Article

β–Ulam–Hyers Stability and Existence of Solutions for Non-Instantaneous Impulsive Fractional Integral Equations

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1. Introduction and Preliminaries

The impulsive differential equations describe evolution processes characterized by the fact that at certain moments they experience an abrupt change of state, i.e., these processes are subject to short-term perturbations whose duration is negligible compared with the duration of the whole process (see, e.g., [1–6] for more details on the subject). Hence, there is a substantial level of interest in assessing the quality of solutions for these impulsive systems. In a broader context, the classical instantaneous impulses fall short in capturing certain aspects of evolution processes. The establishment of hemodynamic equilibrium in an individual exemplifies this principle, wherein drugs are introduced into the bloodstream and assimilated by the body over a prolonged and continuous duration (see [7] (Example 1.1.3)). This particular scenario depicts a unique manifestation of impulsive behavior, commencing at a defined moment and enduring for a limited duration thereafter. In ecological studies, the population dynamics of species are often subject to various forms of disturbances, such as natural disasters, climate changes, and human interventions [1,7,8]. Traditional ecological models, which rely on instantaneous perturbations, may struggle to represent the gradual and continuous impacts of these disturbances. As a general theory, non-instantaneous impulsive fractional differential equations have significant potential to be applied in these areas to better capture the continuous and memory-like characteristics of drug absorption, ecology, and population dynamics. For example, this approach provides a more accurate representation of how drugs are gradually absorbed into the bloodstream, making it an essential tool for pharmaceutical research and dosage optimization.
We recall the definition of a $\beta$-Banach space from [9]. Let $X$ be a vector space over the field $\mathbb{K}$ and $\beta \in (0, 1]$. A function $\| \cdot \|_\beta : X \to [0, +\infty)$ is called a $\beta$-norm if and only if the following holds:

(i) $\|w\|_\beta = 0$ if and only if $w = 0$;
(ii) $\|\lambda w\|_\beta = |\lambda|\|w\|_\beta$ for all $\lambda \in \mathbb{K}$ and $w \in X$;
(iii) $\|w + v\|_\beta \leq \|w\|_\beta + \|v\|_\beta$.

The pair $(X, \| \cdot \|_\beta)$ is called a $\beta$-normed space. A complete $\beta$-normed space is called a $\beta$-Banach space. Let $I = [0, 1]$. The space $C(I : \mathbb{R})$ signifies the $\beta$-Banach space comprising continuous functions $w : I \to \mathbb{R}$, endowed with the $\beta$-norm

$$\|w\|_\beta := \max\{|w(s)|^\beta : s \in I, 0 < \beta \leq 1\}.$$ 

The $\beta$-Banach space $PC(I : \mathbb{R})$ is given by $PC(I : \mathbb{R}) = \{f : I \to \mathbb{R}, f \in C([t_i, t_{i+1}] : \mathbb{R}), f(t_i^-) = f(t_i)$ and $f(t_i^+)$ exists for any $i = 0, 1, 2, \ldots, m\}$, where the symbols $f(t_i^-)$ and $f(t_i^+)$ denote the left and the right limits of the function $f(t)$ at the point $t = t_i$, $i = 0, 1, 2, \ldots, m$, and it is endowed with the $P\beta$-norm $\|f\|_{P\beta} = \sup\{|f(s)|^\beta : s \in I, 0 < \beta \leq 1\}$.

Moreover, we denote $\|f\|_{PC} = \max\{\sup_{s \in J} f(s^+), \sup_{s \in J} f(s^-)\},$ because $f \in PC(I : \mathbb{R})$.

Let $w \in L^1(I)$, $\gamma > 0$. Recall that the definition of the Riemann–Liouville fractional integral of $w$ of order $\gamma$ is given by

$$I_0^\alpha w(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} w(s) \, ds.$$

For a more detailed approach to fractional calculus, interested readers may consult [1,10–12] and the references therein.

In light of more generalized theories, E. Hernández and D. O’Regan [13] originally embarked on a study of a novel class of abstract cases involving semilinear impulsive differential modeling that excluded instantaneous impulses; cf. also the research article [14] by M. Pierri, D. O’Regan, and V. Rolnik. Nevertheless, it is noteworthy that drug absorption differential modeling that excluded instantaneous impulses; cf. also the research article [14] by M. Pierri, D. O’Regan, and V. Rolnik. Nevertheless, it is noteworthy that drug absorption exhibits a memory effect. Furthermore, fractional calculus, with its memory and hereditary properties, offers a more realistic approach to modeling of the aforementioned ecological systems. As indicated by I. Podlubny [11], fractional-order differential equations can effectively describe complex ecological dynamics by accounting for the lasting effects of past disturbances.

The famous problem for the stability of functional equations originally posed by Ulam [15] was the following:

**(Ulam)** Assume that $Z_1$ is a group and $Z_2$ is a metric group with the metric $\rho$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $U : Z_1 \to Z_2$ satisfies the inequality

$$\rho(U(ab), U(a)U(b)) < \delta$$

for all $a, b \in Z_1$, then there exists a homomorphism $P : Z_1 \to Z_2$ with

$$\rho(U(c), P(c)) < \varepsilon$$

for all $c \in Z_1$?

The problem of Ulam stability [15], as extensively explored by many mathematicians, has garnered considerable attention. For a more comprehensive understanding of this topic, interested readers are encouraged to consult the remarkable monographs by D.H. Hyers et al. [16], S.-M. Jung [9], and M.T. Rassias [17], as well as the recent contributions [18–24] in normed spaces. The corresponding results in the realm of $\beta$-normed spaces can be found in [25–27].

In recent studies, significant advancements have been made in the theory and applications of non-instantaneous impulsive fractional differential equations. Kostić et al. [28]
studied generalized ρ-almost periodic sequences and established several new results on the existence and uniqueness of solutions for the abstract impulsive Volterra integro-differential equations. In [29], the existence and uniqueness of continuous solutions for a class of the fractional quadratic functional integro-differential equations with nonlocal fractional-order integro-differential condition were studied. Some recent results on the β-Ulam–Hyers stability of fractional differential equations and also on the existence and uniqueness of piecewise continuous solutions on non-instantaneous impulsive fractional differential equations are given in [27]. The results on the solvability of nonlocal fractional integro-differential equations are presented in [30,31], while the impulsive integro–differential equations are considered in [32]. Agarwal et al. extensively covered both basic and advanced theoretical aspects of non-instantaneous impulsive Caputo fractional differential equations in their works [7,33]. Shah et al. investigated the Hyers–Ulam stability and exponential dichotomy of impulsive linear systems of the first order [34], while Luo et al. considered the existence and Hyers–Ulam stability of solutions for fractional differential equations involving time-varying delays and non-instantaneous impulses [35]. Abbas et al. provided unique insights into Ulam-type stability concepts for the Darboux problem of partial functional differential equations with non-instantaneous impulses in Banach spaces [36]. In a related study, Ibrahim examined the generalized Hyers–Ulam stability for certain types of fractional differential equations in complex Banach spaces [37]. Furthermore, Parthasarathy explored the existence and Hyers–Ulam stability of nonlinear impulsive differential equations with nonlocal conditions [38].

Agarwal et al. also provided sufficient conditions for various types of Ulam–Hyers stability in the context of boundary-value problems for scalar nonlinear differential equations with delays and generalized proportional Caputo fractional derivatives [39]. Dhyal and his collaborators discussed the stability and exact controllability of non-instantaneous ϕ-Caputo fractional systems [40]. The Ulam–Hyers stability of nonlinear ϕ-Hilfer fractional differential equations has been addressed in several studies [41–43].

In another significant contribution, Agarwal et al. obtained uniqueness results using the Banach contraction mapping principle and established existence results with Krasnoselkii’s fixed-point theorem and Leray–Schauder’s nonlinear alternative, while also discussing Ulam–Hyers stability for the considered problem [44]. Qian et al. focused on establishing stability theorems for fractional differential systems with the Riemann–Liouville derivative, covering linear, perturbed, and time-delayed systems [45]. Alam and Shah investigated the Hyers–Ulam stability of coupled implicit fractional integro-differential equations with Riemann–Liouville derivatives under specific conditions [46].

In [27], inspired by the investigation in [14], the following class of non-instantaneous impulsive Caputo fractional differential equations was considered:

\[
\begin{cases}
C_{\Delta_{0+}}D_{s_i}^{\alpha_1}w(t) = -cw(t) + f_i(t, w(t)), & s_i < t \leq t_{i+1}, i = 1, 2, \ldots, m, c > 0, \\
w(t) = a + \int_{s_i}^{t_1}g_i(t, w(t)) - \int_{s_i}^{t_1}f_i(s, w(s))dt, & t_i < t \leq s_i, i = 1, 2, \ldots, m,
\end{cases}
\]

where \(C_{\Delta_{0+}}D_{s_i}^{\alpha_1}\) is the Caputo fractional derivative of the order \(\alpha_1, \alpha_1 \in (0, 1), \alpha_1 \neq \alpha_3\), with the lower limit \(s_i\); \(0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots < t_m \leq s_m \leq t_{m+1} = T\) are prefixed numbers; \(f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous; and \(g_i : [t_i, s_i] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous for all \(i = 1, 2, \ldots, m\), \(a \in \mathbb{R}\), \(\int_{s_i}^{t_1}g_i(t, w(t))dt\) and \(\int_{s_i}^{t_1}f_i(s, w(s))\).

