Valuation of Currency Option Based on Uncertain Fractional Differential Equation

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Abstract: Uncertain fractional differential equations (UFDEs) are excellent tools for describing complicated dynamic systems. This study analyzes the valuation problems of currency options based on UFDE under the optimistic value criterion. Firstly, a new uncertain fractional currency model is formulated to describe the dynamics of the foreign exchange rate. Then, the pricing formulae of European, American, and Asian currency options are obtained under the optimistic value criterion. Numerical simulations are performed to discuss the properties of the option prices with respect to some parameters. Finally, a real-world example is provided to show that the uncertain fractional currency model is superior to the classical stochastic model.

Keywords: fractional-order differential equation; uncertainty theory; currency option; optimistic value; uncertain hypothesis test

1. Introduction

With the development of economic globalization, the volume of transactions in the forex market has been increasing dramatically over recent decades. As an efficient tool for firms or individuals to hedge foreign exchange risks, the currency option has been widely accepted and has drawn much attention from researchers. A currency option is a contract that offers its holder the entitlement to purchase a certain amount of foreign currency with a predetermined exchange rate at maturity or within the validity period. That is, investors can buy a call option against the appreciation of the foreign currency and purchase the put option against the depreciation of the foreign currency. As a result, establishing an appropriate pricing model for currency options is extremely important in modern mathematical finance.

Traditional pricing studies on currency options can be divided into two categories overall. The first class of models assumes the interest rates to be constants, while the spot exchange rate follows a stochastic differential equation. For example, Garman and Kohlhagen [1] supposed that the dynamics of exchange rate were governed by a geometric Brownian motion with constant drift and volatility, and proposed a G-K model for the valuation of the European currency option. However, the G-K model may not be entirely satisfactory due to the differences between currency and stock and the fact that geometric Brownian motion cannot precisely capture the dynamics of currency return. To avoid these drawbacks, many methodologies of pricing currency options were introduced by modifying the G-K model, such as Lian and Chen [2], and Figueiredo et al. [3]. The second class of models involves stochastic interest rates. For example, Grabbe [4] assumed that interest rates and exchange rate were stochastic and found that the prices of American currency options were higher than those of European counterparts. After that, numerous extensions of the Grabbe model were introduced, such as Kim et al. [5] and Dammak et al. [6]. Each model has contributed to the refinement of currency option valuation.
As mentioned above, all currency option pricing models are based on probability theory. A precondition of using probabilistic theory is that the distribution function is close enough to the frequency in the future. Still, numerous empirical studies show that the real world is far from frequency stability. This fact means that the distribution function obtained in practice usually deviates from the frequency. Based on this situation, we cannot but rely on the belief degree that domain experts have in each indeterminate event happening. To deal with belief degree reasonably, Liu [7] established the uncertainty theory based on normality, duality, subadditivity, and product axioms. Due to its ability to handle imprecise information such as subjective judgment, uncertainty theory has become an important branch of axiomatic mathematics to simulate human uncertainty. For more information, readers can consult references [8,9].

Owing to some antinomies in random theory [10], uncertain financial theory based on uncertain differential equations (UDEs) was developed as an alternative. Liu et al. [11] first applied an UDE to model the dynamics of exchange rates and put forward an uncertain currency model. Then, many other types of uncertain currency models were proposed one after another, such as the mean-reverting currency model (Shen and Yao [12]), a currency model with uncertain volatility (Li et al. [13]), and an exponential Ornstein–Uhlenbeck currency model (Li and Sheng [14]). In addition to the foreign exchange market, scholars applied UDEs to other financial fields. Sabahat and Farshid [15] inserted an uncertain model into the stock market and proposed a multifactor uncertain volatility model for pricing European options. Cheng et al. [16] presented the concept of semi-variance for uncertain random variables and employed it to solve portfolio selection problems.

Scholars from different countries have conducted many studies on applying UDEs to handle financial problems. However, in realistic financial markets, the future asset price not only relies on the current price but also correlates with the past condition. Obviously, an UDE cannot accurately reflect this attribute. Different from integer-order differential equations, fractional differential equations (FDEs) have typical characteristics of long memory and non-locality, which can better simulate the fluctuations in asset prices. Therefore, FDEs are more in line with actual financial markets and more suitable for dealing with practical problems.

In 2015, Zhu [17] first introduced uncertain factors into FDEs and presented two types of UFDEs. Then, the mathematical properties of UFDEs were further explored by scholars, such as the existence and uniqueness of solutions [18], the reliability [19] and the stability [20]. Lu et al. [21] utilized an UFDE to simulate the dynamics of stock prices and presented an uncertain fractional stock model, which initiated the research on UFDEs in the finance field. Due to the possibility of drastic fluctuations in asset price, investors do not always take the expected value criterion as their sole guideline. Instead, they usually consider the best case at a certain confidence level. Inspired by these ideas above, Lu et al. [22] investigated the prices of Asian options under the expected value criterion and optimistic value criterion, and compared the expected value models with optimistic value models using numerical experiments.

Currency options are becoming increasingly significant in modern financial markets, but only some researchers are employing UFDEs to price currency options. Although Liu et al. [23] gave an uncertain fractional currency model, they just analyzed short-run fluctuations in the exchange rate. In fact, the foreign exchange rate fluctuates around an average level in the long term. Furthermore, they only considered the expected value criterion and ignored the importance of the optimistic value criterion for investors. To address the shortcomings of the current literature, this study presents a new mean-reverting currency model based on UFDE. It investigates the valuation of currency options under the new model using the optimistic criterion. To the best of our knowledge, research has yet to be conducted on this topic.

This paper is organized as follows. Section 2 recalls some useful results on UFDEs and an uncertain fractional currency model. Section 3 presents a new currency model and computes the prices of European, American, and Asian currency options based on
the optimistic value criterion. Section 4 performs numerical experiments to study the properties of the option prices with respect to some parameters. Section 5 provides a real-world example to illustrate that the uncertain fractional currency model is better than the classical stochastic model. Section 6 gives a concise conclusion.

2. Preliminaries

In this part, we recall some important results on UFDEs and an uncertain fractional currency model. For more information about uncertainty theory, readers may refer to Liu [9] for a better understanding. Unless stated otherwise, the fractional order $p$ is always assumed to satisfy $n - 1 < p \leq n$, $n \in \mathbb{N}_+$.

2.1. UFDEs with Caputo Type

Zhu [17] proposed two kinds of UFDEs: Riemann–Liouville and Caputo. As previously discussed [24,25], the Caputo fractional derivative is more appropriate to model uncertain dynamic systems than the Riemann–Liouville derivative. Thus, this study only focuses on the Caputo type of UFDE.

Definition 1 (Zhu [17]). Assume $g_1, g_2$ are two functions defined on $[0, +\infty) \times \mathbb{R}$ and $C_t$ is a Liu process. Then,

$$cD^pY_t = g_1(t, Y_t) + g_2(t, Y_t)\frac{dC_t}{dt}$$

(1)

is called the UFDE of Caputo type.

