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Numerical Analysis and Computation of the Finite Volume Element Method for the Nonlinear Coupled Time-Fractional Schrödinger Equations

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Abstract: In this article, our aim is to consider an efficient finite volume element method combined with the L2 − 1, σ formula for solving the coupled Schrödinger equations with nonlinear terms and time-fractional derivative terms. We design the fully discrete scheme, where the space direction is approximated using the finite volume element method and the time direction is discretized making use of the L2 − 1, σ formula. We then prove the stability for the fully discrete scheme, and derive the optimal convergence result, from which one can see that our scheme has second-order accuracy in both the temporal and spatial directions. We carry out numerical experiments with different examples to verify the optimal convergence result.

Keywords: coupled time-fractional Schrödinger equations; finite volume element method; L2 − 1, σ formula; optimal error estimates

1. Introduction

As a generalization of the standard Schrödinger equation, the time-fractional Schrödinger equation is often employed to describe quantum systems with non-Markovian processes or memory effects. When dealing with multiple interacting particles or systems, we typically need to employ coupled time-fractional Schrödinger equations to describe their collective behavior [1]. These coupled equations provide a novel theoretical framework for studying quantum systems with memory effects, non-locality, or long-range interactions. They can be used to simulate complex interactions in quantum multi-body systems, such as quantum entanglement and quantum decoherence. In this article, we propose a finite volume element method to solve the following coupled Schrödinger equations with nonlinear and time-fractional derivative terms:

\[
\begin{aligned}
&i \frac{\partial}{\partial t} u + \Delta u + f(|u|^2 + |v|^2) u = g_1, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \\
&i \frac{\partial}{\partial t} v + \Delta v + f(|u|^2 + |v|^2) v = g_2, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \\
u(x, y, t) = 0, \quad v(x, y, t) = 0, \quad (x, y) \in \partial \Omega, \quad 0 < t \leq T, \\
u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y), \quad (x, y) \in \Omega,
\end{aligned}
\]

(1)

where \(i = \sqrt{-1}\) is the imaginary unit, \(\Omega = (a, b) \times (c, d) \subset \mathbb{R}^2\) is a bounded space domain, \(u = u(x, y, t)\) and \(v = v(x, y, t)\) are unknown complex functions, \(f \in C^1(\mathbb{R}^+)\) is a given real function, \(u_0\) and \(v_0\) are the given complex initial functions, and \(\frac{\partial}{\partial t} u^\alpha\) with \(\alpha \in (0, 1)\) represents the Caputo fractional derivative defined by

\[
\frac{\partial}{\partial t} u^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial \omega(x, y, \eta)}{\partial \eta} \frac{1}{(t-\eta)^\alpha} d\eta,
\]

(2)

where \(\Gamma(\gamma) = \int_0^{+\infty} t^{\gamma-1} e^{-t} dt\) denotes the standard Gamma function.
Because Schrödinger equations have great significance in physics, more and more people have developed some numerical methods to solve these models. Gao et al. [2] presented a Padé compact high-order finite volume scheme for carrying out the numerical calculation for the one-dimensional nonlinear Schrödinger equation. Chen et al. proposed a two-grid finite volume element method and constructed a Crank-Nicolson finite volume element scheme for solving the time-dependent Schrödinger equation in [3,4]. With the progress of research, the numerical methods for Schrödinger equations with nonlinear terms and fractional derivative terms have attracted a lot of attention. Zhao et al. provided an optimal error analysis of the Alikhanov formula for a time-fractional Schrödinger equation in [5]. In [6], Wang et al. adopted the \( L^2 - 1_\tau \) method [7] to approximate the Caputo time-fractional derivative and utilized the Galerkin finite element method in space to derive two numerical methods with second-order accuracy in time direction for solving a nonlinear time-fractional Schrödinger equation. Jia et al. derived an \( L^1 \) Legendre-Galerkin spectral method to solve the two-dimensional nonlinear coupled time-fractional Schrödinger equations in [8]. For these numerical methods, the finite volume method and finite volume element method constructed by combining the finite volume method and finite element method are two kinds of efficient numerical algorithms for solving fractional PDEs, which has attracted a lot of attention. In [9], Zhang et al. presented a finite volume method for solving the two-dimensional Bloch-Torrey equation with space-time-fractional derivatives. In [10], Zhang and Guo studied a finite volume method with a finite difference approximation for solving the fractional diffusion equation. Fang et al. used the finite volume element methods on triangular grids and the classical \( L^1 \)-formula to solve time-fractional reaction-diffusion equations with the Caputo fractional derivative in [11,12]. Karaa and Mustapha proposed a finite volume element method with the convergence result \( O(\tau^{1+\alpha} + h^2) \) for two-dimensional fractional subdiffusion problems in [13]. From the literature, it can be seen that there are relevant studies on finite volume element methods based on the \( L^1 \) formula for simple fractional PDE models and other numerical methods for time-fractional Schrödinger-type equations.