The motivation for investigating problem Equation (1) stems from the above problem and its close connection to the quadratic fractional functional integro-differential equation with a nonlocal fractional integro-differential condition studied in [29], having significant potential for applications. Inspired by the aforementioned papers, this paper will explore the following non-instantaneous impulsive fractional integral problem:
\[
\begin{cases}
  w(t) = w_0 - \int_0^1 f_t(s, w(s), f_{0,i}^{\alpha_1-a_4}v(s)) \, ds + \int_{0}^{t} f_{0,i} v(t), & 0 < t \leq t_1, \\
  w(t) = a + \int_{t_i}^{t} g_i(t, w(t)) - \int_{0}^{t_i} f_i s, w(s)) \, ds, & t_i < t \leq s_i, i = 1, 2, \ldots, m, \alpha_1 \neq \alpha_3, \\
  w(t) = a + \int_{t_i}^{t} g_i(s, w(s)) - \int_{0}^{s_i} f_i s, w(s)) - \int_{0}^{1} f_1 s, w(s), f_{0,i}^{\alpha_1-a_4}v(s) \, ds + \int_{s_i}^{t} v(t), & s_i < t \leq t_{i+1},
\end{cases}
\]

where

\[
v(t) := \int_{0}^{1} f_{0,i}^{-1\alpha_1} \left( t, v(t) - \int_{0}^{t} (t - \theta)^{a_2-1} G \left( \theta, \int_{0}^{\theta} (\theta - \nu)^{a_1-a_3-1} v(\nu) \, d\nu \right) d\theta \right)
\]

and \(a, w_0 \in \mathbb{R}, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (0, 1], \alpha_3, \alpha_4 \leq \alpha_1, \alpha_1 \neq \alpha_3,\) and \(0 \leq s_i \leq t_{i+1} \leq 1\) for \(i = 0, 1, 2, \ldots, m\) are prefixed numbers, defining the intervals where non-instantaneous impulses occur, and \(f_{0,i}^\alpha\) is the fractional Riemann–Liouville integral of order \(\alpha; f : I \times \mathbb{R} \rightarrow \mathbb{R}, g_i : [t_i, s_i] \times \mathbb{R} \rightarrow \mathbb{R}\) for all \(i = 1, 2, \ldots, m\), and \(f_1 : I \times \mathbb{R} \rightarrow \mathbb{R}\) are functions. Note that,

\[
f_{0,i}^{\alpha_1} g_i(t, w(t)) = \frac{1}{\Gamma(\alpha_3)} \int_{t_i}^{t} (t - s)^{\alpha_3-1} g_i(s, w(s)) \, ds
\]

and

\[
f_{0,i}^{\alpha_1} f_i s, w(s)) = \frac{1}{\Gamma(\alpha_1)} \int_{0}^{s_i} (s_i - s)^{\alpha_1-1} f_i s, w(s)) \, ds
\]

for all \(i = 1, 2, \ldots, m\).

As far as we are aware, there has been no prior exploration of the existence of a solution or examination of Ulam-type stability for the Equation (1), nor for analogous instances of non-instantaneous impulsive fractional integral equations in \(\beta\)-normed spaces. In this work, we shall employ techniques from fractional differential equations, non-instantaneous impulsive differential equations, and fixed-point theory within \(\beta\)-Banach spaces to investigate our problem under consideration. The provided example emphasizes the importance of the obtained results and potential for application in the aforementioned and many other areas.

The structure of the paper can be outlined as follows. Following an initial presentation of preliminary definitions and results, the paper proceeds to establish results pertaining to the existence and uniqueness of solutions for the class of non-instantaneous impulsive fractional integral equations under consideration. In Section 3, the focus shifts to investigating the \(\beta\)-Ulam–Hyers stability for non-instantaneous impulsive fractional integral equations. Additionally, in Section 4, the paper incorporates two illustrative examples that underscores the significance of the derived results. The paper is ended by summarizing the obtained outcomes and outlining prospective research directions.

2. The Existence and Uniqueness of Solutions

In this study, we employ a set of conditions that will be specified subsequently. These conditions can be broadly classified into three groups: (C1)–(C3) represent Lipschitz conditions, (C4)–(C6) denote boundedness conditions, and (C7)–(C8) are technical conditions, which are closely related to the considered problem. The conditions (C1)–(C6) are widely recognized in the literature devoted to the existence and uniqueness of the solutions of various types of differential equations.

(C1) The function \(f : I \times \mathbb{R} \rightarrow \mathbb{R}\) is measurable in \(t \in I\) for all \(y \in \mathbb{R}\) and there is a constant \(L_f > 0\), such that \(|f(t, y_1) - f(t, y_2)| \leq L_f |y_1 - y_2|\) for all \(t \in I\) and all \(y_1, y_2 \in \mathbb{R}\);
(C2) The function $g_i : [t_i, s_i] \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to $t \in [t_i, s_i]$, $i = 1, 2, \ldots, m$ for all $y \in \mathbb{R}$ and there are constants $L_{g_i} > 0$, such that $|g_i(t, y_1) - g_i(t, y_2)| \leq L_{g_i}|y_1 - y_2|$ for all $t \in [t_i, s_i]$, $i = 1, 2, \ldots, m$ and all $y_1, y_2 \in \mathbb{R}$;

(C3) Suppose $f_1 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to $t \in I$, for all $(y_1, z_1) \in \mathbb{R} \times \mathbb{R}$ and there is a constant $L_{f_1} > 0$, such that $|f_1(t, y_1, z_1) - f_1(t, y_2, z_2)| \leq L_{f_1}|y_1 - y_2| + |z_1 - z_2|$ for all $t \in I$ and all $y_1, y_2, z_1, z_2 \in \mathbb{R}$;

(C4) Suppose $f : I \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to $t \in I$ for all $y \in \mathbb{R}$ and continuous with respect to $y \in \mathbb{R}$ for all $t \in I$. Additionally, there exists a bounded measurable function $a_f : I \to \mathbb{R}$ and a constant $b_f$, such that $|f(t, y)| \leq |a_f(t)| + b_f|y| \leq a_f^* + b_f|y|$, where $a_f^* = \sup_{t \in I} f_0^{a_f} |a_f(t)|$;

(C5) Suppose $g : I \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to $t \in I$ for all $y \in \mathbb{R}$, continuous with respect to $y \in \mathbb{R}$ for all $t \in I$ and $g_i : [t_i, s_i] \times \mathbb{R} \to \mathbb{R}$ is measurable in $t \in [t_i, s_i]$, $i = 1, 2, \ldots, m$ for all $y \in \mathbb{R}$ and continuous with respect to $y \in \mathbb{R}$ for all $t \in [t_i, s_i]$, $i = 1, 2, \ldots, m$. Additionally, there exists a bounded measurable function $a_g : I \to \mathbb{R}$ and a constant $b_g$, such that $|g(t, y)| \leq |a_g(t)| + b_g|y| \leq a_g^* + b_g|y|$, where $a_g^* = \sup_{t \in I} f_0^{a_g} |a_g(t)|$ and $|g_i(t, y)| \leq |a_{g_i}(t)| + b_{g_i}|y|$ for all $i = 1, 2, \ldots, m$;

(C6) Suppose $f_1 : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to $t \in I$ for all $(y_1, y_2) \in \mathbb{R} \times \mathbb{R}$ and continuous with respect to $y_1, y_2$ for all $t \in I$. Additionally, there exists a bounded measurable function $a_{f_1} : I \to \mathbb{R}$ and a constant $b_{f_1}$, such that $|f_1(t, y_1, y_2)| \leq |a_{f_1}(t)| + b_{f_1}|y_1 + y_2|$, for all $t \in I, y_1, y_2 \in \mathbb{R}$, where $\sup_{t \in I} f_0^{a_{f_1}} |a_{f_1}(t)| \leq N$;

(C7) The number $r_1$ is a positive solution of the equation

$$a_f^* + \left(\frac{b_f a_f}{\Gamma(2 - a_1)} - 1\right) r_1 + \frac{b_f b_f r_f^2}{\Gamma(2 - a_1) \Gamma(a_1 + a_2 - a_3 + 1)} = 0;$$

(C8) It holds $b_f f_1 < 1$.

Employing the Banach contraction principle and the Schauder fixed-point theorem, we will present two theorems concerning the existence and uniqueness of the solutions of the non-instantaneous impulsive fractional integral problem Equation (1).