Definition 2 (Lu et al. [21]). Assume $g_1, g_2$ are two functions defined on $[0, +\infty) \times \mathbb{R}$. Then, the solution of the related equation with initial conditions

$$\begin{align*}
\left\{ & cD^pY_t = g_1(t, Y_t) + g_2(t, Y_t)\frac{dC_t}{dt} \\
& Y^{(l)}(0) = y_l, l = 0, 1, \ldots, n - 1
\end{align*}$$

(2)

is an uncertain process $Y_t$ that satisfies

$$Y_t = \sum_{l=0}^{n-1} \frac{y_l t^l}{\Gamma(l+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-v)^{p-1}g_1(v, Y_v)dv + \frac{1}{\Gamma(p)} \int_0^t (t-v)^{p-1}g_2(v, Y_v)dC_v.$$  

(3)

Definition 3 (Lu and Zhu [26]). An UFDE (2) is considered to have an $\alpha$-path $Y^\alpha_t (0 < \alpha < 1)$ if it is the solution of the associated FDE:

$$\begin{align*}
\left\{ & cD^pY^\alpha_t = g_1(t, Y^\alpha_t) + |g_2(t, Y^\alpha_t)|^{\gamma^{-1}(\alpha)} \\
& Y^{(l)}(0) = y_l, l = 0, 1, \ldots, n - 1
\end{align*}$$

(4)

where $\gamma^{-1}(\alpha) = \frac{\sqrt{\pi} \ln \frac{\alpha}{1-\alpha}}{\pi}.$

Theorem 1 (Lu and Zhu [26]). Assume UFDE (2) has a solution $Y_t$ and an $\alpha$-path $Y^\alpha_t$. Then,

$$M\{Y_t \leq Y^\alpha_t, \forall t\} = \alpha,$$

(5)

$$M\{Y_t > Y^\alpha_t, \forall t\} = 1 - \alpha.$$

(6)

Moreover, $Y_t$ has an inverse uncertainty distribution (IUD)

$$I^{-1}_\alpha(t) = Y^\alpha_t.$$  

(7)
Theorem 2 (Jin et al. [27]). Assume UFDE (2) has a solution $Y_t$ and $\alpha$-path $Y_\alpha^t$. Then, the supremum $\sup_{0 \leq t \leq T} H(Y_t)$ has an IUD
\[
\Psi^{-1}_T (\alpha) = \sup_{0 \leq t \leq T} H(Y_t^\alpha) \left( \overline{\zeta}^{-1}_T (\alpha) = \sup_{0 \leq t \leq T} H(Y_t^{1-\alpha}) \right),
\] (8)
where $H(y)$ is a strictly increasing (decreasing) function.

Theorem 3 (Jin et al. [28]). Assume UFDE (2) has a solution $Y_t$ and $\alpha$-path $Y_\alpha^t$. Then, the integral $\int_0^T H(Y_t) \, dt$ has an IUD
\[
\Psi^{-1}_T (\alpha) = \int_0^T H(Y_t^\alpha) \, dt \left( \overline{\zeta}^{-1}_T (\alpha) = \int_0^T H(Y_t^{1-\alpha}) \, dt \right),
\] (9)
where $H(y)$ is a strictly increasing (decreasing) function.

2.2. Uncertain Fractional Currency Model

In realistic financial markets, the future exchange rate not only relies on the current value, it also correlates with the past state. Unlike integer-order differential equations, UFDEs have typical characteristics of long memory and non-locality to model the dynamics of exchange rates better. Therefore, UFDEs are more suitable for pricing currency options than UDEs. Based on the reasons above, Liu et al. [23] introduced an uncertain fractional currency model. Assuming that $X_t$ and $Y_t$ are the domestic riskless currency and foreign riskless currency, respectively, the exchange rate $Z_t$ obeys an UFDE of Caputo type, and the currency model is defined as follows:

\[
\begin{aligned}
\frac{dX_t}{r_d X_t} &= dt \\
\frac{dY_t}{r_f Y_t} &= dt \\
\frac{cD^\alpha Z_t}{Z_t} &= aZ_t + bZ_t \frac{dC_t}{dt} \\
Z_t^{(l)}|_{t=0} &= z_l, l = 0, 1, \ldots, n - 1
\end{aligned}
\] (10)

where $r_d$ and $r_f$ denote the domestic interest rate and foreign interest rate, respectively, $a$ and $b$ are the drift and diffusion of the foreign exchange rate, respectively, and $C_t$ is a canonical Liu process.

3. Currency Option Pricing

In real financial markets, investors do not always take the expected value criterion as their sole guideline due to the possibility of drastic fluctuations in asset prices. Investors usually consider the maximum benefit at a certain confidence level based on confidence in the changing trend in future asset prices. Lu et al. [22] first introduced the idea of VaR into the research of option pricing and discussed the valuation of Asian options based on the optimistic criterion. Motivated by these ideas, this section studies the prices of currency options under the optimistic criterion.

3.1. Mean-Reverting Uncertain Fractional Currency Model

In model (10), Liu et al. [23] analyzed the short-run fluctuations in the exchange rate. However, the actual exchange rate oscillates around an average level in the long run. Considering this situation, we present a mean-reverting currency model for the long term:
Theorem 4. Suppose a European currency option for model (11) has a striking price \( K \) and maturity date \( T \). Set \( f_1 \) in the domestic currency as the contract price. At initial time 0, the investor pays \( f_1 \) for purchasing this contract and receives the maximum benefit at time \( T \) based on the optimistic criterion \( [(Z_T - K)^+]_{\text{sup}}(\delta) \), where \( \delta \) is the confidence level. Hence, the investor obtains the net return at time 0:

\[-f_1 + \exp(-r_d T) [(Z_T - K)^+]_{\text{sup}}(\delta).\]

On the other side, the seller obtains profit \( f_1 \) for giving up this contract at time 0, and pays \( \left(1 - \frac{K}{Z_T}\right)^+ \) \( (\delta) \) in foreign currency at time \( T \). Hence, the seller can receive the net earning at time 0:

\[f_1 - Z_0 \exp(-r_f T) \left(1 - \frac{K}{Z_T}\right)^+ \sup (\delta).\]

According to the principle of fair pricing, the definition of a European currency option price is given as below.

**Definition 4.** Assume that a European currency option’s striking price is \( K \) and the expiry date is \( T \). Then, the option price under confidence level \( \delta \) based on the optimistic criterion is

\[f_1 = \frac{1}{2} \exp(-r_d T) \left[(Z_T - K)^+\right]_{\text{sup}}(\delta) + \frac{Z_0}{2} \exp(-r_f T) \left[(1 - \frac{K}{Z_T})^+\right]_{\text{sup}}(\delta).\] (12)

**Theorem 4.** Suppose a European currency option for model (11) has a striking price \( K \) and maturity date \( T \). Then, the European currency option price under confidence level \( \delta \) based on the optimistic criterion is

\[f_1 = \frac{1}{2} \exp(-r_d T) \left[\Psi_T^{-1}(1 - \delta) - K\right]^+ + \frac{Z_0}{2} \exp(-r_f T) \left[1 - \frac{K}{\Psi_T^{-1}(1 - \delta)}\right]^+ ,\] (13)

where

\[\Psi_T^{-1}(1 - \delta) = \sum_{l=0}^{n-1} z_l E_p(l+1)(-bTP) + \left(a + \frac{c\sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta}\right) TPE_p(l+1)(-bTP),\]

\[E_{x,y}(q) = \sum_{u=0}^{\infty} q^u (ux + y).\]