However, as far as we know, there are limited analyses on the finite volume element method and the \( L^2 - 1_\tau \) formula for nonlinear coupled time-fractional Schrödinger equations. The finite volume element method has good conservation in physics. In [14], Richard et al. studied finite volume element algorithms for solving nonlocal reactive flows in porous media. Wang et al. considered the finite volume element method for diffusion problems with general simplicial meshes in [15] and constructed the finite volume element schemes with the optimal \( L^2 \) convergence rate in [16]. Other than that, the \( L^2 - 1_\tau \) formula [7] has a higher time convergence order and has been applied to solve some fractional PDE models. In [17], Gao et al. developed temporal second-order difference schemes based on the \( L^2 - 1_\tau \) formula for solving the time-distributed-order and multi-term fractional sub-diffusion equations. In [18], Wang et al. proposed a mixed finite element algorithm with the \( L^2 - 1_\tau \) formula for solving a nonlinear time-fractional wave model. In [19], Qin et al. used Alikhanov linearized Galerkin finite element methods for solving the nonlinear time-fractional Schrödinger equation. In this article, our aim is to present the finite volume element method with the \( L^2 - 1_\tau \) formula to establish a numerical algorithm for solving the nonlinear coupled time-fractional Schrödinger equations in one and two dimensions. We derive the stability of the high-order fully discrete finite volume element scheme, prove an optimal error result, and implement the numerical calculations through several examples. In summary, the main contributions and research contents are provided as follows:

- The finite volume element method combined with the \( L^2 - 1_\tau \) approximation formula is established, which has second-order accuracy in both temporal and spatial directions.
- The stability for the fully discrete finite volume element scheme and the optimal error estimate result are proven based on the discrete fractional Gronwall inequality.
- Numerical experiments by choosing 1D and 2D coupled time-fractional Schrödinger equations with several different nonlinear terms are carried out to illustrate the accuracy and efficiency of our scheme.
The remaining structures of this article are as follows. In Section 2, we establish the numerical scheme for the finite volume element method, and prove the stability for the fully discrete scheme. In Section 3, we provide the error analysis for this scheme. In Section 4, we show some numerical examples to verify the theoretical results. Finally, we present the conclusion in Section 5.

2. Numerical Scheme and Stability

For the positive integers $M$ and $N$, we choose the time step length size $\tau = \frac{1}{M}$ and space mesh size $h_x = \frac{b-a}{M^2}$, $h_y = \frac{d-c}{(M^2)^{1/2}}$. We introduce the $L2 - l_1$ formula given in [7] to approximate the Caputo derivative. Denote $\omega(t_n) (n = 0, 1, \cdots, M)$ as $\omega^n$.