**Theorem 1.** Let (C1)–(C3) be satisfied and let $0 < \rho < 1$, where

$$\rho = \max_{1 \leq i \leq m} \left(\left(\frac{L_{g_i}(s_i - t_i)^{a_3}}{\Gamma(a_3 + 1)}\right)^{\beta} + \left(\frac{L_{f_1} s_i^{a_1}}{\Gamma(a_1 + 1)}\right)^{\beta} + (L_{f_1})^{\beta}\right).$$

Then, the Equation (1) has a unique solution.

**Proof.** We define the operator $R : \mathcal{P}(I) \to \mathcal{P}(I : \mathbb{R})$ by

$$(R \mu)(t) := \left\{\begin{array}{ll}
\mu_0 - \int_0^t f_1(s, \mu(s), t_0^{a_1 - a_1} \mu(s)) \ ds + \int_0^t v(t), & 0 < t \leq t_1, \\
\left(a + \int_{t_{i-1}}^{t_i} f_1(s, \mu(s), t_0^{a_1 - a_1} \mu(s)) \ ds \right) + \int_{t_{i-1}}^{t_i} v(t), & t_i < t \leq s_i, i = 1, 2, \ldots, m, \\
\left(a + \int_{t_{i-1}}^{t_i} f_1(s, \mu(s), t_0^{a_1 - a_1} \mu(s)) \ ds \right) + \int_{t_{i-1}}^{t_i} v(t), & s_i < t \leq t_{i+1}, i = 1, 2, \ldots, m,
\end{array}\right.$$

where

$$v(t) = f_0^{a_1} f \left( t, \mu(t), \int_0^t \frac{(t - \theta)^{a_2 - 1}}{\Gamma(a_2)} g(\theta, \int_0^\theta (\theta - \nu)^{(a_1 - a_3) - 1} v(\nu) \ d\nu) \ d\theta.\right)$$
It is obvious that the mapping $\mathcal{R}$ is well defined. Hence, we have the following cases:

**Case 1.** Let $w_1, w_2 \in \mathcal{PC}(I : \mathbb{R})$ and $0 < t \leq t_1$. Then,

$$
|\mathcal{R}(w_1)(t) - \mathcal{R}(w_2)(t)| = |w_0 - \int_0^1 f_1(s, w_1(s), f_{0,j}^{a_1-a_4}v(s))\, ds \, + \, \int_0^1 f_1(s, w_2(s), f_{0,j}^{a_1-a_4}v(s))\, ds - f_{0,j}^{a_1}v(t)| \\
\leq \int_0^1 |f_1(s, w_1(s), f_{0,j}^{a_1-a_4}v(s)) - f_1(s, w_2(s), f_{0,j}^{a_1-a_4}v(s))|\, ds \\
\leq \int_0^1 L_{f_1}|w_1(s) - w_2(s)|\, ds \leq L_{f_1}\|w_1 - w_2\|_{\mathcal{PC}}.
$$

Now, since

$$
|\mathcal{R}(w_1)(t) - \mathcal{R}(w_2)(t)|^\beta \leq (L_{f_1})^\beta\|w_1 - w_2\|_{p_\beta},
$$

we obtain

$$
\|\mathcal{R}w_1 - \mathcal{R}w_2\|_{p_\beta} \leq L_{f_1}^\beta\|w_1 - w_2\|_{p_\beta}.
$$

**Case 2.** Let $w_1, w_2 \in \mathcal{PC}(I : \mathbb{R})$ and $t_i < t \leq s_i$, $i = 1, 2, \ldots, m$. Since

$$
|\mathcal{R}(w_1)(t) - \mathcal{R}(w_2)(t)| \leq \frac{1}{\Gamma(a_3)} \int_{t_i}^t (t - s)^{a_3-1}|g_i(s, w_1(s)) - g_i(s, w_2(s))|\, ds \\
+ \frac{1}{\Gamma(a_1)} \int_0^{s_i}(s - s)^{a_1-1}|f(s, w_1(s)) - f(s, w_2(s))|\, ds \\
\leq \frac{L_{g_i}}{\Gamma(a_3)} \int_{t_i}^1 (t - s)^{a_3-1}|w_1(s) - w_2(s)|\, ds + \frac{L_{f_i}}{\Gamma(a_1)} \int_0^{s_i}(s - s)^{a_1-1}|w_1(s) - w_2(s)|\, ds \\
\leq \left(\frac{L_{g_i}(s_i - t_i)^{a_3}}{\Gamma(a_3 + 1)} + \frac{L_{f_i}s_i^{a_1}}{\Gamma(a_1 + 1)}\right)\|w_1 - w_2\|_{\mathcal{PC}},
$$

we obtain

$$
|\mathcal{R}(w_1)(t) - \mathcal{R}(w_2)(t)|^\beta \leq \left(\left(\frac{L_{g_i}(s_i - t_i)^{a_3}}{\Gamma(a_3 + 1)}\right)^\beta + \left(\frac{L_{f_i}s_i^{a_1}}{\Gamma(a_1 + 1)}\right)^\beta\right)\|w_1 - w_2\|_{p_\beta}.
$$

Hence,

$$
\|\mathcal{R}w_1 - \mathcal{R}w_2\| \leq \left(\left(\frac{L_{g_i}(s_i - t_i)^{a_3}}{\Gamma(a_3 + 1)}\right)^\beta + \left(\frac{L_{f_i}s_i^{a_1}}{\Gamma(a_1 + 1)}\right)^\beta\right)\|w_1 - w_2\|_{p_\beta}.
$$

**Case 3.** Let $w_1, w_2 \in \mathcal{PC}(I : \mathbb{R})$ and $s_i < t \leq t_{i+1}$, $i = 1, 2, \ldots, m$. Since

$$
|\mathcal{R}(w_1)(t) - \mathcal{R}(w_2)(t)| \leq \frac{1}{\Gamma(a_3)} \int_{t_i}^t (t - s)^{a_3-1}|g_i(s, w_1(s)) - g_i(s, w_2(s))|\, ds \\
+ \frac{1}{\Gamma(a_1)} \int_0^{s_i}(s - s)^{a_1-1}|f(s, w_1(s)) - f(s, w_2(s))|\, ds \\
+ \frac{1}{\Gamma(a_1)} \int_0^1 |f_1(s, w_1(s), f_{0,j}^{a_1-a_4}v(s)) - f_1(s, w_2(s), f_{0,j}^{a_1-a_4}v(s))|\, ds \\
\leq \frac{L_{g_i}}{\Gamma(a_3)} \int_{t_i}^1 (t - s)^{a_3-1}|w_1(s) - w_2(s)|\, ds + \frac{L_{f_i}}{\Gamma(a_1)} \int_0^{s_i}(s - s)^{a_1-1}|w_1(s) - w_2(s)|\, ds \\
+ \frac{1}{\Gamma(a_1)} \int_0^1 L_{f_1}|w_1(s) - w_2(s)|\, ds \leq \left(\frac{L_{g_i}(s_i - t_i)^{a_3}}{\Gamma(a_3 + 1)} + \frac{L_{f_i}s_i^{a_1}}{\Gamma(a_1 + 1)} + L_{f_1}\right)\|w_1 - w_2\|_{\mathcal{PC}},
$$

we obtain

$$
|\mathcal{R}(w_1)(t) - \mathcal{R}(w_2)(t)|^\beta \leq \left(\left(\frac{L_{g_i}(s_i - t_i)^{a_3}}{\Gamma(a_3 + 1)}\right)^\beta + \left(\frac{L_{f_i}s_i^{a_1}}{\Gamma(a_1 + 1)} + L_{f_1}\right)^\beta\right)\|w_1 - w_2\|_{p_\beta}.
$$

Hence,

$$
\|\mathcal{R}w_1 - \mathcal{R}w_2\| \leq \left(\left(\frac{L_{g_i}(s_i - t_i)^{a_3}}{\Gamma(a_3 + 1)}\right)^\beta + \left(\frac{L_{f_i}s_i^{a_1}}{\Gamma(a_1 + 1)} + L_{f_1}\right)^\beta\right)\|w_1 - w_2\|_{p_\beta}.
$$
we have

$$\| (Rw_1)(t) - (Rw_2)(t) \|_{\beta} \leq \left( \frac{L_0 (s_i - t_i)^{\alpha_2}}{\Gamma (a_3)} \right)^{\beta} + \left( \frac{L_1 s_i^{a_1}}{\Gamma (a_1 + 1)} \right)^{\beta} L_{f_1},$$

which implies

$$\| Rw_1 - Rw_2 \|_{\beta} \leq \left( \frac{L_0 (s_i - t_i)^{\alpha_2}}{\Gamma (a_3)} \right)^{\beta} + \left( \frac{L_1 s_i^{a_1}}{\Gamma (a_1 + 1)} \right)^{\beta} L_{f_1} \| w_1 - w_2 \|_{\rho \beta}.$$  