**Proof.** Based on Lu and Zhu [26], we obtain the IUD of the foreign exchange rate \( Z_t \)

\[\Psi_t^{-1}(\alpha) = Z_t^e = \sum_{l=0}^{n-1} z_l E_p(l+1)(-bTP) + \left(a + \frac{c\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}\right) TPE_p(l+1)(-bTP).\]
According to the definition of optimistic value for an uncertain variable, we obtain
\[
\left[(Z_T - K)^+ \right]_{\sup} (\delta) = \sup_{\eta \geq 0} \left\{ \eta \left| \mathbb{M} \left\{ (Z_T - K)^+ \geq \eta \right\} \right. \geq \delta \right\} \\
= \sup_{\eta > 0} \left\{ \eta \left| \mathbb{M} \left\{ Z_T - K \geq \eta \right\} \right. \geq \delta \right\} \lor 0 \\
= \sup_{\eta > 0} \left\{ \eta \left| \mathbb{M} \left\{ Z_T < K + \eta \right\} \right. \geq \delta \right\} \lor 0 \\
= \sup_{\eta > 0} \left\{ \eta \left| \Psi_T(K + \eta) \leq \Psi_T^{-1}(1 - \delta) \right\} \lor 0 \\
= \left[ \Psi_T^{-1}(1 - \delta) - K \right]^+.
\]

By using a similar method, we can obtain
\[
\left[ \left(1 - \frac{K}{Z_T} \right)^+ \right]_{\sup} (\delta) = \sup_{\xi \geq 0} \left\{ \xi \left| \mathbb{M} \left\{ \left(1 - \frac{K}{Z_T} \right)^+ \geq \xi \right\} \right. \geq \delta \right\} \\
= \sup_{\xi > 0} \left\{ \xi \left| \mathbb{M} \left\{ 1 - \frac{\xi}{Z_T} \geq \xi \right\} \right. \geq \delta \right\} \lor 0 \\
= \sup_{\xi > 0} \left\{ \xi \left| \mathbb{M} \left\{ (1 - \xi)Z_T \geq K \right\} \right. \geq \delta \right\} \lor 0 \\
= \sup_{0 < \xi < 1} \left\{ \xi \left| \mathbb{M} \left\{ Z_T \geq \frac{K}{1 - \xi} \right\} \right. \geq \delta \right\} \lor 0 \\
= \sup_{0 < \xi < 1} \left\{ \xi \left| \Psi_T \left( \frac{K}{1 - \xi} \right) \right. \leq 1 - \delta \right\} \lor 0 \\
= \sup_{0 < \xi < 1} \left\{ \xi \left| \Psi_T^{-1} \left( \frac{K}{1 - \xi} \right) \right. \right\} \lor 0 \\
= \sup_{0 < \xi < 1} \left\{ \xi \left| \xi \leq 1 - \frac{K}{\Psi_T^{-1}(1 - \delta)} \right\} \lor 0 \\
= \left[ 1 - \frac{K}{\Psi_T^{-1}(1 - \delta)} \right]^+.
\]

It follows from Definition 4 that we finally have
\[
f_1 = \frac{1}{t} \exp(-r_dT) \left[ (Z_T - K)^+ \right]_{\sup} (\delta) + \frac{Z_T}{T} \exp(-r_fT) \left[ \left(1 - \frac{K}{Z_T} \right)^+ \right]_{\sup} (\delta) \\
= \frac{1}{t} \exp(-r_dT) \left[ \Psi_T^{-1}(1 - \delta) - K \right]^+ + \frac{Z_T}{T} \exp(-r_fT) \left[ 1 - \frac{K}{\Psi_T^{-1}(1 - \delta)} \right]^+,
\]

where
\[
\Psi_T^{-1}(1 - \delta) = \sum_{l=0}^{n-1} z_l T^lE_{p,(l+1)}(-bTP) + \left( a + \frac{c \sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta} \right) T^PE_{p,(p+1)}(-bTP).
\]
Hence, the result (13) is proved. □

Based on Theorem 4, we design the numerical algorithm (Algorithm 1) of calculating the European currency option price for model (11).

**Algorithm 1 European currency option price based on optimistic value criteria**

1. Set the values of all parameters \( r_d, r_f, a, b, c, z_I, K, T \), and fix a confidence level \( \delta \) and fractional order \( p \).
2. Calculate the \((1 - \delta)\)-path of the foreign exchange rate \( Z_T \)
   \[
   \Psi_T^{-1}(1 - \delta) = \sum_{i=0}^{n-1} z_i T^i E_{p,i+1}(-bT^p) + \left( a + \frac{\sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta} \right) T^p E_{p,(p+1)}(-bT^p).
   \]
3. Calculate
   \[
   [(Z_T - K)^+]_{\sup}(\delta) = \max \left( 0, \Psi_T^{-1}(1 - \delta) - K \right)
   \]
   and
   \[
   \left(1 - \frac{K}{Z_T}\right)^+ \sup(\delta) = \max \left( 0, 1 - \frac{K}{\Psi_T^{-1}(1 - \delta)} \right).
   \]
4. Calculate the present values of the benefits, and set
   \[ F_d = \exp(-r_d T) [(Z_T - K)^+]_{\sup}(\delta), \]
   and
   \[ Q_f = \exp(-r_f T) \left(1 - \frac{K}{Z_T}\right)^+ \sup(\delta). \]
5. Calculate the European currency option price
   \[ f_1 \leftarrow \frac{1}{2} F_d + \frac{Z_0}{2} Q_f. \]

### 3.3. American Currency Option

Suppose an American currency option’s striking price is \( K \) and that the maturity time is \( T \). Set \( f_2 \) in the domestic currency as the contract price. At time 0, the investor pays \( f_2 \) for purchasing this contract and receives the maximum benefit based on optimistic value criterion
\[
\sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+ \sup(\delta),
\]
where \( \delta \) is the confidence level. Hence, the investor obtains the net return at time 0
\[
-f_2 + \sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+ \sup(\delta).
\]

Meanwhile, the bank obtains profit \( f_2 \) for giving up this contract at time 0 and pays
\[
\sup_{0 \leq t \leq T} \exp(-r_f t) Z_0 \left(1 - \frac{K}{Z_t}\right)^+ \sup(\delta). \]
Hence, the bank can receive the net earning at time 0
\[
f_2 - \sup_{0 \leq t \leq T} \exp(-r_f t) Z_0 \left(1 - \frac{K}{Z_t}\right)^+ \sup(\delta).
\]

According to the principle of fair pricing, the definition of an American currency option price is given as shown below.

**Definition 5.** Suppose that an American currency option’s striking price is \( K \) and its expiry date is \( T \). Then, the option price under confidence level \( \delta \) based on the optimistic criterion is
Theorem 5. Suppose that an American currency option of model (11) has a striking price $K$ and an expiration time $T$. Then, the American currency option price under confidence level $\delta$ based on the optimistic criterion is

$$f_2 = \frac{1}{2} \left[ \sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+ \right] \left( \delta \right) + \frac{Z_0}{2} \left[ \sup_{0 \leq t \leq T} \exp(-r_d t) \left( 1 - \frac{K}{Z_t} \right)^+ \right] \left( \delta \right). \quad (14)$$

Proof. Based on Lu and Zhu [26], we can easily obtain that $Z_t$ has an IUD

$$\Psi_t^{-1}(1 - \delta) = \sum_{l=0}^{n-1} z_l t^l E_{p(l+1)}(-bt^p) + \left( a + \frac{c\sqrt{3}}{\pi} \ln \frac{1-\delta}{1-\alpha} \right) t^p E_{p(p+1)}(-bt^p).$$