$$D_0^\alpha\omega^{n+\sigma} = \frac{1}{\mu} \left( C_0^{(n)} \omega^{n+1} - \sum_{s=1}^{n} (C_n^{(s)} - C_{n-s+1}^{(s)}) \omega^s - C_n^{(0)} \omega^0 \right) = \frac{1}{\mu} \sum_{s=0}^{n} c_{n-s}^{(n)} (\omega^{s+1} - \omega^s),$$

where $\mu = \tau^\alpha (2 - \alpha)$, $\sigma = 1 - \frac{2}{M^2}$, and

$$C_l^{(n)} = \begin{cases} a_0^\sigma + b_0^\sigma, & \text{if } l = 0, \\ a_l^\sigma + b_{l+1}^\sigma - b_l^\sigma, & \text{if } l = 1, 2, \cdots, n-1, \\ a_n^\sigma + b_{n+1}^\sigma, & \text{if } l = n, \end{cases}$$

with

$$a_0^\sigma = \sigma^{1-a},$$

$$a_l^\sigma = (l + \sigma)^{-1-a} - (l + \sigma - 1)^{-1-a}, \quad l \geq 1,$$

$$b_l^\sigma = \frac{1}{2} (l + \sigma)^{1-a} - (l + \sigma - 1)^{1-a} - \frac{1}{2} ((l + \sigma)^{-1-a} + (l + \sigma - 1)^{-1-a}), \quad l \geq 1.$$

Make a triangulation on the spatial domain $\Omega$, which is denoted as $\Gamma_h$. Let $K$ be one of the triangular cells. Make the dual partition of barycenter $\Gamma_h^*$ on $\Gamma_h$. Let $K^*$ be one of the dual units. Let $\bar{\Omega}_h$ represent the node set of $\Gamma_h$ and $\Omega_h = \bar{\Omega}_h \\partial \Omega$ represent the inner node set of $\Gamma_h$. We take the trial functions $Z_h$ as the piecewise linear finite element space corresponding to $\Gamma_h$, i.e., $Z_h = \{ z_h | z_h \in C(\Omega) \}$, $z_h|_K$ is a primary polynomial, completely determined by the values on the three vertices, $\forall K \in \Gamma_h$, $z_h|_Q = 0 \subset Z = H^0_1(\Omega)$.

The test function $Q_h$ is a piecewise lower order polynomial space. Let $\psi_{P_0}$ be the basis function corresponding to the point $P_0 \in \Omega_h$. It is obvious that $Z_h = \text{span} \{ \phi_{P_0} \}$ and $Q_h = \text{span} \{ \psi_{P_0} \}$. Any $q_h \in Q_h$ can be uniquely tabulated as $q_h = \sum_{P_0 \in \Omega_h} q_{P_0}(P_0) \psi_{P_0}$. For any $q \in Z$, let $I_n^h q$ be the interpolation projection of $q$ to the test function $Q_h$

$$I_n^h q = \sum_{P_0 \in \Omega_h} q(P_0) \psi_{P_0}. \quad (5)$$

From interpolation theory, we obtain: $\| q - I_n^h q \|_0 \leq C h \| q \|_1$.

Using the $L2 - l_1$ formula, (1) is discretized as

$$\begin{pmatrix} iD_0^\alpha U^{n+\sigma} + \Delta U^{n+\sigma} + f(|\nabla^{n+\sigma}|^2 + |V^{n+\sigma}|^2) \nabla^{n+\sigma} = g_1^{n+\sigma}, & (x, y) \in \Omega, \\ iD_0^\alpha V^{n+\sigma} + \Delta V^{n+\sigma} + f(|\nabla^{n+\sigma}|^2 + |V^{n+\sigma}|^2) V^{n+\sigma} = g_2^{n+\sigma}, & (x, y) \in \Omega, \end{pmatrix}$$

where $U^n$ and $V^n$ is the time semi-discrete solution pair of $u^n$ and $v^n$ at time $t_n$, $\omega^{n+\sigma} := (1 - \sigma)\omega^n + \sigma \omega^{n+1}$, and $\omega^{n+\sigma} := (1 + \sigma)\omega^n - \sigma \omega^{n+1}$.