Let \( \rho := \max_{1 \leq i \leq m} \{ \left( \frac{L_0 (s_i - t_i)^{\alpha_2}}{\Gamma (a_3)} \right)^{\beta} + \left( \frac{L_1 s_i^{a_1}}{\Gamma (a_1 + 1)} \right)^{\beta} \} \). Accordingly, by cases 1, 2, and 3, we have

$$\| Rw_1 - Rw_2 \|_{\beta} \leq \rho \| w_1 - w_2 \|_{\rho \beta},$$

which shows that \( R \) is a contraction mapping on \( \mathcal{P}C(I : \mathbb{R}) \). Therefore, by the Banach contraction principle, we can prove that Equation (1) has a unique solution. \( \square \)

**Theorem 2.** Let (C4)–(C8) be fulfilled. Then, the Equation (1) possesses at least one solution.

**Proof.** Define the closed ball \( B_{r_2} \) by

$$B_{r_2} := \{ w \in \mathcal{P}C(I : \mathbb{R}) : \| w \| \leq r_2 \},$$

where

$$\max_{1 \leq i \leq m} | a_{g_i} (s) | = a_{g_i}^{s_i},$$

$$\max_{1 \leq i \leq m} | a_f (s) | = a_f^{s_i}$$

and

$$r_2 := \max_{1 \leq i \leq m} \left\{ \begin{array}{c}
| w_0 | + N + \frac{b_{t_1} r_1}{\Gamma (a_1 + 1)} + \frac{r_1}{\Gamma (a_1 + 1)} \frac{L_{f_1}}{1 - b_{f_1}} + \frac{N}{\Gamma (a_1 + 1)} \frac{L_{f_1}}{1 - b_{f_1}} + \frac{r_1}{\Gamma (a_1 + 1)} \frac{L_{f_1}}{1 - b_{f_1}} \\
| a | + \left( \frac{(s_i - t_i)^{\alpha_2} a_{g_i}^{s_i}}{\Gamma (a_1 + 1)} \right) + \frac{a_{g_i}^{s_i} b_{g_i}}{\Gamma (a_1 + 1)} + \frac{N}{\Gamma (a_1 + 1)} \frac{L_{f_1}}{1 - b_{f_1}} + \frac{r_1}{\Gamma (a_1 + 1)} \frac{L_{f_1}}{1 - b_{f_1}} \\
\end{array} \right\}.$$  

We claim that there exists \( r_1 > 0 \), such that \( | v(t) | \leq r_1 \) for all \( t \in I \). To calculate \( r_1 \), we have

$$| v(t) | = \int_{0,t}^{1-\alpha_1} | f (t, v(t), \int_{0,t} (a_{g_i}^2 + b_{g_i}^2 \nu(t))) |$$

$$\leq \int_{0,t}^{1-\alpha_1} \left( | a_f (t) | + b_f | v(t) | \right) \left( \int_{0,t} (a_{g_i}^2 + b_{g_i}^2 \nu(t)) \right)$$

$$\leq \int_{0,t}^{1-\alpha_1} \left( | a_f (t) | + b_f \| v \| (a_{g_i}^2 + b_{g_i}^2 \nu(t)) \right)$$

$$\leq \int_{0,t}^{1-\alpha_1} \left( | a_f (t) | + b_f r_1 \left( a_{g_i}^2 + \frac{b_f r_1^2}{\Gamma (a_1 + a_2 - \alpha_3 + 1)} \right) \right),$$

so

$$r_1 = a_f^s + \frac{1}{\Gamma (2 - \alpha_1)} \left( r_1 b_f a_{g_i}^s + \frac{b_f r_1^2}{\Gamma (a_1 + a_2 - \alpha_3 + 1)} \right).$$
We define the operator $S$ by

$$
(Sw)(t) := \begin{cases}
  a + \int_{I_{0,i}}^{t} g_i(t, w(t)) - f_{0,i}^a(t, w(t), f_{0,i}^a(t, w(t))), & t_i < t \leq s_i, i = 1, 2, \ldots, m, \\
  w_0 - \int_{0}^{t} f_1(s, w(s), f_{0,i}^{s_1-a_1}v(s)) \, ds + f_{0,i}^a(t), & 0 < t \leq t_1,
\end{cases}
$$

where

$$
v(t) := \int_{0}^{t} f(t, v(t)) \cdot \frac{1}{\Gamma(a_2)} \int_{0}^{t} \frac{(1 - \theta)^{a_2 - 1}}{\Gamma(a_1 - a_3)} \, dg \left( \theta, \int_{0}^{\theta} (\theta - v)^{(a_1 - a_3) - 1} \, dv \right) \, d\theta.
$$

Let $w \in B_{2,2}$. First, we are going to prove that $S : B_{2,2} \to B_{2,2}$ and $\{Sw\}$ are uniformly bounded on $B_{2,2}$. We use (C4)–(C6) when considering the following cases:

**Case 1.** Let $0 < t \leq t_1$. Then, we have

$$
|Sw(t)| = |w_0 - \int_{0}^{t} f_1(s, w(s), f_{0,i}^{s_1-a_1}v(s)) \, ds + f_{0,i}^a(t)|
\leq |w_0| + \int_{0}^{t} |f_1(s, w(s), f_{0,i}^{s_1-a_1}v(s))| \, ds + f_{0,i}^a|v(t)|
\leq |w_0| + \int_{0}^{t} \left( |a f_1(s)| + b f_1 \left( |w(s)| + f_{0,i}^{s_1-a_1}|v(s)| \right) \right) \, ds + f_{0,i}^a|v(t)|
\leq |w_0| + \int_{0}^{t} |a f_1(s)| \, ds + b f_1 \int_{0}^{t} |w(s)| \, ds + \frac{b f_1 |v||}{\Gamma(a_1 - a_4 + 1)} \int_{0}^{t} \frac{|v|}{\Gamma(a_1 + 1)} \, ds
\leq |w_0| + N + \frac{b f_1 r_1}{\Gamma(a_1 - a_4 + 1)} + \frac{r_1}{\Gamma(a_1 + 1)} \leq r_2.
$$

**Case 2.** Let $t_i < t \leq s_i, i = 1, 2, \ldots, m$. Then,

$$
|Sw(t)| = |a + \int_{I_{0,i}}^{t} g_i(t, w(t)) - f_{0,i}^a(t, w(t))| 
\leq |a| + \int_{t_i}^{t} \frac{(t - s)^{a_1-1}}{\Gamma(a_1)} \left| g_i(s, w(s)) \right| \, ds + \int_{0}^{s_i} \frac{(s_i - s)^{a_1-1}}{\Gamma(a_1)} |f(s, w(s))| \, ds
\leq |a| + \int_{t_i}^{t} \frac{(t - s)^{a_1-1}}{\Gamma(a_1)} \left( |a f_1(s)| + b f_1 \left| w(s) \right| \right) \, ds
+ \int_{0}^{s_i} \frac{(s_i - s)^{a_1-1}}{\Gamma(a_1)} \left( |a f_1(s)| + b f_1 \left| w(s) \right| \right) \, ds
\leq |a| + \frac{(s_i - t_i)^{a_1}}{\Gamma(a_1)} (a f_1^* + b f_1 r_2) + \frac{s_i^{a_1}}{\Gamma(a_1 + 1)} (a f_1^* + b f_1 r_2) \leq r_2.
$$

where $\max_{1 \leq i \leq m} |a f_1(s)| = a f_1^*$ and $\max_{1 \leq i \leq m} |a f_1(s)| = a f_1^*$.

**Case 3.** Let $s_i < t \leq t_{i+1}$, for $i = 1, 2, \ldots, m$. Then, we have

$$
|Sw(t)| = |a + \int_{I_{0,i}}^{t} g_i(t, w(t)) - f_{0,i}^a(t, w(t))| 
- \int_{0}^{t} f_1(s, w(s), f_{0,i}^{s_1-a_1}v(s)) \, ds + f_{0,i}^a|v(t)|
\leq |a| + \int_{t_i}^{t} \frac{(t - s)^{a_1-1}}{\Gamma(a_1)} \left| g_i(s, w(s)) \right| \, ds + \int_{0}^{s_i} \frac{(s_i - s)^{a_1-1}}{\Gamma(a_1)} |f(s, w(s))| \, ds
+ \int_{0}^{1} \left| f_1(s, w(s), f_{0,i}^{s_1-a_1}v(s)) \right| \, ds + f_{0,i}^a|v(t)|
$$
\[
\leq |a| + \frac{(s_i - t_i)^{a_2}}{\Gamma(a_3 + 1)}(a_{\delta_1}^* + b_{\delta_1}r_2) + \frac{s_i^{a_1}}{\Gamma(a_1 + 1)}(a_{\delta_1}^* + b_f r_2)
\]
\[
+ \int_0^1 |a_{f_1}(s)| \, ds + b_{f_1} \int_0^1 |w(s)| \, ds + b_{f_1} \int_0^1 f_{0, t}^{a_1 - a_4} |v(s)| \, ds + f_{s_f, t}^{a_1} |v(t)|
\]
\[
\leq |a| + \frac{(s_i - t_i)^{a_2}}{\Gamma(a_3 + 1)}(a_{\delta_1}^* + b_{\delta_1}r_2) + \frac{s_i^{a_1}}{\Gamma(a_1 + 1)}(a_{\delta_1}^* + b_f r_2)
\]
\[
+ N + b_{f_1} r_2 + \frac{b_{f_1} r_1}{\Gamma(a_1 - a_4 + 1)} + \frac{r_1}{\Gamma(a_1 + 1)} \leq r_2.
\]

