fractional calculus, which is, in essence, a generalisation of the well-known integer-order calculus, can be traced back to 1695 when Guillaume de L’Hopital wrote a letter addressed to Sir Gottfried Wilhelm Leibniz, asking him about a notation he used in his publication for the n th-derivative of a linear function. L’Hopital posed the following question to Leibniz: What would the result be when $n = 1/2$? Leibniz’s response was, “An apparent paradox, from which one day many useful consequences will be drawn”. With these words, fractional calculus was born.

Falling under chaos theory is the Fractal Market Hypothesis (FMH), which presents an appropriate approach for modelling unusual market anomalies. The FMH concepts primarily emanate from Mandelbrot’s theory on fractal geometry [6,7]. In [8], Bourke defined fractals as fragmented geometric structures, which, when broken down into smaller parts, would still maintain the shape and structure of the whole. Repeated self-similar patterns have long been observed in financial markets data (see, for example, Refs. [4,9,10] and the references therein). From a financial point of view, one may regard stock price movements as mathematical fractals that, once broken down into smaller time frames, would still replicate the actual structures of markets in bigger time frames.

In recent years, several researchers have dedicated considerable attention to the study of fractional differential equations and their applications to solving various problems occurring in physics and biochemistry [11–13], engineering [6,14], and finance [4,9,10,15–22], among others.

There is scientific evidence to show that fractional calculus provides an efficient and reliable framework for capturing the unusual dynamics of processes and systems occurring in various scientific fields (see, for example, Refs. [9,10,15,20–25] and the references therein). In finance, it is imperative that fractional calculus-based models are better suited to modelling black swan-type dynamics, which cannot be appropriately modelled using classical approaches.

The theory of no arbitrage has always been one of the most central and unifying concepts in financial modelling and literature. However, with market evolutions, classical pricing theory has been confronted with a number of challenges. In an attempt to fully capture the unusual features in markets and dynamics beyond what classical theory can predict, behavioural finance and empirical evidence show, to some extent, that there is a need to surpass the concept of no arbitrage. Since fractional Brownian motions (fBMs) are neither martingale nor semi-martingale (see [11,23,26]), using fBM-based models in asset pricing may allow for arbitrage opportunities under a complete and frictionless market setup. However, a considerable number of arbitrage strategies for fBM-based models have been proposed. Strategies ranging from incorporating no-good deal bounds and

transactional cost structures, as well as incorporating dividend yields in the pricing models, have been suggested.

In [27], while extending the Leland [28] approach to fractional Black–Scholes models, Guasoni demonstrated that arbitrage opportunities vanish when the proportional transaction costs are incorporated into the fractional Black–Scholes model structure [15]. For a detailed account of this and other numerous reviews on arbitrage-free strategies for fractional models, see [27–31] and the references therein. In [15], fractional dynamics, under which the discounting stock price movements follow a martingale asymptotically, were chosen to eliminate possible arbitrage opportunities. Since trading is always associated with some types of costs and restraints, such as transactional costs, commissions, and taxes, the frictionless assumption under the no-arbitrage principle may not hold in practice. Therefore, relaxing this assumption, as in our case, can provide for a robust and practical valuation approach.

Among other properties, fractional models have two vital important features: self-similarity and memory. For detailed discussions on these features, see, e.g., [10,23]. These two features of fractional models allow for a better interpretation of the asymmetric effects of isolated extreme price fluctuations. According to Panas [10], as far as the subject of memory is concerned, there are two kinds of memories: noise memory and trend memory. Incorporating a fractional Brownian motion as the underlying process for the pricing dynamics only helps capture the noise memory effects. To capture the trend memory effects and effects due to irregular price movements, one replaces the integer derivatives with their corresponding fractional-order derivatives (see, for example, Refs. [10–12,14] for detailed discussions).

The PDEs arising from the fractional transformed models are generically referred to as fractional partial differential equations (fPDEs). In the Black–Scholes setup, there are three families of fPDEs: time-fractional Black–Scholes (tfBS) PDEs, space-fractional Black–Scholes (sfBS) PDEs, and time-space-fractional Black–Scholes (tsfBS) PDEs. In the tfBS PDEs, the time derivative is replaced with a corresponding fractional derivative of order α ($0 < \alpha \leq 1$), whereas in the sfBS case, only the space derivatives are replaced with their corresponding fractional derivatives of orders α for the first-order derivative and β for the second-order derivative, where ($0 < \alpha \leq 1$ and $1 < \beta \leq 2$). In the case of tsfBS, one has a combination of both tfBS and sfBS. By assuming that the stock price dynamics are driven by a non-Gaussian fractional dynamic process, Jumarie [32] designed two sets of fractional BS equations: the time-fractional (tfBS) and the time-and-space fractional (tsfBS) PDEs. His approach has been replicated by other researchers. Some designed numerical schemes for solving the fPDEs, whereas others used the fPDEs in practical asset-pricing scenarios. Among others, Chen et al. [33] used an approach almost similar to that of Jumarie [32] but instead maintained the usual Gaussian dynamics and only analogously replaced the integer derivatives in the BS PDE with their corresponding fractional-order derivatives α and β ($0 < \alpha \leq 1$ and $1 < \beta \leq 2$). They indicated that by using either approach, one obtains similar results.

The fractional approach is very attractive for asset pricing, as it provides a somewhat unique framework that has the potential to transform conventional thinking on asset pricing. The involved non-local derivatives and integral operators, as well as the accompanying fractional stochastic dynamics, provide the best tools for explaining the trend and noise memory effects, as well as providing non-localised information about stock price movements, something classical models may fail to explain.

Since the fractional derivative operators involved in this approach are of a non-local nature, there is hardly any literature about analytic solutions to fractional BS models. Hence, the numerical approaches are the only possibly available avenues to help understand the nature of the solutions to these models. The aim of this paper is, therefore, twofold. Firstly, to transform a tfBS diffusion equation into a tfBS heat equation, easing the mathematical complexity involved in solving the full model. Secondly, to construct a robust and high-order numerical scheme for the model.

The rest of this paper is organised as follows. Section 2 presents a review of some contemporary ideas on fractional calculus and its definitions, as well as a brief discussion on the development of the involved model. Section 3 deals with the construction and analysis of the finite-difference scheme. To validate our theoretical observations, Section 4 presents some practical experiments on pricing European options on continuous dividend-paying stocks. The conclusions and recommendations drawn from this study, as well as some prospects for future research, are presented in Section 5.

2. Model Specification

In this section, we present some preliminaries on fractional differentiation, specifically focusing on three common fractional derivative operators: the Caputo, Riemann–Liouville, and Jumarie fractional derivatives. We further present a brief discussion on the derivation of the proposed model.

2.1. Fractional Differentiation Revisited

In the most recent literature (see, for example, Refs. [4,15,20–22,34]), the derivatives of fractional orders and their applications to finance are commonly defined in the Caputo, Riemann–Liouville, and Jumarie (modified Riemann–Liouville) sense.

Definition 1. Caputo Fractional Derivative

Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous but not necessarily differentiable function. The Caputo fractional derivative of order α is defined as follows:

$$D_C^\alpha u(t) = \frac{1}{\Gamma(\eta - \alpha)} \int_0^t \frac{d^\eta u(\tau)}{dt^\eta} \frac{1}{(t - \tau)^{\alpha - \eta + 1}} d\tau, \quad \eta - 1 < \alpha \leq \eta, \quad \eta \in \mathbb{N}. \quad (1)$$

Definition 2. Riemann–Liouville Fractional Derivative

Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous but not necessarily differentiable function. Then, the Riemann–Liouville fractional derivative of order α is given by

$$D_{RL}^\alpha u(t) = \frac{1}{\Gamma(\eta - \alpha)} \frac{d^\eta}{dt^\eta} \int_0^t \frac{u(\tau)}{(t - \tau)^{\alpha - \eta + 1}} d\tau, \quad \eta - 1 < \alpha \leq \eta, \quad \eta \in \mathbb{N}. \quad (2)$$

In [35], Jumarie successfully derived some interesting modifications of the Riemann–Liouville definition with the aid of the so-called fractional differencing theory coupled with the generalised Taylor-series theory. The Jumarie definition takes into account the existence of a fractional derivative at $t = 0$, which is undefined for non-differentiable functions in the Caputo and Riemann–Liouville sense.

