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Existence and Stability of Solutions for p -Proportional ω -Weighted κ -Hilfer Fractional Differential Inclusions in the Presence of Non-Instantaneous Impulses in Banach Spaces

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Abstract: In this work, we introduce a new definition for the fractional differential operator that generalizes several well-known fractional differential operators. In fact, we introduce the notion of the p -proportional ω -weighted κ -Hilfer derivative includes an exponential function, $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$, and then we consider a non-instantaneous impulse differential inclusion containing $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$ with order $\sigma \in (1, 2)$ and of kind $\rho \in [0, 1]$ in Banach spaces. We deduce the relevant relationship between any solution to the studied problem and the integral equation that corresponds to it, and then, by using an appropriate fixed-point theorem for multi-valued functions, we give two results for the existence of these solutions. In the first result, we show the compactness of the solution set. Next, we introduce the concept of the (p, ω, κ) -generalized Ulam-Hyeres stability of solutions, and, using the properties of the multi-valued weakly Picard operator, we present a result regarding the (p, ω, κ) -generalized Ulam-Rassias stability of the objective problem. Since many fractional differential operators are particular cases of the operator $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$, our work generalizes a number of recent findings. In addition, there are no past works on this kind of fractional differential inclusion, so this work is original and enjoyable. In the last section, we present examples to support our findings.

Keywords: fractional differential inclusions; p -proportional ω -weighted κ -Hilfer fractional derivative; weakly Picard operators; measure of non-compactness; Ulam-Hyeres stability; weakly Picard operators

MSC: 26A33; 34A08; 34A60



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1. Introduction

Fractional calculus has been used in many fields that no one expected, such as engineering, life and social sciences [1], fluid mechanics [2], medicine [3–6], physics [7], applied mathematics [8], industry [9], polymer science [10], biology [11], and chemical engineering [12,13]. Differential inclusions include differential equations and appear in many situations; see [14,15].

Many real-life processes and phenomena are distinguished by sudden state changes, and these are modeled by impulsive differential equations and impulsive differential inclusions. There are two kinds of such changes. In the first kind, the changes occur instantaneously, such as the movement of an elastic ball bouncing vertically on a surface. The moments of the impulses are in the time when the ball touches the surface and its velocity is rapidly changed. Mathematical models in such cases are created using instantaneous impulsive differential equations or instantaneous impulsive differential inclusions. In the second type, the changes remain active over certain time intervals, such as the injection of drugs in the bloodstream and their consequent absorption in the body as a random, periodic, gradual, and continuous process. It is shown that the non-instantaneous impulse has more advantages in describing the human body's absorption, diffusion, and

metabolism of drugs. Mathematical models of these situations use non-instantaneous impulsive differential equations and non-instantaneous impulsive differential inclusions. Other examples of the first kind: [16] operation of a damper subjected to the percussive effects, change of the valve shutter speed in its transition from an open to a closed state, fluctuations of the pendulum system in the case of external impulsive effects, percussive model of a clock mechanism, percussive systems with vibrations, relaxation oscillations of the electromechanical systems, electronic schemes, remittent oscillator, subjected to the impulsive effects, dynamic of a system with automatic regulation, the passage of the solid body from a given fluid density to another fluid density, a change in the speed of a chemical reaction in the addition or removal of a catalyst, disturbances in cellular neural networks in cellular neural networks, impulsive external intervention and optimization problems in the dynamics of isolated populations, and death in the populations as a result of impulsive effects. Kevin [17] presented a study about the application of instantaneous impulsive differential equations to the control of malaria outbreaks, and Ballinger et al. [18] applied impulsive delay differential equations in population growth models. Benchohra et al. [19] published recently a book about instantaneous impulsive fractional differential equations, while Terzieva [20], Wang et al. [21], and Agarwal et al. [22] published books on non-instantaneous impulsive fractional differential equations. In the papers [23–26], there are very recently studies about instantaneous and non-instantaneous impulsive differential inclusions.

Many researchers introduced new concepts of fractional differential operators, which contributed to the development of fractional calculus and its increased application in more fields.

Sousa et al. [27] introduced the concept of the κ -Hilfer fractional derivative, which generalizes the κ -Caputo fractional derivative presented by Almeida [28], where $\kappa : \mathcal{h} = [0, \mathcal{L}] \rightarrow \mathbb{R}$ be a strictly increasing differentiable mapping such that κ' is continuous and $\kappa'(\vartheta) \neq 0$, for any $\vartheta \in \mathcal{h}$.

Since a lot of integral equations are solved by efficient methods, using weighted fractional derivatives, therefore Alsheekhussain et al. [29] presented the notion of the ω -weighted κ -Hilfer fractional derivative, that generalizes many types of known fractional derivatives, such as Riemann–Liouville, Caputo, Hadamard, κ -Riemann–Liouville, κ -Caputo, Katugampola, Hilfer–Hadamard, Hilfer, Hilfer–Katugampola, and κ -Hilfer fractional derivatives, where $\omega : \mathcal{h} = [0, \mathcal{L}] \rightarrow]0, \infty[$, $\omega^{-1}(\lambda) = \frac{1}{\omega(\lambda)}$, and then the existence of solutions to a non-instantaneous impulsive differential inclusion that includes this differential operator of order $\sigma \in (1, 2)$ was studied. In [30–34], fractional differentials and fractional differential inclusions involving weighted fractional derivatives were studied.

In [35–37], Jarad et al. presented the concepts of p -proportional fractional integrals and differentials, where $p \in (0, 1]$, and this allows to a more accurate description and better representation of fractional order systems [38,39].

Agarwal et al. [40] considered a fractional differential equation (FDE) involving the proportional fractional Caputo derivative in the presence of non-instantaneous impulses, when the lower limit of the derivative is preserved at the initial time, as well as being changeable at each impulsive time point. Then the stability of the solutions was studied in the sense of Ulam. In [41], Abbas considered κ -Riemann Liouville integral differential equations with the influence of non-instantaneous impulses. Ahmed et al. [42] presented the notion of the proportional Hilfer fractional derivative and investigated the existence of a solution to a FDE involving this differential operator. Mallah et al. [43] instituted the notion of the proportional κ -Hilfer fractional derivative and showed that there is a solution to a FDE involving this differential operator. For further work on differential equations involving proportional fractional derivatives, we cite [42,44–46].

The idea of stability of a system is a very important issue because if a system is stable, then very important features can be found about the exact solution. There are a large number of studies on the stability of FDEs and FDIs. Let us mention the most recent ones related to the topic of our research. Wang et al. [47] investigated the generalized Ulam–Hyers stabil-

ity for a Caputo differential inclusions with the effect of non-instantaneous impulses, Abbas et al. [48] studied the existence and Ulam stability of solutions for FDEs involving Hilfer-Hadamard, Awad et al. [49] presented the Ulam-Hyers stability of κ -Caputo implicit FDEs, Elsayed et al. [50] showed the existence and stability of FDEs with the Hilfer-Katugampola derivative, Vanterler et al. [51,52] studied the stability in the sense of Ulam-Hyers and δ -Ulam-Hyers-Rassias, of FDEs containing the κ -Hilfer derivative, Ibrahim et al. [53] investigated the existence and stability, in the sense of generalized Ulam-Hyers, of solutions of non-instantaneous impulsive FDEs containing κ -Hilfer derivative of order $\delta \in (0, 1)$ with delay, and Hammad et al. [54] studied the existence and stability results for a coupled of impulsive FDEs involving Hadamard fractional derivatives. In [55–59], there are further recent studies on the stability of different problems.

In this paper, motivated by the above-mentioned works, especially [29,40–42,47,48,53], we introduce for the first time a new fractional derivative that generalizes many known fractional differential operators, including those used in all the above-mentioned works except [55]. This operator is called the p -proportional ω -weighted κ -Hilfer fractional derivative. Naturally, there are no studies on the existence of solutions to differential embeddings involving this differential operator nor on the stability of these solutions. We will achieve the following goals in this paper:

- 1 Introducing the definitions of the p -proportional ω -weighted κ -Hilfer differential operators of order σ and of kind v , where $\sigma \in (1, 2)$, $p \in (0, 1]$, $\rho \in [0, 1]$, $\omega : \mathbb{R} \rightarrow]0, \infty[$ and $\omega^{-1}(\lambda) = \frac{1}{\omega(\lambda)}$ (Definition 7). Then, we demonstrate some properties that we will use to prove the main results (Lemmas 4–7).
- 2 Deriving the accurate relationship between any solution of the following non-instantaneous impulsive FDI with the p -proportional ω -weighted κ -Hilfer fractional derivative of order σ and of kind ρ :

$$\left\{ \begin{array}{l} D_{b_i, \lambda}^{\sigma, \rho, p, \kappa, \omega} u(\lambda) \in F(\lambda, u(\lambda)), \text{ a.e. for } \lambda \in \mathbb{I}_i, i \in A_0, \\ u(l_i^+) = \pi_i(l_i, u(l_i^-)), i \in A_1, \\ u(\lambda) = \pi_i(\lambda, u(l_i^-)), \lambda \in B_i, i \in A_1, \\ \lim_{\lambda \rightarrow 0^+} I_{0, \lambda}^{2-\mu, p, \kappa} (u\omega)(\lambda) = u_0 \\ \lim_{\lambda \rightarrow 0^+} (D_{0, \lambda}^{1, p, \kappa} (I_{0, \lambda}^{2-\mu, p, \kappa} (u\omega)(\lambda))) = u_1, \\ \lim_{\lambda \rightarrow b_i^+} I_{b_i, \lambda}^{2-\mu, p, \kappa} (u\omega)(\lambda) = \pi_i(b_i, u(b_i^-)), i \in A_1, \\ \lim_{\lambda \rightarrow b_i^+} (D_{b_i, \lambda}^{1, p, \kappa} (I_{b_i, \lambda}^{2-\mu, p, \kappa} (u\omega)(\lambda))) = \pi_i^*(b_i, u(b_i^-)), i \in A_1, \end{array} \right. \quad (1)$$

and the integral equation that corresponds to it (Lemma 8), where, $\mu = \sigma + 2\rho - \rho\sigma$, $\mathcal{L} > 0$, $\mathcal{U}$ is a Banach space and $P_{ck}(\mathcal{U}) := \{Z \subseteq \mathcal{U} : Z \text{ is not empty, compact, and convex}\}$, $n \in \mathbb{N}$, $F : \mathbb{R} \times \mathcal{U} \rightarrow P_{ck}(\mathcal{U})$, $A_0 = \{0, 1, 2, \dots, n\}$, $A_1 = \{1, 2, \dots, n\}$, $b_0 = 0 < l_1 \leq b_1 < l_2 \leq b_2 < \dots < l_n \leq b_n < l_{n+1} = \mathcal{L}$, $\mathbb{I}_k = (b_k, l_{k+1}]$, $k \in A_0$ and $B_i = (l_i, b_i]$, $i \in A_1$. Moreover, $D_{b_i, \lambda}^{\sigma, \rho, p, \kappa, \omega}$ is the p -proportional ω -weighted κ -Hilfer fractional differential operator of order σ and of kind ρ and with lower limit at b_i ; $I_{a, \lambda}^{2-\mu, p, \kappa}$ is the p -proportional κ -Riemann-Liouville fractional integral operator of order $2 - \sigma$ and with lower limit at a (The definitions of $D_{b_i, \lambda}^{\sigma, \rho, p, \kappa, \omega}$ and $I_{a, \lambda}^{2-\mu, p, \kappa}$ will be presented in the next section); $\pi_i, \pi_i^* : [l_i, b_i] \times \mathcal{U} \rightarrow \mathcal{U}$; $i \in A_1$ and $u_0, u_1 \in \mathcal{U}$ are two fixed two points.

- 3 Definition of the concept of solutions to problem (1) (Definition 8); and then we specify the formula for any solution to Problem (1) (Lemma 8).
- 4 Employing a suitable fixed point for multi-valued functions, we prove that the set of solutions to Problem (1) is not empty and compact in the Banach space $PC_{2-\mu, \kappa, \omega}(\mathbb{I}, \mathcal{U})$, which will be specified later (Theorem 1).
- 5 Utilizing the Covitz-Nadler fixed point for contraction multi-valued functions, we provide one more existence result for the solutions to Problem (1) (Theorem 2).

- 6 Employing properties of the weakly Picard operators, we demonstrate the stability of Problem (1) in the sense of (p, ω, κ) generalized Ulam–Rassias (Theorem 3).
- 7 Give two examples to support our findings (Examples 1 and 2).

Remark 1. *In order to explain the novelty of this work and its distinguished contributions to the field of fractional calculus and its relationship with the recent relevant results, we mention the following:*

- 1 In a very recent paper [29], the existence of solutions to Problem (1) was studied in the special case $p = 1$.
- 2 In the particular cases, $\omega(\lambda) = 1$, $\kappa(\lambda) = \lambda$; $\lambda \in [0, \mathcal{L}]$, $\sigma \in (0, 1)$, $\rho = 1$, F is a single valued function, and $\mathcal{U} = \mathbb{R}^m$. Theories 2 and 3 were demonstrated in Theorem 4 and (i) of Theorem 6 in [40].
- 3 The existence and generalized Ulam–Hyers solutions to Problem (1) were investigated by [47] in the particular cases $p = 1$, $\omega(\lambda) = 1$, $\kappa(\lambda) = \lambda$; $\lambda \in [0, \mathcal{L}]$ and $\sigma \in (0, 1)$.
- 4 Abbas et al. [48] studied the existence of Ulam–Hyers–Rassias stability of solutions to Problem (1) in the special cases $p = 1$, $\omega(\lambda) = 1$, $\kappa(\lambda) = \log \lambda$; $\lambda \in [0, \mathcal{L}]$ and $\sigma \in (0, 1)$, F is a single-valued function, $l_i = b_i = \mathcal{L}$, $\forall i \in A_1$ and $\mathcal{U} = \mathbb{R}$.
- 5 The existence and generalized Ulam–Hyers of solutions of Problem (1) with delay were investigated by [53] in the special cases $p = 1$, $\omega(\lambda) = 1$; $\lambda \in [0, \mathcal{L}]$ and $\sigma \in (0, 1)$.
- 6 Since the differential operators, which are named in the literature by Caputo, (proportional, Hilfer–Hadamard, Hilfer–Katugampola, κ -Riemann Liouville, κ -Caputo, κ -Hilfer, proportional κ -Riemann Liouville) are special cases of $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$, then our technique allows us to extend the majority of the above mentioned works after replacing the considered differential operator in these works by $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$.

We recommend the next future research directions related to our work:

- 1 Using our method and that used in the recently published paper [58] to study the Hyers–Ulam–Rassias of a delay differential inclusion with multiple variable time delays involving proportional weighted κ -Hilfer on bounded and unbounded intervals.
- 2 Empolying our technique as well as that used in the recently published paper [54] to investigate the existence and stability of solutions for a coupled system of impulsive fractional differential inclusions with proportional weighted κ -Hilfer fractional derivative.
- 3 Study Problem (1) in the presence of finite delay as well as infinite delay.
- 4 Using the arguments in the recently published papers [24,60], to study the existence of anti-periodic solutions to Problem (1).

The paper is formed of six sections. In Section 2, we offer some well-known symbols and information. Section 3 is concerned with introducing the definition of the p -proportional ω -weighted κ -Hilfer derivative and deducing some properties. After that, we obtain in detail the relevance between the solutions of Problem (1) and the corresponding integral equations, and deduce the formula of the solutions to Problem (1). Furthermore, we present two consequences of the existence of solutions to Problem (1), and as a first consequence, we show that the set of these solutions is not empty and compact. Section 4 is devoted to studying the (p, ω, κ) -generalized Ulam–Hyers stability of Problem (1). Section 5 contains discussion and conclusion. In Section 6, we give examples to support our findings.