We now prove that $S$ is continuous. Let $(w_n) \in B_{\varepsilon_2}$ and $w_n \rightarrow w$, when $n \rightarrow \infty$. By the assumptions in the statement of the theorem, we note that the conditions for Lebesgue dominated convergence theorem in the fractional calculations in the corresponding domains are fulfilled. Then, we have the following cases:

**Case (a).** Let $0 < t \leq t_1$. Then,

\[
S w_n(t) = w_0 - \int_0^1 f_1(s, w_n(s), f_{0, t}^{a_1 - a_4} v(s)) \, ds + f_{0, t}^{a_1} v(t).
\]

By using the Lebesgue dominated convergence theorem, we obtain

\[
\lim_{n \to \infty} S w_n(t) = \lim_{n \to \infty} \left( w_0 - \int_0^1 f_1(s, w_n(s), f_{0, t}^{a_1 - a_4} v(s)) \, ds + f_{0, t}^{a_1} v(t) \right)
\]
\[
= w_0 - \int_0^1 f_1(s, \lim_{n \to \infty} w_n(s), f_{0, t}^{a_1 - a_4} v(s)) \, ds + f_{0, t}^{a_1} v(t)
\]
\[
= w_0 - \int_0^1 f_1(s, w(s), f_{0, t}^{a_1 - a_4} v(s)) \, ds + f_{0, t}^{a_1} v(t) = S w(t).
\]

**Case (b).** Let $t_i < t \leq s_i$, $i = 1, 2, \ldots, m$. So,

\[
S w_n(t) = a + f_{t, s}^{a_3} g_i(t, w_n(t)) - f_{0, s}^{a_1} f(s_i, w_n(s_i)).
\]

Therefore, it follows that

\[
\lim_{n \to \infty} S w_n(t) = \lim_{n \to \infty} \left( a + f_{t, s}^{a_3} g_i(t, w_n(t)) - f_{0, s}^{a_1} f(s_i, w_n(s_i)) \right)
\]
\[
= a + \frac{1}{\Gamma(a_3)} \int_t^{s_i} (t - s)^{a_3 - 1} g_i(s) \, ds - \frac{1}{\Gamma(a_1)} \int_0^{s_i} (s_i - s)^{a_1} f(s, \lim_{n \to \infty} w_n(s)) \, ds
\]
\[
= a + \frac{1}{\Gamma(a_3)} \int_t^{s_i} (t - s)^{a_3 - 1} g_i(s, w(s)) \, ds - \frac{1}{\Gamma(a_1)} \int_0^{s_i} (s_i - s)^{a_1} f(s, w(s)) \, ds = S w(t).
\]

**Case (c).** Let $s_i < t \leq t_{i+1}$, $i = 1, 2, \ldots, m$. Since

\[
S w_n(t) = a + f_{s, s}^{a_3} g_i(s_i, w_n(s_i)) - f_{0, s}^{a_1} f(s_i, w_n(s_i))
\]
\[
- \int_0^1 f_1(s, w_n(s), f_{0, t}^{a_1 - a_4} v(s)) \, ds + f_{0, t}^{a_1} v(t),
\]

we acquire

\[
\lim_{n \to \infty} S w_n(t) = \lim_{n \to \infty} \left( a + f_{s, s}^{a_3} g_i(s_i, w_n(s_i)) - f_{0, s}^{a_1} f(s_i, w_n(s_i))
\right.
\]
\[
- \int_0^1 f_1(s, w_n(s), f_{0, t}^{a_1 - a_4} v(s)) \, ds + f_{0, t}^{a_1} v(t)
\]
\[
= a + f_{s, s}^{a_3} g_i(s_i, \lim_{n \to \infty} w_n(s_i)) - f_{0, s}^{a_1} f(s_i, \lim_{n \to \infty} w_n(s_i))
\]
\[
- \int_0^1 f_1(s, \lim_{n \to \infty} w_n(s), f_{0, t}^{a_1 - a_4} v(s)) \, ds + f_{0, t}^{a_1} v(t)
\]
\[ S(t) = a + f_{t,w}^3(s_i, w(s_i)) - f_{0,0}^3 f(s_i, w(s_i)) \]
\[ - \int_0^t f_1(s, w(s), f_{0,0}^{t_2} v(s)) \, ds + f_{0,0}^3 v(t) = S w(t). \]

Hence, by cases (a), (b), and (c), we show that the operator \( S \) is continuous.

Next, we are going to prove that the family \( \{ S w \} \) is equicontinuous and the operator \( S \) is relatively compact. Let \( w \in B_{\delta} \) and let \( t', t'' \in I, t'' > t' \) and \( |t' - t''| \leq \delta \), for \( \delta > 0 \). Hence, we need to consider the following possible cases:

**Case (i).** Let \( t', t'' \in (0, t_1) \). Then, we have

\[
|S w(t'') - S w(t')| = \left| w_0 - \int_0^{t'} f_1(s, w(s), f_{0,0}^{t_2} v(s)) \, ds + f_{0,0}^3 v(t') \right|
\]
\[
- w_0 + \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds - \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds
\]
\[
\leq \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds - \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds
\]
\[
= \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds - \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds
\]
\[
= \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds
\]
\[
+ \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds.
\]

By \( t' \rightarrow t'' \), we obtain that \( S w(t'') \rightarrow S w(t') \).

**Case (ii).** Let \( t', t'' \in (t_i, t_{i+1}) \), \( i = 1, 2, \ldots, m \). Then,

\[
|S w(t'') - S w(t')| = \left| a + f_{t,w}^3(t', w(t'')) - f_{0,0}^3 f(s_i, w(s_i)) \right|
\]
\[
- a + f_{t,w}^3(t', w(t')) + f_{0,0}^3 f(s_i, w(s_i)) \right|
\]
\[
\leq \frac{1}{\Gamma(\alpha_3)} \int_{t_i}^{t''} (t-s)^{\alpha_3-1} |g_i(s, w(s))| \, ds - \frac{1}{\Gamma(\alpha_3)} \int_{t_i}^{t'} (t-s)^{\alpha_3-1} |g_i(s, w(s))| \, ds
\]
\[
= \frac{1}{\Gamma(\alpha_3)} \int_{t_i}^{t''} (t-s)^{\alpha_3-1} |g_i(s, w(s))| \, ds \rightarrow 0 \quad \text{as} \ t' \rightarrow t''.
\]

**Case (iii).** Let \( t', t'' \in (s_i, t_{i+1}) \), \( i = 1, 2, \ldots, m \). Hence, we have

\[
|S w(t'') - S w(t')| = \left| a + f_{t,w}^3(s_i, w(s_i)) - f_{0,0}^3 f(s_i, w(s_i)) \right|
\]
\[
- \int_0^{t'} f_1(s, w(s), f_{0,0}^{t_2} v(s)) \, ds + f_{0,0}^3 v(t') \right|
\]
\[
+ \int_0^{t''} f_1(s, w(s), f_{0,0}^{t_2} v(s)) \, ds - f_{0,0}^3 v(t') \right|
\]
\[
\leq \int_0^{t''} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds - \int_0^{t'} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds
\]
\[
= \int_0^{t''} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds
\]
\[
+ \int_0^{t''} |f(s, v(s), f_{0,0}^{t_2} v(s))| \, ds.
\]
Keeping in mind that $f$ is a continuous function with respect to $w$ for all $t \in I$ (by (C4)), we find that $S\omega(t^n) \to S\omega(t')$ when $t' \to t^n$.

Finally, by applying the Schauder fixed-point theorem, we conclude the existence of at least one solution of Equation (1).

3. $\beta$–Ulam–Hyers Stability of Solutions

Let $0 < \beta < 1$ and $\varepsilon > 0$. Let us consider the following inequality:

$$
\begin{align*}
|w(t) - w_0 + \frac{1}{0} \int_0^1 f_1(s, w(s), \int_0^{a_1-a_4} v(s)) \, ds - \int_0^{a_1-a_4} v(t)| &\leq \varepsilon, \quad 0 < t \leq t_1 \\
|w(t) - a + \int_0^{a_1-a_4} v(t)| &\leq \varepsilon, \quad s_i < t \leq t_{i+1}, \ i = 0, 1, 2, \ldots, m \\
|w(t) - a + \int_0^{a_1-a_4} v(t) - \int_0^{a_1-a_4} f_2(s, w(s))| &\leq \varepsilon, \quad t_i < t \leq s_i, \ i = 1, 2, \ldots, m,
\end{align*}
$$

where $v(t)$ is defined as in Equation (1). The definition provided bears a resemblance to the definition of solution stability in the context of $\beta$–Ulam–Hyers, as outlined in [27].