Definition 3. Fractional Differencing

Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous but not necessarily differentiable function, and let $\kappa > 0$ denote the discretization step in t . Define the forward operator $FW(\kappa)$ by

$$FW(\kappa)u(t) = u(t + \kappa), \quad (3)$$

then, the fractional difference of order $\eta - 1 < \alpha \leq \eta, \eta \in \mathbb{N}$ of $u(t)$ is defined by the expression

$$\begin{aligned} \Delta^\alpha u(t) &= (FW - 1)^\alpha u(t), \\ &= \sum_{\zeta=0}^{\infty} (-1)^\zeta \binom{\alpha}{\zeta} u(t + (\alpha - \zeta)\kappa). \end{aligned} \quad (4)$$

This definition allows for a proper interpretation of the fractional derivation of a constant function, which aligns fractional calculus theory consistently well with its classical counterpart.

Definition 4. Jumarie (Modified Riemann–Liouville) Derivative

Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous but not necessarily differentiable function, and suppose that $u(t)$ is as follows:

(i) A constant K , and then its Jumarie fractional derivative of order α is defined by

$$D_J^\alpha u(t) = \begin{cases} \frac{K}{\Gamma(\eta-\alpha)} t^{-\alpha+1-\eta}, & \alpha \leq \eta - 1, \\ 0, & \alpha > \eta - 1, \quad \eta \in \mathbb{N} \end{cases} \quad (5)$$

(ii) Not a constant, and then

$$D_J^\alpha u(t) = \frac{1}{\Gamma(\eta-\alpha)} \frac{d^\eta}{dt^\eta} \int_0^t \frac{\{u(\tau) - u(0)\}}{(t-\tau)^\alpha} d\tau, \quad \eta - 1 < \alpha \leq \eta, \quad \eta \in \mathbb{N}. \quad (6)$$

Definition 5. Generalised (Fractional) Taylor Series

Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $u(t)$ has a fractional derivative of order $k\alpha$ for some positive integer k , and $0 < \alpha \leq 1$. Then, the following equality holds:

$$u(t+h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\Gamma(1+k\alpha)} u^{(k\alpha)}(t), \quad 0 < \alpha \leq 1, \quad (7)$$

where $u^{(k\alpha)}(t)$ represents the fractional derivative of the $k\alpha$ -th order of $u(t)$.

2.2. The Time-Fractional Black–Scholes Equation

To the best of our knowledge, Mandelbrot and Cioczek–Georges ([6]) introduced fractional Brownian motion (fBM) to mainstream research with financial applications in the 1960s. As a generalisation of standard Brownian motion, fractional Brownian motion (fBM) is a stochastic process that possesses hereditary properties, and unlike standard Brownian motion, the fBM process has no independent increments. Instead, fBM and models based on fBM exhibit self-similarity, power-law properties, and properties that are dependent on a wide range (see [15,30], among others). These are some of the features appropriate for modelling market anomalies observed in a number of financial data. Some empirical evidence of this effect can be found in [9,10,23–25].

To begin, we let the stock price (S) dynamics follow a non-random fractional stochastic process given by

$$dS = (r - \delta)Sdt + \sigma S\omega(t)(dt)^{\alpha/2}, \quad 0 < \alpha \leq 1, \quad (8)$$

where $\omega(t)$ is the Gaussian white noise, with a mean of zero and a standard deviation of one, and r , σ^2 , and δ are the risk-free interest rate, the volatility, and the continuous dividend yield of the underlying stock, respectively.

If we denote standard Brownian motion as used in the classical BS model with $B(t)$, Equation (8) can be viewed as a generalisation of $B(t)$ by replacing

$$dB(t) = \omega(t)dt^{1/2} \quad (9)$$

with

$$dB(t) = \omega(t)(dt)^{\alpha/2}, \text{ for some } 0 < \alpha \leq 1 \quad (10)$$

The fractional stochastic process (8) is more appropriate compared to the standard one because it does not make any prior distributional assumptions about asset prices/returns. The model-free assumption provides a better approach to modelling the power-law properties of markets with greater flexibility.

Consider

$$dS = \frac{S^{(1-\alpha)}}{\Gamma(1+\alpha)\Gamma(2-\alpha)}(dS)^\alpha, \quad 0 < \alpha \leq 1. \quad (11)$$

Let $V(S, t)$ be the value of a European put option, and suppose that $V(S, t)$ is sufficiently smooth with respect to S , and its α -derivative with respect to time exists. Then, the theory of interest suggests that

$$dV = rVdt. \quad (12)$$

The above leads to the variational fractional increment formula

$$d^\alpha V = \Gamma(1+\alpha)rVdt. \quad (13)$$

Using (11) and (13), we obtain the following fractional interest rate dynamic equation

$$d^\alpha V = \frac{rV}{\Gamma(2-\alpha)}t^{1-\alpha}(dt)^\alpha. \quad (14)$$

The assumption about the differentiability and continuity of $V(S, t)$ implies that (14) satisfies the following generalised Taylor series of order α , up to some remaining error terms,

$$dV = \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha V}{\partial t^\alpha} (dt)^\alpha + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2. \quad (15)$$

The Itô's lemma applied to (8) and (15) yields

$$dV = \frac{1}{\Gamma(1+\alpha)} \frac{\partial^\alpha V}{\partial t^\alpha} (dt)^\alpha + (r-\delta)S \frac{\partial V}{\partial S} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} (dt)^\alpha. \quad (16)$$

After some algebraic manipulations and changes of derivatives, we obtain the following:

$$\begin{aligned} d^\alpha V &= \frac{rV}{\Gamma(2-\alpha)}t^{1-\alpha}(dt)^\alpha, \\ &= \left(\frac{\partial^\alpha V}{\partial t^\alpha} + \frac{(r-\delta)}{\Gamma(2-\alpha)}St^{1-\alpha} \frac{\partial V}{\partial S} + \frac{\Gamma(1+\alpha)}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) (dt)^\alpha, \end{aligned} \quad (17)$$

which implies

$$\frac{rV}{\Gamma(2-\alpha)}t^{1-\alpha} = \frac{\partial^\alpha V}{\partial t^\alpha} + \frac{(r-\delta)}{\Gamma(2-\alpha)}St^{1-\alpha} \frac{\partial V}{\partial S} + \frac{\Gamma(1+\alpha)}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}. \quad (18)$$

Equation (18) can be further simplified into the following tfBS-PDE:

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \left(rV - qS \frac{\partial V}{\partial S} \right) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, \quad q = r - \delta; \quad 0 < \alpha \leq 1. \quad (19)$$

To ease the computational difficulties involved in computing the solution to the tfBS-PDE in (19), we transform (19) into a heat equation by changing the variables. Then, we use the known solution of the heat equation to represent the solution. Thereafter, we change the variables back to the original variable $V(S, t)$.

First, we want to eliminate the term rV in (19). To do so, we assume that $V(S, t)$ is differentiable with respect to t . Then, considering the following change in the variables

$$V(S, t) = e^{-r(T-t)}v(S, t), \quad (20)$$

and applying the Caputo derivative to (20) yields,

$$V_t^\alpha = D_t^\alpha(e^{-r(T-t)})v(S, t) + e^{-r(T-t)}v_t^\alpha, \quad 0 < \alpha \leq 1, \quad (21)$$

where D_t^α is the Caputo derivative with respect to t of order α .

From the generalised Taylor-series expansion (7), we have

$$D_t^\alpha(e^{-r(T-t)}) = re^{-r(T-t)} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \quad (22)$$

By substituting (22) into (21), we obtain

$$V_t^\alpha(S, t) = re^{-r(T-t)} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} v(S, t) + e^{-r(T-t)} v_t^\alpha(S, t), \quad 0 < \alpha \leq 1. \quad (23)$$

The above equation, along with (19), yields

$$\begin{aligned} re^{-r(T-t)} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} v(S, t) + e^{-r(T-t)} v_t^\alpha(S, t) &= \left(e^{-r(T-t)} v(S, t) - qSe^{-r(T-t)} v_S(S, t) \right) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \\ &\quad - \frac{\Gamma(1+\alpha)}{2} \sigma^2 S^2 e^{-r(T-t)} v_{SS}(S, t), \end{aligned} \quad (24)$$

which, after simplifying, yields

$$v_t^\alpha(S, t) = -qSv_S(S, t) \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2} \sigma^2 S^2 v_{SS}(S, t), \quad 0 < \alpha \leq 1, \quad (25)$$

with the terminal condition

$$v(S, T) = V(S, T).$$

Now, we want to have an fPDE with constant coefficients. To eliminate the non-linear terms $Sv_S(S, t)$ and $S^2v_{SS}(S, t)$ from (25), we consider the transformation

$$s = e^{S-\tau}, \text{ for some constant } \tau, \quad (26)$$

resulting in an fPDE with a solution of the form

$$v(S, t) \equiv v(s, t),$$

and the terminal condition

$$v(s, T) = v(S, T) = V(S, T).$$

Therefore,

$$v_t^\alpha(s, t) = \left(\frac{\Gamma(1+\alpha)}{2} \sigma^2 - q \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \right) v_s(s, t) - \frac{\Gamma(1+\alpha)}{2} \sigma^2 v_{ss}(s, t). \quad (27)$$

According to Jumarie ([19]), equations of the form in (27) have a general solution of the form

$$v(s, t) \equiv p(s, t) = \Phi\left(s - qt + \frac{1}{2}\sigma^2 t^2\right), \quad (28)$$

where Φ represents the cumulative distribution function (CDF) of the standard normal distribution.