2. Preliminaries and Notations

To make it easier to read the rest of the work, we mention below the symbols that we will use:

- $AC(\hbar, \mathcal{U})$ is the Banach space of absolutely continuous functions from $\hbar = [0, \mathcal{L}]$ to $\mathcal{U}$.
- $q \in [1, \infty)$ and

$$L_{\omega}^{q,\kappa}((0, \mathcal{L}), \mathcal{U}) := \{z : \hbar \rightarrow \mathcal{U} : z \text{ is Lebesgue measurable and } z\omega(\kappa')^{\frac{1}{q}} \in L^q((0, \mathcal{L}), \mathcal{U})\}.$$

- For any function $z : [a, \mathcal{L}] \rightarrow \mathcal{U}$, define

$$D_{\lambda}^{1,p,\kappa} z(\lambda) := (1-p)f(\lambda) + \frac{p}{\kappa'(\lambda)} \frac{d}{d\lambda} f(\lambda),$$

$$D_{\lambda}^{k,p,\kappa} f(\lambda) := D_{\lambda}^{k-1,p,\kappa} (D_{\lambda}^{1,p,\kappa} f(\lambda)),$$

$$D_{\lambda}^{1,p,\kappa,\omega} f(\lambda) := \omega^{-1}(\lambda) D_{\lambda}^{1,p,\kappa} (f(\lambda) \omega(\lambda)),$$

and

$$D_{\lambda}^{k,p,\kappa,\omega} f(\lambda) := D_{\lambda}^{k-1,p,\kappa,\omega} D_{\lambda}^{1,p,\kappa,\omega} f(\lambda), k \in \mathbb{N} - \{1\}.$$

Notice that

$$\begin{aligned} D_{\lambda}^{2,p,\kappa,\omega} f(\lambda) &= D_{\lambda}^{1,p,\kappa,\omega} (D_{\lambda}^{1,p,\kappa,\omega} f(\lambda)) \\ &= \omega^{-1}(\lambda) D_{\lambda}^{1,p,\kappa} (\omega(\lambda) (D_{\lambda}^{1,p,\kappa,\omega} f(\lambda))) \\ &= \omega^{-1}(\lambda) D_{\lambda}^{1,p,\kappa} D_{\lambda}^{1,p,\kappa} (f(\lambda) \omega(\lambda)) \\ &= \omega^{-1}(\lambda) D_{\lambda}^{2,p,\kappa} (f(\lambda) \omega(\lambda)). \end{aligned}$$

By induction, it results for any natural number $k \geq 2$,

$$D_{\lambda}^{k,p,\kappa,\omega} f(\lambda) = \omega^{-1}(\lambda) D_{\lambda}^{k,p,\kappa} (f(\lambda) \omega(\lambda)).$$

Let us consider the Banach spaces:

- $C_{\omega}(\hbar, \mathcal{U}) := \{u : \hbar \rightarrow \mathcal{U} : \omega u \in C(\hbar, \mathcal{U})\}$, where $\|u\| = \max_{b \in \hbar} \|\omega(b)u(b)\|$.
- $PC_{2-\mu,\kappa,\omega}(\mathfrak{S}, \mathcal{U}) := \{u : \hbar \rightarrow \mathcal{U}, (\kappa(\cdot) - \kappa(b_k))^{2-\mu} u(\cdot) \in C_{\omega}(\hbar_k, \mathcal{U}), u|_{B_i} \in C(B_i, \mathcal{U}), \lim_{\lambda \rightarrow b_k^+} (\kappa(\lambda) - \kappa(b_k))^{2-\mu} u(b) \text{ and } \lim_{b \rightarrow b_i^+} u(b) \text{ exists}, k \in A_0, i \in A_1\}$, where

$$\|u\|_{PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})} := \max\left\{\sup_{\substack{\lambda \in \hbar_k \\ k \in A_0}} (\kappa(\lambda) - \kappa(b_k))^{2-\mu} \|\omega(\lambda)u(\lambda)\|_{\mathcal{U}}, \sup_{\substack{\lambda \in \overline{B_k} \\ k \in A_1}} \|u(\lambda)\|_{\mathcal{U}}\right\},$$

- $AC^{1,\omega}(\hbar, \mathcal{U}) := \{u : \hbar \rightarrow \mathcal{U}, u\omega \in AC(\hbar, \mathcal{U})\}$, where, $\|u\|_{AC^{1,\omega}(\hbar, \mathcal{U})} = \|u\omega\|_{AC(\hbar, \mathcal{U})}$.
- $C_{2-\mu,\kappa,\omega}^1(\hbar, \mathcal{U}) := \{u \in C_{2-\sigma,\kappa,\omega}(\hbar, \mathcal{U}) : D^{1,\kappa,\omega} u \in C_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})\}$.
- $C_{2-\mu,\kappa,\omega}^n(\hbar, \mathcal{U}) := \{u \in C_{2-\mu,\kappa,\omega}^{n-1}(\hbar, \mathcal{U}) : D^{n,\kappa,\omega} u \in C_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})\}, n \in \mathbb{N} - \{1\}$,

where

$$\|u\|_{C_{2-\mu,\kappa,\omega}^n(\hbar, \mathcal{U})} := \sum_{k=1}^{n-1} \|D^{k,\kappa,\omega} u\|_{C_{2-\mu,\kappa,\omega}^k(\hbar, \mathcal{U})} + \|D^{n,\kappa,\omega} u\|_{C_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})}.$$

- $AC^{n,\kappa,\omega}(\hbar, \mathcal{U}) := \{u : \hbar \rightarrow \mathcal{U}, D^{n-1,\kappa,\omega} u \in AC^{1,\omega}(\hbar, \mathcal{U}), D^{n,\kappa,\omega} u \in L_{\omega}^{1,\kappa}((0, \mathcal{L}), \mathcal{U})\}; n \in \mathbb{N} - \{1\}$, where $\|u\|_{AC^{n,\kappa,\omega}(\hbar, \mathcal{U})} := \|D^{n-1,\kappa,\omega} u\|_{AC(\mathfrak{S}, \mathcal{U})}$.
- $C_{2-\mu,\kappa,\omega}^{\gamma}(\hbar, \mathcal{U}) := \{x \in C_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U}), D_{0,t}^{\gamma,\kappa,w} x \in C_{2-\mu,\psi,w}(\hbar, \mathcal{U})\}; \gamma > 0$.
-

$$PC_{2-\mu,\kappa,\omega}^{\gamma}(\hbar, \mathcal{U}) := \{x \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U}), D_{b_k^+, t}^{\gamma,\psi} x|_{\hbar_k} \in C_{2-\mu,\kappa,\omega}(\hbar_k, \mathcal{U}), k \in A_0\}; \gamma > 0.$$

Now, the function $\chi_{PC_{2-\mu,\kappa,\omega}(\mathfrak{S}, \mathcal{U})} : P_b(PC_{2-\sigma,\kappa,\omega}(\hbar, \mathcal{U})) \rightarrow [0, \infty)$, which is given by :

$$\chi_{PC_{2-\mu,\kappa,\omega}(\mathfrak{S}, \mathcal{U})}(D) := \max\left\{\max_{k \in A_0} \chi_{C(\hbar_k, \mathcal{U})}(D|_{\hbar_k}), \max_{i \in A_1} \chi_{C(\overline{B_i}, \mathcal{U})}(D|_{\overline{B_i}})\right\},$$

is a measure of non-compactness on $PC_{2-\mu,\kappa,\omega}(\mathfrak{S},\mathfrak{U})$, where

$$D_{|\overline{h_k}} := \{f^* \in C(\overline{\mathfrak{S}_k}, \mathfrak{U}) : f^*(\lambda) = (\kappa(\lambda) - \kappa(b_k))^{2-\mu} \omega(\lambda) f(\lambda), \lambda \in \mathfrak{h}_k, \\ f^*(b_k) = \lim_{\lambda \rightarrow b_k^+} f^*(\lambda), f \in D\},$$

and

$$D_{|\overline{B_i}} := \{f^* \in C(\overline{B_i}, \mathfrak{U}) : f^*(\lambda) = f(t), t \in B_i, f^*(t_i) = f(t_i^+), f \in D\}.$$

In what follows, $\gamma \in (k-1, k)$, $k \in \mathbb{N}$, and $g \in L_{\omega}^{1,\kappa}((a, \mathcal{L}), \mathfrak{U})$, $a \geq 0$.

Definition 1 (Definition 3.2, [35]). *the p -proportional κ -Riemann-Liouville fractional integral of order γ for g is given by:*

$$(I_{a,\lambda}^{\gamma,p,\kappa} g)(\lambda) := \frac{1}{p^\gamma \Gamma(\gamma)} \int_a^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda) - \kappa(\vartheta))^{\gamma-1} \kappa'(\vartheta) g(\vartheta) d\vartheta.$$

Definition 2 (Definition 3.3, [35]). *The p -proportional κ -Riemann-Liouville fractional derivative whose order γ , where the lower limit at a of g is defined by:*

$$(D_{a,\lambda}^{\gamma,p,\kappa} g)(\lambda) := D^{k,p,\kappa}(I_{a,\lambda}^{k-\gamma,\kappa,\omega} f g)(\lambda) \\ = \frac{1}{p^{k-\gamma} \Gamma(k-\gamma)} D^{n,p,\kappa} \left(\int_a^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda) - \kappa(\vartheta))^{k-\gamma-1} \kappa'(\vartheta) g(\vartheta) d\vartheta \right),$$

as long as the right side has a definite value.

Proposition 1 (Prop. 3.1, Th.3.1, Th. 3.2, [35]). *See also [43]. Let $\delta > 0$ and $g : [a, \mathcal{L}] \rightarrow \mathfrak{U}$ be continuous.*

- i $(I_{a,\lambda}^{\gamma,p,\kappa} I_{a,\lambda}^{\delta,p,\kappa} g)(\lambda) = (I_{a,\lambda}^{\gamma+\delta,p,\kappa} g)(\lambda); \lambda \in [a, \mathcal{L}].$
ii

$$I_{a,\lambda}^{\gamma,p,\kappa} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta-1} \\ = \frac{p^{-\gamma} \Gamma(\delta)}{\Gamma(\delta + \gamma)} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta+\gamma-1}; \forall \lambda \in \mathfrak{S}. \quad (2)$$

- iii If $\gamma < \delta$, then

$$D_{a,\lambda}^{\gamma,p,\kappa} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta-1} \\ = \frac{p^\gamma \lambda^\gamma \Gamma(\delta)}{\Gamma(\delta - \gamma)} e^{\frac{p-1}{p}\kappa(\lambda)} (\kappa(\lambda) - \kappa(a))^{\delta-1-\gamma}. \quad (3)$$

- iv For any $r \in \{0, 1, \dots, k\}$,

$$D_{a,\lambda}^{\gamma,p,\kappa} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\gamma-r} = 0. \quad (4)$$

- v If $c \in \mathbb{N}$ with $c \leq k-1$, then $D_{a,\lambda}^{c,p,\kappa} I_{a,\lambda}^{\gamma,p,\kappa} g(\lambda) = I_{a,\lambda}^{\gamma-c,p,\kappa} g(\lambda), \forall \lambda \in (a, \mathcal{L}].$

- vi If $\delta \leq \gamma$, then $D_{a,\lambda}^{\delta,p,\kappa} I_{a,\lambda}^{\gamma,p,\kappa} g(\lambda) = I_{a,\lambda}^{\gamma-\delta,p,\kappa} g(\lambda), \forall \lambda \in (a, \mathcal{L}].$

- vii $D_{a,\lambda}^{\gamma,p,\kappa} I_{a,\lambda}^{\gamma,p,\kappa} g(\lambda) = f(\lambda), \forall \lambda \in (a, \mathcal{L}].$

Lemma 1 (Theorem 3.1, [36]).

Let $g : [a, \mathcal{L}] \rightarrow \mathcal{U}$ such that $I_{a,\lambda}^{\gamma,p,\kappa} f \in AC^k([a, \mathcal{L}], \mathcal{U})$, then

$$\begin{aligned} & I_{a,\lambda}^{\gamma,p,\kappa} (D_{a,\lambda}^{\gamma,p,\kappa} g)(\lambda) \\ &= g(\lambda) - e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} \sum_{r=1}^k \frac{(\kappa(\lambda) - \kappa(a))^{\gamma-r}}{p^{\gamma-k}\Gamma(\gamma-k+1)} \lim_{\lambda \rightarrow a} (D^{k-r,p,\kappa} (I_{a,\lambda}^{k-\gamma,p,\kappa} g)(\lambda)). \end{aligned} \quad (5)$$

Definition 3 ([36]). The p -proportional κ -Caputo fractional derivative of order γ with the lower limit at a of g is given by:

$$\begin{aligned} ({}^c D_{a,\lambda}^{\gamma,p,\kappa} g)(\lambda) &:= I_{a,\lambda}^{k-\gamma,p,\kappa} (D^{k,p,\kappa} f)(\lambda) \\ &= \frac{1}{p^{k-\gamma}\Gamma(k-\gamma)} \left(\int_a^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda) - \kappa(\vartheta))^{k-\gamma-1} \kappa'(\vartheta) (D^{k,p,\kappa} g(\vartheta)) d\vartheta, \right. \end{aligned}$$

as long as the right side has a definite value.

Remark 2. If $c \in \mathbb{R}$, then $D^{k,p,\kappa}(c) = c(1-p)^k$.

Lemma 2 (Prop. 4.1, [36]). See also [43]. Let $\delta > 0$.

i

$$\begin{aligned} & {}^c D_{a,\lambda}^{\gamma,p,\kappa} (e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta-1}) \\ &= \frac{\lambda^\gamma \Gamma(\delta)}{\Gamma(\delta-\gamma)} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta-1-\gamma}. \end{aligned}$$

ii For any $r \in \{0, 1, \dots, k-1\}$,

$$({}^c D_{a,\lambda}^{\gamma,p,\kappa} (e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^r) = 0.$$

To drive our results, we need to use the following fixed point for multi-valued functions. Let $U \in P_{cc}(\mathcal{U}) = \{C \subseteq \mathcal{U} : C \text{ be not empty, closed, and convex}\}$ and χ be a non-singular measure of noncompactness on the bounded subsets of U . The following lemma can be found in Corollary 3.3.1, Proposition 3.5.1 [61–63].

Lemma 3. If $Y : U \rightarrow P_{ck}(\mathcal{U})$ has a closed graph and χ -condensing, then $\text{Fix}(Y) \neq \emptyset$. If in addition, χ is monotone and $\text{Fix}(Y)$ is bounded, then it is compact.

3. New Concepts, Integral Representations, and Existence of Solutions

In this section, we introduce the following concepts: p -proportional ω -weighted κ -Riemann-Liouville fractional integral, the p -proportional ω -weighted κ -Riemann-Liouville fractional derivative, and the p -proportional ω -weighted κ -Hilfer fractional derivative, then we conduct some of its properties. Furthermore, we construct the formula for the solution function for the objective Problem (1). Moreover, we demonstrate two consequences of the existence of solutions, and in the first one, we show that the set of such solutions is not empty and compact. Let $\gamma \in (k-1, k)$, $k \in \mathbb{N}$, and $g \in L_{\omega}^{1,\kappa}((a, \mathcal{L}), \mathcal{U})$, $a \geq 0$.

Definition 4. The p -proportional ω -weighted κ -Riemann-Liouville fractional integral of order γ , where the lower limit at $a < \mathcal{L}$ is defined by:

$$\begin{aligned} (I_{a,\lambda}^{\gamma,p,\kappa,\omega} g)(\lambda) &:= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma,p,\kappa} (g(\lambda)\omega(\lambda)) \\ &= \frac{\omega^{-1}(\lambda)}{p^\gamma \Gamma(\gamma)} \int_a^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda) - \kappa(\vartheta))^{\gamma-1} \kappa'(\vartheta) g(\vartheta) \omega(\vartheta) d\vartheta, \end{aligned}$$

Remark 3. $f \in L_{\omega}^{1,\kappa}([a, \mathcal{L}], \mathcal{U}) \implies \omega(\lambda) I_{a,\lambda}^{\gamma,p,\kappa,\omega} f(\lambda) = I_{a,\lambda}^{\gamma,p,\kappa}(f(\lambda)\omega(\lambda)) \iff I_{a,\lambda}^{\gamma,p,\kappa,\omega}(\omega^{-1}(\lambda)f(\lambda)) = \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma,p,\kappa} f(\lambda), \forall \lambda \in [a, \mathcal{L}]$.

Lemma 4. Let $\delta > 0$ and $f : [a, \mathcal{L}] \rightarrow \mathcal{U}$ is continuous.

- i $I_{a,\lambda}^{\gamma,\kappa,\omega} I_{a,\lambda}^{\delta,\kappa,\omega} f = I_{a,\lambda}^{\gamma+\delta,\kappa,\omega} f$.
 ii

$$\begin{aligned} & I_{a,\lambda}^{\gamma,p,\kappa,\omega}(\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1}) \\ &= \omega^{-1}(\lambda) \frac{p^{-\gamma}\Gamma(\delta)}{\Gamma(\delta+\gamma)} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda)-\kappa(a))^{\delta+\gamma-1}; \forall \lambda \in [a, \mathcal{L}]. \end{aligned} \quad (6)$$

Proof. i It yields from Remark (3) and (i) in Proposition (1),

$$\begin{aligned} (I_{a,\lambda}^{\gamma,p,\kappa,\omega} I_{a,\lambda}^{\delta,p,\kappa,\omega} f)(\lambda) &= I_{a,\lambda}^{\gamma,p,\kappa,\omega} (I_{a,\lambda}^{\delta,p,\kappa,\omega} f)(\lambda) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma,p,\kappa}(\omega(\lambda) I_{a,\lambda}^{\delta,p,\kappa,\omega} f(\lambda)) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma,p,\kappa} I_{a,\lambda}^{\delta,p,\kappa}(f(\lambda)\omega(\lambda)) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma+\delta,p,\kappa}(f(\lambda)\omega(\lambda)) \\ &= I_{a,\lambda}^{\gamma+\delta,p,\kappa,\omega} f(\lambda). \end{aligned}$$

ii Remark (3) and (2) tell us,

$$\begin{aligned} & I_{a,\lambda}^{\gamma,p,\kappa,\omega}(\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1}) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma,p,\kappa,\omega}(e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1}) \\ &= \omega^{-1}(\lambda) \frac{p^{-\gamma}\Gamma(\delta)}{\Gamma(\delta+\gamma)} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda)-\kappa(a))^{\delta+\gamma-1}; \forall \lambda \in [a, \mathcal{L}]. \end{aligned}$$

□

Definition 5. The p -proportional ω -weighted κ -Riemann-Liouville fractional derivative whose order γ , where the lower limit at a of g is given by:

$$\begin{aligned} (D_{a,\lambda}^{\gamma,p,\kappa,\omega} f)(\lambda) &:= D_{\lambda}^{k,p,\kappa,\omega} (I_{a,\lambda}^{k-\gamma,p,\kappa,\omega} f)(\lambda) \\ &= \frac{1}{\omega(\lambda) p^{k-\gamma} \Gamma(k-\gamma)} \\ &\quad \times D_{\lambda}^{k,p,\kappa,\omega} \left(\int_a^{\lambda} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda)-\kappa(\vartheta))^{k-\gamma-1} \kappa'(\vartheta) f(\vartheta) \omega(\vartheta) d\vartheta \right), \end{aligned}$$

as long as the right side has a definite value.