**Definition 1.** The Equation (1) is $\beta$–Ulam–Hyers-stable if there exists a positive real number $C_{a_1,a_2,a_3,a_4,f,g,i_1} > 0$, such that for each $\varepsilon > 0$ and for each solution $w \in PC(I : \mathbb{R})$ of the inequality (2) there exists a solution $u \in PC(I : \mathbb{R})$ of (1) whereby

$$
|u(t) - w(t)|^\beta \leq C_{a_1,a_2,a_3,a_4,f,g,i_1} \varepsilon^\beta
$$

for all $t \in I$.

In terms of Ulam’s stability, the mapping and homomorphism are defined on the respective intervals. Therefore, we consider several cases.

**Remark 1.** It is worth noting that $w \in PC(I : \mathbb{R})$ is a solution of Equation (2) and is equivalent to saying that there exists a real number $K > 0$, such that

(i) $|K| \leq \varepsilon$;

(ii) $w(t) = w_0 - \int_0^1 f_1(s, w(s), \int_0^{a_1-a_4} v(s)) \, ds + \int_0^{a_1-a_4} v(t)$, $0 < t \leq t_1$;

(iii) $w(t) = a + \int_0^{a_1-a_4} v(t)$, $s_i < t \leq t_{i+1}$;

(iv) $w(t) = a + \int_0^{a_1-a_4} v(t) - \int_0^{a_1-a_4} f_2(s, w(s)) + K, t_i < t \leq s_i, i = 1, 2, \ldots, m$, where

$$
\nu_1(t) = \int_0^{a_1-a_4} \left( f(t,v(t) \cdot \int_0^1 (t - \theta)^{\beta_2-1} G \left( \theta, \int_0^\theta (v(t) - \nu(t)) d\theta \right) + K) \right).
$$

We continue by stating the following result:

**Proposition 1.** Let $w(t)$ be a solution of inequality Equation (2). Then, $w(t)$ is a solution of the following inequality:

\begin{align*}
\left\{ \begin{array}{l}
|w(t) - w_0 + \int_0^1 f_1(s, w(s), \int \alpha_1^{-a_4} v(s)) ds - \int \alpha_1 v(t)| \leq \frac{\varepsilon \alpha_1}{\alpha_1 \pi}, \quad 0 < t \leq t_1 \\
|w(t) - a - \int \alpha_1 \alpha_4 g_i(t, w(t)) + \int \alpha_1 f(s_i, w(s_i))| \leq \varepsilon, \quad t_i < t \leq s_i, \quad i = 1, 2, \ldots, m \\
|w(t) - a - \int \alpha_1 \alpha_4 g_i(s_i, w(s_i)) + \int \alpha_1 f(s_i, w(s_i)) \\
+ \int_0^1 f_1(s, w(s), \int \alpha_1^{-a_4} v(s)) ds - \int \alpha_1 v(t)| \leq \varepsilon(1 + \frac{(t_{i+1} - s_i)^a_1}{\alpha_1 \pi}),
\end{array} \right.
\end{align*}

where

\begin{align*}
s_i < t \leq t_{i+1}, \quad i = 1, 2, \ldots, m.
\end{align*}

**Proof.** We have

\begin{align*}
w(t) = w_0 - \int_0^1 f_1(s, w(s), \int \alpha_1^{-a_4} v(s)) ds + \int \alpha_1 v_K(t) \quad \text{for} \quad 0 < t \leq t_1,
\end{align*}

\begin{align*}
w(t) = a + \int \alpha_1 \alpha_4 g_i(t, w(t)) - \int \alpha_1 \alpha_4 f(s_i, w(s_i)) + K \quad \text{for} \quad t_i < t \leq s_i, \quad i = 1, 2, \ldots, m,
\end{align*}

and

\begin{align*}
w(t) = a + \int \alpha_1 \alpha_4 g_i(s_i, w(s_i)) - \int \alpha_1 \alpha_4 f(s_i, w(s_i)) \\
- \int_0^1 f_1(s, w(s), \int \alpha_1^{-a_4} v(s)) ds + \int \alpha_1 v_K(t) \quad \text{for} \quad s_i < t \leq t_{i+1}, \quad i = 1, 2, \ldots, m,
\end{align*}

where

\begin{align*}
v_K(t) = \int \alpha_1 \alpha_4 \left( f(t, v(t)) \cdot \int_0^1 \frac{(t-\theta)^a_1-1}{\Gamma(a_2)} \Theta \left( \int_0^\theta \frac{(\theta-\nu)^a_1-a_3 - 1}{\Gamma(a_1-a_3)} v(\nu) d\nu \right) d\theta \right) + K.
\end{align*}

Let \( 0 < t \leq t_1 \). Then, we have

\begin{align*}
\left| w(t) - w_0 + \int_0^1 f_1(s, w(s), \int \alpha_1^{-a_4} v(s)) ds - \int \alpha_1 v(t) \right|
= \int \alpha_1 \alpha_4 \left( \frac{1}{\Gamma(1-a_1)} \int_0^t (t-s)^{a_1-1} |K| ds \right)
\leq \int \alpha_1 \alpha_4 \left( \frac{|K|}{\Gamma(1-a_1)} \right) = \frac{|K|}{\Gamma(a_1)} \cdot \int_0^t (t-s)^{a_1-1} ds
\leq \frac{|K| \alpha_1^a \sin(\pi a_1)}{\pi a_1} \leq \frac{\varepsilon \alpha_1}{\pi a_1}.
\end{align*}

For \( t_i < t \leq s_i, \quad i = 1, 2, \ldots, m \), we obtain

\begin{align*}
\left| w(t) - a - \int \alpha_1 \alpha_4 g_i(t, w(t)) + \int \alpha_1 f(s_i, w(s_i)) \right| \leq |K| \leq \varepsilon.
\end{align*}

Finally, for \( s_i < t \leq t_{i+1}, \quad i = 1, 2, \ldots, m \), we acquire

\begin{align*}
\left| w(t) - a - \int \alpha_1 \alpha_4 g_i(s_i, w(s_i)) + \int \alpha_1 f(s_i, w(s_i)) \\
+ \int_0^1 f_1(s, w(s), \int \alpha_1^{-a_4} v(s)) ds - \int \alpha_1 v(t) \right|
\leq |K| + \int \alpha_1 \alpha_4 \left( \frac{1}{\Gamma(1-a_1)} \int_0^t (t-s)^{a_1-1} |K| ds \right)
\leq |K| + \int \alpha_1 \alpha_4 \left( \frac{|K|}{\Gamma(a_1)} \cdot \int_0^t (t-s)^{a_1-1} ds \right)
\leq |K| + \frac{|K|}{\Gamma(1-a_1)} \cdot \frac{1}{a_1} (t_{i+1} - s_i)^{a_1} \leq |K| \left( 1 + \frac{(t_{i+1} - s_i)^{a_1}}{\alpha_1 \pi} \right).
\end{align*}
\[ \leq \epsilon \left( 1 + \frac{(t_{i+1} - s_i)^{\alpha_1}}{\alpha_1 \pi} \right). \]

The proof is completed. \( \square \)

We now present the following \( \beta \)-Ulam–Hyers stability result, which is one of the main results of this paper.

**Theorem 3.** Let the conditions from Theorem 1 be satisfied. Suppose that

\[ 1 - \left( \frac{L_{\beta}(s_i - t_{i+1})^{\alpha_3}}{\Gamma(\alpha_3 + 1)} \right) - \left( \frac{L_{\beta}^{s_i^{\alpha_1}}}{\Gamma(\alpha_1 + 1)} \right) - L_{f_{i+1}}^\beta > 0 \]

and \( 1 - C_1^\beta - L_{f_{i+1}}^\beta > 0 \) hold, where \( |u_0 - w_0|^\beta \leq C_1 \|u - w\|_{p\beta} \) and \( w(0) = u(0) \). Then, Equation (1) is \( \beta \)-Ulam–Hyers-stable with respect to \( \epsilon \).