These suggest the following transformation:

$$\hat{s} = s - \ln(c) + r(T-t) - \frac{1}{2}\sigma^2(T^\alpha - t^\alpha) \quad (29)$$

and the following terminal condition:

$$v(\hat{s}, T) = c(e^{\hat{s}} - 1) = c\left(e^{\hat{s} - \ln(c) + r(T-t) - \frac{1}{2}\sigma^2(T^\alpha - t^\alpha)} - 1\right).$$

Choosing the constant τ in (26) as $\tau = -\ln(c) + rT - \frac{1}{2}\sigma^2 T^\alpha$ for some constant c and substituting the new variable ($\hat{s}$) from (28) back into (27) eliminates the term $v_s(s, t)$ from (27), resulting in the following fractional heat equation:

$$p_t^\alpha(\hat{s}, t) = -\frac{\Gamma(1+\alpha)\sigma^2}{2} p_{\hat{s}\hat{s}}(\hat{s}, t), \quad (30)$$

which can be simplified to

$$\frac{\partial^\alpha p(\hat{s}, t)}{\partial t^\alpha} = \psi \frac{\partial^2 p(\hat{s}, t)}{\partial \hat{s}^2}, \quad (31)$$

where

$$\psi = -\frac{\Gamma(1+\alpha)\sigma^2}{2},$$

with European put initial and boundary conditions in terms of the original variable

$$\left. \begin{aligned} p(\hat{s}, 0) &= V(S, 0) = \max(K - Ke^{rt}, 0), \\ p(0, t) &= V(0, T) = Ke^{-rt}, \\ \lim_{s \rightarrow \infty} p(s, t) &= \lim_{S \rightarrow \infty} V(S, t) = 0, \end{aligned} \right\} \quad (32)$$

where K is the strike price of the option and T is the maturity time.

For the remainder of this text, Equation (31) is referred to as the time-fractional Black–Scholes heat equation.

3. Numerical Method

This section presents the fundamental aspects and design of the proposed numerical scheme for solving Equation (31).

To begin, let L and N be some positive integers, and define $h = S_{max}/L$ and $k = T/N$ as the spatial and temporal grid sizes. Further, denote $s_l \in [0, S_{max}]$ and $t_n \in [0, T]$ as the grid points in the asset and time directions, such that $s_l = lh$ and $t_n = nk$ for $l = 0, 1, 2, 3, \dots, L$ and $n = 0, 1, 2, \dots, N - 1$.

Denote the value of the option p at grid point (t_n, s_l) by

$$p_l^n = p(t_n, s_l), \quad (33)$$

and also, define

$$C(t, s) := \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{p(\tau, s)}{(t-\tau)} d\tau, \quad (34)$$

such that

$$\frac{\partial^\alpha p(t_{n+1/2}, s_l)}{\partial t^\alpha} = \frac{\partial C(t_{n+1/2}, s_l)}{\partial t}, \quad (35)$$

which is defined in the Caputo sense. Then,

$$\frac{\partial^\alpha p(t_{n+1/2}, s_l)}{\partial t^\alpha} = \frac{\partial C(t_{n+1/2}, s_l)}{\partial t} = \frac{C(t_{n+1}, s_l) - C(t_n, s_l)}{k} + \mathcal{O}(k^2). \quad (36)$$

Using the Caputo derivative in Definition 1, $C(t_{n+1}, s_l)$ can be represented as follows:

$$\begin{aligned}
 C(t_{n+1}, s_l) &= \frac{1}{\Gamma(1+\alpha)} \int_0^{t_{n+1}} \frac{p(\tau, s_l)}{(t_{n+1} - \tau)^\alpha} dz, \\
 &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n+1} \int_{(j-1)k}^{jk} \frac{p(\tau, s_l)}{(t_{n+1} - \tau)^\alpha} dz, \\
 &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n+1} \left(\frac{(\tau - t_j)}{-k} p_l^{j-1} + \frac{(\tau - t_{j-1})}{k} p_l^j + \mathcal{O}(k^2) \right) \frac{1}{(t_{n+1} - \tau)^\alpha} dz, \\
 &= \sum_{j=0}^n (\delta_j - j\beta_j) p_l^{n-j} - k \sum_{j=0}^n (\delta_j - (j+1)\beta_j) p_l^{n-j+1} + E_{n+1},
 \end{aligned} \tag{37}$$

where

$$\begin{aligned}
 E_{n+1} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n+1} \int_{(j-1)k}^{jk} \mathcal{O}(k^2) \frac{dz}{(t_{n+1} - \tau)^\alpha} \\
 &= \frac{1}{\Gamma(2-\alpha)} \mathcal{O}(k^2) \sum_{j=1}^{n+1} \left((n+2-j)^{1-\alpha} - (n+1-j)^{1-\alpha} \right) k^{1-\alpha} \\
 &= \frac{(n+1)^{1-\alpha}}{\Gamma(2-\alpha)} \mathcal{O}(k^{3-\alpha}),
 \end{aligned} \tag{38}$$

$$\delta_j = \frac{1}{k^\alpha (2-\alpha) \Gamma(1-\alpha)} \left((j+1)^{2-\alpha} - j^{2-\alpha} \right), \tag{39}$$

and

$$\beta_j = \frac{1}{k^\alpha \Gamma(2-\alpha)} \left((j+1)^{1-\alpha} - j^{1-\alpha} \right). \tag{40}$$

Similarly,

$$\begin{aligned}
 C(t_n, s_l) &= \frac{1}{\Gamma(1+\alpha)} \int_0^{t_n} \frac{p(\tau, s_l)}{(t_n - \tau)^\alpha} dz, \\
 &= \sum_{j=1}^n (\delta_{j-1} - (j-1)\beta_{j-1}) p_l^{n-j} - k \sum_{j=1}^n (\delta_{j-1} - j\beta_{j-1}) p_l^{n-j+1} + E_n.
 \end{aligned} \tag{41}$$

We can, therefore, approximate $\frac{\partial^\alpha p(t_{n+1/2}, s_l)}{\partial t^\alpha}$ as follows:

$$\begin{aligned}
 \frac{\partial^\alpha p(t_{n+1/2}, s_l)}{\partial t^\alpha} &= \frac{C(t_{n+1}, s_l) - C(t_n, s_l)}{k} + \mathcal{O}(k^2), \\
 &= \sum_{j=0}^n (\delta_j - j\beta_j) p_l^{n-j} - \sum_{j=0}^n (\delta_j - (j+1)\beta_j) p_l^{n-j+1} \\
 &\quad - \sum_{j=1}^n (\delta_{j-1} - (j-1)\beta_{j-1}) p_l^{n-j} + \sum_{j=1}^n (\delta_j - j\beta_{j-1}) p_l^{n-j+1} \\
 &\quad + \frac{E_{n+1} - E_n}{k} + \mathcal{O}(k^2), \\
 &= \sum_{j=0}^n \{(\delta_j - \delta_{j-1}) + ((j-1)\beta_{j-1} - j\beta_j)\} p_l^{n-j} \\
 &\quad + \sum_{j=0}^n \{(\delta_{j-1} - \delta_j) + ((j+1)\beta_{j+1} - j\beta_j)\} p_l^{n-j+1} \\
 &\quad + \frac{E_{n+1} - E_n}{k} + \mathcal{O}(k^2), \\
 &= \sum_{j=0}^n \sigma_j p_l^{n-j+1} + \frac{1}{\Gamma(2-\alpha)} \left(\frac{(n+1)^{1-\alpha} - n^{1-\alpha}}{k} \right) \mathcal{O}(k^{3-\alpha}) + \mathcal{O}(k^2), \\
 &= \sum_{j=0}^n \sigma_j p_l^{n-j+1} + \mathcal{O}(k^{3-\alpha}), \\
 &< \sum_{j=0}^n \sigma_j p_l^{n-j+1} + \mathcal{O}(k^2), \text{ since } \mathcal{O}(k^{3-\alpha}) < \mathcal{O}(k^2),
 \end{aligned} \tag{42}$$