Lemma 5. Let $\delta > 0$ and $g : [a, \mathcal{L}] \rightarrow \mathcal{U}$ be continuous.

i For any $\lambda \in [a, \mathcal{L}]$,

$$(D_{a,\lambda}^{\gamma,p,\kappa,\omega} g)(\lambda) = \omega^{-1}(\lambda) (D_{a,\lambda}^{\gamma,p,\kappa} g\omega)(\lambda). \quad (7)$$

ii

$$\begin{aligned} D_{a,\lambda}^{\gamma,p,\kappa,\omega}(\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1}) \\ = \frac{\omega^{-1}(\lambda)\lambda\gamma\Gamma(\delta)}{\Gamma(\delta-\gamma)}e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1-\gamma}. \end{aligned}$$

iii For any $r \in \{0, 1, \dots, k\}$,

$$D_{a,\lambda}^{\gamma,p,\kappa,\omega}(\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\gamma-r}) = 0. \quad (8)$$

iv If $c \in \mathbb{N}$ and $c \leq k-1$, then

$$D_{a,\lambda}^{c,p,\kappa,\omega}I_{a,\lambda}^{\gamma,p,\kappa,\omega}g(\lambda) = I_{a,\lambda}^{\gamma-c,p,\kappa,\omega}g(\lambda), \forall \lambda \in [a, \mathcal{L}].$$

v If $\delta < \gamma$, then

$$D_{a,\lambda}^{\delta,p,\kappa,\omega}I_{a,\lambda}^{\gamma,p,\kappa,\omega}g(\lambda) = I_{a,\lambda}^{\gamma-\delta,p,\kappa,\omega}g(\lambda), \forall \lambda \in [a, \mathcal{L}].$$

vi $D_{a,\lambda}^{\gamma,p,\kappa,\omega}I_{a,\lambda}^{\gamma,p,\kappa,\omega}f(\lambda) = f(\lambda), \forall \lambda \in \mathfrak{S}$.vii If $I_{a,\lambda}^{\gamma,p,\kappa}g\omega \in AC^n([a, \mathcal{L}], \mathfrak{U})$, then

$$\begin{aligned} I_{a,\lambda}^{\gamma,p,\kappa,\omega}(D_{a,\lambda}^{\gamma,p,\kappa,\omega}g)(\lambda) \\ = g(\lambda) - \omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} \sum_{r=1}^k \frac{(\kappa(\lambda)-\kappa(a))^{\gamma-k}}{p^{\gamma-k}\Gamma(\gamma-k+1)} \lim_{\lambda \rightarrow a} (D_{a,\lambda}^{k-r,p,\kappa}(I_{a,\lambda}^{k-\gamma,p,\kappa}(g\omega))(\lambda)). \end{aligned} \quad (9)$$

Proof. i Making use of Definition 5 and Remark 3,

$$\begin{aligned} (D_{a,\lambda}^{\gamma,p,\kappa,\omega}g)(\lambda) &= D_{a,\lambda}^{\gamma,p,\kappa,\omega}(I_{a,\lambda}^{n-\gamma,p,\kappa,\omega}g)(\lambda) \\ &= \omega^{-1}(\lambda)D_{a,\lambda}^{n,p,\kappa}\omega(\lambda)(I_{a,\lambda}^{n-\gamma,p,\kappa,\omega}g)(\lambda) \\ &= \omega^{-1}(\lambda)D_{a,\lambda}^{n,p,\kappa}\omega(\lambda)\omega^{-1}(\lambda)(I_{a,\lambda}^{n-\gamma,p,\kappa}g\omega)(\lambda) \\ &= \omega^{-1}(\lambda)D_{a,\lambda}^{n,p,\kappa}(I_{a,\lambda}^{n-\gamma,p,\kappa}g\omega)(\lambda) \\ &= \omega^{-1}(\lambda)(D_{a,\lambda}^{\gamma,p,\kappa}g\omega)(\lambda). \end{aligned}$$

ii Taking advantage of (3) and (7),

$$\begin{aligned} D_{a,\lambda}^{\gamma,p,\kappa,\omega}(\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1-\gamma}) \\ = \omega^{-1}(\lambda)D_{a,\lambda}^{\gamma,p,\kappa}(e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1-\gamma}) \\ = \frac{\omega^{-1}(\lambda)p\gamma\Gamma(\delta)}{\Gamma(\delta-\gamma)}e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\delta-1-\gamma} \end{aligned}$$

iii Relations (4) and (7) tell us

$$\begin{aligned} D_{a,\lambda}^{\gamma,p,\kappa,\omega}\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\gamma-k} \\ = \omega^{-1}(\lambda)D_{a,\lambda}^{\gamma,p,\kappa}(e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))}(\kappa(\lambda)-\kappa(a))^{\gamma-k}) = 0. \end{aligned}$$

iv It follows from Remark 3, (7) and (iv) of Proposition (1) that,

$$\begin{aligned} & D_{a,\lambda}^{c,p,\kappa,\omega} I_{a,\lambda}^{\gamma,p,\kappa,\omega} g(\lambda) \\ &= \omega^{-1}(\lambda) D_{a,\lambda}^{c,p,\kappa} \omega(\lambda) I_{a,\lambda}^{\gamma,p,\kappa,\omega} g(\lambda) \\ &= \omega^{-1}(\lambda) D_{a,\lambda}^{c,p,\kappa} I_{a,\lambda}^{\gamma,p,\kappa} (g(\lambda) \omega(\lambda)) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma-c,p,\kappa} (g(\lambda) \omega(\lambda)) \\ &= I_{a,\lambda}^{\gamma-c,p,\kappa,\omega} g(\lambda), \forall \lambda \in \mathfrak{I}. \end{aligned}$$

v It yields from Remark 3, (7) and (v) of Proposition 1 that,

$$\begin{aligned} & D_{a,\lambda}^{\delta,p,\kappa,\omega} I_{a,\lambda}^{\gamma,p,\kappa,\omega} g(\lambda) \\ &= \omega^{-1}(\lambda) D_{a,\lambda}^{\delta,p,\kappa} \omega(\lambda) I_{a,\lambda}^{\gamma,p,\kappa,\omega} g(\lambda) \\ &= \omega^{-1}(\lambda) D_{a,\lambda}^{\delta,p,\kappa} I_{a,\lambda}^{\gamma,p,\kappa} (g(\lambda) \omega(\lambda)) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma-\delta,p,\kappa} (g(\lambda) \omega(\lambda)) \\ &= I_{a,\lambda}^{\gamma-\delta,p,\kappa,\omega} g(\lambda), \forall \lambda \in \mathfrak{I}. \end{aligned}$$

vi It is a direct consequence of (iv).

vii In view of Remarks 3, (7), and (5), we get

$$\begin{aligned} & I_{a,\lambda}^{\gamma,p,\kappa,\omega} (D_{a,\lambda}^{\gamma,p,\kappa,\omega} g)(\lambda) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma,p,\kappa} (\omega(\lambda) D_{a,\lambda}^{\gamma,p,\kappa,\omega} g(\lambda)) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\gamma,p,\kappa} D_{a,\lambda}^{\gamma,p,\kappa} g\omega(\lambda)) \\ &= g(\lambda) - \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} \sum_{r=1}^k \frac{(\kappa(\lambda) - \kappa(a))^{\gamma-r}}{p^{\gamma-k} \Gamma(\gamma - k + 1)} \lim_{\lambda \rightarrow a} (D^{k-r,p,\kappa} (I_{a,\lambda}^{k-\gamma,p,\kappa} g\omega(\lambda))). \end{aligned}$$

□

Remark 4. If we put $\omega(\lambda) = 1; \lambda \in [a, \mathcal{L}]$ in (9), we obtain the formula (5); if we put $p = 1$ in (9), we obtain formula (3.7) in [37].

Definition 6. The p -proportional ω -weighted κ -Caputo fractional derivative of order γ , where the lower limit at a of g is given by:

$$\begin{aligned} ({}^c D_{a,\lambda}^{\gamma,p,\kappa,\omega} f)(\lambda) &:= I_{a,\lambda}^{k-\gamma,p,\kappa,\omega} (D_{a,\lambda}^{k,p,\kappa,\omega} g)(\lambda) \\ &= \frac{\omega^{-1}(\lambda)}{p^{k-\gamma} \Gamma(k-\gamma)} \\ &\times \int_a^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda) - \kappa(\vartheta))^{k-\gamma-1} \kappa'(\vartheta) \omega(\vartheta) (D_{a,\lambda}^{k,p,\kappa,\omega} g(\vartheta)) d\vartheta \\ &= \frac{\omega^{-1}(\lambda)}{p^{k-\gamma} \Gamma(n-\gamma)} \\ &\times \int_a^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda) - \kappa(\vartheta))^{k-\gamma-1} \kappa'(\vartheta) (D_{a,\lambda}^{k,p,\kappa} g\omega(\vartheta)) d\vartheta, \end{aligned}$$

as long as the right side has a definite value.

Using similar arguments in the proof of Lemma 4, one can demonstrate the following Lemma.

Lemma 6.

i If $\delta > 0$, then

$$\begin{aligned} & {}^c D_{a,\lambda}^{\gamma,p,\kappa,\omega} (\omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta-1-\gamma}) \\ &= \omega^{-1}(\lambda) {}^c D_{a,\lambda}^{\gamma,p,\kappa} (e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta-1-\gamma}) \\ &= \omega^{-1}(\lambda) \frac{\lambda^\gamma \Gamma(\delta)}{\Gamma(\delta-\gamma)} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^{\delta-1-\gamma}. \end{aligned} \quad (10)$$

ii For any $r \in \{0, 1, \dots, k-1\}$,

$$\begin{aligned} & {}^c D_{a,\lambda}^{\gamma,p,\kappa,\omega} (\omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^r) \\ &= \omega^{-1}(\lambda) {}^c D_{a,\lambda}^{\gamma,p,\kappa} (e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(a))^r) = 0. \end{aligned} \quad (11)$$

Definition 7. The p -proportional ω -weighted κ -Hilfer derivative of order $\sigma \in (1, 2)$ and of kind $\rho \in (0, 1)$ with lower limit at a for g is given by

$$D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega} f(\lambda) := I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{\lambda}^{2,p,\kappa,\omega} I_{a,\lambda}^{(2-\sigma)(1-\rho),p,\kappa,\omega} f(\lambda),$$

as long as the right side has a definite value.

Remark 5.

1 let $\mu = \sigma + 2\rho - \sigma\rho$. If $f \in C^{2,\kappa,\omega}([a, \mathcal{L}], \mathbb{U})$, then $D_{a,\lambda}^{\mu,\kappa,\omega} f(\lambda)$ exists a.e., and consequently $D_{a,\lambda}^{\sigma,\rho,\kappa,\omega} f(\lambda)$ exists for $\lambda \in (a, \mathcal{L}]$. In fact, from Definition 7 and (7), it results

$$\begin{aligned} D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega} f(\lambda) &:= I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{\lambda}^{2,p,\kappa,\omega} I_{a,\lambda}^{(2-\sigma)(1-\rho),p,\kappa,\omega} f(\lambda), \\ &= I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{\lambda}^{2,p,\kappa,\omega} I_{a,\lambda}^{2-\mu,p,\kappa,\omega} f(\lambda) \\ &= I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{a,\lambda}^{\mu,p,\kappa,\omega} f(\lambda). \end{aligned} \quad (12)$$

2 If $I_{a,\lambda}^{2-\mu,\kappa,\omega} f(\lambda) \in C_{2-\sigma,\kappa,\omega}^{\mu}([a, \mathcal{L}], \mathbb{U})$, then $D_{a,\lambda}^{\mu,\kappa,\omega} f(\lambda) \in C_{2-\sigma,\kappa,\omega}([a, \mathcal{L}], \mathbb{U})$, and consequently $D_{a,\lambda}^{\sigma,\rho,\kappa,\omega} f(\lambda)$ exists for $\lambda \in (a, \mathcal{L}]$. Moreover,

$$\begin{aligned} D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega} f(\lambda) &:= I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{\lambda}^{2,p,\kappa,\omega} I_{a,\lambda}^{(2-\sigma)(1-\rho),p,\kappa,\omega} f(\lambda), \\ &= I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{\lambda}^{2,p,\kappa,\omega} I_{a,\lambda}^{2-\mu,p,\kappa,\omega} f(\lambda) \\ &= I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{a,\lambda}^{\mu,p,\kappa,\omega} f(\lambda) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\rho(2-\sigma),p,\kappa} \omega(\lambda) D_{a,\lambda}^{\mu,p,\kappa,\omega} f(\lambda) \\ &= \omega^{-1}(\lambda) I_{a,\lambda}^{\rho(2-\sigma),p,\kappa} D_{a,\lambda}^{\mu,p,\kappa,\omega} (f\omega(\lambda)), \\ &= \omega^{-1}(\lambda) D_{a,\lambda}^{\sigma,\rho,p,\kappa} (f\omega)(\lambda). \end{aligned}$$

3 Let $f \in L_{\omega}^{q,\kappa}((a, \mathcal{L}], E)$ be such that $I_{a,\lambda}^{2-\mu,p,\kappa,\omega} u \in C_{2-\mu,\kappa,\omega}^2([a, \mathcal{L}], E)$. Since $C_{2-\mu,\kappa,\omega}^2([a, \mathcal{L}], E) \subseteq C^{2,\kappa,\omega}([a, \mathcal{L}], E)$, then, $I_{a,\lambda}^{2-\mu,\kappa,\omega} f \in C^{2,\kappa,\omega}([a, \mathcal{L}], E)$. Therefore, by (i) in Lemma 3 and (9), we get for $\lambda \in (a, \mathcal{L}]$,

$$\begin{aligned}
I_{a,\lambda}^{\sigma,p,\kappa,\omega}(D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}f)(\lambda) &= I_{a,\lambda}^{\sigma,p,\kappa,\omega}I_{a,\lambda}^{\rho(2-\sigma),p,\kappa,\omega}D_{a,\lambda}^{\mu,p,\kappa,\omega}f(\lambda) \\
&= I_{a,\lambda}^{\mu,p,\kappa,\omega}D_{a,\lambda}^{\mu,p,\kappa,\omega}f(\lambda) \\
&= f(\lambda) - \omega^{-1}(\lambda)e^{\frac{p}{p-1}(\kappa(\lambda)-\kappa(a))}\left[\frac{(\kappa(\lambda)-\kappa(a))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)}\right. \\
&\quad \times \lim_{\lambda \rightarrow a}(D^{1,p,\kappa}(I_{a,\lambda}^{2-\mu,p,\kappa}f\omega)(\lambda)) \\
&\quad \left. - \frac{(\kappa(\lambda)-\kappa(a))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)}\lim_{\lambda \rightarrow a^+}(I^{2-\mu,p,\kappa}f\omega(\lambda))\right]. \tag{13}
\end{aligned}$$

Lemma 7. If $f \in C_{2-\mu,\kappa,\omega}([a, \mathcal{L}], \mathfrak{U})$, then $\lim_{\lambda \rightarrow a} \|I_{a,\lambda}^{1-\rho(2-\sigma),p,\kappa}f(\lambda)\omega(\lambda)\| = 0$.

Proof. From the assumption that $f \in C_{2-\mu,\kappa,\omega}(\hbar, \mathfrak{U})$, there is $M > 0$ with

$$\|\omega(\lambda)f(\lambda)\| \leq M(\kappa(\lambda) - \kappa(a))^{\mu-2}, \forall \lambda \in \hbar.$$

So, from (2), we get

$$\begin{aligned}
&\lim_{\lambda \rightarrow a} \|I_{a,\lambda}^{1-\rho(2-\sigma),p,\kappa}f(\lambda)\omega(\lambda)\| \\
&= \lim_{\lambda \rightarrow a} \frac{M}{\Gamma(1-\rho(2-\sigma))} \int_a^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} (\kappa(\lambda) - \kappa(b))^{\rho(2-\sigma)} \times \\
&\quad \kappa'(b)(\kappa(\lambda) - \kappa(a))^{\mu-2} \kappa'(b) db \\
&= M \lim_{\lambda \rightarrow a} |I_{a,\lambda}^{1-\rho(2-\sigma),p,\kappa}(\lambda)(\kappa(\lambda) - \kappa(a))^{\mu-2}(\lambda)| \\
&= Mp^{-(1-\rho(2-\sigma))} \lim_{\lambda \rightarrow a} \left| \frac{\Gamma(\mu-1)}{\Gamma(\mu-\rho(2-\sigma))} (\kappa(\lambda) - \kappa(a))^{1+\mu-\rho(2-\sigma)}(\lambda) \right| \\
&= 0.
\end{aligned}$$

□

Remark 6. If $f \in C_{2-\mu,\kappa,\omega}^\mu([a, \mathcal{L}], \mathfrak{U})$, then $D_{a,\lambda}^{\mu,\kappa,\omega}f(\lambda)$ exists, and consequently, by (4), $D_{a,\lambda}^{\lambda,\rho,\kappa,\omega}f(\lambda)$ exists. Therefore, $C_{2-\mu,\kappa,\omega}^\mu([a, \mathcal{L}], \mathfrak{U}) \subseteq C_{2-\mu,\kappa,\omega}^{\lambda,\rho}([a, \mathcal{L}], \mathfrak{U})$. likewise, $PC_{2-\mu,\kappa,\omega}^\mu([a, \mathcal{L}], \mathfrak{U}) \subseteq PC_{2-\mu,\kappa,\omega}^{\lambda,\rho}([a, \mathcal{L}], E)$.

Definition 8. A function $u \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathfrak{U})$ is a solution to Problem (1) if $u(\lambda) = \pi_i(\lambda, u(l_i^-))$, $\forall \lambda \in (l_i, b_i]; i \in A_1$, $D_{b_k,\lambda}^{\lambda,\rho,\kappa,\omega}u(\lambda)$ exists for $\lambda \in (b_k, l_{k+1}]; k \in A_0$ and u verifies (1).

Lemma 8.

1 Assume that $u \in PC_{2-\mu,\kappa,\omega}^\mu(\hbar, \mathfrak{U})$ and $f \in L_\omega^{q,\kappa}(\hbar, E)$, $q \geq 1$. If u and f satisfy the p -proportional ω -weighted κ -differential equation of order λ and of kind ρ :

$$\begin{cases} D_{b_i,\lambda}^{\sigma,\rho,p,\kappa,\omega}u(\lambda) = f(\lambda), \lambda \in \hbar_i, i \in A_0, \\ u(l_i^+) = \pi_i(l_i, u(l_i^-)), i \in A_1, \\ u(\lambda) = \pi_i(\lambda, u(l_i^-)), \lambda \in B_i, i \in A_1, \\ \lim_{\lambda \rightarrow 0^+} I^{2-\mu,p,\kappa}(u\omega)(\lambda) = u_0 \\ \lim_{\lambda \rightarrow 0^+} (D^{1,p,\kappa}(I_{a,\lambda}^{2-\mu,p,\kappa}(u\omega)(\lambda))) = u_1, \\ \lim_{\lambda \rightarrow b_i^+} I^{2-\mu,p,\kappa}u\omega(\lambda) = \pi_i(b_i, u(b_i^-)), i \in A_1, \\ \lim_{\lambda \rightarrow b_i^+} (D^{1,p,\kappa}(I_{a,\lambda}^{2-\mu,p,\kappa}(u\omega)(\lambda))) = \pi_i^*(b_i, u(b_i^-)), i \in A_1, \end{cases} \tag{14}$$

then,

$$u(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda), \lambda \in (b_0, l_1], \\ \pi_i(\lambda, u(l_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda), \lambda \in \bar{h}_i, i \in A_1. \end{cases} \quad (15)$$

- 2 If $f \in PC_{2-\mu,\kappa,\omega}^{\rho(2-\sigma)}(\bar{h}, \bar{U})$, then the function u , given by (15) belongs to $PC_{2-\mu,\kappa,\omega}^{\mu}(\bar{h}, \bar{U})$, $D_{b_k,\lambda}^{\sigma,\rho,\kappa,\omega} u(\lambda)$ exists for $\lambda \in (b_k, l_{k+1}]$, $\forall k \in A_0$, and it is a solution to (14).

Proof.