It follows from Theorem 2 that the IUD of $\sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+$ is

$$\Phi_t^{-1}(a) = \sup_{0 \leq t \leq T} \exp(-r_d t) \left( \Psi_t^{-1}(a) - K \right)^+,$$

and the IUD of $\sup_{0 \leq t \leq T} \exp(-r_d t) \left( 1 - \frac{K}{Z_t} \right)^+$ is

$$\Omega_t^{-1}(a) = \sup_{0 \leq t \leq T} \exp(-r_d t) \left( 1 - \frac{K}{\Psi_t^{-1}(a)} \right)^+. $$

According to the definition of optimistic value for an uncertain variable, we have

$$\left[ \sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+ \right] \left( \delta \right) = \sup_{\omega \geq 0} \left\{ \omega \left[ \sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+ \geq \omega \right] \geq \delta \right\}$$

$$= \sup_{\omega \geq 0} \left\{ \omega \left[ \sup_{0 \leq t \leq T} \exp(-r_d t)(Z_t - K)^+ < \omega \right] \geq \delta \right\}$$

$$= \sup_{\omega \geq 0} \left\{ \omega \left[ 1 - \Phi_t^{-1}(\omega) \geq \delta \right] \right\}$$

$$= \sup_{\omega \geq 0} \left\{ \omega \left[ \phi_t^{-1}(\Phi_t(\omega)) \leq \phi_t^{-1}(1-\delta) \right] \right\}$$

$$= \sup_{\omega \geq 0} \left\{ \omega \left[ \omega \leq \phi_t^{-1}(1-\delta) \right] \right\}$$

$$= \phi_t^{-1}(1-\delta).$$

By using a similar method, we can obtain
\[
\sup_{0 \leq t \leq T} \exp \left( -r_d t \right) \left( 1 - \frac{K}{Z_t} \right)^+ \sup_{0 \leq t \leq T} \frac{\beta}{1 - M \left( \sup_{0 \leq t \leq T} \left( 1 - \frac{K}{Z_t} \right)^+ \right)} \geq \delta = \sup_{\beta \geq 0} \beta \left( \sup_{0 \leq t \leq T} \left( 1 - \frac{K}{Z_t} \right)^+ \right) \geq \delta
\]

It follows from Definition 5 that we finally have

\[
f_2 = \frac{1}{2} \left( \sup_{0 \leq t \leq T} \exp \left( -r_d t \left( Z_t - K \right)^+ \right) \right) + \frac{z_0}{T} \left( \sup_{0 \leq t \leq T} \exp \left( -r_d t \left( 1 - \frac{K}{Z_t} \right)^+ \right) \right)
\]

\[
= \frac{1}{2} \phi_T^{-1} \left( 1 - \delta \right) + \frac{z_0}{T} \Omega_T^{-1} \left( 1 - \delta \right)
\]

where

\[
\Psi_T^{-1} \left( 1 - \delta \right) = \sum_{i=0}^{n-1} z_i t_i^d E_{p,i+1} \left( -b t^p \right) + \left( a + \frac{c \sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta} \right) t^p E_{p,i+1} \left( -b t^p \right).
\]

Hence, the result (15) is proved. \(\square\)

Based on Theorem 5, we design the numerical algorithm (Algorithm 2) of calculating the American currency option price for model (11).

\begin{algorithm}
\textbf{Algorithm 2} American currency option price based on optimistic value criteria
\begin{enumerate}
\item Step 1: Choose a number \(N\) based on the desired precision degree. Set the values of all parameters \(r_d, r_f, a, b, c, z_t, K, T, \delta, p,\) and \(t_j = \frac{2 \pi j}{N}, j = 0, 1, 2, \ldots, N.\)
\item Step 2: Set \(j = 0.\)
\item Step 3: Set \(j \leftarrow j + 1.\)
\item Step 4: Calculate the \((1 - \delta)\)-path of the foreign exchange rate \(Z_t\)
\[
\Psi_T^{-1} \left( 1 - \delta \right) = \sum_{i=0}^{n-1} z_i t_i^d E_{p,i+1} \left( -b t^p \right) + \left( a + \frac{c \sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta} \right) t^p E_{p,i+1} \left( -b t^p \right).
\]
\item Step 5: Calculate the positive deviations
\[
\left[ \left( \Psi_T^{-1} \left( 1 - \delta \right) - K \right)^+ \right] = \max \left( 0, \Psi_T^{-1} \left( 1 - \delta \right) - K \right)
\]
and
\[
\left[ \left( 1 - \frac{K}{\Psi_T^{-1} \left( 1 - \delta \right)} \right)^+ \right] = \max \left( 0, 1 - \frac{K}{\Psi_T^{-1} \left( 1 - \delta \right)} \right).
\]
\item Step 6: Calculate the present values of benefits, and set
\[
F_{t_i} = \exp \left( -r_d t_i \right) \left[ \left( \Psi_T^{-1} \left( 1 - \delta \right) - K \right)^+ \right],
\]
\end{enumerate}
\end{algorithm}
Algorithm 2 cont.

and

\[ Q_t = \exp(-r_f t) \left[ \left( 1 - \frac{K}{\Psi_t^{-1}(1 - \delta)} \right)^+ \right]. \]

If \( j < N \), return to Step 3.

Step 7: Find and set

\[ F_d = \max_{0 \leq j \leq N} F_{t_j} Q_f = \max_{0 \leq j \leq N} Q_{t_j}. \]

Step 8: Calculate the American currency option price

\[ f_2 \leftarrow \frac{1}{2} F_d + \frac{Z_0}{2} Q_f. \]

3.4. Asian Currency Option

Suppose an Asian currency option’s striking price is \( K \) and that its maturity time is \( T \). Set \( f_3 \) in the domestic currency as the contract price. At initial time 0, the investor pays \( f_3 \) for purchasing this contract and receives the maximum benefit at time \( T \) based on optimistic criterion

\[ \left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]_{\sup} (\delta), \]

where \( \delta \) is the confidence level. Hence, the investor obtains the net return at time 0

\[ -f_3 + \exp(-r_d T) \left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]_{\sup} (\delta). \]

On the other side, the seller obtains profit \( f_3 \) for giving up this contract at time 0 and pays

\[ \left[ \left( 1 - K/ \frac{1}{T} \int_0^T Z_t dt \right)^+ \right]_{\sup} (\delta) \]

in foreign currency at time \( T \). Hence, the seller can receive the net earning at time 0:

\[ f_3 - Z_0 \exp(-r_f T) \left[ \left( 1 - K/ \frac{1}{T} \int_0^T Z_t dt \right)^+ \right]_{\sup} (\delta). \]

It follows from the principle of fair pricing that an Asian currency option price is defined as shown below.

