Editorial

Mathematical Inequalities in Fractional Calculus and Applications

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All types of inequalities play a very important role in various aspects of mathematical analysis, such as approximation theory and differential equation theory. The theory of fractional calculus, which deals with the study and applications of derivatives and integrals of arbitrary orders, has received considerable attention due to its many applications in the applied sciences. Recently, a wide range of applied sciences have relied heavily on fractional differential equations for modeling real systems. Fractional inequalities, also known as the inequalities involving derivatives and integrals of arbitrary orders, have been used to investigate the existence, uniqueness, and stability of solutions to a system of fractional differential equations. Upper and lower bounds of solutions to a system of fractional differential equations are sometimes found by using fractional inequalities. Furthermore, numerical quadrature, probability, and many other related areas also use fractional inequalities. Numerous writers have developed numerous extensions of the different classical inequalities to fractional calculus in the literature over time.

This Special Issue is a compilation of original research on mathematical inequalities involving fractional derivatives and fractional integral operators and their many applications in various mathematical and related fields. The papers in this Special Issue offer deep insight into the interplay between inequalities and fractional calculus, shedding light on new mathematical results, analytical techniques, and computational methods. The Special Issue contains fifteen articles with novel results, and below is an overview of the contributions.

In the first article by Stojiljković et al. [1], a new type of convexity for set-valued function is introduced. Utilizing the definition of this new convexity, the authors prove the Hadamard inequalities involving the Katugampola fractional integrals. These inequalities generalize the non-integer Hadamard inequalities for a convex IVM, $(p,h)$-convex IVM, $p$-convex IVM, $h$-convex, and $s$-convex in the second sense and many other related well-known classes of functions implicitly. In addition, some numerical examples are provided as supplements to the derived results.

In the second article by Ciurdariu and Grecu [2], several improved quantum Hermite–Hadamard-type integral inequalities for convex functions using a parameter are presented. For this, a new quantum identity is established as the main tool in the proof of the main results. Consequently, in some special cases, several new quantum estimations for $q$-midpoints and $q$-trapezoidal-type inequalities are derived with an example.

In [3], Yıldız et al. derived several inequalities for convex functions by using the monotonicity properties of functions and a generalized weighted-type fractional integral operator, which allows the integration of a function with respect to another function in fractional order. The results obtained herein are generalizations of some previously presented findings.

In [4], Vivas-Cortez et al. studied a new class of mean-type inequalities by incorporating Riemann-type fractional integrals. By doing so, a novel set of such inequalities is discovered and analyzed by using different mathematical identities. This class of inequalities is introduced by employing a generalized concept of convexity known as $h$-convexity. To validate the work, the authors created visual graphs and a table of values using specific...
functions to represent the various parts of the inequalities. This approach presents the opportunity to demonstrate the validity of the findings and further solidify the conclusions. Moreover, some previously published results emerged as special consequences of the main findings. This research serves as a catalyst for future investigations, encouraging researchers to explore more comprehensive outcomes by using generalized fractional operators and expanding the concept of convexity.

Fuzzy-interval-valued functions (FIVFs) are the generalization of interval-valued and real-valued functions, which contribute significantly to resolving the problems arising in the theory of interval analysis. In [5], Vivas-Cortez et al. explored the convexities and pre-invexities of FIVFs and defined some fuzzy fractional integral operators (FFIOs) with a generalized Bessel–Maitland function as their kernel. Using the class of convexities and pre-invexities of FIVFs, some Hermite–Hadamard and trapezoid-type inequalities via the FFIOs are derived. In [6], Yang established certain novel reverse Hölder- and Minkowski-type inequalities for modified unified fractional integral operators (FIOS) with extended unified Mittag–Leffler functions (MLFs). The predominant results of this article generalize and extend the existing fractional Hölder- and Minkowski-type integral inequalities in the literature. As applications, the reverse versions of weighted Radon-, Jensen-, and power-mean-type inequalities for modified unified generalized FIOs with extended unified MLFs are also investigated.

Farid et al. studied and established some bounds of $k$-integral operators with the Mittag–Leffler kernel in a unified form in [7]. These bounds are obtained by applying the definition of exponentially $(\alpha, h-\rho)$-convexity. The presented results provide a large number of new bounds of several integral operators for various kinds of convexities by using appropriate substitutions. In addition, a fractional Hadamard-type inequality that shows the upper and lower bounds of $k$-integral operators exponentially $(\alpha, h-\rho)$-convexity is presented. The results in this paper have some direct links with the results of many published articles.