where

$$\sigma_j = \delta_{j+1} - 2\delta_j + \delta_{j-1}, \text{ for } j > 1, \tag{43}$$

since

$$j\beta_j = \delta_j.$$

Therefore, using an appropriate second-order approximation in the asset direction, (31) yields

$$\sum_{j=0}^n \sigma_j p_l^{n-j+1} + \mathcal{O}(k^2) = \frac{\psi}{2h^2} \{p_{l+1}^{n+1} - 2p_l^{n+1} + p_{l-1}^{n+1} + p_{l+1}^n - 2p_l^n + p_{l-1}^n\} + \mathcal{O}(h^2), \tag{44}$$

$$0 \leq n \leq N-1, \text{ and } 1 \leq l \leq L-1.$$

This can be simplified to the following difference scheme

$$-\frac{\psi}{2h^2} p_{l+1}^1 + \left(\frac{\psi}{h^2} + \sigma_0 \right) p_l^1 - \frac{\psi}{2h^2} p_{l-1}^1 = \frac{\psi}{2h^2} p_{l+1}^0 - \frac{\psi}{h^2} p_l^0 + \frac{\psi}{2h^2} p_{l-1}^0, \tag{45}$$

for $n = 0$, and

$$\begin{aligned}
 -\frac{\psi}{2h^2} p_{l+1}^{n+1} + \left(\frac{\psi}{h^2} + \sigma_0 \right) p_l^{n+1} - \frac{\psi}{2h^2} p_{l-1}^{n+1} &= \frac{\psi}{2h^2} p_{l+1}^n - \left(\frac{\psi}{h^2} + \sigma_0 \right) p_l^n + \frac{\psi}{2h^2} p_{l-1}^n \\
 &\quad - \sum_{j=1}^n \sigma_j p_l^{n-j+1}
 \end{aligned} \tag{46}$$

for $n \geq 1$, with European put initial and boundary conditions

$$\left. \begin{aligned} p_l^0 &= \max(K - Ke^{rt}, 0), \\ p_0^n &= Ke^{-rt}, \\ p_L^n &= 0, \quad 0 \leq n \leq N, \end{aligned} \right\} \quad (47)$$

Now, we prove the stability of the scheme we derived above.

3.1. Stability of the Difference Scheme

Let $P_l^n; l = 0, 1, 2, \dots, L; n = 0, 1, 2, \dots, N$; be an approximate solution to the difference scheme in (45) and (46), and define $\epsilon_l^n = p_l^n - P_l^n$ as the truncation error such that $\epsilon_0^n = \epsilon_L^n = 0$ for all n . By substituting ϵ_l^n into (45) and (46), we obtain the following error equations:

$$-\frac{\psi}{2h^2}\epsilon_{l+1}^1 + \left(\frac{\psi}{h^2} + \sigma_0\right)\epsilon_l^1 - \frac{\psi}{2h^2}\epsilon_{l-1}^1 = \frac{\psi}{2h^2}\epsilon_{l+1}^0 - \frac{\psi}{h^2}\epsilon_l^0 + \frac{\psi}{2h^2}\epsilon_{l-1}^0 = 0 \quad (48)$$

for $n = 0$, and

$$\begin{aligned} -\frac{\psi}{2h^2}\epsilon_{l+1}^{n+1} + \left(\frac{\psi}{h^2} + \sigma_0\right)\epsilon_l^{n+1} - \frac{\psi}{2h^2}\epsilon_{l-1}^{n+1} &= \frac{\psi}{2h^2}\epsilon_{l+1}^n - \left(\frac{\psi}{h^2} + \sigma_0\right)\epsilon_l^n + \frac{\psi}{2h^2}\epsilon_{l-1}^n \\ &\quad - \sum_{j=1}^n \sigma_j \epsilon_l^{n-j+1}, \quad n \geq 1. \end{aligned} \quad (49)$$

Let us define the following grid functions:

$$\epsilon^n(S) = \begin{cases} \epsilon_l^n, & \text{when } S_l - \frac{h}{2} < S \leq S_l + \frac{h}{2}, \quad l = 1, 2, \dots, L-1, \\ 0, & \text{when } 0 \leq S < \frac{h}{2} \quad \text{or} \quad S_{\max} - \frac{h}{2} < S \leq S_{\max} + \frac{h}{2}, \end{cases} \quad (50)$$

Then, $\epsilon^n(S)$ can be represented by the following Fourier series:

$$\epsilon^n(S) = \sum_{j=1}^{\infty} q_n(j) e^{i2\pi j S / S_{\max}}, \quad n = 1, 2, \dots, N, \quad (51)$$

where

$$q_n(j) = \frac{1}{S_{\max}} \int_0^{S_{\max}} \epsilon^n(S) e^{-i2\pi j S / S_{\max}} dS, \quad n = 1, 2, \dots, N. \quad (52)$$

Let $\epsilon^n = (\epsilon_1^n, \epsilon_2^n, \dots, \epsilon_{L-1}^n)^T$ and its norm is defined as follows:

$$\|\epsilon^n\|_2 = \left(\sum_{l=1}^{L-1} h |\epsilon_l^n|^2 \right)^{1/2} = \left(\int_0^{S_{\max}} |\epsilon^n(S)|^2 dS \right)^{1/2}. \quad (53)$$

Applying the Parseval equality

$$\int_0^{S_{\max}} |\epsilon^n(S)|^2 dS = \sum_{j=-\infty}^{\infty} |q_n(j)|^2, \quad (54)$$

we obtain

$$\|\epsilon^n\|_2^2 = \int_0^{S_{\max}} |\epsilon^n(S)|^2 dS = \sum_{j=-\infty}^{\infty} |q_n(j)|^2. \quad (55)$$

Therefore, we can propose that the solution to (48) and (49) takes the following form:

$$\epsilon^n = q_n e^{i\beta lh}, \quad (56)$$

where $\beta = 2\pi j/S_{max}$ and $i = \sqrt{-1}$. By substituting the expression for ϵ^n into (48) and (49), we obtain

$$\begin{aligned} -\frac{\psi}{2h^2} q_1 e^{i\beta(l+1)h} + \left(\frac{\psi}{h^2} \sigma_0\right) &+ q_1 e^{i\beta lh} - \frac{\psi}{2h^2} q_1 e^{i\beta(l-1)h} \\ &= \frac{\psi}{2h^2} q_0 e^{i\beta(l+1)h} - \left(\frac{\psi}{h^2} + \sigma_0\right) q_0 e^{i\beta lh} \\ &+ \frac{\psi}{2h^2} q_0 e^{i\beta(l-1)h}, \quad n = 0, \end{aligned} \quad (57)$$

and

$$\begin{aligned} -\frac{\psi}{2h^2} q_{n+1} e^{i\beta(l+1)h} &+ \left(\frac{\psi}{h^2} + \sigma_0\right) q_{n+1} e^{i\beta lh} - \frac{\psi}{2h^2} q_{n+1} e^{i\beta(l-1)h} \\ &= \frac{\psi}{2h^2} q_n e^{i\beta(l+1)h} - \left(\frac{\psi}{h^2} + \sigma_0\right) q_n e^{i\beta lh} \\ &+ \frac{\psi}{2h^2} q_n e^{i\beta(l-1)h} - \sum_{j=1}^n \sigma_j q_{n-j+1} e^{i\beta lh}, \quad n \geq 1, \end{aligned} \quad (58)$$

which, after simplifications, leads to

$$q_1 \left(-\frac{\psi}{2h^2} (e^{i\beta h} + e^{-i\beta h}) + \left(\frac{\psi}{h^2} + \sigma_0\right) \right) = q_0 \left(\frac{\psi}{2h^2} (e^{i\beta h} + e^{-i\beta h}) - \left(\frac{\psi}{h^2} + \sigma_0\right) \right), \quad n = 0 \quad (59)$$

and

$$\begin{aligned} q_{n+1} \left(-\frac{\psi}{2h^2} (e^{i\beta h} + e^{-i\beta h}) + \left(\frac{\psi}{h^2} + \sigma_0\right) \right) &= q_n \left(\frac{\psi}{2h^2} (e^{i\beta h} + e^{-i\beta h}) - \left(\frac{\psi}{h^2} + \sigma_0\right) \right) \\ &- \sum_{j=1}^n \sigma_j q_{n-j+1}, \quad n \geq 1. \end{aligned} \quad (60)$$

These imply that

$$q_1 \left(-\frac{\psi}{h^2} \cos \beta h + \frac{\psi}{h^2} + \sigma_0 \right) = q_0 \left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0\right) \right), \quad n = 0, \quad (61)$$

and

$$q_{n+1} \left(-\frac{\psi}{h^2} \cos \beta h + \frac{\psi}{h^2} + \sigma_0 \right) = q_n \left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0\right) \right) - \sum_{j=1}^n \sigma_j q_{n-j+1}, \quad n \geq 1. \quad (62)$$

Proposition 1. Suppose that q_{n+1} satisfies (61) and (62), and then $|q_{n+1}| \leq |q_0|$ for all $n = 0, 1, 2, \dots, N$.