**Definition 6.** Assuming that an Asian currency option’s striking price is \( K \) and its expiry time is \( T \). Then, the option price under confidence level \( \delta \) based on the optimistic criterion is

\[ f_3 = \frac{1}{2} \exp(-r_d T) \left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]_{\sup} (\delta) + \frac{Z_0}{2} \exp(-r_f T) \left[ \left( 1 - K/ \frac{1}{T} \int_0^T Z_t dt \right)^+ \right]_{\sup} (\delta). \]  \( (16) \)

**Theorem 6.** Suppose an Asian currency option for model (11) has a striking price \( K \) and an expiry date \( T \). Then, the Asian currency option price under confidence level \( \delta \) based on the optimistic criterion is

\[ f_3 = \frac{1}{2} \exp(-r_d T) \left[ t^{-1}(1 - \delta) - K \right]^+ + \frac{Z_0}{2} \exp(-r_f T) \left[ 1 - \frac{K}{t^{-1}(1 - \delta)} \right]^+, \]  \( (17) \)
where
\[
I^{-1}_T(1 - \delta) = \sum_{i=0}^{n-1} z_i T_i E_{p_i(i+2)}(-bT^p) + \left( a + \frac{c\sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta} \right) T^p E_{p,i(p+2)}(-bT^p).
\]

**Proof.** Based on Lu and Zhu [26], we can easily obtain that \( Z_t \) has an IUD
\[
\Psi_t^{-1}(\alpha) = Z_t^p = \sum_{i=0}^{n-1} z_i T_i E_{p_i(i+1)}(-bT^p) + \left( a + \frac{c\sqrt{3}}{\pi} \ln \frac{a}{1 - \alpha} \right) T^p E_{p,i(p+1)}(-bT^p).
\]

It follows from Theorem 3 that the IUD of \( \frac{1}{T} \int_0^T Z_t dt \) is
\[
I^{-1}_T(\alpha) = \frac{1}{T} \int_0^T Z_t^p dt = \frac{1}{T} \int_0^T \left[ \sum_{i=0}^{n-1} z_i T_i E_{p_i(i+1)}(-bT^p) + \left( a + \frac{c\sqrt{3}}{\pi} \ln \frac{a}{1 - \alpha} \right) T^p E_{p,i(p+1)}(-bT^p) \right] dt = \sum_{i=0}^{n-1} z_i T_i E_{p_i(i+2)}(-bT^p) + \left( a + \frac{c\sqrt{3}}{\pi} \ln \frac{a}{1 - \alpha} \right) T^p E_{p,i(p+2)}(-bT^p).
\]

According to the definition of optimistic value for an uncertain variable, we have
\[
\left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]^- (\delta) = \sup_{\tau \geq 0} \left\{ \tau \left| \left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right] \geq \tau \right\} \geq \delta \right\} = \sup_{\tau > 0} \left\{ \tau \left| \left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right] \leq \tau \right\} \geq \delta \right\} \cup 0 = \sup_{\tau > 0} \left\{ \tau \left| \left[ \left( \frac{1}{T} \int_0^T Z_t dt < K + \tau \right) \geq \delta \right\} \right\} \cup 0 = \sup_{\tau > 0} \left\{ \left| \left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right) \geq \delta \right\} \right\} \cup 0 = \sup_{\tau > 0} \left\{ \left| \left( I^{-1}_T(K + \tau) \right) \leq I^{-1}_T(1 - \delta) \right\} \right\} \cup 0 = \sup_{\tau > 0} \left\{ \left| \left( K + \tau \leq I^{-1}_T(1 - \delta) \right) \right\} \right\} \cup 0 = \sup_{\tau > 0} \left\{ \left| \left( \left( K + \tau \leq I^{-1}_T(1 - \delta) \right) \right\} \right\} \cup 0 = \left[ I^{-1}_T(1 - \delta) - K \right]^+,
By using a similar method, we have

\[
\left[ \left( 1 - \frac{K}{\int_0^T Z_i dt} \right) \right]^+ = \sup_{\delta \geq 0} \left\{ v \left\| \left( 1 - K/\frac{1}{T} \int_0^T Z_i dt \right) \right\| \geq \delta \right\}
\]

\[
= \sup_{\delta \geq 0} \left\{ v \left\| M \left( 1 - K/\frac{1}{T} \int_0^T Z_i dt \right) \right\| \geq \delta \right\} \vee 0
\]

\[
= \sup_{\delta \geq 0} \left\{ v \left\| M \left( 1 - K/\frac{1}{T} \int_0^T Z_i dt \right) \right\| \geq \delta \right\} \vee 0
\]

\[
= \sup_{\delta \geq 0} \left\{ v \left\| M \left( 1 - K/\frac{1}{T} \int_0^T Z_i dt \right) \right\| \geq \delta \right\} \vee 0
\]

It follows from Definition 6 that we finally have

\[
f_3 = \frac{1}{2} \exp(-r_dT) \left[ \left( \frac{1}{T} \int_0^T Z_i dt - K \right) \right]^+ \left( \delta \right) + \frac{Z_0}{2} \exp\left( -r_fT \right) \left[ \left( 1 - K/\frac{1}{T} \int_0^T Z_i dt \right) \right]^+ \left( \delta \right)
\]

where

\[
I_T^{-1}(1 - \delta) = \sum_{l=0}^{n-1} z_i T_i E_{p_{l+2}}(-bT^p) + \left( a + \frac{c\sqrt{3}}{\pi} \ln \frac{1 - \delta}{\delta} \right) T^p E_{p_{l+2}}(-bT^p).
\]

Thus, the result (17) is proved. □

Based on Theorem 6, we design the numerical algorithm (Algorithm 3) of calculating the Asian currency option price for model (11).
Algorithm 3 Asian currency option price based on optimistic value criteria

Step 1: Set the values of all parameters \( r_d, r_f, a, b, c, z_1, K, T, \) and fix a confidence level \( \delta \) and fractional order \( p. \)

Step 2: Calculate the \((1 - \delta)\)-path of the integral for the foreign exchange rate \( Z_t \)

\[
I_T^{-1}(1 - \delta) = \sum_{l=0}^{n-1} z_l T^l E_{p,(l+2)}(-bTP) + \left( a + \frac{c\sqrt{3}}{\pi} \ln \frac{1}{\delta} \right) \frac{T^p E_{p,(p+2)}}{(p+2)}(-bTP).
\]

Step 3: Calculate

\[
\left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]_{\sup} (\delta) = \max \left( 0, I_T^{-1}(1 - \delta) - K \right)
\]

and

\[
\left[ \left( 1 - \frac{K}{T} \int_0^T Z_t dt \right)^+ \right]_{\sup} (\delta) = \max \left( 0, 1 - \frac{K}{I_T^{-1}(1 - \delta)} \right).
\]

Step 4: Calculate the present values of benefits, and set

\[
F_d = \exp(-r_dT) \left[ \left( \frac{1}{T} \int_0^T Z_t dt - K \right)^+ \right]_{\sup} (\delta),
\]

and

\[
Q_f = \exp(-r_fT) \left[ \left( 1 - \frac{K}{T} \int_0^T Z_t dt \right)^+ \right]_{\sup} (\delta).
\]

Step 5: Calculate the Asian currency option price

\[
f_3 \leftarrow \frac{1}{2} F_d + \frac{z_0}{2} Q_f.
\]

4. Numerical Experiments

This section performs numerical simulations to further study the properties of the option prices with respect to some parameters.

Example 1. Assume \( r_d = 0.05, r_f = 0.04, a = 6, b = 1, c = 0.1, z_0 = 4, z_1 = 0.5, K = 5, \) and \( T = 1. \) Using the pricing Formula (13), we calculate the price of the European currency option with different order \( p(0 < p \leq 2) \) and \( \delta \) under an optimistic criterion, as presented in Table 1.

From Table 1, we first discover that when the confidence level \( \delta \) remains unchanged, the price \( f_1 \) increases monotonically with respect to \( p \) when \( p \in (0,1.2] \), but decreases when \( p \in (1.2,2] \). The price jumps when \( p \) converts 1 into 1.1, and this is because the instantaneous growth rate of exchange rate \( z_1 \) works at this time. Meanwhile, we find the price \( f_1 \) decreases with the increase in confidence level \( \delta \) when \( p \) is fixed, which is in accord with real financial markets: the more confident investors are in future changes in the exchange rate, the less willing they are to spend more to diminish latent risks.