- 1 Let $i = 0$. From (14), we have

$$D_{0,\lambda}^{\sigma,\rho,p,\kappa,\omega} u(\lambda) = f(\lambda), \lambda \in \bar{h}_0. \quad (16)$$

Since $C_{2-\mu,\kappa,\omega}^2(\bar{h}_0, \bar{U}) \subseteq AC_{2-\mu,\kappa,\omega}^{2-\mu,p,\kappa,\omega}(\bar{h}_0, \bar{U})$, then $I_{0,\lambda}^{2-\mu,p,\kappa,\omega} u \in AC_{2-\mu,\kappa,\omega}^{2-\mu,p,\kappa,\omega}(\bar{h}_0, \bar{U})$. Applying $I_{0,\lambda}^{\sigma,p,\kappa,\omega}$ to both sides of (16) and using (13), we obtain for $\lambda \in \bar{h}_0$,

$$\begin{aligned} I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda) &= I_{0,\lambda}^{\sigma,p,\kappa,\omega} (D_{0,\lambda}^{\sigma,\rho,p,\kappa,\omega} u)(\lambda) = I_{0,\lambda}^{\sigma,p,\kappa,\omega} I_{b_i,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{0,\lambda}^{\mu,\kappa,\omega} u(\lambda) \\ &= u(\lambda) - \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \left[\frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \lim_{\lambda \rightarrow 0^+} (D^{1,p,\kappa}(I_{a,\lambda}^{2-\mu,p,\kappa} u \omega)(\lambda)) \right. \\ &\quad \left. - \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \lim_{\lambda \rightarrow 0^+} (I^{2-\mu,p,\kappa} u \omega)(\lambda) \right], \end{aligned}$$

which implies

$$\begin{aligned} u(\lambda) &= \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ &\quad + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 + I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda). \end{aligned}$$

Analogously, we get for $\lambda \in (b_i, l_{i+1}]$, $i \in L_1$,

$$\begin{aligned} u(\lambda) &= \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \left[\frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \lim_{\lambda \rightarrow b_i^+} (D^{1,p,\kappa}(I_{a,\lambda}^{2-\mu,p,\kappa} u \omega)(\lambda)) \right. \\ &\quad \left. + \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \lim_{\lambda \rightarrow b_i^+} (I^{2-\mu,p,\kappa} u \omega)(\lambda) \right] + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda) \\ &= \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u(b_i^-)) \\ &\quad + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu-1)} \pi_i^*(b_i, u(b_i^-)) + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda). \end{aligned}$$

Therefore, (15) is verified.

2 Assume that (15) is true and let $\lambda \in (0, l_1]$. Then,

$$\begin{aligned} u(\lambda) &= \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ &\quad + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 + I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda). \end{aligned} \quad (17)$$

Applying $D_{0,\lambda}^{\mu,p,\kappa,\omega}$ on both sides of (17) and making use of $1 < \sigma < \mu \leq 2$, $\rho(2-\sigma) < 1$ (8), Definition (5), Lemma 3, (i), and Lemma 4 (iv), it follows for $\lambda \in \bar{h}_0$

$$\begin{aligned} D_{0,\lambda}^{\mu,p,\kappa,\omega} u(\lambda) &= D_{0,\lambda}^{\mu,p,\kappa,\omega} I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda) \\ &= D_{\lambda}^{2,p,\kappa,\omega} I_{0,\lambda}^{2-\mu,p,\kappa,\omega} I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda) \\ &= D_{\lambda}^{2,p,\kappa,\omega} I_{0,\lambda}^{2-\rho(2-\sigma),p,\kappa,\omega} f(\lambda) \\ &= D_{\lambda}^{1,p,\kappa,\omega} (D_{\lambda}^{1,p,\kappa,\omega} I_{0,\lambda}^{2-\rho(2-\sigma),p,\kappa,\omega}) f(\lambda) \\ &= D_{\lambda}^{1,\kappa,p,\omega} I_{0,\lambda}^{1-\rho(2-\sigma),p,\kappa,\omega} f(\lambda) \\ &= D_{0,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} f(\lambda). \end{aligned} \quad (18)$$

Now, the assumption $f \in PC_{2-\mu,\kappa,\omega}^{\rho(2-\sigma)}(\bar{h}, E)$ means $D_{0,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} f|_{\bar{h}_0} \in C_{2-\mu,\kappa,\omega}(\bar{h}_0, \bar{U})$, and hence by (18) $D_{0,\lambda}^{\mu,p,\kappa,\omega} u|_{\bar{h}_0} \in C_{2-\mu,\kappa,\omega}(\bar{h}_0, \bar{U})$. But this implies $u|_{\bar{h}_0} \in C_{2-\mu,\kappa,\omega}^{\mu}(\bar{h}_0, \bar{U})$. Therefore, by Remark 5, $D_{0,\lambda}^{\sigma,p,\kappa,\omega} u(\lambda)$ exists for $\lambda \in \bar{h}_0$. It remains to show that $D_{0,\lambda}^{\sigma,p,\kappa,\omega} u(\lambda) = f(\lambda)$, $\forall \lambda \in \bar{h}_0$. Let $\lambda \in \bar{h}_0$. Since $D_{0,\lambda}^{\rho(2-\sigma),\kappa,\omega} f|_{\bar{h}_0} \in C_{2-\mu,\kappa,\omega}(\bar{h}_0, \bar{U})$, then $D_{\lambda}^{1,\kappa,\omega} I_{0,\lambda}^{1-\rho(2-\sigma),\kappa,\omega} f|_{\bar{h}_0} \in C_{2-\mu,\kappa,\omega}(\bar{h}_0, \bar{U})$; consequently, $I_{b_0,\lambda}^{1-\rho(2-\sigma),\kappa,\omega} f|_{\bar{h}_0} \in C_{2-\mu,\kappa,\omega}^1(\bar{h}_0, \bar{U})$. Therefore, from (12) and (18), we get

$$D_{0,\lambda}^{\sigma,p,\kappa,\omega} u(\lambda) = I_{0,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{0,\lambda}^{\mu,p,\kappa,\omega} u(\lambda) = I_{0,\lambda}^{\rho(2-\sigma),p,\kappa,\omega} D_{0,\lambda}^{\rho(2-\sigma),\kappa,\omega} f(\lambda); \lambda \in \bar{h}_0,$$

Making use of (9) and noting that $0 < \rho(2-\sigma) < 1$, we obtain

$$\begin{aligned} D_{0,\lambda}^{\sigma,p,\kappa,\omega} u(\lambda) &= f(\lambda) - \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(a))} \frac{(\kappa(\lambda)-\kappa(0))^{\rho(2-\sigma)-1}}{p^{\rho(2-\sigma)-1}\Gamma(\rho(2-\sigma))} \lim_{\lambda \rightarrow 0} (I_{0,\lambda}^{1-(\rho(2-\sigma)),p,\kappa} f \omega(\lambda)). \end{aligned}$$

Using Lemma 6 we obtain $\lim_{\lambda \rightarrow 0} (I_{0,\lambda}^{1-\rho(2-\sigma),p,\kappa} f(\lambda) \omega(\lambda)) = 0$. Thus, $D_{0,\lambda}^{\sigma,p,\kappa,\omega} u(\lambda) = f(\lambda)$, for any $\lambda \in \bar{h}_0$. Likewise, it can be shown that $D_{b_1,\lambda}^{\sigma,p,\kappa,\omega} u(\lambda)$ exists and $D_{b_i,\lambda}^{\sigma,p,\kappa,\omega} u(\lambda) = f(\lambda)$, $\forall \lambda \in \bar{h}_i = (b_i, l_{i+1}]$, $i \in A_1$. $\square$

Based on Lemma 7 and Definition 8, we present the following corollary, in which the solution function to Problem (1) is specified.

Corollary 1. A function $u : \hbar \rightarrow \mathfrak{U}$ defined by

$$u(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda), \lambda \in (b_0, l_1], \\ \pi_i(\lambda, u(I_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda), \lambda \in \hbar_i, i \in A_1. \end{cases} \quad (19)$$

is a solution to Problem (1), where $f \in C_{2-\mu,\kappa,\omega}^{\rho(2-\sigma)}(\hbar, E)$ with $f(\lambda) \in F(\lambda, u(\lambda))$, a.e.

Remark 7. The solution function u defined by (19) has the following properties:

- i $u \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathfrak{U})$.
- ii If $p = 1$, then the formula (19) is reduced to (10) in [29].
- iii u is not continuous at $b_i, \forall i \in L_1$.
- iv If $\rho = 1, \omega(\lambda) = 1; \lambda \in [0, \mathcal{L}]$, i.e., the case of p -proportional κ -Caputo of order σ , then u is not continuous at $b_i, \forall i \in A_1$. In fact, in this case $\mu = 2, D_{b_i,\lambda}^{\sigma,\rho,p,\kappa,\omega} u(\lambda) = {}^c D_{b_i,\lambda}^{\sigma,p,\kappa} u(\lambda); \lambda \in (b_i, l_{i+1}], i \in A_1$, and (19) is reduced to

$$u(\lambda) = \begin{cases} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} u_0 \\ + e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))}{p} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda), \lambda \in (b_0, l_1], \\ \pi_i(\lambda, u(I_i^-)), \lambda \in B_i, i \in A_1, \\ e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \pi_i(b_i, u(b_i^-)) \\ + e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))}{p} \pi_i^*(b_i, u(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} f(\lambda), \lambda \in \hbar_i, i \in A_1, \end{cases} \quad (20)$$

Note that the function given in (20) is a solution to the p -proportional κ -Caputo differential inclusion in the presence of non-instantaneous impulses:

$$\begin{cases} {}^c D_{b_i,\lambda}^{\sigma,p,\kappa} u(\lambda) \in F(\lambda, u(\lambda)), \text{ a.e. } \lambda \in (b_i, l_{i+1}], i \in L_0, \\ u(l_i^+) = \pi_i(l_i, u(I_i^-)), i \in L_1 \\ u(\lambda) = \pi_i(\lambda, u(I_i^-)), \lambda \in (l_i, b_i], i \in L_1, \\ u(0) = u_0, \lim_{\lambda \rightarrow 0^+} \frac{1}{\kappa(\lambda)} \frac{d}{d\lambda} u(\lambda) = u_1, \\ \lim_{\lambda \rightarrow b_i^-} \frac{1}{\kappa(\lambda)} \frac{d}{d\lambda} u(\lambda) = \pi_i^*(b_i, u(b_i^-)), i \in L_1. \end{cases}$$

Now, let $q > \frac{1}{2-\sigma} > 1$ be a fixed real number and for any $z \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathfrak{U})$, set

$$S_{F(\cdot, u)}^q := \{f \in L_{\omega}^{q,\kappa}(\hbar, \mathfrak{U}) : f(\lambda) \in F(\lambda, z(\lambda)), \text{ a.e. for } \lambda \in \hbar\},$$

and

$$I^{\rho(2-\sigma),p,\kappa,\omega}(PC_{2-\mu,\kappa,\omega}(\hbar, \mathfrak{U})) = \{f : \hbar \rightarrow \mathfrak{U}, \text{ there is } \tau \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathfrak{U}) \text{ such that} \\ f(t) = I_{b_i+}^{\rho(2-\sigma),p,\kappa} \tau(t), t \in \hbar_k, k \in L_0\}.$$

Remark 8. If $S_{F(.,u)}^q \subseteq I^{\rho(2-\sigma),p,\kappa,\omega}(PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U}))$, then for any $f \in S_{F(.,u)}^q$, there is $\tau \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})$ such that $f(\lambda) = I_{b_i+,\lambda}^{\rho(2-\sigma),\kappa}\tau(\lambda), \forall \lambda \in \hbar_k, k \in L_0$, and consequently

$$D_{b_i+,\lambda}^{\rho(2-\sigma),\kappa,\omega}f(\lambda) = D_{b_i+,\lambda}^{\rho(2-\sigma),\kappa,\omega}I_{b_i+,\lambda}^{\rho(2-\sigma),\kappa}\tau(\lambda) = \tau(\lambda), \forall \lambda \in \hbar_k, k \in A_0,$$

which implies $f \in PC_{2-\mu,\kappa,\omega}^{\rho(2-\sigma)}(\hbar, \mathcal{U})$.

Now, let $F : \hbar \times \mathcal{U} \rightarrow P_{ck}(\mathcal{U})$, $q > \frac{1}{2-\sigma}$ and $\pi_i, \pi_i^* : [l_i, b_i] \times \mathcal{U} \rightarrow \mathcal{U}; i \in A_1$. We make the following assumptions:

(HF₁) For any $z \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})$, the multi-valued function $\lambda \rightarrow F(\lambda, z(\lambda))$ is measurable.

(HF₂) For any $z \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})$, the set $S_{F(.,z)}^q$ is not an empty subset of $I^{\rho(2-\sigma),p,\kappa,\omega}(PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U}))$.

(HF₃) There is a $\xi \in L_{\omega}^{q,\kappa}(\hbar, \mathbb{R}^+)$ and $\Theta : [0, \infty) \rightarrow [0, \infty)$ such that for any $z \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})$,

$$\|F(\lambda, z(\lambda))\| = \sup_{y \in F(\lambda, z(\lambda))} \|y\| \leq \xi(\lambda)\Theta(\|z\|_{PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})}), \text{ for almost } \lambda \in \cup_{k \in A_1}(s_i, l_{i+1}]. \quad (21)$$

and

$$\limsup_{n \rightarrow \infty} \frac{\Theta(n)}{n} = \delta < \infty \quad (22)$$

(HF₄) If $z_n \in PC_{2-\mu,\Psi,\omega}(\hbar, \mathcal{U})$, $z_n \rightarrow z \in PC_{2-\mu,\Psi,\omega}(\hbar, \mathcal{U})$, $\varsigma_n \in S_{F(.,z_n)}^{\rho(2-\sigma),q}$ and $\varsigma_n \rightarrow \varsigma \in L_{\omega}^{q,\Psi}(\hbar, \mathcal{U})$, then $\varsigma \in S_{F(.,z)}^{\rho(2-\sigma),q}$.

(HF₅) There is a $\beta \in L_{\omega}^{1,\kappa}(\hbar, \mathbb{R}^+)$ such that, for any bounded subset $D \subseteq PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})$ and any $k \in A_0$, we get

$$\begin{aligned} \chi_{\mathcal{U}}(F(\lambda, z(\lambda)) : z \in D) \\ \leq (\kappa(\lambda) - \kappa(s_k))^{2-\mu} \delta(\lambda) \chi_{\mathcal{U}}\{z(\lambda) : z \in D\}, a.e. \lambda \in \hbar_k. \end{aligned} \quad (23)$$

where χ is the Hausdorff measure of non-compactness on $\mathcal{U}$.

(HF₆) There is a $\xi \in L_{\omega}^{q,\kappa}(\hbar, \mathbb{R}^+)$, $q > 1$ such that for any $z \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})$,

$$\|F(\lambda, z(\lambda))\| = \sup_{y \in F(\lambda, z(\lambda))} \|y\| \leq \xi(\lambda)(1 + \|z\|_{PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})}), \text{ for almost } \lambda \in \cup_{k \in A_0}(b_i, l_{i+1}].$$

(HF₇) There is a $\xi \in L_{\omega}^{q,\kappa}(\hbar, \mathbb{R}^+)$, $q \geq 1$ such that

(i) For any $u_1, u_2 \in PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})$ and any $\varsigma_1 \in S_{F(.,u_1)}^q$, there is $\varsigma_2 \in S_{F(.,u_1)}^q$ with

$$\|\varsigma_1(\lambda) - \varsigma_2(\lambda)\| \leq \xi(\lambda)\|u_1 - u_2\|_{PC_{2-\mu,\kappa,\omega}(\hbar, \mathcal{U})}, a.e., \text{ for } \lambda \in \cup_{k \in A_0}(b_i, l_{i+1}], \quad (24)$$

(ii) For almost $\lambda \in \hbar$

$$h(F(\lambda, 0), \{0\}) \leq \xi(\lambda), \text{ for almost } \lambda \in \cup_{k \in A_0}(b_i, l_{i+1}], \quad (25)$$

where h is the Hausdorff distance.

(H₁) For any $i \in A_1$, $\pi_i : [l_i, b_i] \times \mathcal{U} \rightarrow \mathcal{U}$, $i \in A_1$ is uniformly continuous on bounded sets $\pi_i(\lambda, .); \lambda \in \hbar$ is compact, and there is $r_i > 0$ with

$$\|\pi_i(\lambda, u)\| \leq r_i\|u\|, \forall \lambda \in (l_i, b_i] \text{ and } \forall u \in \mathcal{U}. \quad (26)$$

(H₂) For any $i \in A_1$, the function $\pi_i^* : [l_i, b_i] \times \mathcal{U} \rightarrow \mathcal{U}$ is compact and there is $r_i^* > 0$ with

$$\|\pi_i^*(\lambda, u)\| \leq r_i^* \|u\|, \forall \lambda \in [l_i, b_i] \text{ and } \forall u \in \mathcal{U}. \quad (27)$$

(H₃) There is $\tau > 0$ such that for any $z_1, z_2 \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$ one has

$$\|\pi_i(\lambda, z_1(t)) - \pi_i(\lambda, z_2(t))\| \leq \tau \|z_1(t) - z_2(t)\|, \forall \lambda, t \in \hbar. \quad (28)$$

and

$$\|\pi_i^*(\lambda, z_1(t)) - \pi_i^*(\lambda, z_2(t))\| \leq \tau \|z_1(t) - z_2(t)\|, \forall \lambda, t \in \hbar. \quad (29)$$

(HF $\pi\pi^*$) There are $\xi \in L_{\omega}^{q, \kappa}(\hbar, \mathbb{R}^+)$, $q \geq 1, \tau > 0$ and a strictly comparison function $\eta : [0, \infty) \rightarrow [0, \infty)$ such that

(i) For any $z_1, z_2 \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$, and any $\varsigma_1 \in S_{F(., u_1)}^q$, there is $\varsigma_2 \in S_{F(., u_2)}^q$ with

$$\|\varsigma_1(\lambda) - \varsigma_2(\lambda)\| \leq \xi(\lambda) \eta(\|z_1 - z_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})}), a.e, \text{ for } \lambda \in \cup_{k \in A_0} (s_i, l_{i+1}], \quad (30)$$

(ii) For almost $\lambda \in \hbar$

$$h(F(\lambda, 0), \{0\}) \leq \xi(\lambda), \text{ for almost } \lambda \in \cup_{k \in A_0} (b_i, A_{i+1}]$$

(iii) For any $z_1, z_2 \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$ one has

$$\|\pi_i(\lambda, z_1(t)) - \pi_i(\lambda, z_2(t))\| \leq \tau \eta(\|z_1(t) - z_2(t)\|), \forall \lambda, t \in \hbar. \quad (31)$$

and

$$\|\pi_i^*(\lambda, z_1(t)) - \pi_i^*(\lambda, z_2(t))\| \leq \tau \eta(\|z_1(t) - z_2(t)\|), \forall \lambda, t \in \hbar. \quad (32)$$

Remark 9. (HF₆) implies (HF₃) with $\delta = 1$.