Proof. Suppose that $n = 0$. From (61), we have

$$|q_1| = \left| \frac{-\frac{\psi}{h^2} \cos \beta h + (\frac{\psi}{h^2} + \sigma_0)}{\frac{\psi}{h^2} \cos \beta h - (\frac{\psi}{h^2} + \sigma_0)} q_0 \right| = \left| \frac{-(\frac{\psi}{h^2} \cos \beta h - (\frac{\psi}{h^2} + \sigma_0))}{\frac{\psi}{h^2} \cos \beta h - (\frac{\psi}{h^2} + \sigma_0)} q_0 \right| = |-q_0| \leq |q_0|. \quad (63)$$

Now, suppose that $|q_n| \leq |q_0|$ for $n = 1, 2, 3, \dots, N$. We need to show that $|q_{n+1}| \leq |q_0|$. From (62), we have

$$\begin{aligned}
 |q_{n+1}| &\leq \left| \frac{q_n \left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)}{-\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)} \right| + \left| \frac{-\sum_{j=1}^n \sigma_j q_{n-j+1}}{-\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)} \right|, \\
 &\leq \left| \frac{q_n \left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)}{\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)} \right| + \left| \frac{\sum_{j=1}^n \sigma_j q_{n-j+1}}{\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)} \right|, \\
 &= |q_n| + \frac{1}{\left| \left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right) \right|} \sum_{j=1}^n |\sigma_j q_{n-j+1}|, \\
 &\leq |q_n| + \sum_{j=1}^n |\sigma_j q_{n-j+1}|, \left(\because \frac{1}{\left| \frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right|} \leq 1 \right), \\
 &= |q_n| + \sigma_1 |q_n| + \sigma_2 |q_{n-1}| + \dots + \sigma_n |q_1| \\
 &\leq |q_0| + \sigma_1 |q_0| + \sigma_2 |q_0| + \dots + \sigma_n |q_0| \\
 &= \sum_{j=1}^n \sigma_j |q_0| = |q_0| \left(\text{since } \sum_{j=1}^n \sigma_j = 1 \right). \tag{64}
 \end{aligned}$$

From Proposition 1, coupled with the Parseval equality, we obtain, $\|\epsilon^n\|_2 \leq \|\epsilon^0\|_2$ for all $n = 1, 2, \dots, N-1$, which leads us to the following theorem. $\square$

Theorem 1. The difference scheme in (45) and (46) is unconditionally stable.

The proof of Theorem 1 follows from the results from Equation (48) to Equation (64).

3.2. Convergence of the Difference Scheme

To analyse for convergence, we use the concept of Fourier analysis. To begin, let us denote the truncation error at grid point (t_{n+1}, s_l) by R_l^{n+1} . Then, from Equation (37) and part of (44), we have

$$|R_l^n| \leq C(k^2 + h^2), \quad l = 1, 2, \dots, L-1; \quad n = 0, 1, 2, \dots, N-1, \tag{65}$$

where C is a constant given by

$$C = \max_{1 \leq l \leq L-1, 0 \leq n \leq N-1} \{C_l^n\},$$

for some constants C_l^n independent of h and k .

Let $\xi_l^n = p(s_l, t_n) - p_l^n$ denote the approximation error at grid point (t_n, s_l) such that $\xi_L^n = 0$ for $n = 1, 2, \dots, N$, and $\xi_l^0 = 0$ for $l = 0, 1, \dots, L$. By substituting ξ_l^n into the scheme in (45) and (46), we obtain

$$-\frac{\psi}{2h^2} \xi_{l+1}^1 + \left(\frac{\psi}{h^2} + \sigma_0 \right) \xi_l^1 - \frac{\psi}{2h^2} \xi_{l-1}^1 = R_l^1, \tag{66}$$

for $n = 0$ and,

$$\begin{aligned}
 -\frac{\psi}{2h^2} \xi_{l+1}^{n+1} + \left(\frac{\psi}{h^2} + \sigma_0 \right) \xi_l^{n+1} - \frac{\psi}{2h^2} \xi_{l-1}^{n+1} &= \frac{\psi}{2h^2} \xi_{l+1}^n - \left(\frac{\psi}{h^2} + \sigma_0 \right) \xi_l^n + \frac{\psi}{2h^2} \xi_{l-1}^n \\
 &\quad - \sum_{j=1}^n \sigma_j \xi_l^{n-j+1} + R_l^{n+1}; \quad n \geq 1. \tag{67}
 \end{aligned}$$

Similar to the stability analysis, we define the following grid functions:

$$\xi^n(S) = \begin{cases} \xi_l^n, & \text{when } S_l - \frac{h}{2} < S \leq S_l + \frac{h}{2}, \quad l = 1, 2, \dots, L-1, \\ 0, & \text{when } 0 \leq S < \frac{h}{2} \quad \text{or} \quad S_{\max} - \frac{h}{2} < S \leq S_{\max} + \frac{h}{2}, \end{cases} \quad (68)$$

$$R^n(S) = \begin{cases} R_l^n, & \text{when } S_l - \frac{h}{2} < S \leq S_l + \frac{h}{2}, \quad l = 1, 2, \dots, L-1, \\ 0, & \text{when } 0 \leq S < \frac{h}{2} \quad \text{or} \quad S_{\max} - \frac{h}{2} < S \leq S_{\max} + \frac{h}{2}, \end{cases} \quad (69)$$

which implies that $\xi^n(S)$ and R_l^n have the following Fourier-series representations:

$$\xi^n(S) = \sum_{j=1}^{\infty} \tau_n(j) e^{i2\pi j S / S_{\max}}; \quad n = 1, 2, \dots, N, \quad (70)$$

$$R^n(S) = \sum_{j=1}^{\infty} \nu_n(j) e^{i2\pi j S / S_{\max}}; \quad n = 1, 2, \dots, N, \quad (71)$$

where

$$\tau_n(j) = \frac{1}{S_{\max}} \int_0^{S_{\max}} \xi^n(S) e^{-i2\pi j S / S_{\max}} dS; \quad n = 1, 2, \dots, N. \quad (72)$$

$$\nu_n(j) = \frac{1}{S_{\max}} \int_0^{S_{\max}} R^n(S) e^{-i2\pi j S / S_{\max}} dS; \quad n = 1, 2, \dots, N. \quad (73)$$

Let $\xi^n = (\xi_1^n, \xi_2^n, \dots, \xi_{L-1}^n)^T$ and $R^n = (R_1^n, R_2^n, \dots, R_{L-1}^n)^T$, and let us define their norms as follows:

$$\|\xi^n\|_2 = \left(\sum_{l=1}^{L-1} h |\xi_l^n|^2 \right)^{1/2} = \left(\int_0^{S_{\max}} |\xi^n(S)|^2 dS \right)^{1/2}, \quad (74)$$

$$\|R^n\|_2 = \left(\sum_{l=1}^{L-1} h |R_l^n|^2 \right)^{1/2} = \left(\int_0^{S_{\max}} |R^n(S)|^2 dS \right)^{1/2}, \quad (75)$$

and apply the following Parseval equalities:

$$\int_0^{S_{\max}} |\xi^n(S)|^2 dS = \sum_{j=-\infty}^{\infty} |\tau_n(j)|^2; \quad n = 1, 2, \dots, N; \quad (76)$$

$$\int_0^{S_{\max}} |R^n(S)|^2 dS = \sum_{j=-\infty}^{\infty} |\nu_n(j)|^2; \quad n = 1, 2, \dots, N; \quad (77)$$

to obtain

$$\|\xi^n\|_2^2 = \sum_{j=-\infty}^{\infty} |\tau_n(j)|^2; \quad n = 1, 2, \dots, N. \quad (78)$$

$$\left\| R^n \right\|_2^2 = \sum_{j=-\infty}^{\infty} |\nu_n(j)|^2; \quad n = 1, 2, \dots, N. \quad (79)$$