**Proof.** Let us denote by \( w(\cdot) \) the unique solution of Equation (1). Let \( u \in PC(I : \mathbb{R}) \) be a solution of the inequality Equation (2). By virtue of Proposition 1, we conclude the following:

(1) For \( 0 < t \leq t_1 \),

\[ |u(t) - u(0) - \int_0^t v(s) \, ds| \leq \frac{\epsilon L_{f_{i+1}}^{s_i^{\alpha_1}}}{\alpha_1 \pi}. \]

(2) For \( t_i < t \leq s_i, i = 1, 2, \ldots, m \),

\[ |u(t) - a - \int_{t_i}^{s_i} f(t, u(t)) + \int_{t_i}^{s_i} f(s_i, u(s_i)) \, ds - \int_{0}^{t} v(s) \, ds| \leq \epsilon. \]

(3) For \( s_i < t \leq t_{i+1}, i = 1, 2, \ldots, m \),

\[ |u(t) - a - \int_{t_i}^{s_i} f(t, u(t)) - \int_{0}^{t} v(s) \, ds - \int_{s_i}^{s_i} v(t) \, ds| \leq \epsilon \left( 1 + \frac{(t_{i+1} - s_i)^{\alpha_1}}{\alpha_1 \pi} \right). \]

So, we need to verify the following cases:

**Case 1.** Let \( 0 < t \leq t_1 \). Then,

\[ |u(t) - w(t)|^\beta = |u(t) - w_0 + \int_0^t f_1(s, w(s), f_{0,1}^{s_1 - s_i} v(s)) \, ds - f_{0,1}^{s_1} v(t)|^\beta \]

\[ \leq |u(t) - u_0 + \int_0^1 f_1(s, u(s), f_{0,1}^{s_1 - s_i} v(s)) \, ds - f_{0,1}^{s_1} v(t)|^\beta \]

\[ + |w_0 - w_0| + \left| \int_0^1 \left( f_1(s, w(s), f_{0,1}^{s_1 - s_i} v(s)) - f_1(s, w(s), f_{0,1}^{s_1 - s_i} v(s)) \right) \, ds \right|^\beta \]

\[ \leq \left( \frac{\epsilon L_{f_{i+1}}^{s_i^{\alpha_1}}}{\alpha_1 \pi} \right)^\beta + \left( C_1 + L_{f_{i+1}} \int_0^1 ds \right) \|u - w\|_{p\beta}, \]

where \( |w_0 - w_0|^\beta \leq C_1 \|u - w\|_{p\beta} \) with \( C_1 \in (0, 1) \). Accordingly, we obtain

\[ (1 - C_1^\beta - L_{f_{i+1}}^\beta) |u(t) - w(t)|^\beta \leq \left( \frac{\epsilon L_{f_{i+1}}^{s_i^{\alpha_1}}}{\alpha_1 \pi} \right)^\beta \cdot \epsilon^\beta, \]

which implies

\[ |u(t) - w(t)|^\beta \leq C_{\alpha_1, a, a_3, f_{0,1}, f_{i+1}, f_{i,1}} \cdot \epsilon^\beta, \]
where
\[
C_{a_1,a_2,a_3,f_0,g_0,f_1} = \frac{\left(\frac{a_1^{a_1}}{a_2^{a_2}}\right)^\beta}{1 - C_1^{\beta} - L_{f_1}^{\beta}}.
\]

**Case 2.** Let \( t_i < t \leq s_i, i = 1,2,\ldots,m \). Since
\[
|u(t) - w(t)|^\beta = |u(t) - a - \int_{t_i}^{a_1} g_i(t,u(t)) \, dt + f_{0,s_i}^a(s_i,u(s_i))|^\beta
\leq |u(t) - a - \int_{t_i}^{a_1} g_i(t,u(t)) \, dt + f_{0,s_i}^a(s_i,u(s_i))|^\beta
\leq |\int_{t_i}^{a_1} g_i(t,u(t)) \, dt - \int_{t_i}^{a_1} g_i(t,w(t)) \, dt + f_{0,s_i}^a(s_i,u(s_i)) - f_{0,s_i}^a(s_i,u(s_i))|^\beta
\leq \epsilon^\beta + \left( \frac{1}{F(a_3)} \int_{t_i}^{a_1} (t - s)^{a_3-1} g_i(t,u(t)) - g_i(t,w(t)) \, ds \right)^\beta
\leq \epsilon^\beta + \left( \left( \frac{L_f s_{i_1}^{a_1}}{F(a_1 + 1)} \right)^\beta + \left( \frac{L_g(s_i - t_i)^{a_3}}{F(a_3 + 1)} \right)^\beta \right) ||u - w||_{p^\beta},
\]
we derive
\[
\left(1 - \left( \frac{L_f s_{i_1}^{a_1}}{F(a_1 + 1)} \right)^\beta - \left( \frac{L_g(s_i - t_i)^{a_3}}{F(a_3 + 1)} \right)^\beta \right) ||u - w||_{p^\beta} \leq \epsilon^\beta.
\]
Hence, we obtain
\[
|u(t) - w(t)|^\beta \leq C_{a_1,a_2,a_3,f_0,g_0,f_1} \cdot \epsilon^\beta, \quad t_i < t \leq s_i, i = 1,2,\ldots,m,
\]
where
\[
C_{a_1,a_2,a_3,f_0,g_0,f_1} = \frac{1}{1 - \left( \frac{L_f s_{i_1}^{a_1}}{F(a_1 + 1)} \right)^\beta - \left( \frac{L_g(s_i - t_i)^{a_3}}{F(a_3 + 1)} \right)^\beta}.
\]

**Case 3.** Let \( s_i < t \leq t_{i+1}, i = 1,2,\ldots,m \). Then,
\[
|u(t) - w(t)|^\beta = |u(t) - a - \int_{t}^{s_i} g_i(s,u(s)) \, ds - f_{0,s_i}^a(s_i,u(s_i))|^\beta
\leq |u(t) - a - \int_{t}^{s_i} g_i(s,u(s)) \, ds - f_{0,s_i}^a(s_i,u(s_i))|^\beta
\leq |\int_{t}^{s_i} g_i(s,u(s)) \, ds - \int_{t}^{s_i} g_i(s,w(s)) \, ds + f_{0,s_i}^a(s_i,u(s_i)) - f_{0,s_i}^a(s_i,u(s_i))|^\beta
\leq \epsilon^\beta \left(1 + \left( \frac{t_{i+1} - s_i}{a_1 \pi} \right)^\beta \right) + \left( \frac{L_g(s_i - t_i)^{a_3}}{F(a_3 + 1)} \right)^\beta
\leq \left( \frac{L_f s_{i_1}^{a_1}}{F(a_1 + 1)} \right)^\beta + L_{f_1}^{\beta} ||u - w||_{p^\beta},
\]
Hence,
\[
\left(1 - \left( \frac{L_{l_1}(s_i - t_i)_{a_3}}{\Gamma(a_3 + 1)} \right)^\beta \right) - \left( \frac{L_{f_1}(s_{i_1})_{a_1}}{\Gamma(a_1 + 1)} \right)^\beta - L_{f_1}^\beta \right) |u(t) - w(t)|^\beta \leq \varepsilon^\beta \left(1 + \frac{(t_{i+1} - s_i)^{a_1}}{\alpha_1^2} \right)^\beta,
\]
implies that
\[
|u(t) - w(t)|^\beta \leq C_{\alpha_1,\alpha_2,\alpha_3,\mathcal{g},\mathcal{f},f_1} \cdot \varepsilon^\beta, \quad s_i < t \leq t_{i+1}, \quad i = 1, 2, \ldots, m,
\]
where
\[
C_{\alpha_1,\alpha_2,\alpha_3,\mathcal{g},\mathcal{f},f_1} = \frac{\left(1 + \frac{(t_{i+1} - s_i)^{a_1}}{\alpha_1^2} \right)^\beta}{1 - \left( \frac{L_{l_1}(s_i - t_i)_{a_3}}{\Gamma(a_3 + 1)} \right)^\beta - \left( \frac{L_{f_1}(s_{i_1})_{a_1}}{\Gamma(a_1 + 1)} \right)^\beta - L_{f_1}^\beta}.
\]

Finally, keeping in mind the considered three cases, we showed in each case the existence of a constant
\[
\mathcal{C}_{\alpha_1,\alpha_2,\alpha_3,\mathcal{g},\mathcal{f},f_1} > 0,
\]
such that \(|u(t) - w(t)|^\beta \leq \mathcal{C}_{\alpha_1,\alpha_2,\alpha_3,\mathcal{g},\mathcal{f},f_1} \cdot \varepsilon^\beta\) for all \(t \in I\), i.e., Equation (1) is \(\beta\)-Ulam–Hyers-stable, with respect to \(\varepsilon\). \(\square\)

4. Nontrivial Example of Application of Theorem 1

In this section, we give a nontrivial example illustrating the application of Theorem 1.

Example 1. Let us consider the following equation:

\[
\begin{cases}
  w(t) = 1 - \int_0^t \left( \left( \frac{s}{2} \right)^3 + \frac{1}{2} \left( w(s) + \int_0^s v(s) ds \right) \right) ds + \int_0^t v(t) ds,
  & 0 < t \leq \frac{1}{2}, \\
  w(t) = a + \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t - s)^{-\frac{3}{2}} \frac{\left| w(s) \right|}{16(1 + |w(s)|)} ds \\
  - \frac{1}{\Gamma(\frac{1}{2})} \int_0^1 (1 - s)^{-\frac{3}{2}} \arctan(s^2 + w(s)) ds + \frac{1}{2} \left( w(s) + \int_0^s v(s) \right) ds, & \frac{1}{2} < t \leq 1,
  \\
  w(0) = 1 - \int_0^1 \left( \left( \frac{s}{2} \right)^3 + \frac{1}{2} \left( w(s) + \int_0^s v(s) \right) \right) ds,
\end{cases}
\]

where

\[
v(s) = \int_0^s \left( \frac{1}{8 + e^s + s^2} \arctan(s^2 + v(s)) \cdot \int_0^s \left( s - \theta \right)^{-\frac{1}{2}} \left( \frac{\sqrt{\theta}}{3} \right) + \frac{1}{2} \int_0^\theta \left( \frac{\theta - \nu}{\Gamma(\frac{1}{2})} \right)^{\frac{1}{2}} \nu(v) d\nu \right) d\theta \)
\]

and \(a \in \mathbb{R}\).