Theorem 1. If the conditions (HF₁) – (HF₅), (H₁) and (H₂) are satisfied, then the set of solutions to Problem (1) is non-empty and compact in $PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$ provided that

$$r_1 + \frac{r_1}{p^{\mu-2}\Gamma(\mu-1)} + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} r_2 + \frac{\delta(\kappa(\mathcal{L}) - \kappa(0))^{\sigma-1} \|\xi\|_{L_{\omega}^{1, \kappa}}}{p^{\sigma}\Gamma(\sigma)} < 1, \quad (33)$$

and

$$\frac{(\kappa(\mathcal{L}) - \kappa(0))^{\sigma-\mu+1}}{p^{\sigma}\Gamma(\sigma)} \|\beta\|_{L_{\omega}^{1, \kappa}(\hbar, \mathbb{R}^+)} < 1, \quad (34)$$

where $r_1 = \sum_{i=0}^{i=n} r_i$ and $r_2 = \sum_{i=0}^{i=n} r_i^*$.

Proof. We define a multi-valued operator $Y : PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U}) \rightarrow \{A \subseteq PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U}) : A \neq \emptyset\}$ in the following way: Let $u \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$. Assumptions (HF₁), (HF₂) assure the existence of $\zeta \in S_{F(., u)}^q$. Then, we define $v \in Y(u)$ if and only if

$$v(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta(\lambda), \lambda \in (s_0, l_1], \\ \pi_i(\lambda, u(I_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} \zeta(\lambda), \lambda \in \tilde{h}_i, i \in A_1. \end{cases} \quad (35)$$

Notice that, thanks to Remark 8, $\zeta \in PC_{2-\mu,\kappa,\omega}^{v(2-\sigma)}(\tilde{h}, \tilde{U})$ and by Lemma 8 and Corollary 1, $v \in PC_{2-\mu,\kappa,\omega}^\mu(\tilde{h}, \tilde{U}) \subseteq PC_{2-\mu,\kappa,\omega}(\tilde{h}, \tilde{U})$ and $D_{b_k,\lambda}^{\sigma,\nu,\kappa,\omega} u(\lambda)$ exist for $\lambda \in (b_k, l_{k+1}]$, $\forall k \in A_0$. Therefore, any fixed point for Y is a solution to Problem (1). So we will be using Lemma 3 to demonstrate that Y has a fixed point. For every $n \in \mathbb{N}$, let $D_n = \{u \in PC_{2-\mu,\kappa,\omega}(\tilde{h}, \tilde{U}) : \|u\|_{PC_{2-\mu,\kappa,\omega}(\tilde{h}, \tilde{U})} \leq n\}$. The proof will be carried out through the following steps:

Step 1. We illustrate the existence of an $n_0 \in \mathbb{N}$ with $Y(D_{n_0}) \subseteq D_{n_0}$. If this is not right, then for every $n \in \mathbb{N}$, there are $v_n, u_n \in PC_{2-\mu,\kappa,\omega}(\tilde{h}, \tilde{U})$ with $v_n \in Y(u_n)$, $\|u_n\|_{PC_{2-\mu,\kappa,\omega}} \leq n$ and $\|v_n\|_{PC_{2-\mu,\kappa,\omega}} > n$. Then, for any $n \in \mathbb{N}$, there is $\zeta_n \in S_{F(.,u)}^q$ with

$$v_n(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta_n(\lambda), \lambda \in (s_0, l_1], \\ \pi_i(\lambda, u_n(I_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u_n(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u_n(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} \zeta_n(\lambda), \lambda \in \tilde{h}_i, i \in A_1. \end{cases} \quad (36)$$

From (HF_3) , it follows

$$\|\zeta_n(\lambda)\| \leq \xi(\lambda) \Theta(n), a.e., \forall n \in \mathbb{N}. \quad (37)$$

Let $\lambda \in (0, l_1]$. Since $|e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b))}| \leq 1, \forall b \in [0, \lambda]$, we get from (36) and (37) that

$$\begin{aligned} & (\kappa(\lambda) - \kappa(0))^{2-\mu} \|\omega(\lambda) v_n(\lambda)\| \\ & \leq \frac{|e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}|}{p^{\mu-2}\Gamma(\mu-1)} \|u_0\| + \frac{\kappa(\lambda) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} \|u_1\| |e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}| \\ & + \frac{\Theta(n)}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \xi(b) \omega(b) db \\ & \leq \frac{1}{p^{\mu-2}\Gamma(\mu-1)} \|u_0\| + \frac{\kappa(\lambda) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} \|u_1\| \\ & + \frac{\Theta(n)}{p^\sigma \Gamma(\sigma)} (\kappa(\mathcal{L}) - \kappa(0))^{\sigma-1} \int_0^\lambda \kappa'(b) \xi(b) \omega(b) db \\ & \leq \frac{\Theta(n)}{p^{\mu-2}\Gamma(\mu-1)} \|u_0\| + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} \|u_1\| \\ & + \frac{\Theta(n)}{p^\sigma \Gamma(\sigma)} (\kappa(\mathcal{L}) - \kappa(0))^{\sigma-1} \|\xi\|_{L_\omega^{1,\kappa}}. \end{aligned} \quad (38)$$

Let $\lambda \in (l_i, b_i], i \in L_1$. It yields from (H_1)

$$\|u_n(\lambda)\| = \|\pi_i(\lambda, u_n(l_i^-))\| \leq r_1 \|u_n(l_i^-)\| \leq r_1 \|u_n\|_{PC_{2-\mu, \kappa, \omega}(\hbar, E)}. \quad (39)$$

If $\lambda \in (b_i, l_{i+1}], i \in L_1$, it results by arguing as above and using (H_1) and (H_2)

$$\begin{aligned} & (\kappa(\lambda) - \kappa(b_i))^{2-\mu} \|\omega(\lambda) v_n(\lambda)\| \\ & \leq \frac{1}{p^{\mu-2}\Gamma(\mu-1)} r_1 \|u_n\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} r_2 \|u_n\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} \\ & + \frac{\Theta(n)}{p^\sigma \Gamma(\sigma)} (\kappa(\mathcal{L}) - \kappa(0))^{\sigma-1} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)}, \end{aligned}$$

which with (38) and (39) leads to

$$\begin{aligned} \|v_n\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} & \leq \frac{\|u_0\|}{p^{\mu-2}\Gamma(\mu-1)} + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} \|u_1\| + r_1 \|u_n\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} \\ & + \frac{1}{p^{\mu-2}\Gamma(\mu-1)} r_1 \|u_n\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} \\ & + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} r_2 \|u_n\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} \\ & + \frac{\Theta(n)}{p^\sigma \Gamma(\sigma)} (\kappa(\mathcal{L}) - \kappa(0))^{\sigma-1} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)} \end{aligned}$$

Thus,

$$\begin{aligned} n & < \frac{\|u_0\|}{p^{\mu-2}\Gamma(\mu-1)} \|u_0\| + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} \|u_1\| + nr_1 + \frac{nr_1}{p^{\mu-2}\Gamma(\mu-1)} \\ & + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} nr_2 + \frac{\Theta(n)}{p^\sigma \Gamma(\sigma)} (\kappa(\mathcal{L}) - \kappa(0))^{\sigma-1} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)}. \end{aligned}$$

Dividing this inequality by n and taking the limit as $n \rightarrow \infty$, it yields

$$1 < r_1 + \frac{r_1}{p^{\mu-2}\Gamma(\mu-1)} + \frac{\kappa(\mathcal{L}) - \kappa(0)}{p^{\mu-1}\Gamma(\mu)} r_2 + \frac{\delta(\kappa(\mathcal{L}) - \kappa(0))^{\sigma-1} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)}}{p^\sigma \Gamma(\sigma)}.$$

But this relation counters with the relationship (33), and our goal in this step is fulfilled.

Step 2. The graph of Y is closed on D_{n_0} .

Let $(v_n)_{n \geq 1}, (u_n)_{n \geq 1}$ be two sequences in D_{n_0} with $v_n \rightarrow v, u_n \rightarrow u$ in D_{n_0} and $v_n \in Y(u_n); n \geq 1$. Then, there is $\zeta_n \in S_{F(\cdot, u_n)}^q$ such that (36) is verified. From (22), we get $\|\zeta_n(\lambda)\| \leq \zeta(\lambda)\Theta(n_0)$, and consequently, $\{\zeta_n\}_{n \geq 1}$ is weakly compact in $L_\omega^{q, \kappa}(\hbar, \mathcal{U})$. Making use of Mazur's lemma, there is a subsequence $\{\zeta_{n_k}\}_{k \geq 1}$ that converges strongly to a function $\zeta \in L_\omega^{1, \kappa}(\hbar, \mathcal{U})$. By Lebesgue convergence theorem, we get $\lim_{n \rightarrow \infty} I_{s_0, \lambda}^{\sigma, p, \kappa, \omega} \zeta_n(\lambda) = I_{b_0, \lambda}^{\sigma, p, \kappa, \omega} \zeta(\lambda), \forall \lambda \in \hbar$. Moreover, assumptions (H_1) and (H_2) imply $\lim_{n \rightarrow \infty} \pi_i(\lambda, z_n(l_i^-)) = \pi_i(\lambda, z(\lambda))$ and $\lim_{n \rightarrow \infty} \pi_i^*(b_{i, n}(l_i^-)) = \pi_i^*(b_i, z(\lambda)), \forall \lambda \in \hbar$. Therefore, $\lim_{n \rightarrow \infty} v_n(\lambda) = v(\lambda)$, where

$$v(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta_n(\lambda), \lambda \in (b_0, l_1], \\ \pi_i(\lambda, u(I_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} \zeta(\lambda), \lambda \in \bar{h}_i, i \in A_1, \end{cases}$$

and since assumption (HF_4) implies to $\varsigma(\lambda) \in F(\lambda, u(\lambda))$, a.e., then $v \in Y(u)$.

Step 3. Let $\Delta = Y(D_{n_0})$; for any $k \in A_0$ and $i \in A_1$, we set

$$\Delta_{|\bar{h}_k} = \{v^* : v^* : \bar{h}_k \rightarrow \mathcal{U}, v^*(\lambda) = (\kappa(\lambda) - \kappa(b_k))^{2-\mu} \omega(\lambda) v(\lambda), \lambda \in \bar{h}_k, \\ v^*(b_k) = \lim_{\lambda \rightarrow b_k^+} v^*(\lambda), v \in \Delta\},$$

and

$$\Delta_{|\bar{B}_i} = \{p^* \in C(\bar{B}_i, \mathcal{U}) : p^*(t) = v(t), t \in (l_i, b_i], \\ p^*(l_i) = p(l_i^+), v \in \Delta\}.$$

Here, we show the equicontinuity of $\{\Delta_{|\bar{h}_k}\}; k \in L_0$ and $\Delta_{|\bar{B}_i}; i \in A_1$ in the Banach space $PC_{2-\mu,\kappa,\omega}(\bar{h}, \mathcal{U})$.

Case 1. Let $v^* \in \Delta_{|\bar{h}_k}$. Then, there are $u \in D_{n_0}$ and $v \in Y(u)$ such that, for $\lambda \in (0, l_1]$,

$$\begin{aligned} v(\lambda) &= (\kappa(\lambda) - \kappa(0))^{2-\mu} \omega(\lambda) [\omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda) - \kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ &+ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda) - \kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta_n(\lambda)] \\ &= \frac{u_0 e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}}{p^{\mu-2}\Gamma(\mu-1)} + \frac{(\kappa(\lambda) - \kappa(0)) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ &+ \frac{(\kappa(\lambda) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \varsigma(b) \omega(b) db, \end{aligned}$$

where $\varsigma \in b_{F(.,u)}^q$. Let $\delta > 0$ with $\lambda + \delta \in (0, l_1]$. We have

$$\begin{aligned} &\|v(\lambda + \delta) - v(\lambda)\| \\ &\leq \frac{u_0}{p^{\mu-2}\Gamma(\mu-1)} |e^{\frac{p-1}{p}(\kappa(\lambda+\delta)-\kappa(0))} - e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}| \\ &+ \frac{u_1}{p^{\mu-1}\Gamma(\mu)} |\kappa(\lambda + \delta) e^{\frac{p-1}{p}(\kappa(\lambda+\delta)-\kappa(0))} - \kappa(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}| \\ &+ \left\| \frac{(\kappa(\lambda + \delta) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^{\lambda+\delta} e^{\frac{p-1}{p}(\kappa(\lambda+\delta)-\kappa(0))} (\kappa(\lambda + \delta) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \varsigma(b) db \right. \\ &\left. - \frac{(\kappa(\lambda) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \varsigma(b) db \right\|. \end{aligned} \quad (40)$$

Obviously,

$$\lim_{\delta \rightarrow 0} |e^{\frac{p-1}{p}(\kappa(\lambda+\delta)-\kappa(0))} - e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}| = 0, \quad (41)$$

and

$$\lim_{\delta \rightarrow 0} |\kappa(\lambda + \delta) e^{\frac{p-1}{p}(\kappa(\lambda + \delta) - \kappa(0))} - \kappa(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))}| = 0. \quad (42)$$

Furthermore,

$$\begin{aligned} & \left| \frac{(\kappa(\lambda + \delta) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^{\lambda+\delta} e^{\frac{p-1}{p}(\kappa(\lambda + \delta) - \kappa(0))} (\kappa(\lambda + \delta) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right. \\ & \quad \left. - \frac{(\kappa(\lambda) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right| \\ & \leq \left| \frac{(\kappa(\lambda + \delta) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_\lambda^{\lambda+\delta} e^{\frac{p-1}{p}(\kappa(\lambda + \delta) - \kappa(0))} (\kappa(\lambda + \delta) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right| \\ & \quad + \left| \frac{(\kappa(\lambda + \delta) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda + \delta) - \kappa(0))} (\kappa(\lambda + \delta) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right. \\ & \quad \left. - \frac{(\kappa(\lambda + \delta) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right| \\ & \quad + \left| \frac{(\kappa(\lambda + \delta) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right. \\ & \quad \left. - \frac{(\kappa(\lambda) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right| \\ & = I_1 + I_2 + I_3. \end{aligned} \quad (43)$$

Now since $\sigma > 1, \mu < 2$, $|\zeta(b)| \leq \xi(b)\Theta(n_0)$, a.e. and κ are increasing and continuous, it results in

$$\begin{aligned} \lim_{\delta \rightarrow 0} I_1 & \leq \frac{\kappa(\mathcal{L})^{2-\mu}\Theta(n_0)}{p^\sigma \Gamma(\sigma)} \\ & \times \lim_{\delta \rightarrow 0} \left| \int_\lambda^{\lambda+\delta} e^{\frac{p-1}{p}(\kappa(\lambda + \delta) - \kappa(0))} (\kappa(\lambda + \delta) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \right| \\ & = 0, \end{aligned} \quad (44)$$

$$\begin{aligned} \lim_{\delta \rightarrow 0} I_2 & \leq \frac{\kappa(\mathcal{L})^{2-\mu}\Theta(n_0)}{p^\sigma \Gamma(\sigma)} \\ & \times \lim_{\delta \rightarrow 0} \left[\int_0^\lambda \left| e^{\frac{p-1}{p}(\kappa(\lambda + \delta) - \kappa(0))} (\kappa(\lambda + \delta) - \kappa(b))^{\sigma-1} - e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} \right| \right. \\ & \quad \left. \times (\kappa(\lambda + \delta) - \kappa(b))^{\sigma-1} |\kappa'(b) \omega(b) \zeta(b) db| \right] \\ & = 0, \end{aligned} \quad (45)$$

and

$$\begin{aligned} \lim_{\delta \rightarrow 0} I_3 & \leq \lim_{\delta \rightarrow 0} |(\kappa(\lambda + \delta) - \kappa(0))^{2-\mu} - (\kappa(\lambda) - \kappa(0))^{2-\mu}| \\ & \times \frac{\Theta(n_0)}{p^\sigma \Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \zeta(b) db \\ & = 0. \end{aligned} \quad (46)$$

We conclude from (40)–(46) that $\lim_{\delta \rightarrow 0} \|\nu(t + \delta) - \nu(t)\| = 0$ independently of ν .

Case 2. Let $v^* \in \Delta_{|\overline{B_i}|}; i \in A_1$ and $\lambda + \delta, \lambda \in (l_i, b_i]$. Then, since π_i is uniformly continuous on bounded sets, we have

$$\begin{aligned} \lim_{\delta \rightarrow 0} \|v^*(\lambda + \delta) - v^*(\lambda)\| &= \lim_{\delta \rightarrow 0} \|v(\lambda + \delta) - v(\lambda)\| \\ &= \lim_{\delta \rightarrow 0} \|\pi_i(\lambda + \delta, u(l_i^-)) - \pi_i(\lambda, u(l_i^-))\| = 0, \end{aligned}$$

and

$$\lim_{\delta \rightarrow 0+} \|p^*(l_i + \delta) - p^*(l_i)\| = \lim_{\delta \rightarrow 0+} \lim_{t \rightarrow l_i} \|\pi_i(l_i + \delta) - \pi_i(t)\| = 0.$$

Case 3. Let $v \in \Delta_{|\overline{h_1}|}$. Then, there are $u \in D_{n_0}$ and $v \in Y(u)$ such that

$$\begin{aligned} v(\lambda) &= (\kappa(\lambda) - \kappa(0))^{2-\mu} \omega(\lambda) [\omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(b_1))} \frac{(\kappa(\lambda) - \kappa(b_1))^{\mu-2}}{p^{\mu-2} \Gamma(\mu-1)} \pi_1(b_1, u(b_1^-)) \\ &\quad + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(b_1))} \frac{(\kappa(\lambda) - \kappa(b_1))^{\mu-1}}{p^{\mu-1} \Gamma(\mu)} \pi_1^*(b_1, u(b_1^-)) + I_{b_1, \lambda}^{\sigma, p, \kappa, \omega} \zeta(\lambda)] \\ &= \frac{e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(b_1))}}{p^{\mu-2} \Gamma(\mu-1)} \pi_i(b_1, u(b_1^-)) + e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(b_1))} \frac{\kappa(\lambda) - \kappa(b_1)}{\Gamma(\mu)} \pi_i^*(b_1, u(b_1^-)) \\ &\quad + (\kappa(\lambda) - \kappa(0))^{2-\mu} \omega(\lambda) I_{b_1, \lambda}^{\sigma, p, \kappa, \omega} \zeta(\lambda), t \in (b_1, l_2], \end{aligned}$$

where $\zeta \in S_{F(\cdot, u)}^q$ and $v(b_1) = \lim_{\lambda \rightarrow b_1+} v(\lambda)$. Using similar reasoning to case (1), we can prove that $\lim_{\delta \rightarrow 0} \|v(\lambda + \delta) - v(\lambda)\| = 0$ independently of v . Likewise, we show the equicontinuity of $\{\Delta_{|\overline{h_k}|}\}; k \in A_1$.