Based on these analyses, we can, therefore, propose that

$$\xi^n = \tau_n e^{i\beta l h} \quad \text{and} \quad R^n = \nu_n e^{i\beta l h}, \quad (80)$$

where $\beta = 2\pi j/S_{max}$ and $i = \sqrt{-1}$. By substituting the expressions in (80) into (66) and (67), we obtain

$$-\frac{\psi}{2h^2} \tau_1 e^{i\beta(l+1)h} + \left(\frac{\psi}{h^2} + \sigma_0 \right) \tau_1 e^{i\beta l h} - \frac{\psi}{2h^2} \tau_1 e^{i\beta(l-1)h} = \nu_1 e^{i\beta l h}, \quad n = 0, \quad (81)$$

and

$$\begin{aligned} -\frac{\psi}{2h^2} \tau_{n+1} e^{i\beta(l+1)h} + \left(\frac{\psi}{h^2} + \sigma_0 \right) \tau_{n+1} e^{i\beta l h} - \frac{\psi}{2h^2} \tau_{n+1} e^{i\beta(l-1)h} \\ = \frac{\psi}{2h^2} \tau_n e^{i\beta(l+1)h} - \left(\frac{\psi}{h^2} + \sigma_0 \right) \tau_n e^{i\beta l h} \\ + \frac{\psi}{2h^2} \tau_n e^{i\beta(l-1)h} - \sum_{j=1}^n \sigma_j \tau_{n-j+1} e^{i\beta l h} \\ + \nu_{n+1} e^{i\beta l h}; \quad n \geq 1, \end{aligned} \quad (82)$$

which, after simplification, leads to

$$\tau_1 \left(-\frac{\psi}{2h^2} (e^{i\beta h} + e^{-i\beta h}) + \left(\frac{\psi}{h^2} + \sigma_0 \right) \right) = \nu_1, \quad (83)$$

and

$$\begin{aligned} \tau_{n+1} \left(-\frac{\psi}{2h^2} (e^{i\beta h} + e^{-i\beta h}) + \left(\frac{\psi}{h^2} + \sigma_0 \right) \right) &= \tau_n \left(\frac{\psi}{2h^2} (e^{i\beta h} + e^{-i\beta h}) - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right) \\ &- \sum_{j=1}^n \sigma_j \tau_{n-j+1} + \nu_{n+1}, \quad n \geq 1. \end{aligned} \quad (84)$$

These imply that for $n = 0$,

$$\tau_1 \left(-\frac{\psi}{h^2} \cos \beta h + \frac{\psi}{h^2} + \sigma_0 \right) = \nu_1, \quad (85)$$

and for $n \geq 1$,

$$\tau_{n+1} \left(-\frac{\psi}{h^2} \cos \beta h + \frac{\psi}{h^2} + \sigma_0 \right) = \tau_n \left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right) - \sum_{j=1}^n \sigma_j \tau_{n-j+1} + \nu_{n+1}, \quad (86)$$

which leads to the following equalities:

$$\tau_1 = -\frac{\nu_1}{\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)}, \quad (87)$$

and

$$\tau_{n+1} = -\left(\tau_n + \frac{\sum_{j=1}^n \sigma_j \tau_{n-j+1} + \nu_{n+1}}{\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0 \right) \right)} \right). \quad (88)$$

Now, we have the following proposition.

Proposition 2. Suppose that τ_n for $n = 0, 1, \dots, N$ is a solution to Equations (87) and (88). Then, there exists a positive constant C_1 such that $|\tau_n| \leq C_1|v_1|$ for all n .

Proof. Notice that when $n = 0$, from (87), we have

$$|\tau_1| \leq \left| \frac{v_1}{\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0\right)\right)} \right| \leq C_1|v_1|. \quad (89)$$

Suppose that $|\tau_n| \leq C_0|v_1|$ for $n = 1, 2, \dots, N$, for some constant C independent of h and k . Then,

$$\begin{aligned} |\tau_{n+1}| &\leq \left| \tau_n + \frac{\sum_{j=1}^n \sigma_j \tau_{n-j+1} + v_{n+1}}{\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0\right)\right)} \right|, \\ &\leq |\tau_n| + \left| \frac{\sum_{j=1}^n \sigma_j \tau_{n-j+1} + v_{n+1}}{\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0\right)\right)} \right|, \\ &\leq C_0|v_1| + \sum_{j=1}^n \frac{1}{\left|\left(\frac{\psi}{h^2} \cos \beta h - \left(\frac{\psi}{h^2} + \sigma_0\right)\right)\right|} (\sigma_j |\tau_{n-j+1}| + |v_{n+1}|), \\ &\leq C_0|v_1| + \sum_{j=1}^n C_j (\sigma_j |\tau_{n-j+1}| + |v_{n+1}|), \\ &\leq C_0|v_1| + \sum_{j=1}^n \sigma_j C_j |\tau_{n-j+1}| + C_{n+1} |v_1|, \\ &\leq C_0|v_1| + \sum_{j=1}^n \sigma_j C_j |v_1| + C_{n+1} |v_1|, \\ &= C_0|v_1| + \sigma_1 C_1 |v_1| + \sigma_2 C_2 |v_1| + \dots + \sigma_n C_n |v_1| + C_{n+1} |v_1|, \\ &\leq \hat{C} (|v_1| + \sum_{j=1}^n \sigma_j |v_1| + |v_1|), \left(\hat{C} = \max_{0 \leq j \leq n+1} \{C_j\} \right) \\ &= \hat{C} \left(2 + \sum_{j=1}^n \sigma_j \right) |v_1| \\ &= C |v_1|. \end{aligned} \quad (90)$$

The following theorem, therefore, holds. $\square$

Theorem 2. The difference scheme in (45) and (46) is convergent and converges with order $\mathcal{O}(k^2, h^2)$.

The proof for Theorem 2 follows from Equation (66) to Equation (90).

4. Numerical Experiments

In this section, we numerically investigate the pricing of European put options using the tfBS-PDE (31), implemented via schemes (45) and (46), with the initial and boundary conditions in (47). We consider two distinct examples with varying dividend yields, and the orders of the fractional derivative (α) are fixed at 0.3, 0.5, 0.7, and 0.9. The numerical solutions obtained using the proposed method show that the option maturity payoff curves obtained are in very good agreement with the European put option intrinsic payoff curves obtained under all possible parameter settings.

Example 1. Consider Equation (31) subject to the conditions in (47) for pricing a European put option with the following parameters: $K = 150$, $r = 0.08$, $\sigma = 0.1$, $T = 1$, $S_{\max} = 450$, $L = 100$, $N = 200$, and $\delta = 0.035, 0.05$, and 0.10 .

Example 2. Consider (31), again, subject to the conditions in (47), but under the following parameters: $K = 150$, $r = 0.065$, $\sigma = 0.13$, $T = 1$, $S_{\max} = 600$, $L = 30$, $N = 100$, and $\delta = 0.045, 0.085$, and 0.11 .

The tabular results for Example 1 and Example 2, with N starting from 30 for Example 1 and 50 for Example 2, are presented in Tables 1–4. The numerical results herein confirm that our results are in excellent agreement with our theoretical deductions presented in Theorem 1 and Theorem 2 under Section 3.1 and Section 3.2, respectively. That is, our numerical scheme in (45) and (46) is unconditionally stable and converges with order $\mathcal{O}(k^2, h^2)$.

Similar to Example 1, for Example 2, we only present the payoff results when $\delta = 0.045$ for all four considered values of α . The maturity payoff curves for $\delta = 0.045$ for all four values of α are presented in Figure 1, and their respective general payoffs are presented in Figure 2.

Due to space limitations and the fact that the results are quite similar, of course, not in terms of their numerical values but in terms of the shapes of their curves, for the three considered dividend yields (δ), herein, we only present the results for the case when $\delta = 0.035$ for all four values of α . The maturity payoff curves for $\delta = 0.035$ are shown in Figure 3, and the general payoffs throughout the lifespan of the option for all considered values of α are shown in Figure 4.

Table 1. Maximum absolute errors for Example 1 with $r = 0.08$ and $\delta = 0.035$.