Here, we set

\[
f(t, s) = \frac{1}{8 + e^t + t^2} \arctan(t^2 + s),
\]

\[
g_1(t, s) = \frac{|s|}{16(1 + |s|)}
\]

\[
g(t, s) = \frac{\sqrt{t}}{3} + \frac{s}{2},
\]

\[
f_1(t, s, p) = \left( \frac{t}{2} \right)^3 + \frac{1}{2} (s + p),
\]
\( \alpha_1 = \frac{3}{4}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{1}{3}, \alpha_4 = \frac{1}{2}, w_0 = 1, \) and \( a \in \mathbb{R}. \) Moreover, \( I = [0, 1], \) where \( 0 = t_0 = s_0 < t_1 < s_1 = 1, \) with \( t_1 = \frac{1}{2} \) and \( s_1 = 1. \)

It is clear that (C1)–(C3) hold, with \( L_f = \frac{1}{3}, L_{g_1} = \frac{1}{16}, L_g = \frac{1}{2}, \) and \( L_{f_1} = \frac{1}{2}. \) Since

\[
\rho = \left( \frac{1}{16} \left( \frac{1}{2} \right)^{\frac{3}{2}} \right)^{\beta} + \left( \frac{1}{4} \cdot \frac{1}{16} \right)^{\beta} + \left( \frac{1}{2} \right)^{\beta} < 1,
\]

for \( \beta > 0.688, \) by Theorem 1, the considered non-instantaneous impulsive fractional integral problem has a unique solution.

Let \( u \in \mathcal{PC}(I : \mathbb{R}) \) be a solution of inequality Equation (2). Put \( \varepsilon = 1. \) Then, there exists a constant \( K \in \mathbb{R}, \) such that \( |K| \leq \varepsilon = 1 \) and

\[
u_K(s) = \int_0^s \left( \frac{s}{2} \right)^3 + \frac{1}{2} (u(s) + J_{0+}^{1/2}v(s)) \right) ds + J^{3/4}_{0+} v(t), \quad 0 < t \leq \frac{1}{2},
\]

where

\[
u_K(s) = \int_0^s \left( \frac{s}{2} \right)^3 + \frac{1}{2} (u(s) + J_{0+}^{1/2}v(s)) \right) ds + J^{3/4}_{0+} v(t), \quad 0 < t \leq \frac{1}{2},
\]

and

\[
u_K(s) = \int_0^s \left( \frac{s}{2} \right)^3 + \frac{1}{2} (u(s) + J_{0+}^{1/2}v(s)) \right) ds + J^{3/4}_{0+} v(t), \quad 0 < t \leq \frac{1}{2},
\]

From the previous discussion,

\[
0 < t \leq \frac{1}{2}, \quad w(t) = 1 - \int_0^t \left( \frac{s}{2} \right)^3 + \frac{1}{2} (w(s) + J_{0+}^{1/2}v(s)) \right) ds + J^{3/4}_{0+} v(t),
\]

\[
0 < t \leq \frac{1}{2}, \quad w(t) = 1 - \int_0^t \left( \frac{s}{2} \right)^3 + \frac{1}{2} (w(s) + J_{0+}^{1/2}v(s)) \right) ds + J^{3/4}_{0+} v(t),
\]

has a unique solution for \( \beta = \frac{9}{4}. \) Now, for \( 0 < t \leq \frac{1}{2}, \) we obtain

\[
|u(t) - w(t)|^2 \leq \left( \frac{1}{2} \right)^\beta \cdot \left( \frac{1}{2} \right)^\beta \cdot \varepsilon^2 = 54.3934 \cdot 1^2 = 54.3934.
\]
For $\frac{1}{2} < t \leq 1$, we have

$$|u(t) - w(t)|^2 \leq \frac{1}{1 - \left(\frac{1}{2} \frac{1}{\Gamma\left(\frac{3}{2}\right)}\right)^{\frac{3}{2}}} \cdot \epsilon^2 = 1.0105 \cdot 1^2 = 1.0105.$$ 

Hence, we obtain

$$|u(t) - w(t)|^2 \leq 54.3934 = 54.3934 \cdot 1^2,$$

for all $t \in [0, 1]$, implying that the considered problem is $\frac{3}{4}$-Ulam–Hyers-stable with respect to $\epsilon = 1$.

**Example 2.** Consider the following equation:

$$w(t) = \begin{cases} w(t) = 1 - \int_0^1 \left(\sqrt[3]{5} + \frac{1}{4} (w(s) + \int_{0,t}^1 v(s))\right) ds + \int_0^1 v(t), \\ 0 < t \leq \frac{1}{2}, \\ w(t) = a + \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-s)^{-\frac{1}{2}} \frac{w(s)^2}{8(1+w(s))} ds \\ - \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 \frac{1}{2} (1-s)^{-\frac{3}{2}} \epsilon^{-s}(\sin(s^2 + w(s))) ds, \quad \frac{1}{2} < t \leq 1, \\ w(0) = 1 - \int_0^1 \left(\sqrt[3]{5} + \frac{1}{4} (w(s) + \int_{0,t}^1 v(s))\right) ds, \end{cases}$$

where

$$v(s) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^s \left(\frac{s}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{4} \int_0^\theta (\theta - 1) (\theta - 1)^{-\frac{3}{2}} v(\theta) d\theta\right) d\theta,$$

and $a \in \mathbb{R}$.

Now,

$$f(t,s) = \frac{1}{3} e^{-t} \sin(t^2 + s),$$

$$g_1(t,s) = \frac{\epsilon^2}{8(1+s^2)},$$

$$g(t,s) = t^3 + \frac{1}{4}s,$$

$$f_1(t,s,p) = \sqrt[3]{7} + \frac{1}{4}(s + p).$$

Also, $a_1 = \frac{2}{7}, \ a_2 = \frac{1}{7}, \ a_3 = \frac{1}{2}, \ a_4 = \frac{1}{2}, \ w_0 = 1, \ I = [0,1], \ and \ 0 = t_0 = s_0 < t_1 < s_1 = 1$, with $t_1 = \frac{1}{2}$ and $s_1 = 1$. We have $L_f = \frac{5}{7}, \ L_{g_1} = \frac{1}{2}, \ L_s = \frac{1}{2}, \ and \ L_{f_1} = \frac{1}{2}$, hence the conditions (C1)–(C3) hold and for $\beta > 0.717$, $\rho < 1$. So, the conditions of Theorem 1 are fulfilled, hence the considered problem has a unique solution.
Like in the previous example, the considered problem has a unique solution for $\beta = \frac{3}{2}$.

For $0 < t \leq \frac{1}{2}$, we have

$$|u(t) - w(t)|^2 \leq \left( \frac{\left( \frac{1}{2} \right)^{\frac{3}{2}}}{1 - 1 - 0.9 \frac{1}{2} - \frac{1}{4}} \cdot \epsilon^2 \right)^{\frac{1}{2}} = \frac{7.7867 \cdot 1^2}{1} = 7.7867.$$  

For $\frac{1}{2} < t \leq 1$, we have

$$|u(t) - w(t)|^2 \leq \frac{1}{1 - \left( \frac{1 + 1}{1 + \left( \frac{3}{2} \right)} \right)^{\frac{3}{2}} - \left( \frac{1}{1 - \left( \frac{3}{2} \right)} \right)^{\frac{3}{2}}} \cdot \epsilon^2 = 1.3012 \cdot 1^2 = 1.3012.$$  

Hence, we obtain

$$|u(t) - w(t)|^2 \leq 7.7867 \cdot 1^2 = 7.7867,$$

for all $t \in [0, 1]$, implying that the considered problem is $\frac{3}{2}$–Ulam–Hyers-stable with respect to $\epsilon = 1$.

5. Conclusions

In this research, we focused on investigating a class of non-instantaneous impulsive fractional integral equations. We have examined the existence and uniqueness of solutions and the $\beta$–Ulam–Hyers stability of solutions for this class of integral impulsive equations. Finally, we provided examples to illustrate the validity and effectiveness of our results. Our findings have unveiled the phenomenon of $\beta$–Ulam–Hyers stability among these solutions, providing a valuable perspective for both researchers and practitioners in their pursuit of dependable and resilient mathematical models in future studies.

It remains an open question whether the imposed conditions on Lipschitz continuity or boundedness can be replaced with weaker ones, like the ones proposed in [47], while still obtaining the same results. This issue warrants further investigation. Additionally, we propose exploring the semigroup approach (see, for example, [48] and references therein), which may provide new tools and perspectives for further generalizations.


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