In the study conducted by Vivas-Cortez et al. [8], the researchers explore some new aspects of Ostrowski-type integral inequalities by implementing the generalized Jensen–Mercer inequality established for generalized $s$-convexity in fractal space. In this light, the authors construct a new generalized integral equality for first-order local differentiable functions, which serves as an auxiliary result for the main results. The desired results are obtained by employing the new generalized integral equality and some renowned generalized integral inequalities like Hölder’s, power mean, and Yang–Hölder’s under the conditions of boundedness and generalized $s$-convexity of the functions. Moreover, in support of the main findings, specific applications to means, numerical integration, and some graphical visualizations are presented.

In the study carried out by Nazir et al. [9], a new filtration class associated with the filtration of infinitesimal generators by using the nonlinear fractional differential operator is studied. The authors establish a connection between the Fekete–Szegö quadratic functional and the class of infinitesimal generators. Certain properties, like sharp Fekete–Szegö inequalities and filtration problems, are also considered.

Kermausuor and Nwaeze [10] propose the definitions of some new fractional integral operators called $k$-Atangana–Baleanu fractional integral operators. These newly proposed operators are generalizations of the well-known Atangana–Baleanu fractional integral operators. As an application, a generalization of the Hermite–Hadamard inequality is established. Additionally, some new identities involving these new integral operators and new fractional integral inequalities of the midpoint and trapezoidal type for functions whose derivatives are bounded or convex are obtained.

In the article by Ramzan et al. [11], the authors present a novel parameterized fractional integral identity. By using this auxiliary result and the s-convexity property of the function, a series of fractional variants of certain classical inequalities, including Simpson’s, midpoint, and trapezoidal-type inequalities, are derived. Additionally, some applications of the main results to special means of real numbers are also explored. Moreover, a new generic
numerical scheme for solving nonlinear equations is derived to demonstrate an application of the main results in numerical analysis.

Saeed et al., in [12], propose some new concepts of coordinated up and down convex mappings with fuzzy-number values (coordinated UD-convex FNVMs). Thereafter, Hermite–Hadamard-type inequalities via coordinated up and down convex fuzzy-number-valued mappings are introduced. By taking the products of two coordinated UD-convex FNVMs, Pachpatte-type inequalities are also obtained. Some nontrivial examples are further presented to show the validity of the main results.

The study by Chiheb et al. in [13] deals with the Newton–Cotes-type inequalities involving three points for geometrically arithmetic-convex (GA-convex) functions. The authors first established a new integral identity involving the Hadamard integral operators. Using this new identity, they established some new Maclaurin-type inequalities for functions whose first derivatives in absolute value are GA-convex functions.

Symmetric derivatives and integrals are extensively studied to overcome the limitations of classical derivatives and integral operators. In the study by Vivas-Cortez et al. [14], they explored the quantum symmetric derivatives on finite intervals. They introduce the idea of right quantum symmetric derivatives and integral operators and study various properties of both operators as well. Using these concepts, new variants of Young’s inequality, Hölder’s inequality, Minkowski’s inequality, Hermite–Hadamard’s inequality, Ostrowski’s inequality, and Gruss–Chebysev inequality are obtained. Furthermore, the Hermite–Hadamard’s inequalities, by considering the differentiability of convex mappings, are established. These fundamental results are pivotal to studying the various other problems in the field of inequalities. The authors provide some numerical and visual examples to verify the correctness of the main results.

Finally, in [15], Saeed et al. introduce a new extension of interval-valued convexity, namely, coordinated LR-h-convexity. By using the double Riemann–Liouville fractional integrals, they derive fractional forms of the Hermite–Hadamard inequality for the newly defined class of convex mappings. By taking the product of two LR-h-convex functions, some new versions of fractional integral inequalities are also obtained. Moreover, some new and classical exceptional cases are also discussed by taking some restrictions on endpoint functions of interval-valued functions that can be seen as applications of these new outcomes.

In conclusion, this Special Issue serves as a testament to the vibrancy of research in this burgeoning field. We trust that the findings presented here will inspire further exploration and collaboration, leading to new breakthroughs in both theoretical developments and practical applications of fractional calculus. We invite readers to delve into the rich array of articles contained within this Special Issue and to join us in advancing the frontiers of knowledge in fractional calculus and mathematical inequalities.

As editors, we extend our sincere gratitude to all of the authors who have contributed their innovative research to this issue. Their dedication and intellectual rigor have enriched the discourse surrounding mathematical inequalities in fractional calculus. We also thank the reviewers for their constructive feedback and insightful comments, which have been crucial in maintaining the quality and relevance of the articles.

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**References**


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