α	$N = 30$	$N = 60$	$N = 120$	$N = 240$	$N = 480$
0.1	7.5592×10^{-3}	1.8952×10^{-3}	4.7712×10^{-4}	1.1987×10^{-4}	3.0128×10^{-5}
0.2	6.8088×10^{-3}	1.8694×10^{-3}	4.7170×10^{-4}	1.1825×10^{-4}	3.3784×10^{-5}
0.3	6.2147×10^{-3}	1.6191×10^{-3}	4.3368×10^{-4}	1.1698×10^{-4}	3.1352×10^{-5}
0.4	5.7443×10^{-3}	1.5001×10^{-3}	4.0358×10^{-4}	1.1684×10^{-4}	2.9426×10^{-5}
0.5	5.3746×10^{-3}	1.4966×10^{-3}	3.7993×10^{-4}	1.1651×10^{-4}	2.7912×10^{-5}
0.6	5.0893×10^{-3}	1.4044×10^{-3}	3.6167×10^{-4}	1.1619×10^{-4}	2.1544×10^{-5}
0.7	4.8773×10^{-3}	1.3880×10^{-3}	3.4810×10^{-4}	1.1607×10^{-4}	2.1676×10^{-5}
0.8	4.7318×10^{-3}	1.2500×10^{-3}	3.3879×10^{-4}	1.1405×10^{-4}	1.9950×10^{-5}
0.9	4.6499×10^{-3}	1.1682×10^{-3}	2.9355×10^{-4}	7.3797×10^{-5}	1.8575×10^{-5}
1.0	4.6328×10^{-3}	1.1645×10^{-3}	3.0246×10^{-4}	7.5988×10^{-5}	1.9670×10^{-5}

Table 2. Convergence rates for Example 1 with $r = 0.08$ and $\delta = 0.035$.

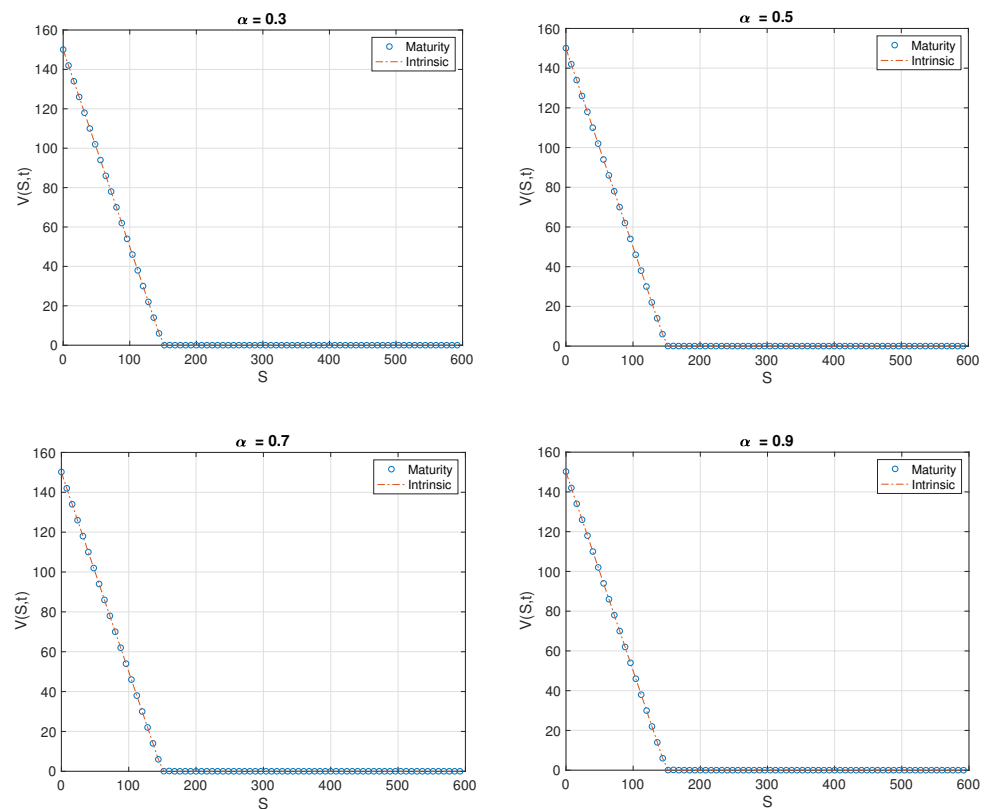
α	$N = 60$	$N = 120$	$N = 240$	$N = 480$
0.1	1.67	1.86	1.88	1.89
0.2	1.66	1.87	1.88	1.89
0.3	1.68	1.87	1.89	1.90
0.4	1.68	1.87	1.89	1.91
0.5	1.68	1.88	1.89	1.95
0.6	1.68	1.88	1.89	1.96
0.7	1.78	1.88	1.89	1.98
0.8	1.89	1.88	1.89	1.98
0.9	1.89	1.88	1.89	1.98
1.0	1.89	1.87	1.89	1.98

Table 3. Maximum absolute errors for Example 2 with $r = 0.065$ and $\delta = 0.045$.

α	$N = 50$	$N = 100$	$N = 200$	$N = 400$	$N = 800$
0.1	1.4714×10^{-3}	3.5691×10^{-4}	8.8754×10^{-5}	2.2298×10^{-5}	5.6053×10^{-6}
0.2	1.3394×10^{-3}	3.2352×10^{-4}	7.9308×10^{-5}	2.0162×10^{-5}	5.0648×10^{-6}
0.3	1.1162×10^{-3}	2.6506×10^{-4}	6.6025×10^{-5}	1.6549×10^{-5}	4.1509×10^{-6}
0.4	9.6623×10^{-4}	2.5089×10^{-4}	6.2188×10^{-5}	1.5578×10^{-5}	3.7253×10^{-6}
0.5	8.6515×10^{-4}	2.1632×10^{-4}	5.5720×10^{-5}	1.3942×10^{-5}	3.5015×10^{-6}
0.6	8.1315×10^{-4}	2.1216×10^{-4}	5.2393×10^{-5}	1.3100×10^{-5}	3.2885×10^{-6}
0.7	7.6609×10^{-4}	2.0126×10^{-4}	4.9381×10^{-5}	1.2439×10^{-5}	3.0859×10^{-6}
0.8	7.4065×10^{-4}	1.9183×10^{-4}	4.7753×10^{-5}	1.1927×10^{-5}	2.9817×10^{-6}
0.9	7.2436×10^{-4}	1.9170×10^{-4}	4.6711×10^{-5}	1.1663×10^{-5}	2.9350×10^{-6}
1.0	7.2202×10^{-4}	1.8911×10^{-4}	4.6561×10^{-5}	1.1625×10^{-5}	2.9254×10^{-6}

Table 4. Convergence rates for Example 2 with $r = 0.065$ and $\delta = 0.045$.

α	$N = 100$	$N = 200$	$N = 400$	$N = 800$
0.1	1.78	1.89	1.97	1.97
0.2	1.79	1.92	1.96	1.98
0.3	1.79	1.93	1.97	1.98
0.4	1.80	1.94	1.97	1.99
0.5	1.88	1.95	1.98	1.99
0.6	1.88	1.96	1.98	1.99
0.7	1.88	1.96	1.98	1.99
0.8	1.88	1.97	1.98	1.99
0.9	1.89	1.97	1.98	1.99
1.0	1.89	1.97	1.98	1.99

**Figure 1.** Maturity payoffs for $\delta = 0.045$ with $\alpha = (0.3, 0.5, 0.7, \text{ and } 0.9)$, respectively.

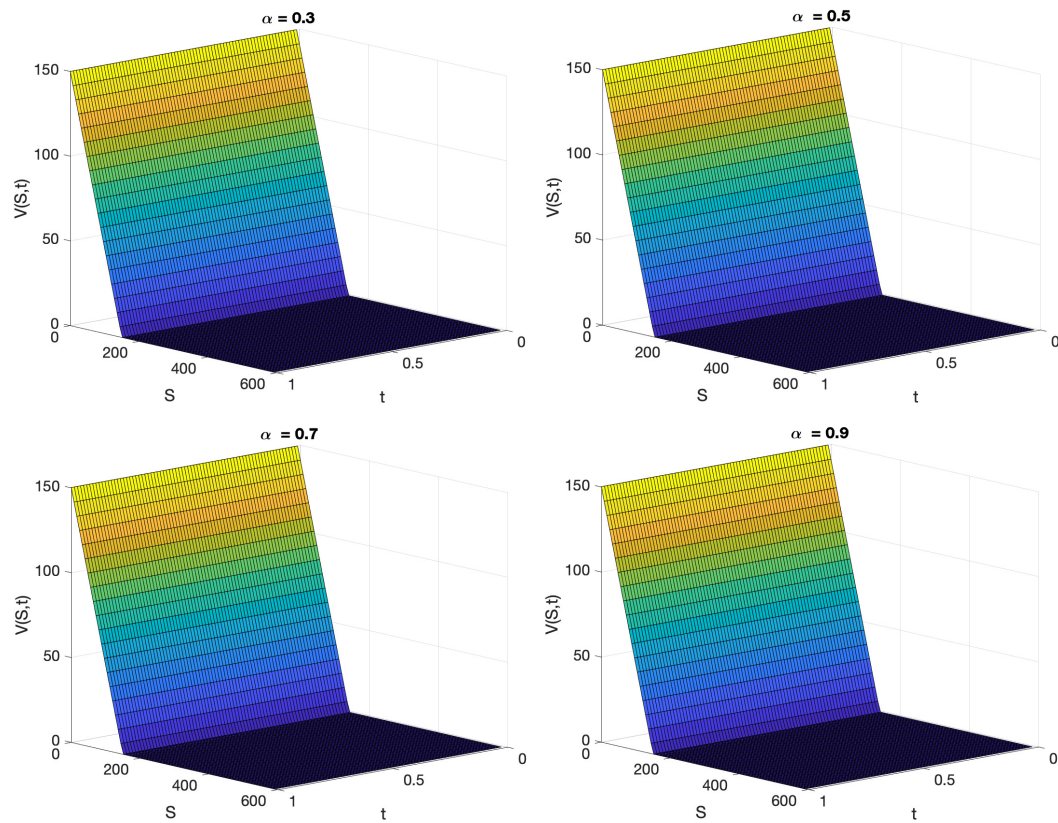


Figure 2. General payoffs for $\delta = 0.045$ with $\alpha = (0.3, 0.5, 0.7, \text{ and } 0.9)$, respectively.