Step 4. Set $\mathfrak{R}_0 = D_{n_0}$, $\mathfrak{R}_n = u(\mathfrak{R}_{n-1}); n \geq 1$. In this step, we will show that

$$\lim_{n \rightarrow \infty} \chi_{PC_{2-\mu, \kappa, \omega}(\tilde{h}, \tilde{U})}(\mathfrak{R}_n) = 0, \quad (47)$$

where $\chi_{PC_{2-\mu, \kappa, \omega}(\tilde{h}, \tilde{U})}(\mathfrak{R}_n)$ is the measure of noncompactness on $PC_{2-\mu, \kappa, \omega}(\tilde{h}, \tilde{U})$.

Let $n \in \mathbb{N}$ be fixed and $\varepsilon > 0$. From Lemma 5 in [64], there is a sequence $(v_\alpha)_{\alpha \in \mathbb{N}}$ in $\mathfrak{R}_n = Y(\mathfrak{R}_{n-1})$ such that

$$\begin{aligned} \chi_{PC_{2-\mu, \kappa, \omega}(\tilde{h}, \tilde{U})}(\mathfrak{R}_n) &\leq 2\{\chi_{PC_{2-\mu, \kappa, \omega}(\tilde{h}, \tilde{U})} v_\alpha : \alpha \geq 1\} + \varepsilon \\ &= \max\{\max_{k \in A_0} \chi_{C_\omega(\overline{h_k}, \tilde{U})}(\Lambda_{|\overline{h_k}|}), \max_{i \in A_1} \chi_{C(\overline{B_i}, \tilde{U})}(\Lambda_{|\overline{B_i}|})\} + \varepsilon, \end{aligned} \quad (48)$$

where $\Lambda = \{v_\alpha : \alpha \in \mathbb{N}\} \subseteq \mathfrak{R}_n = Y(\mathfrak{R}_{n-1})$,

$$\begin{aligned} \Lambda_{|\overline{h_k}|} &:= \{v_\alpha^* \in C(\overline{h_k}, \tilde{U}) : v_\alpha^*(\lambda) = (\kappa(\lambda) - \kappa(b_k))^{2-\mu} \omega(\lambda) v_\alpha(\lambda), \lambda \in \tilde{h}_k, \\ v_\alpha^*(b_k) &= \lim_{\lambda \rightarrow b_k^+} v_\alpha^*(\lambda), u \in \Lambda\}, \end{aligned}$$

and

$$\Lambda_{|\overline{B_i}|} := \{z_\alpha^* \in C(\overline{B_i}, \tilde{U}) : z_\alpha^*(\lambda) = v_\alpha^*(\lambda), \lambda \in B_i, z_\alpha^*(l_i) = z_\alpha^*(l_i^+), v_\alpha^* \in \Lambda\}.$$

Now, thanks to Step (3), the sets $\Lambda_{|\overline{h_k}|}, k \in A_0$ and $\Lambda_{|\overline{B_i}|}, i \in A_1$ are equicontinuous, so (48) becomes

$$\chi_{PC_{2-\mu, \kappa, \omega}(\tilde{h}, \tilde{U})}(\mathfrak{R}_n) \leq \max\{\aleph_1, \aleph_2\} + \varepsilon, \quad (49)$$

where

$$\aleph_1 = \max_{k \in L_0} \max_{\lambda \in \tilde{h}_k} \chi\{(\kappa(\lambda) - \kappa(b_k))^{2-\mu} \omega(\lambda) v_\alpha(\lambda) : \alpha \in \mathbb{N}\}, \quad (50)$$

$$\aleph_2 = \max_{i \in L_1} \max_{\lambda \in \overline{B_i}} \chi\{v_\alpha(\lambda) : \alpha \in \mathbb{N}\}. \quad (51)$$

We will estimate $\aleph_1$ and $\aleph_2$. Let $\alpha \in \mathbb{N}$ be fixed. Since v_α in $\aleph_n = Y(\aleph_{n-1})$, there is $u_\alpha \in \aleph_{n-1}$ with $v_\alpha \in Y(u_\alpha)$. It results from the definition of Y ,

$$v_\alpha(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta_\alpha(\lambda), \lambda \in \hbar_0 = (b_0, l_1], \\ \pi_i(\lambda, u_n(l_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u_\alpha(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u_\alpha(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} \zeta_\alpha(\lambda), \lambda \in \hbar_i, i \in A_1, \end{cases}$$

where $\zeta_\alpha \in S_{F(\cdot, u_\alpha)}^q$. From the assumption $\pi_i(\lambda, \cdot)$ and $\pi_i^*(\lambda, \cdot); \lambda \in \hbar$, maps bounded subsets to relatively compact subsets, it yields that $\aleph_2 = 0$. To estimate $\aleph_1$, for any $\lambda \in \hbar_0$, put

$$B_1(\lambda) = \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(s))} (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \chi\{\zeta_\alpha(b) : \alpha \in \mathbb{N}\} db. \quad (52)$$

and

$$B_2(\lambda) = \frac{(\kappa(\lambda) - \kappa(0))^{2-\mu}}{p^\sigma \Gamma(\sigma)} B_1(\lambda). \quad (53)$$

In view of (HF_5) , it holds for $b \in (0, \lambda]$

$$\begin{aligned} \chi\{\zeta_\alpha(b) : \alpha \in \mathbb{N}\} &\leq \chi_E(\cup_{\alpha \in \mathbb{N}} F(b, z_\alpha(b))) \\ &\leq \beta(b) (\kappa(b) - \kappa(0))^{2-\mu} \omega(b) \chi\{u_\alpha(b) : \alpha \in \mathbb{N}\} \\ &\leq \beta(b) \chi\{(\kappa(b) - \kappa(0))^{2-\mu} \omega(b) u_\alpha(b) : \alpha \in \mathbb{N}\} \\ &\leq \beta(b) \chi_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}(\aleph_{n-1}). \end{aligned} \quad (54)$$

Since $\sigma > 1$ and $|e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\theta))}| \leq 1$, it yields from (52) and (54)

$$\begin{aligned} B_1(\lambda) &\leq \kappa(\lambda)^{\sigma-1} \int_0^\lambda \kappa'(b) \omega(b) \chi\{\zeta_\alpha(b) : \alpha \in \mathbb{N}\} db \\ &\leq \kappa(\mathcal{L})^{\sigma-1} \chi_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}(\aleph_{n-1}) \int_0^\lambda \kappa'(b) \beta(b) \omega(b) db \\ &= \kappa(\mathcal{L})^{\sigma-1} \chi_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}(\aleph_{n-1}) \|\beta\|_{L_\omega^{1, \kappa}(\hbar, \mathbb{R}^+)}, \end{aligned}$$

which implies

$$A_2(\lambda) \leq \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \chi_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}(\aleph_{n-1}) \|\beta\|_{L_\omega^{1, \kappa}(\hbar, \mathbb{R}^+)}.$$

Then,

$$\begin{aligned} (\kappa(\lambda) - \kappa(b_0))^{2-\mu} \omega(\lambda) \chi\{I_{b_0,\lambda}^{\sigma,p,\kappa,\omega} \zeta_\alpha(\lambda) : \alpha \in \mathbb{N}\} \\ \leq \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \chi_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}(\aleph_{n-1}) \|\beta\|_{L_\omega^{1, \kappa}(\hbar, \mathbb{R}^+)}. \end{aligned}$$

Next, for $\lambda \in \hbar_i, i \in L_1$,

$$\chi\{\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))}\frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)}\pi_i(b_i, u_\alpha(b_i^-)) : \alpha \geq 1\} = 0,$$

and

$$\chi\{\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))}\frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)}\pi_i^*(b_i, u_\alpha(b_i^-)) : \alpha \geq 1\} = 0.$$

Likewise, for any $\lambda \in \hbar_k, k \in L_1$

$$\begin{aligned} (\kappa(\lambda) - \kappa(b_k))^{2-\mu}\omega(\lambda)\chi\{I_{b_k, \lambda}^{\sigma, \kappa, \omega}\zeta_\alpha(\lambda) : \alpha \in \mathbb{N}\} \\ \leq \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma\Gamma(\sigma)}\chi_{PC_{2-\mu, \kappa, \omega}(\hbar, E)}(\mathfrak{R}_{n-1})\|\zeta\|_{L_\omega^{1, \kappa}(\hbar, \mathbb{R}^+)}. \end{aligned}$$

Then,

$$\aleph_1 \leq \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma\Gamma(\sigma)}\chi_{PC_{2-\mu, \kappa, \omega}(\hbar, E)}(\mathfrak{R}_{n-1})\|\zeta\|_{L_\omega^{1, \kappa}(\hbar, \mathbb{R}^+)}. \quad (55)$$

Since ϵ is arbitrary small, it follows from (48), (49) and (55), one has

$$\begin{aligned} \chi_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})}(\mathfrak{R}_n) \\ \leq \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma\Gamma(\sigma)}\chi_{PC_{2-\mu, \kappa, \omega}(\hbar, E)}(\mathfrak{R}_{n-1})\|\beta\|_{L_\omega^{1, \kappa}(\hbar, \mathbb{R}^+)}, \forall n \in \mathbb{N}. \end{aligned}$$

Then,

$$\begin{aligned} \chi_{PC_{2-\mu, \kappa, \omega}(\hbar, E)}(\mathfrak{R}_n) \\ \leq \left(\frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma\Gamma(\sigma)}\|\beta\|_{L_\omega^{1, \kappa}(\hbar, \mathbb{R}^+)}\right)^{n-1}\chi_{PC(\hbar, E)}(\mathfrak{R}_1), \forall n \geq 1. \end{aligned}$$

This inequality with (34) leads to the validity of (47).

Step 5. By making use of the Cantor intersection property and taking in mind that the sets $\mathfrak{R}_n$ are closed and convex (Step 2), the set $\mathfrak{R} = \bigcap_{n \in \mathbb{N}} \mathfrak{R}_n$ belongs to $P_{ck}(PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U}))$. Making use of Lemma 3, the multi-valued function $Y : \mathfrak{R} \rightarrow \mathfrak{R}$ has a fixed point. Because from step 1, the fixed point set of Y is bounded, we conclude from Lemma 3 that the set of solutions to Problem (1) is a compact subset in $PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$. $\square$

Now, we present one more existence result of solutions to Problem (1).

Theorem 2. Suppose that (HF_1) , (HF_2) , $(HF7)$ and (H_3) are satisfied. Then, Problem (1) has a solution under the condition

$$\frac{\tau}{p^{\mu-2}\Gamma(\mu-1)} + \frac{(\kappa(\mathcal{L})\tau)}{p^{\mu-1}\Gamma(\mu)} + \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma\Gamma(\sigma)}\|\zeta\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)} < 1. \quad (56)$$

Proof. Define Y as in Theorem1. Let $u \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$. In view of (HF_7) , we get for almost $\lambda \in \bigcup_{k \in A_0} (b_i, l_{i+1}]$

$$\begin{aligned} \|F(\lambda, u(\lambda))\| &= h(F(\lambda, u(\lambda)), F(\lambda, 0)) + h(F(\lambda, 0), \{0\}) \\ &\leq \zeta(\lambda)\|u\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} + \zeta(\lambda) = \zeta(\lambda)(1 + \|u\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})}), a.e., \end{aligned}$$

and so (HF_6) holds; by Remark 9, (HF_3) of Theorem1 is satisfied. Since (HF_7) implies (HF_4) , then, by the same arguments as in Step 2 of Theorem 1, one can show that the values of Y are not empty and closed subsets in $PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$. Next, we prove that

Y is a contraction. For this aim, let $u_1, u_2 \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$, $v_1 \in Y(u_1)$. Then, there is $\varsigma_1 \in S_{F(\cdot, u_1)}^q$ with

$$v_1(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta_1(\lambda), \lambda \in (b_0, l_1], \\ \pi_i(\lambda, u_1(I_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u_1(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u_1(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} \zeta_1(\lambda), \lambda \in \hbar_i, i \in A_1. \end{cases} \quad (57)$$

Now, we define a multi-valued function $\Psi : J \rightarrow 2^E$ by:

$$\Psi(\lambda) = \{u \in F(\lambda, u_2(\lambda)) : \|\varsigma_1(\lambda) - u\| = d(\varsigma_1(\lambda), F(\lambda, u_2(\lambda)))\}.$$

Since $F(\lambda, u_2(\lambda)) \in P_{ck}(\mathcal{U})$, the values of Ψ are not empty. By applying Theorem III.41 in [65], there is $\varsigma_2 \in S_{F(\cdot, u_2)}^q$ such that

$$\|\varsigma_1(\lambda) - \varsigma_2(\lambda)\| = d(\varsigma_1(\lambda), F(\lambda, u_2(\lambda))), \forall \lambda \in \hbar.$$

But from $(HF_7)(i)$.

$$\begin{aligned} & d(\varsigma_1(\lambda), F(\lambda, u_2(\lambda))) \\ & \leq h(F(\lambda, u_1(\lambda)), F(\lambda, u_2(\lambda))) \\ & \leq \xi(\lambda) \|u_1(\lambda) - u_2(\lambda)\|, a.e. \end{aligned}$$

Then,

$$\|\varsigma_1(\lambda) - \varsigma_2(\lambda)\| \leq \xi(\lambda) \|u_1(\lambda) - u_2(\lambda)\|, a.e., \forall \lambda \in \cup_{k \in A_0} (b_i, l_{i+1}]. \quad (58)$$

We define $v_2 : \hbar \rightarrow \mathcal{U}$ by:

$$v_2(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta_2(\lambda), \lambda \in (b_0, l_1], \\ \omega^{-1}(\lambda) \pi_i(\lambda, u_2(I_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, u_2(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, u_2(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} \zeta_2(\lambda), \lambda \in \hbar_i, i \in A_1. \end{cases} \quad (59)$$

In view of (57)–(59), we get:

(i) If $\lambda \in (b_0, l_1]$, it yields

$$\begin{aligned}
 & |(\kappa(\lambda) - \kappa(0))^{2-\mu} \omega(\lambda)(v_1(\lambda) - v_2(\lambda))| \\
 & \leq \frac{\kappa(\mathcal{L})^{2-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^\lambda |e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(b))}| (\kappa(\lambda) - \kappa(b))^{\sigma-1} \kappa'(b) \omega(b) \\
 & \times \|\varsigma_1(b) - \varsigma_2(b)\| db \\
 & \leq \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})} \int_0^\lambda \kappa'(b) \omega(b) \xi(b) db \\
 & = \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)} \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}. \quad (60)
 \end{aligned}$$

(ii) If $\lambda \in B_i, i \in A_1$, then, by (H_3) , it results

$$\begin{aligned}
 & \|v_1(\lambda) - v_2(\lambda)\| \\
 & \leq \|\pi_i(\lambda, u_1(l_i^-)) - \pi_i(\lambda, u_2(l_i^-))\| \\
 & \leq \tau \|u_1(l_i^-) - u_2(l_i^-)\| \\
 & \leq \tau \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}. \quad (61)
 \end{aligned}$$

(iii) If $\lambda \in \hbar_i, i \in A_1$, it follows

$$\begin{aligned}
 & |(\kappa(\lambda) - \kappa(0))^{2-\mu} \omega(\lambda)(v_1(\lambda) - v_2(\lambda))| \\
 & \leq \frac{|e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(b_i))}|}{p^{\mu-2} \Gamma(\mu-1)} \tau \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})} \\
 & + |e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(b_i))}| \frac{(\kappa(\lambda) - \kappa(b_i))}{p^{\mu-1} \Gamma(\mu)} \tau \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})} \\
 & + \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})} \int_0^\lambda \kappa'(b) \omega(b) \xi(b) db \\
 & \leq \left[\frac{\tau}{p^{\mu-2} \Gamma(\mu-1)} + \frac{(\kappa(\mathcal{L}) \tau}{p^{\mu-1} \Gamma(\mu)} + \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)} \right] \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})}. \quad (62)
 \end{aligned}$$

By (60)–(62), it results

$$\begin{aligned}
 & \|v_1 - v_2\| \\
 & < \left[\frac{\tau}{p^{\mu-2} \Gamma(\mu-1)} + \frac{(\kappa(\mathcal{L}) \tau}{p^{\mu-1} \Gamma(\mu)} + \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)} \right] \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathfrak{U})},
 \end{aligned}$$

Interchanging the rules of v_1 and v_2 , we deduce that

$$\begin{aligned}
 & h(Y(u_1), Y(u_2)) \\
 & < \left[\frac{\tau}{p^{\mu-2} \Gamma(\mu-1)} + \frac{(\kappa(\mathcal{L}) \tau}{p^{\mu-1} \Gamma(\mu)} + \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^\sigma \Gamma(\sigma)} \|\xi\|_{L_\omega^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)} \right] \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, E)}, \quad (63)
 \end{aligned}$$

which, with (56), implies to Y is contraction. Then, Y has a fixed point [66], which will be a solution to Problem (1). $\square$

4. (p, κ, ω) -Generalized Ulam-Hyers Stability of Problem (1)

In this part, we provide with an additional outcome for the existence of solutions and demonstrate the (p, κ, ω) -generalized Ulam-Hyers stability of (1). We need the following information about multi-valued weakly Picard operators. Let Y be a Banach, $\Phi : Y \rightarrow P_{cc}(Y)$, $Fix(\Phi) = \{z \in Y : z \in \Phi(z)\}$ and $graph \Phi = \{(u, y) \in Y \times Y : y \in \Phi(u)\}$.