Furthermore, we discuss the properties of the option price \( f_1 \) with regard to some parameters \( K, z_0, c, r_d, r_f \). It is worth noting that when we analyze the sensitivity of one parameter on the price \( f_1 \), other parameters remain unchanged. According to Figure 1, we notice that \( f_1 \) is increasing monotonically regarding \( z_0 \), while it is decreasing with respect to \( K, c, r_d, r_f \), respectively. These results coincide with the realities in the financial market, which proves the accuracy of our results.
Furthermore, we discuss the properties of the option price \( f \) with \( \delta \) as the parameter. The price jumps when \( p \) converts 1 into 1.1, and this is because the instantaneous growth rate of exchange rate \( z_t \) works at this time. Meanwhile, we find the price \( f_2 \) decreases with the increase in confidence level \( \delta \) when \( p \) is fixed, which is in accord with real financial markets.

**Example 2.** Assume \( r_d = 0.05, r_f = 0.04, a = 6, b = 0.8, c = 0.1, z_0 = 5, z_1 = 0.5, K = 6, \) and \( T = 1. \) Using the pricing formula (15), we calculate the price of the American currency option with different orders \( p (0 < p \leq 2) \) and \( \delta \) values under an optimistic criterion, as presented in Table 2.

From Table 2, we first discover that when the confidence level \( \delta \) remains unchanged, the price \( f_2 \) is monotonously increasing by \( p \) when \( p \in (0, 1], \) but it is monotonously decreasing when \( p \in (1, 2). \) The price jumps when \( p \) converts 1 into 1.1, and this is because the instantaneous growth rate of exchange rate \( z_t \) works at this time. Meanwhile, we find the price \( f_2 \) decreases with the increase in confidence level \( \delta \) when \( p \) is fixed, which is in accord with real financial markets.

**Table 1.** Price \( f_1 \) with different confidence levels (\( \delta \)) and orders (\( p \)).

<table>
<thead>
<tr>
<th>Confidence Level ( \delta )</th>
<th>Price of European Currency Option ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta = 0.1 )</td>
<td>( p = 0.5 ) 0.1808 ( p = 0.8 ) 0.2518 ( p = 1.0 ) 0.2847 ( p = 1.1 ) 0.2847 ( p = 1.2 ) 0.2847 ( p = 1.5 ) 0.2847 ( p = 1.8 ) 0.2847 ( p = 2.0 ) 0.2847</td>
</tr>
<tr>
<td>( \delta = 0.2 )</td>
<td>( p = 0.5 ) 0.1595 ( p = 0.8 ) 0.2293 ( p = 1.0 ) 0.2617 ( p = 1.1 ) 0.2617 ( p = 1.2 ) 0.2617 ( p = 1.5 ) 0.2617 ( p = 1.8 ) 0.2617 ( p = 2.0 ) 0.2617</td>
</tr>
<tr>
<td>( \delta = 0.3 )</td>
<td>( p = 0.5 ) 0.1454 ( p = 0.8 ) 0.2143 ( p = 1.0 ) 0.2464 ( p = 1.1 ) 0.2464 ( p = 1.2 ) 0.2464 ( p = 1.5 ) 0.2464 ( p = 1.8 ) 0.2464 ( p = 2.0 ) 0.2464</td>
</tr>
<tr>
<td>( \delta = 0.4 )</td>
<td>( p = 0.5 ) 0.1337 ( p = 0.8 ) 0.2020 ( p = 1.0 ) 0.2337 ( p = 1.1 ) 0.2337 ( p = 1.2 ) 0.2337 ( p = 1.5 ) 0.2337 ( p = 1.8 ) 0.2337 ( p = 2.0 ) 0.2337</td>
</tr>
<tr>
<td>( \delta = 0.5 )</td>
<td>( p = 0.5 ) 0.1230 ( p = 0.8 ) 0.1907 ( p = 1.0 ) 0.2221 ( p = 1.1 ) 0.2221 ( p = 1.2 ) 0.2221 ( p = 1.5 ) 0.2221 ( p = 1.8 ) 0.2221 ( p = 2.0 ) 0.2221</td>
</tr>
<tr>
<td>( \delta = 0.6 )</td>
<td>( p = 0.5 ) 0.1122 ( p = 0.8 ) 0.1793 ( p = 1.0 ) 0.2105 ( p = 1.1 ) 0.2105 ( p = 1.2 ) 0.2105 ( p = 1.5 ) 0.2105 ( p = 1.8 ) 0.2105 ( p = 2.0 ) 0.2105</td>
</tr>
<tr>
<td>( \delta = 0.7 )</td>
<td>( p = 0.5 ) 0.1005 ( p = 0.8 ) 0.1669 ( p = 1.0 ) 0.1978 ( p = 1.1 ) 0.1978 ( p = 1.2 ) 0.1978 ( p = 1.5 ) 0.1978 ( p = 1.8 ) 0.1978 ( p = 2.0 ) 0.1978</td>
</tr>
<tr>
<td>( \delta = 0.8 )</td>
<td>( p = 0.5 ) 0.0862 ( p = 0.8 ) 0.1518 ( p = 1.0 ) 0.1822 ( p = 1.1 ) 0.1822 ( p = 1.2 ) 0.1822 ( p = 1.5 ) 0.1822 ( p = 1.8 ) 0.1822 ( p = 2.0 ) 0.1822</td>
</tr>
<tr>
<td>( \delta = 0.9 )</td>
<td>( p = 0.5 ) 0.0645 ( p = 0.8 ) 0.1289 ( p = 1.0 ) 0.1588 ( p = 1.1 ) 0.1588 ( p = 1.2 ) 0.1588 ( p = 1.5 ) 0.1588 ( p = 1.8 ) 0.1588 ( p = 2.0 ) 0.1588</td>
</tr>
</tbody>
</table>

**Figure 1.** Price \( f_1 \) with different values of \( K \) for various other parameters while \( p = 1.5, \delta = 0.9. \)
the more confident investors are in future changes in the exchange rate, the less willing they are to spend more to diminish latent risks. Furthermore, we discuss the properties of the option price $f_2$ with regard to some parameters $(K, z_0, c, r_d, r_f)$. It is worth noting that when we analyze the sensitivity of one parameter on the price $f_2$, other parameters remain unchanged. According to Figure 2, we notice that $f_2$ is increasing monotonically regarding $z_0$, while it is decreasing with respect to $K$, $c$, $r_d$, $r_f$, respectively. These results coincide with the realities in the financial market, which again proves the accuracy and effectiveness of our results.