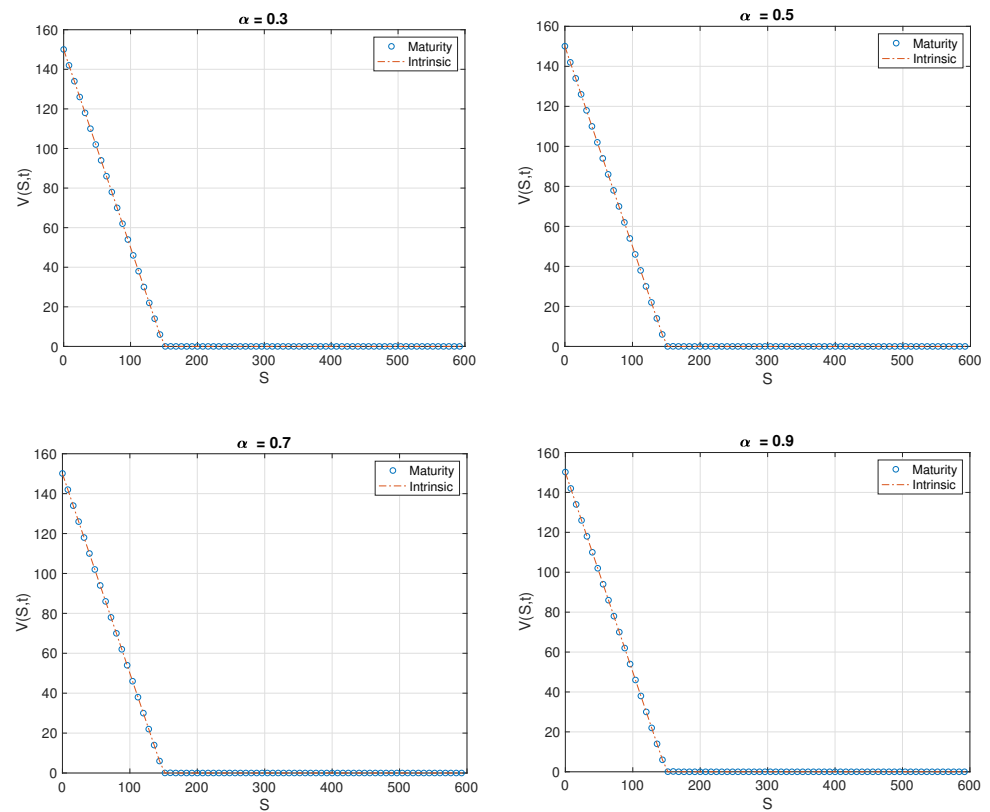


Figure 3. Maturity payoffs for $\delta = 0.035$ with $\alpha = (0.3, 0.5, 0.7, \text{ and } 0.9)$, respectively,

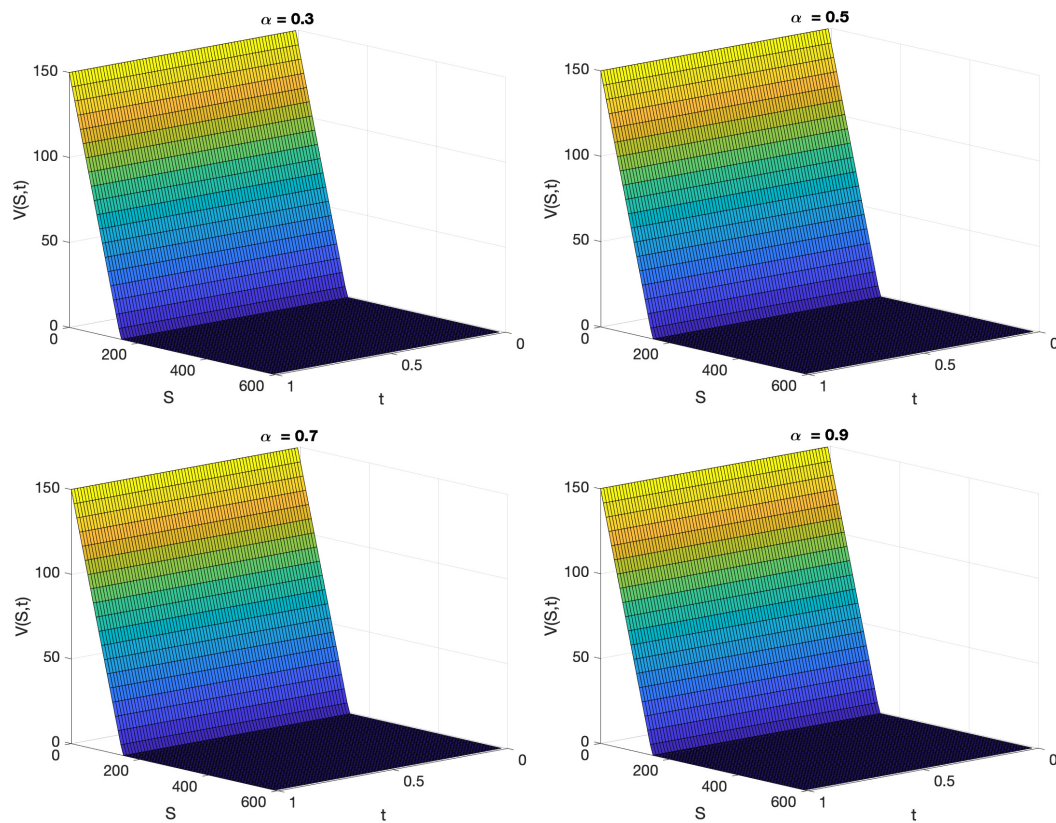


Figure 4. General payoffs for $\delta = 0.035$ with $\alpha = (0.3, 0.5, 0.7, \text{ and } 0.9)$, respectively.

5. Concluding Remarks and Future Directions

The fractional approach is a very effective approach for asset pricing, as it provides a unique framework that has the potential to transform conventional thinking on asset-pricing theory and applications. The involved non-local derivatives and integral operators, as well as the accompanying fractional stochastic dynamics, provide the best tools for explaining trend and noise memory effects, as well as providing non-localised information about stock price movements, something classical models may fail to explain. Since the fractional derivative operators are of a non-local nature, there is little to no existing knowledge of analytic solutions to fractional BS models. As such, numerical methods are the only tractable available avenues to help understand the nature of the solutions to these models.

In this paper, we have successfully transformed a standard tfBS-PDE into a solvable tfBS-PDE in the form of a heat equation. In general, this transformation is necessary, as it helps ease the mathematical difficulties and complexities involved in solving the original tfBS-PDE using the high-order numerical method presented herein. We have constructed a robust and high-order numerical scheme for solving the resulting model. From a simulation point of view, we have considered two examples, and the numerical results in Tables 1–4 are in agreement with theoretical observations, indicating that the method is unconditionally stable and converges up to order two. Besides our results being in agreement with classical theory, we further observed that our approach provides a very effective, convenient, and powerful mathematical tool for asset pricing.

Although the asymptotic long-term behaviours of markets tend to be similar whether the dynamics are driven by typical Gaussian processes or non-Gaussian processes (i.e., fractal processes), we have observed that incorporating fractional parameters often describes the dynamics much more accurately and with greater flexibility. This fact holds particularly true in markets exhibiting empirical evidence of memory or those presenting non-random power-law properties, which are often not predictable using ordinary random processes.

The application of the proposed approach to other models and their calibration to real-time market data remain the subject of future research.

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