Definition 9 ([67]). Φ is said to be a weakly multi-valued Picard operator, if for any $(u, v) \in \text{graph } \Phi$, there is a sequence (u_n) with $u_1 = u, u_2 = v, u_{n+1} \in \Phi(u_n); n \geq 2$ and $\lim_{n \rightarrow \infty} u_n = z$, where $z \in \text{Fix}(\Phi)$. The sequence (u_n) is called a sequence of consecutive approximations Φ begin from (u, v) that converges to z .

Definition 10 ([67]). Let $\eta : [0, \infty) \rightarrow [0, \infty)$ be continuous at 0, increasing and $\eta(0) = 0$. The weakly multi-valued Picard operator Φ is called η -weakly multi-valued Picard operator when there is a function $\Phi^\infty : \text{graph } \Phi \rightarrow \text{Fix}(\Phi)$ with the following properties:

- (i) $\Phi^\infty(u, y) = z$ if and only if there exists a sequence of successive approximations of Φ starting from (u, y) that converges to z .
- (ii) If $(u, y) \in \text{graph } \Phi$, then $d(u, \Phi^\infty(u, y)) \leq \eta(d(u, y))$.

Definition 11 ([68,69]). An increasing function $\eta : [0, \infty) \rightarrow [0, \infty)$ is said to be a comparison if $\lim_{k \rightarrow \infty} \eta^k(b) = 0, \forall b \in [0, \infty)$, where $\eta^k(b) = \eta^{k-1}(\eta(b))$. If additionally, $\sum_{k=1}^{\infty} \eta^k(b) < \infty, \forall b \in (0, \infty)$, then η is called strictly a comparison.

Lemma 9 ([70]). Let $\eta : [0, \infty) \rightarrow [0, \infty)$ be a strictly comparison function and $\Phi : \mathcal{U} \rightarrow P_{cc}(\mathcal{U})$ be a η -contraction multi-valued operator, that is $h(\Phi(u), \Phi(y)) < \eta(d(u, y)), \forall u, y \in \mathcal{U}$. Then

- (i) Φ is a η -weakly multi-valued Picard operator and hence it has a fixed point.
- (ii) If additionally, there exists $c > 1$ such that $\eta(c\lambda) \leq c\eta(\lambda)$, for every $\lambda \in [0, \infty)$ and $\lambda = 0$ is a point of uniform convergence for the series $\sum_{k=1}^{\infty} \eta^k(t)$, then Φ is a q -weakly multi-valued Picard operator, where $q(\lambda) = \lambda + \sum_{k=1}^{\infty} \eta^k(\lambda); \lambda \in [0, \infty)$.

Now, in order to introduce the concept (p, ω, κ) -generalized Ulam-Hyers stability of Problem (1), we set $\mathbb{Y} := \{u \in PC_{2-\mu, \kappa, \omega}(\bar{h}, E) : D_{b_i, \lambda}^{\sigma, \rho, p, \kappa, \omega} u(t) \text{ exists for any } \lambda \in (b_i, l_{i+1}], i \in L_0\}$

Definition 12. Problem (1) is called (p, ω, κ) -generalized Ulam-Hyers stable if there is a continuous function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(0) = 0$ such that for any $\epsilon > 0$ and any solution $y \in \mathbb{Y}$ of the inequality

$$\begin{cases} d(D_{b_i, \lambda}^{\sigma, \rho, p, \kappa, \omega} y(\lambda), \varsigma(\lambda)) \leq \epsilon, \text{ a.e. } \lambda \in (b_i, l_{i+1}], i \in A_0, \\ \|y(l_i^+) - \pi_i(l_i, y(l_i^-))\| \leq \epsilon, i \in L_0, \\ \|y(\lambda) - \pi_i(\lambda, y(l_i^-))\| \leq \epsilon, \lambda \in (l_i, b_i], i \in A_0, \end{cases} \quad (64)$$

where $\varsigma \in S_{F(\cdot, y)}^q$, there is a solution $u \in PC_{2-\mu, \kappa, \omega}(\bar{h}, \mathcal{U})$ for Problem (1) with

$$\sup_{t \in \bar{h}} \|u(t) - y(t)\| \leq \psi(\epsilon). \quad (65)$$

In the next result, we show the (p, ω, κ) -generalized Ulam-Hyers stability of (1).

Theorem 3. Suppose that (HF_1) , (HF_2) and $(HF\pi\pi^*)$ are met. Then, Problem (1) has a solution if the inequality (56) is verified. If, in addition there, is a $d > 1$ such that $\eta(dt) \leq d\eta(t); t \in [0, \infty)$ and $t = 0$ is a point of uniform convergence for $\sum_{k=1}^{\infty} \eta^k(t)$, then Problem (1) is stable in the sense of (p, ω, κ) -generalized Ulam-Hyers.

Proof. Define Y as in Theorem 2. By using the same arguments that led to proof of relationship (63), we can prove that, if $u_1, u_2 \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$, then

$$h(Y(u_1), Y(u_2)) < \left[\frac{\tau}{p^{\mu-2}\Gamma(\mu-1)} + \frac{(\kappa(\mathcal{L})\tau}{p^{\mu-1}\Gamma(\mu)} + \frac{\kappa(\mathcal{L})^{1-\mu+\sigma}}{p^{\sigma}\Gamma(\sigma)} \|\xi\|_{L_{\omega}^{1,\kappa}((0,\mathcal{L}),\mathbb{R}^+)} \right] \eta(\|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})}),$$

which with (56) implies that Y is η -contraction and since η is strictly comparison, then, by Theorem 3.1(i) in [69], Y has a fixed point and consequently Problem (1) has a solution.

Now, we demonstrate that Problem (1) is a (p, κ, ω) -generalized Ulam-Hyers stable. By applying (i) of Lemma 9, Y is the η -multi-valued weakly Picard operator. Because $\eta(ct) \leq c \eta(t); t \in [0, \infty), c > 1$, and $t = 0$ is a point of uniform convergence for the series $\sum_{k=1}^{\infty} \eta^k(t)$, it results from (ii) of Lemma 9 that Y is the ϱ -multi-valued weakly Picard operator, where $\varrho(t) = t + \sum_{k=1}^{\infty} \eta^k(t)$. Consequently, from Definition 9, there is a function $Y^{\infty} : graph(Y) \rightarrow Fix(Y)$ satisfying

$$\|u - Y^{\infty}(u, v)\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} \leq \varrho(\|u - v\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})}), \forall (u, v) \in graph(Y). \quad (66)$$

Next, let $\epsilon > 0$ and $y \in \mathbb{Y}$ be a solution to inequality (64). So, there are two functions $\varsigma, z \in PC_{2-\mu, \kappa, \omega}^{\rho(2-\sigma)}(\hbar, \mathcal{U})$ such that $\varsigma \in S_{F(\cdot, y)}^q, \|z(\lambda)\| \leq \epsilon, \forall \lambda \in \hbar$ and

$$D_{b_i, \lambda}^{\sigma, \rho, p, \kappa, \omega} y(\lambda) = \varsigma(\lambda) + z(\lambda), \text{ a.e. } \lambda \in \cup_{i=0}^{i=n} (b_i, l_{i+1}].$$

Consequently,

$$\begin{cases} D_{b_i^+, \lambda}^{\theta, \rho, \kappa} y(\lambda) = \varsigma(\lambda) + z(\lambda), \text{ a.e. } t \in \cup_{i=0}^{i=n} (b_i, l_{i+1}], \\ y(l_i^+) = \pi_i(l_i, y(l_i^-)) + \epsilon, i \in A_0, \\ y(\lambda) = \pi_i(\lambda, y(l_i^-)) + \epsilon, \lambda \in (l_i, b_i], i \in A_0, \end{cases} \quad (67)$$

In view of the second assertion of Lemma 8, the solution to the fractional differential Equation (67) is given by

$$y(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0, \lambda}^{\sigma, p, \kappa, \omega} \varsigma(\lambda) + I_{0, \lambda}^{\sigma, p, \kappa, \omega} z(\lambda), \lambda \in (b_0, l_1], \\ \pi_i(\lambda, y(l_i^-)) + \epsilon, \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, y(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, y(b_i^-)) \\ + I_{b_i, \lambda}^{\sigma, p, \kappa, \omega} \varsigma(\lambda) + I_{b_i, \lambda}^{\sigma, p, \kappa, \omega} z(\lambda), \lambda \in (b_i, l_{i+1}], i \in A_1. \end{cases} \quad (68)$$

Let $u : \mathcal{h} \rightarrow \mathcal{U}$ be defined as:

$$u(\lambda) = \begin{cases} \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} u_0 \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} \frac{(\kappa(\lambda)-\kappa(0))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} u_1 \\ + I_{0,\lambda}^{\sigma,p,\kappa,\omega} \zeta(\lambda), \lambda \in (b_0, l_1], \\ \pi_i(\lambda, y(I_i^-)), \lambda \in B_i, i \in A_1, \\ \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-2}}{p^{\mu-2}\Gamma(\mu-1)} \pi_i(b_i, y(b_i^-)) \\ + \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(b_i))} \frac{(\kappa(\lambda)-\kappa(b_i))^{\mu-1}}{p^{\mu-1}\Gamma(\mu)} \pi_i^*(b_i, y(b_i^-)) \\ + I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} \zeta(\lambda), \lambda \in (b_i, l_{i+1}], i \in A_1. \end{cases} \quad (69)$$

Obviously, $(y, u) \in \text{graph}(\mathcal{Y})$, and so by (66),

$$\|y - \mathcal{Y}^\infty(y, u)\|_{PC_{2-\mu,\kappa,\omega}(\mathcal{h},\mathcal{U})} \leq \varrho(\|y - u\|_{PC_{2-\mu,\kappa,\omega}(\mathcal{h},\mathcal{U})}). \quad (70)$$

Since $\mathcal{Y}^\infty(y, u) \in \text{Fiu}(\mathcal{Y})$, it is a solution for Problem(1), so Problem(1) will be a (p, κ, ω) -generalized Ulam-Hyers stable if we show that

$$\varrho(\|y - u\|_{PC_{2-\mu,\kappa,\omega}(\mathcal{h},\mathcal{U})}) < \psi(\epsilon), \quad (71)$$

where

$$\psi(\epsilon) = \varrho(\epsilon(\frac{\kappa(\mathcal{L})^{1+\sigma-\mu}}{p^\sigma\Gamma(\sigma)} \int_0^{\mathcal{L}} \kappa'(\vartheta)\omega(\vartheta)d\vartheta + 1)). \quad (72)$$

From (68) and (69), we get:

(i) If $\lambda \in (b_0, l_1]$, then

$$\begin{aligned} & \|\omega(\lambda)(\kappa(\lambda)-\kappa(0))^{2-\mu}(y(\lambda)-u(\lambda))\| \\ &= \omega(\lambda)(\kappa(\lambda)-\kappa(0))^{2-\mu} \|I_{0,\lambda}^{\sigma,p,\kappa,\omega} z(\lambda)\| \\ &\leq \frac{(\kappa(\lambda)-\kappa(0))^{2-\mu}}{p^\sigma\Gamma(\sigma)} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda)-\kappa(\vartheta))^{\sigma-1} \kappa'(\vartheta) \|z(\vartheta)\| \omega(\vartheta) d\vartheta \\ &\leq \frac{\epsilon\kappa(\mathcal{L})^{1+\sigma-\mu}}{p^\sigma\Gamma(\sigma)} \int_0^\lambda \kappa'(\vartheta)\omega(\vartheta)d\vartheta \\ &= \frac{\epsilon\kappa(\mathcal{L})^{1+\sigma-\mu}}{p^\sigma\Gamma(\sigma)} \int_0^{\mathcal{L}} \kappa'(\vartheta)\omega(\vartheta)d\vartheta. \end{aligned} \quad (73)$$

(ii) If $\lambda \in B_i, i \in A_1$, then

$$\|y(\lambda) - u(\lambda)\| \leq \epsilon. \quad (74)$$

(iii) If $\lambda \in (b_i, l_{i+1}], i \in A_1$, then

$$\begin{aligned} & \|\omega(\lambda)(\kappa(\lambda)-\kappa(b_i))^{2-\mu}(y(\lambda)-u(\lambda))\| \\ &= \omega(\lambda)(\kappa(\lambda)-\kappa(b_i))^{2-\mu} \|I_{b_i,\lambda}^{\sigma,p,\kappa,\omega} z(\lambda)\| \\ &\leq \frac{(\kappa(\lambda)-\kappa(b_i))^{2-\mu}}{p^\sigma\Gamma(\sigma)} \int_{b_i}^\lambda e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(\vartheta))} (\kappa(\lambda)-\kappa(\vartheta))^{\sigma-1} \kappa'(\vartheta) \|z(\vartheta)\| \omega(\vartheta) d\vartheta \\ &\leq \frac{\epsilon\kappa(\mathcal{L})^{1+\sigma-\mu}}{p^\sigma\Gamma(\sigma)} \int_0^{\mathcal{L}} \kappa'(\vartheta)\omega(\vartheta)d\vartheta. \end{aligned} \quad (75)$$

As a consequence of (73)–(75), it yields

$$\begin{aligned} & \varrho(\|y - u\|_{PC_{2-\mu, \kappa, \omega}(\hbar, u)}) \\ & \leq \varrho\left(\epsilon \left(\frac{\kappa(\mathcal{L})^{1+\sigma-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^{\mathcal{L}} \kappa'(\vartheta) \omega(\vartheta) d\vartheta + 1\right)\right) \\ & \leq \varrho\left(\epsilon \left(\frac{\kappa(\mathcal{L})^{1+\sigma-\mu}}{p^\sigma \Gamma(\sigma)} \int_0^{\mathcal{L}} \kappa'(\vartheta) \omega(\vartheta) d\vartheta + 1\right)\right) \\ & = \psi(\epsilon). \end{aligned}$$

Then (71) is true, and the proof is completed. $\square$

5. Discussion and Conclusions

The notion of p -proportional ω -weighted κ -Hilfer fractional differential operator with the lower limit at a point a , $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$, is introduced, and some of its properties are deduced. If $p = 1$, then $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$ becomes the ω -weighted κ -Hilfer fractional differential operator with a lower limit at a point a , which was introduced in the very recent paper [29], and if $\omega(t) = t$; $t \in \hbar$, then $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$ converts to the κ -Hilfer proportional fractional operators given by [43]. Next, a differential inclusion involving $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$ of order $\sigma \in (1, 2)$ with the effect of non-instantaneous impulses in Banach spaces whose dimension is infinite is considered. The relationship between any solution to the studied problem and its corresponding fractional integral equation is obtained in detail, and then the solution function representation of the objective problem is defined. Through the use of suitable fixed point theorems for multi-valued functions, two results for the existence of solutions are proven. In the first, we prove the non-emptiness and compactness of the set of solutions. Next, the concept of (p, κ, ω) -generalized Ulam–Hyers stability for solutions of the studied problem is given, and using the properties of the multi-valued weakly Picard operator, the (p, κ, ω) -generalized Ulam–Hyers stability of solutions for the problem under consideration is studied. Since the presented p -proportional ω -weighted κ -Hilfer fractional differential operator includes many kinds of famous fractional differential operators, our results are a generalization of several results recently published in the literature. In addition, there are no past works on this kind of fractional differential inclusion, so this work is original and enjoyable. Two examples to support our findings are presented. As mentioned in Remark 1, this work generalizes Theorem 4 and (i) of Theorem 6, in [40] and the results obtained in [29,47,48]. Moreover, because the differentiation operators called in the literature (Caputo proportional, Hilfer–Hadamard, Hilfer–Katugampola, κ -Riemann Liouville, κ -Caputo, κ -Hilfer, and κ -Riemann Liouville proportional) are particular cases of $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$, our technique allows extending the majority of the works mentioned in Section 1 after replacing the differential operator considered in these works by $D_{a,\lambda}^{\sigma,\rho,p,\kappa,\omega}$ and making sure the dimension of space infinite and the right hand side is a multi-valued function.

We recommend the next future research directions related to our work:

- 1 Using our method and that used in the recently published paper [58] to study the Hyers–Ulam–Rassias of a delay differential inclusion with multiple variable time delays involving proportional weighted κ -Hilfer on bounded and unbounded intervals.
- 2 Employing our technique as well as that used in the recently published paper [54] to investigate the existence and stability of solutions for a coupled system of impulsive fractional differential inclusions with proportional weighted κ -Hilfer fractional derivative.
- 3 Study Problem (1) in the presence of finite delay as well as infinite delay.
- 4 Using the arguments in the recently published papers [24,60] to study the existence of anti-periodic solutions to Problem (1).