**Table 2.** Price $f_2$ with different confidence levels ($\delta$) and orders ($p$).

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>$p = 0.5$</th>
<th>$p = 0.8$</th>
<th>$p = 1.0$</th>
<th>$p = 1.1$</th>
<th>$p = 1.2$</th>
<th>$p = 1.5$</th>
<th>$p = 1.8$</th>
<th>$p = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = 0.1$</td>
<td>0.3027</td>
<td>0.3695</td>
<td>0.3899</td>
<td>0.6724</td>
<td>0.6699</td>
<td>0.6073</td>
<td>0.4744</td>
<td>0.3632</td>
</tr>
<tr>
<td>$\delta = 0.2$</td>
<td>0.2789</td>
<td>0.3445</td>
<td>0.3646</td>
<td>0.6482</td>
<td>0.6458</td>
<td>0.5849</td>
<td>0.4549</td>
<td>0.3460</td>
</tr>
<tr>
<td>$\delta = 0.3$</td>
<td>0.2630</td>
<td>0.3278</td>
<td>0.3477</td>
<td>0.6320</td>
<td>0.6298</td>
<td>0.5699</td>
<td>0.4418</td>
<td>0.3345</td>
</tr>
<tr>
<td>$\delta = 0.4$</td>
<td>0.2500</td>
<td>0.3142</td>
<td>0.3338</td>
<td>0.6188</td>
<td>0.6167</td>
<td>0.5577</td>
<td>0.4311</td>
<td>0.3250</td>
</tr>
<tr>
<td>$\delta = 0.5$</td>
<td>0.2380</td>
<td>0.3016</td>
<td>0.3210</td>
<td>0.6066</td>
<td>0.6046</td>
<td>0.5464</td>
<td>0.4213</td>
<td>0.3164</td>
</tr>
<tr>
<td>$\delta = 0.6$</td>
<td>0.2259</td>
<td>0.2890</td>
<td>0.3083</td>
<td>0.5944</td>
<td>0.5925</td>
<td>0.5351</td>
<td>0.4114</td>
<td>0.3077</td>
</tr>
<tr>
<td>$\delta = 0.7$</td>
<td>0.2128</td>
<td>0.2752</td>
<td>0.2943</td>
<td>0.5810</td>
<td>0.5793</td>
<td>0.5227</td>
<td>0.4007</td>
<td>0.2982</td>
</tr>
<tr>
<td>$\delta = 0.8$</td>
<td>0.1967</td>
<td>0.2584</td>
<td>0.2772</td>
<td>0.5647</td>
<td>0.5631</td>
<td>0.5077</td>
<td>0.3875</td>
<td>0.2866</td>
</tr>
<tr>
<td>$\delta = 0.9$</td>
<td>0.1725</td>
<td>0.2330</td>
<td>0.2514</td>
<td>0.5401</td>
<td>0.5387</td>
<td>0.4849</td>
<td>0.3677</td>
<td>0.2691</td>
</tr>
</tbody>
</table>

**Figure 2.** Price $f_2$ with different $K$ values for various other parameters while $p = 1.5$, $\delta = 0.9$. 
Example 3. Assume $r_d = 0.05, r_f = 0.04, a = 6, b = 0.8, c = 0.2, z_0 = 5, z_1 = 0.5, K = 5$, and $T = 1$. Using the pricing formula (17), we calculate the price of the Asian currency option with different orders $p (0 < p \leq 2)$ and $\delta$ values under an optimistic criterion, as presented in Table 3.

From Table 3, we first discover that when confidence level $\delta$ remains unchanged, the price $f_3$ decreases as the order $p$ increases in $(0, 1]$ and $[1, 2]$, respectively. The price jumps when $p$ converts 1 into 1.1, and this is because the instantaneous growth rate of exchange rate $z_3$ works at this time. Meanwhile, we find the price $f_3$ decreases with the increase in confidence level $\delta$ when $p$ is fixed, which is in accord with real financial markets: the more confident investors are in future changes in the exchange rate, the less willing they are to spend more to diminish latent risks.

Furthermore, we discuss the properties of the option price $f_3$ with regard to some parameters $(K, z_0, c, r_d, r_f)$. It is worth noting that when we analyze the sensitivity of one parameter on the price $f_3$, other parameters remain unchanged. According to Figure 3, we notice that $f_3$ is increasing monotonically regarding $z_0$, while it is decreasing with respect to $K, c, r_d, r_f$, respectively. These results coincide with the realities in the financial market, again proving the accuracy and effectiveness of our results.

![Figure 3. Price $f_3$ with different values of $K$ for various other parameters while $p = 1.5, \delta = 0.9$.](image)

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>Price of Asian Currency Option</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 0.5$</td>
</tr>
<tr>
<td>$\delta = 0.1$</td>
<td>0.9459</td>
</tr>
<tr>
<td>$\delta = 0.2$</td>
<td>0.9111</td>
</tr>
<tr>
<td>$\delta = 0.3$</td>
<td>0.8879</td>
</tr>
</tbody>
</table>
Table 3. Cont.

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>Price of Asian Currency Option</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p = 0.5 )</td>
</tr>
<tr>
<td>( \delta = 0.4 )</td>
<td>0.8688</td>
</tr>
<tr>
<td>( \delta = 0.5 )</td>
<td>0.8512</td>
</tr>
<tr>
<td>( \delta = 0.6 )</td>
<td>0.8336</td>
</tr>
<tr>
<td>( \delta = 0.7 )</td>
<td>0.8143</td>
</tr>
<tr>
<td>( \delta = 0.8 )</td>
<td>0.7907</td>
</tr>
<tr>
<td>( \delta = 0.9 )</td>
<td>0.7550</td>
</tr>
</tbody>
</table>

5. Empirical Study

This section provides a real-world example to illustrate that the uncertain fractional currency model is superior to the classical stochastic model.

Example 4. Consider the US Dollar to Chinese Yuan (USD-CNY) exchange rates (weekly average) from 1 July 2022 to 31 December 2023, which are shown in Table 4 and Figure 4.

Table 4. USD-CNY exchange rates (weekly average) from 1 July 2022 to 31 December 2023.

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7.0466</td>
<td>7.0824</td>
<td>7.1114</td>
<td>7.1411</td>
<td>7.1531</td>
<td>7.2144</td>
<td>7.2065</td>
<td>7.1684</td>
<td>7.1437</td>
</tr>
<tr>
<td></td>
<td>7.1115</td>
<td>7.1102</td>
<td>7.0969</td>
<td>7.0956</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4. USD-CNY exchange rates (weekly average) from 1 July 2022 to 31 December 2023.
Suppose that \( i = 1, 2, \ldots, 76 \) are the weeks from 1 July 2022 to 31 December 2023, and denote the exchange rates by 
\[ z_1, z_2, \ldots, z_{76}. \]

Assume the exchange rate \( Z_t \) follows an UFDE with the initial condition
\[
\begin{align*}
\frac{c}{D} p Z_t &= (a - b Z_t) + c \frac{dC}{dt} \\
Z(0) &= 6.7032, (18)
\end{align*}
\]

where \( p, a, b \) and \( c \) are unknown parameters to be estimated. Based on the method of moments proposed by He et al. [29], we have the following equations
\[
\begin{align*}
\frac{1}{76} \sum_{i=1}^{76} \Gamma(p) \gamma(p, i, z_i) - (a - b z_{i-1}) &= 0, \\
\frac{1}{76} \sum_{i=1}^{76} \left( \frac{\Gamma(p) \gamma(p, i, z_i) - (a - b z_{i-1})}{c} \right)^2 &= 1, \\
\frac{1}{76} \sum_{i=1}^{76} \left( \frac{\Gamma(p) \gamma(p, i, z_i) - (a - b z_{i-1})}{c} \right)^3 &= 0, \\
\frac{1}{76} \sum_{i=1}^{76} \left( \frac{\Gamma(p) \gamma(p, i, z_i) - (a - b z_{i-1})}{c} \right)^4 &= \frac{21}{5}, (19)
\end{align*}
\]

where 
\[
\gamma(p, i, z_i) = \begin{cases} 
  z_1 - z_0, & i = 1, \\
  z_i - z_0 - \sum_{l=1}^{i-1} (i - l + 1)^{p-1} \gamma(p, l, z_l), & i = 2, 3, \ldots, 76.
\end{cases}
\]