6. Examples

Example 1. Suppose that $\hbar = [0, 3]$, $\mathcal{U}$ is a Hilbert space, $\sigma = \frac{4}{3}$, $\rho = \frac{1}{3}$, $p \in [0, 1]$, $\omega : \hbar \rightarrow]0, \infty[$, $\omega^{-1}(\lambda) = \frac{1}{\omega(\lambda)}$ and $\kappa : \hbar = [0, \mathcal{L}] \rightarrow \mathbb{R}$ is a strictly increasing continuously differentiable function with $\kappa'(\vartheta) \neq 0$, for any $\vartheta \in \hbar$, and κ^{-1} is its inverse. Then, $\mu = \sigma + 2\rho - \sigma\rho = \frac{14}{9}$ and $\rho(2 - \sigma) = \frac{2}{9}$. Let $b_0 = 0, l_1 = 1, b_1 = 2, l_2 = 3, n = 1$ and $q = 2$. Let $F : \hbar \times \mathcal{U} \rightarrow P_{ck}(\mathcal{U})$ be defined such that if $u \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$, then

$$F(\lambda, u(\lambda)) = \begin{cases} q\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}(\kappa(\lambda)-\kappa(0))^{2-\mu}\|u(0)\|\Omega, & \lambda \in [0, 1], \\ q\Omega, & \lambda \in (1, 2], \\ q\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(2))}(\kappa(\lambda)-\kappa(2))^{2-\mu}\|u(0)\|\Omega, & \lambda \in (2, 3], \end{cases} \quad (76)$$

where q is a positive real number and $\Omega \in P_{ck}(\mathcal{U})$. Let $z \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$. Obviously, the multi-valued function $\lambda \rightarrow F(\lambda, z(\lambda))$ is measurable, and i.e. (HF_1) is met. Moreover, if $f \in S_{F(\cdot, z)}^q$, then

$$f(\lambda) = \begin{cases} q\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}(\kappa(\lambda)-\kappa(0))^{\frac{4}{9}}\|z(0)\|q_1, & \lambda \in [0, 1], \\ q q_2, & \lambda \in (1, 2], \\ q\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(2))}(\kappa(\lambda)-\kappa(2))^{\frac{4}{9}}\|z(0)\|q_3, & \lambda \in (2, 3], \end{cases}$$

where $q_i \in \Omega; i = 1, 2, 3$. Consider the function

$$f^*(\lambda) = \begin{cases} q \frac{p^{\frac{2}{9}}\Gamma(\frac{13}{9})}{\Gamma(\frac{11}{9})}\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}(\kappa(\lambda)-\kappa(0))^{\frac{2}{9}}\|z(0)\|q_1, & \lambda \in [0, 1], \\ q q_2, & \lambda \in (1, 2], \\ q \frac{p^{\frac{2}{9}}\Gamma(\frac{13}{9})}{\Gamma(\frac{11}{9})}\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(2))}(\kappa(\lambda)-\kappa(2))^{\frac{2}{9}}\|z(0)\|q_3, & \lambda \in (2, 3], \end{cases}$$

From (ii) of Lemma 3, it follows for $(0, 1]$

$$\begin{aligned} & I_{0, \lambda}^{\rho(2-\sigma), p, \kappa, \omega} f^*(\lambda) \\ &= q \frac{p^{\frac{2}{9}}\Gamma(\frac{13}{9})}{\Gamma(\frac{11}{9})}\|z(0)\|q_1 I_{0, \lambda}^{\frac{2}{9}, p, \kappa, \omega} (\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}(\kappa(\lambda)-\kappa(0))^{\frac{2}{9}}) \\ &= q \frac{p^{\frac{2}{9}}\Gamma(\frac{13}{9})}{\Gamma(\frac{11}{9})}\|z(0)\|q_1 \omega^{-1}(\lambda) \frac{p^{-\frac{2}{9}}\Gamma(\frac{11}{9})}{\Gamma(\frac{13}{9})} e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} (\kappa(\lambda)-\kappa(0))^{\frac{4}{9}} \\ &= q \|z(0)\|q_1 \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))} (\kappa(\lambda)-\kappa(0))^{\frac{4}{9}} \\ &= f(\lambda). \end{aligned}$$

Similarly, one can show that $I_{2, \lambda}^{\rho(2-\sigma), p, \kappa, \omega} f^*(\lambda) = f(\lambda); \lambda \in (2, 3]$. Therefore, condition (HF_2) of Theorem 1 is verified. Moreover, for any $\lambda \in (0, 1] \cup (2, 3]$, we get

$$\begin{aligned} \|f(\lambda)\| &\leq \begin{cases} q\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(0))}(\kappa(\lambda)-\kappa(0))^{\frac{1}{3}}r\|z(0)\|, & \lambda \in [0, 1], \\ q\omega^{-1}(\lambda)e^{\frac{p-1}{p}(\kappa(\lambda)-\kappa(2))}(\kappa(\lambda)-\kappa(2))r\|z(0)\|, & \lambda \in (2, 3], \end{cases} \\ &\leq qr\|z\|. \end{aligned}$$

where $r = \sup\{\|u\| : u \in \Omega\}$. So (HF_3) is satisfied, where $\xi(\lambda) = \varrho r; \lambda \in [0, 3]$ and $\Theta(\lambda) = \lambda; \lambda \in [0, \infty)$. Notice that

$$\delta = \limsup_{n \rightarrow \infty} \frac{\Theta(n)}{n} = 1. \quad (77)$$

Next, let $u_n \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$, $u_n \rightarrow u \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$, $\varsigma_n \in S_{F(\cdot, u_n)}^{\rho(2-\sigma), q}$ and $\varsigma_n \rightarrow \varsigma \in L_{\omega}^{q, \kappa}(\hbar, \mathcal{U})$. It follows from (76)

$$\varsigma_n(\lambda) = \begin{cases} \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(0))^{\frac{4}{9}} \|u_n(0)\| b_n, & \lambda \in [0, 1], \\ \varrho c_n, & \lambda \in (1, 2], \\ \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(2))} (\kappa(\lambda) - \kappa(2))^{\frac{4}{9}} \|u_n(0)\| d_n, & \lambda \in (2, 3], \end{cases}$$

where $b_n, c_n, d_n \in \Omega$, for every $n \in \mathbb{N}$. Since Ω is compact and convex, there are subsequences of (b_n) , (c_n) and (d_n) , denoted again by (b_n) , (c_n) and (d_n) , such that $\lim_{n \rightarrow \infty} b_n = b \in \Omega$, $\lim_{n \rightarrow \infty} c_n = c \in \Omega$ and $\lim_{n \rightarrow \infty} d_n = d \in \Omega$. Set

$$k(\lambda) = \begin{cases} \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(0))^{\frac{4}{9}} \|u(0)\| b, & \lambda \in [0, 1], \\ \varrho c, & \lambda \in (1, 2], \\ \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(2))} (\kappa(\lambda) - \kappa(2))^{\frac{4}{9}} \|u(0)\| d, & \lambda \in (2, 3]. \end{cases}$$

then $\varsigma_n \rightarrow k$ in $PC_{2-\mu, \kappa, \omega}(\hbar, E)$, and consequently, $\varsigma = k \in S_{F(\cdot, u)}^{\rho(2-\sigma), q}$. As a result, (HF_4) is verified. Next, let $B \subseteq PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$ be bounded, $u_1, u_2 \in B$ and $\eta_1 \in F(\lambda, u_1(\lambda))$. Then, there is $q_4, q_5, q_6 \in \Omega$ with

$$\eta_1 = \begin{cases} \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(0))^{2-\mu} \|u_1(0)\| q_4, & \text{if } \lambda \in [0, 1], \\ \varrho q_5, & \text{if } \lambda \in (1, 2], \\ \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(2))} (\kappa(\lambda) - \kappa(2))^{2-\mu} \|u_1(0)\| q_6, & \text{if } \lambda \in (2, 3]. \end{cases}$$

Define

$$\eta_2 = \begin{cases} \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(0))^{2-\mu} \|u_2(0)\| q_4, & \text{if } \lambda \in [0, 1], \\ \varrho q_5, & \text{if } \lambda \in (1, 2], \\ \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(2))} (\kappa(\lambda) - \kappa(2))^{2-\mu} \|u_2(0)\| q_6, & \text{if } \lambda \in (2, 3]. \end{cases}$$

Note that $\eta_2 \in F(\lambda, u_2(\lambda))$ and

$$\begin{aligned} \|\eta_1 - \eta_2\|_{\mathcal{U}} &\leq \begin{cases} \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(0))} (\kappa(\lambda) - \kappa(0))^{2-\mu} \|u_2(0)\| r, & \text{if } \lambda \in [0, 1], \\ \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(2))} (\kappa(\lambda) - \kappa(2))^{2-\mu} \|u_2(0)\| r, & \text{if } \lambda \in (2, 3]. \end{cases} \\ &\leq \begin{cases} \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}\kappa(3)} (\kappa(\lambda) - \kappa(0))^{2-\mu} \|u_2(0)\| r, & \text{if } \lambda \in [0, 1], \\ \varrho \omega^{-1}(\lambda) e^{\frac{p-1}{p}\kappa(3)} (\kappa(\lambda) - \kappa(2))^{2-\mu} \|u_2(0)\| r, & \text{if } \lambda \in (2, 3]. \end{cases} \end{aligned}$$

Then,

$$\begin{aligned} h(F(\lambda, u_1(\lambda)), F(\lambda, u_2(\lambda))) &\leq \omega^{-2}(\lambda) \varrho e^{\frac{p-1}{p}\kappa(3)} \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} \\ &\leq \omega^{-2}(\lambda) \varrho \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})}. \end{aligned}$$

and

$$\begin{aligned} \chi_E(F(\lambda, z(\lambda)) : z \in D)) \\ \leq (\kappa(\lambda) - \kappa(b_k))^{2-\mu} \beta(\lambda) \chi_{\mathcal{U}} \{(z(\lambda) : z \in D\}, a.e. \lambda \in \hbar_k, k = 1, 2, \end{aligned}$$

where $\hbar_1 = (0, 1]$, $\hbar_2 = (2, 3]$. Therefore, (HF_5) is satisfied with $\beta(\lambda) = \varrho \omega^{-2}(\lambda) e^{\frac{p-1}{p}\kappa(3)}$.

Next, define $\pi_1 : [1, 2] \times \mathcal{U} \rightarrow \mathcal{U}$, $\pi_1^* : [1, 2] \times \mathcal{U} \rightarrow \mathcal{U}$ as follows:

$$\pi_1(\lambda, u) = \varrho r_1 \text{Proj}_{\Omega}(u), \pi_1^*(\varrho, u) = \varrho r_2 \text{Proj}_{\Omega}(u), \quad (78)$$

where $r_1, r_2 > 0$ and $\text{Proj}_{\Omega}(u)$ is the projection operators on Ω . It is evident that π_1 and π_2 satisfy (H_1) and (H_2) . By applying Theorem 1, the following problem:

$$\begin{cases} D_{b_i, \lambda}^{\frac{4}{3}, \frac{1}{3}, p, \kappa, \omega} u(\lambda) \in F(\lambda, u(\lambda)), \text{ a.e. for } \lambda \in (0, 1] \cup (2, 3], \\ u(1^+) = \text{Proj}_{\Omega} u(1^-), \\ u(\lambda) = \text{Proj}_{\Omega} u(1^-), \lambda \in (1, 2], \\ \lim_{\lambda \rightarrow 0^+} I_{0, \lambda}^{2-\mu, p, \kappa}(u\omega)(\lambda) = u_0 \\ \lim_{\lambda \rightarrow 0^+} (D^{1, p, \kappa}(I_{0, \lambda}^{2-\mu, p, \kappa}(u\omega)(\lambda)) = u_1, \\ \lim_{\lambda \rightarrow 2^+} I_{2, \lambda}^{2-\mu, p, \kappa}(u\omega)(\lambda) = r_1 \text{Proj}_{\Omega} u(2^-), \\ \lim_{\lambda \rightarrow 2^+} (D^{1, p, \kappa}(I_{2, \lambda}^{2-\mu, p, \kappa}(u\omega)(\lambda)) = r_2 \text{Proj}_{\Omega} u(2^-), \end{cases} \quad (79)$$

where F is defined by (76), $\zeta(\lambda) = \varrho r$ and $\beta(\lambda) = \varrho \omega^{-2}(\lambda) e^{\frac{p-1}{p}\kappa(3)}$; $\lambda \in [0, 3]$, has a solution provided that

$$\frac{(\kappa(3) - \kappa(0))^{\frac{7}{9}}}{p^{\sigma} \Gamma(\frac{3}{4})} \|\beta\|_{L_{\omega}^{1, \kappa}(\hbar, \mathbb{R}^+)} < 1, \quad (80)$$

and

$$\varrho r_1 + \frac{\varrho r_1}{p^{\frac{4}{9}} \Gamma(\frac{5}{9})} + \varrho \frac{\kappa(3) - \kappa(0)}{p^{\frac{5}{9}} \Gamma(\frac{14}{9})} r_2 + \frac{(\kappa(3) - \kappa(0))^{\frac{1}{3}} \|\zeta\|_{L_{\omega}^{1, \kappa}}}{p^{\frac{4}{3}} \Gamma(\frac{4}{4})} < 1. \quad (81)$$

By selecting ϱ, r_1 and r_2 small enough, we can reach to (80) and (81).

Example 2. Suppose $\hbar, \mathcal{U}, b_0, l_1, b_1, l_2, n, q, p, \kappa, \pi_1$ and π_1^* are as in Example 1. Let $\sigma = \frac{5}{3}$ and $\rho = \frac{1}{2}$ and $\omega : \hbar \rightarrow]0, \infty[$ be bounded and $\omega^{-1}(\lambda) = \frac{1}{\omega(\lambda)}$. Then, $\mu = \sigma + 2\rho - \sigma\rho = \frac{11}{6}$ and $\rho(2 - \sigma) = \frac{1}{6}$. Let $\Lambda : \hbar \rightarrow P_{ck}(\mathcal{U})$ be a continuous multi-valued function and $F : \hbar \times \mathcal{U} \rightarrow P_{ck}(\mathcal{U})$ be defined such that if $u \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$, then $\varsigma \in F(\lambda, u(\lambda))$ if and only if

$$\varsigma(\lambda) = \begin{cases} r(\kappa(\lambda) - \kappa(0))^{2-\mu} \omega^2(\lambda) \|u(0)\| I_{0, \lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda), \lambda \in [0, 1], \\ r\Omega, \lambda \in (1, 2], \\ r(\kappa(\lambda) - \kappa(0))^{2-\mu} \omega^2(\lambda) \|u(0)\| I_{2, \lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda), \lambda \in [0, 1], \lambda \in (2, 3], \end{cases} \quad (82)$$

where $f \in S_{\Lambda}^q = \{f \in L_{\omega}^{q, \kappa}(\hbar, \mathcal{U}), f(t) \in \Lambda(\lambda), a.e.\}$, $\Omega = \cup\{\Lambda(\lambda) : \lambda \in \hbar\}$ and $r = \max\{\|z\| : z \in \Omega\}$. It is evident that (HF_1) and (HF_2) are satisfied. Now, let $u_1, u_2 \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$ and $\varsigma_1 \in S_{F(\cdot, u_1)}^q$. Then,

$$\varsigma_1(\lambda) = \begin{cases} r(\kappa(\lambda) - \kappa(0))^{2-\mu} \omega^2(\lambda) \|u_1(0)\| I_{0, \lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda), \lambda \in [0, 1], \\ rz_1, \lambda \in (1, 2], \\ r(\kappa(\lambda) - \kappa(2))^{2-\mu} \omega^2(\lambda) \|u_1(0)\| I_{2, \lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda), \lambda \in (2, 3], \end{cases} \quad (83)$$

where $f \in S_{\Lambda}^q$. Define

$$\varsigma_2(\lambda) = \begin{cases} r(\kappa(\lambda) - \kappa(0))^{2-\mu} \omega^2(\lambda) \|u_2(0)\| I_{0,\lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda), & \lambda \in [0, 1], \\ rz_1, & \lambda \in (1, 2], \\ r(\kappa(\lambda) - \kappa(0))^{2-\mu} \omega^2(\lambda) \|u_2(0)\| I_{2,\lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda), & \lambda \in [0, 1], \lambda \in (2, 3]. \end{cases} \quad (84)$$

Obviously, $\varsigma_2 \in S_{F(., u_2)}^q$ and if $\lambda \in [0, 1]$, then

$$\|\varsigma_1(\lambda) - \varsigma_2(\lambda)\| \leq (\kappa(1) - \kappa(0))^{2-\mu} \omega(\lambda) \|u_1(0) - u_2(0)\| \|I_{0,\lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda)\|. \quad (85)$$

But,

$$\begin{aligned} & \|I_{0,\lambda}^{\frac{1}{6}, p, \kappa, \omega} f(\lambda)\| \\ & \leq \frac{\omega^{-1}(\lambda)}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})} \int_0^\lambda e^{\frac{p-1}{p}(\kappa(\lambda) - \kappa(\vartheta))} (\kappa(\lambda) - \kappa(\vartheta))^{-\frac{5}{6}} \kappa'(\vartheta) \|f(\vartheta)\| \omega(\vartheta) d\vartheta \\ & \leq \frac{\omega^{-1}(\lambda) r \tau}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})} \int_0^\lambda (\kappa(\lambda) - \kappa(\vartheta))^{-\frac{5}{6}} \kappa'(\vartheta) d\vartheta \\ & = \frac{6\omega^{-1}(\lambda) r \tau (\kappa(\lambda) - \kappa(0))^{\frac{1}{6}}}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})} \\ & \leq \frac{6\omega^{-1}(\lambda) r \tau (\kappa(3))^{\frac{1}{6}}}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})}, \end{aligned} \quad (86)$$

where $\tau = \sup\{\omega(\lambda) : \lambda \in \hbar\}$. It results from (83)–(86),

$$\begin{aligned} \|\varsigma_1(\lambda) - \varsigma_2(\lambda)\| & \leq \frac{6r\tau(\kappa(3))^{\frac{1}{6}}}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})} (\kappa(1) - \kappa(0))^{2-\mu} \omega(\lambda) \|u_1(0) - u_2(0)\| \\ & \leq \frac{6r\tau(\kappa(3))^{\frac{1}{6}}}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})} \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})} \end{aligned}$$

Likewise, if $\lambda \in (2, 3]$, then

$$\|\varsigma_1(\lambda) - \varsigma_2(\lambda)\| \leq \frac{6r\tau(\kappa(3))^{\frac{1}{6}}}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})} \|u_1 - u_2\|_{PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})}.$$

Then, (30) is verified with $\xi(\lambda) = \frac{12r\tau(\kappa(3))^{\frac{1}{6}}}{p^{\frac{1}{6}} \Gamma(\frac{1}{6})}$ and $\eta(\lambda) = \frac{\lambda}{2}$. Moreover, $F(\lambda, 0) = \{0\}$, and so (31) is verified also.

Next, it yields from (78), for any $u, v \in PC_{2-\mu, \kappa, \omega}(\hbar, \mathcal{U})$ one has

$$\|\pi_1(\lambda, u(t)) - \pi_1(\lambda, v(t))\| \leq 2\tau\eta(\|u(t) - v(t)\|), \forall \lambda, t \in \hbar.$$

and

$$\|\pi_1^*(\lambda, u(t)) - \pi_1^*(\lambda, v(t))\| \leq 2\tau\eta(\|u(t) - v(t)\|), \forall \lambda, t \in \hbar.$$

Thus, all assumptions of Theorem 3 are satisfied, and hence, by application of Theorem 3, Problem (1) has a solution, and it is stable in the sense of (p, κ, ω) -generalized Ulam-Hyeres under the condition that

$$\frac{2\tau}{p^{-\frac{1}{6}} \Gamma(\frac{5}{6})} + \frac{2\kappa(3)\tau}{p^{\frac{5}{6}} \Gamma(\frac{11}{6})} + \frac{\kappa(3)^{1-\mu+\sigma}}{p^{\frac{5}{3}} \Gamma(\frac{5}{3})} \|\xi\|_{L_{\omega}^{1, \kappa}((0, \mathcal{L}), \mathbb{R}^+)} < 1, \quad (87)$$

where $\sigma = \frac{5}{3}, \nu = \frac{1}{2}, F, \pi_1, \pi_1^*$ are given above and $\zeta(\lambda) = \frac{12r\tau(\kappa(3))^{\frac{1}{6}}}{p^{\frac{1}{6}}\Gamma(\frac{1}{6})}; \lambda \in (0, 3]$. By selecting τ and r small enough, we can reach (87).

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