By calculating the above Equation (19), we have 
\[
p = 0.4208, \quad a = 1.4847, \quad b = 0.1255, \quad c = 0.2850. (20)
\]

Therefore, we obtain an uncertain fractional currency model
\[
\begin{align*}
\frac{c}{D} p^{0.4208} Z_t &= (1.4847 - 0.1255 Z_t) + 0.2850 \frac{dC}{dt} \\
Z(0) &= 6.7032, (21)
\end{align*}
\]

where \( Z_t \) is the foreign exchange rate. Lastly, let us verify whether the model (21) can fit USD-CNY exchange rates well. In other words, we should verify if the standard normal uncertainty distribution \( N(0, 1) \) fits 76 samples of UFDE (21)
\[
\phi_i(0.4208, 1.4847, 0.1255, 0.2850), \quad i = 1, 2, \ldots, 76.
\]

According to He et al. [29], the 76 samples can be obtained, as shown in Table 5. Based on the uncertain hypothesis test proposed by Ye and Liu [30], we consider the following hypotheses:
\[
H_0 : e = 0 \text{ and } \delta = 1 \text{ versus } H_1 : e \neq 0 \text{ or } \delta \neq 1. (22)
\]

Given a significance level \( \alpha = 0.05 \), then
\[
\Phi^{-1} \left( \frac{\alpha}{2} \right) = -2.0198, \quad \Phi^{-1} \left( 1 - \frac{\alpha}{2} \right) = 2.0198.
\]

It follows from \( \alpha \times 76 = 3.8 \) that the test is
\[
W = \{(\varphi_1, \varphi_2, \ldots, \varphi_{76}) : \text{there are at least 4 of indexes } i \text{'s with } 1 \leq i \leq 76 \text{ such that } \varphi_i < -2.0198 \text{ or } \varphi_i > 2.0198 \}. (23)
\]
Since only $\varphi_1, \varphi_{71} \notin [-2.0198, 2.0198]$, we have $\varphi_1, \varphi_2, \ldots, \varphi_{76} \notin W$, indicating that the 76 samples follow a standard normal uncertainty distribution $N(0, 1)$. Thus, the uncertain fractional model (21) can effectively fit USD-CNY exchange rates.

To further compare the fitting effects of the uncertain fractional model and the classical stochastic model on real exchange rate, we assume that the exchange rate $Z_t$ obeys a stochastic differential equation (SDE)

$$dZ_t = (a - bZ_t)dt + cdW_t, \quad (24)$$

where the three parameters $a$, $b$ and $c$ are to be estimated and $W_t$ is a Wiener process. For any fixed $a$, $b$, $c$ and $i$ ($2 \leq i \leq 76$), we calculate the following updated SDE

$$dZ_t = (a - bZ_t)dt + cdW_t, \quad Z_{i-1} = z_{i-1}$$

and obtain the probability distribution of normal random variable $Z_i$:

$$\varphi_i(z) = \frac{1}{v\sqrt{2\pi}} \int_{-\infty}^{z} \exp \left( -\frac{(y - \mu_i)^2}{2v^2} \right) dy, \quad (25)$$

where $\mu_i$ denotes the expected value, i.e.,

$$\mu_i = \frac{a}{b} + \exp(-b) \left( z_{i-1} - \frac{a}{b} \right),$$

and $v^2$ is the variance, i.e.,

$$v^2 = \frac{c^2}{2b} (1 - \exp(-2b)).$$

Table 5. Samples of UFDE (21).

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\varphi_i$</th>
<th>$i$</th>
<th>$\varphi_i$</th>
<th>$i$</th>
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<td>1.7847</td>
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Since

$$P(\varphi_i(Z_i) \leq z) = P \left( Z_i \leq \varphi_i^{-1}(z) \right) = \varphi_i \left( \varphi_i^{-1}(z) \right) = z, \forall z \in (0, 1)$$
we obtain that \( q_i(Z_i) \) is always a uniform random variable \( U(0, 1) \). Substitute \( Z_i \) with the observed value \( z_i \), and define the \( i \)-th residual of SDE (24):

\[
\epsilon_i(a, b, c) := q_i(z_i).
\]

Then, \( \epsilon_i(a, b, c) \) is always a sample of uniform probability distribution \( U(0, 1) \).

Since the number of unknown parameters in SDE (24) is three and the first three moments of the uniform probability distribution are \( 1/2, 1/3, \) and \( 1/4 \), we have the following equation:

\[
\begin{cases}
\frac{1}{75} \sum_{i=2}^{76} \epsilon_i(a, b, c) = \frac{1}{2} \\
\frac{1}{75} \sum_{i=2}^{76} \epsilon_i^2(a, b, c) = \frac{1}{3} \\
\frac{1}{75} \sum_{i=2}^{76} \epsilon_i^3(a, b, c) = \frac{1}{4}
\end{cases}
\]

(26)

whose root is

\[
a = 5.5104, \ b = 0.7847, \ c = 0.1412.
\]

Hence, we obtain a stochastic currency model

\[
dZ_t = (5.5104 - 0.7847Z_t)dt + 0.1412dW_t,
\]

(27)

where \( Z_t \) is the foreign exchange rate. Lastly, let us verify whether the model (27) can fit the USD-CNY exchange rates well. In other words, we should verify if the uniform probability distribution \( U(0, 1) \) fits 75 residuals of SDE (27):

\[
\epsilon_i(5.5104, 0.7847, 0.1412), \ i = 2, 3, \ldots, 76.
\]

see Figure 5. Based on the “Chi-square goodness-of-fit test” with a significance level of 0.05, we find \( p = 9.624 \times 10^{-6} < 0.05 \) by applying the function “chi2gof” in Matlab (2020b), which indicates that 75 residuals are not from the same population \( U(0,1) \). Therefore, the model (27) cannot fit USD-CNY exchange rates. According to the above analysis, we conclude that the uncertain fractional currency model is indeed better than the stochastic model.

![Figure 5](image-url)
6. Conclusions

Previously established uncertain currency models were mainly based on UDEs. In contrast, in actual financial markets, the future exchange rate relies not only on the current but also on the past status. Differently from integer-order differential equations, UFDEs have typical characteristics of long memory and non-locality to model the dynamics of exchange rates better. Moreover, traditional currency option pricing methodologies were confined to the expected value criterion and ignored the importance of the optimistic value criterion for investors. This study analyzed the valuation issues of currency options based on UFDE under the optimistic value criterion. Assuming that the foreign exchange rate followed an uncertain fractional currency model with a mean-reverting process, the pricing formulae of currency options were given using rigorous derivations. Then, the properties of the option prices with respect to some parameters were discussed. Finally, a real-world example illustrated that the uncertain fractional currency model was superior to the classical stochastic model. This research enriches the existing currency option pricing methodologies and offers investors a theoretical basis and behavioral reference. Future work will consider the prices of other financial derivatives based on UFDEs under the optimistic value criterion.

Author Contributions: Conceptualization, W.W. and D.A.R.; Methodology, W.W.; Software, W.W. and X.X.; Validation, W.W., D.A.R. and X.X.; Formal Analysis, D.A.R.; Investigation, W.W.; Writing—Original Draft Preparation, W.W.; Writing—Review and Editing, W.W. and D.A.R.; Supervision, D.A.R.; Funding Acquisition, W.W. and X.X. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China (Grant No. 12201451) and the Natural Science Research Project Foundation of Higher Education Institutions of Jiangsu Province (Grant No. 24KJB110017).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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