The space is discretized by using the finite volume element method. Multiplying the test functions $q_h$ and $p_h$ on both sides of (6) and integrating over the dyadic cell $K^*$, we have
\[
\sum_{K} \int_{K} \left( iD_{h}^{\alpha} u_{h}^{n+\sigma} q_{h} + \Delta u_{h}^{n+\sigma} q_{h} + f(\|u_{h}^{n+\sigma}\|^{2}) \bar{u}_{h}^{n+\sigma} q_{h} \right)dx dy = \sum_{K} \int_{K} s_{1}^{n+\sigma} q_{h} dx dy,
\]
\[
\sum_{K} \int_{K} \left( iD_{h}^{\alpha} v_{h}^{n+\sigma} p_{h} + \Delta v_{h}^{n+\sigma} p_{h} + f(\|u_{h}^{n+\sigma}\|^{2}) \bar{v}_{h}^{n+\sigma} p_{h} \right)dx dy = \sum_{K} \int_{K} s_{2}^{n+\sigma} p_{h} dx dy.
\]

That is, find \( u_{h}, v_{h} \in Z_{h} \), such that
\[
\begin{align*}
(iD_{h}^{\alpha} u_{h}^{n+\sigma}, q_{h}) &+ (u_{h}^{n+\sigma}, q_{h}) + (f(\|u_{h}^{n+\sigma}\|^{2}) \bar{u}_{h}^{n+\sigma}, q_{h}) = (s_{1}^{n+\sigma}, q_{h}), \\
(iD_{h}^{\alpha} v_{h}^{n+\sigma}, p_{h}) &+ (v_{h}^{n+\sigma}, p_{h}) + (f(\|u_{h}^{n+\sigma}\|^{2}) \bar{v}_{h}^{n+\sigma}, p_{h}) = (s_{2}^{n+\sigma}, p_{h}),
\end{align*}
\]
where
\[
a(\bar{z}_{h}^{n+\sigma}, q_{h}) = \sum_{K} \int_{K} \nabla z_{h}^{n+\sigma} \cdot \nabla q_{h} dx dy - \sum_{K} \int_{\partial K} \nabla z_{h}^{n+\sigma} \cdot n q_{h} ds.
\]

In summary, we obtain the fully discrete finite volume element scheme based on the \( L2 - 1_{\sigma} \) formula
\[
\begin{cases}
(iD_{h}^{\alpha} u_{h}^{n+\sigma}, q_{h}) + (u_{h}^{n+\sigma}, q_{h}) + (f(\|u_{h}^{n+\sigma}\|^{2}) \bar{u}_{h}^{n+\sigma}, q_{h}) = (s_{1}^{n+\sigma}, q_{h}), & (x, y) \in \Omega, \\
(iD_{h}^{\alpha} v_{h}^{n+\sigma}, p_{h}) + (v_{h}^{n+\sigma}, p_{h}) + (f(\|u_{h}^{n+\sigma}\|^{2}) \bar{v}_{h}^{n+\sigma}, p_{h}) = (s_{2}^{n+\sigma}, p_{h}), & (x, y) \in \Omega.
\end{cases}
\]

We use the local extrapolation when dealing with nonlinear terms. Therefore, we need to know the initial values \( u_{0}^{\sigma}, v_{0}^{\sigma}, \) and \( v_{1}^{\sigma} \) to start the iteration. Because \( u_{0}^{\sigma} \) and \( v_{0}^{\sigma} \) are known, we only consider \( u_{1}^{\sigma} \) and \( v_{1}^{\sigma} \). To deal with this issue, we adopt the following special technique like the one in \([20]\)
\[
\begin{align*}
\left( i \frac{\alpha^{1-k}}{\mu} (u_{h}^{1-k} - u_{0}^{1-k}), q_{h} \right) + (a_{h}^{1-k}, q_{h}) + (f(\|u_{h}^{1-k}\|^{2}) \bar{u}_{h}^{1-k}, q_{h}) = (s_{1}^{1}, q_{h}), & \quad k = 1, \ldots, n_{a}, \\
\left( i \frac{\alpha^{1-k}}{\mu} (v_{h}^{1-k} - v_{0}^{1-k}), p_{h} \right) + (v_{h}^{1-k}, p_{h}) + (f(\|u_{h}^{1-k}\|^{2}) \bar{v}_{h}^{1-k}, p_{h}) = (s_{2}^{1}, p_{h}), & \quad k = 1, \ldots, n_{a}, \\
u_{h}^{1} = u_{h}^{1}, v_{h}^{1} = v_{h}^{1}, \quad (x, y) \in \partial \Omega, & \quad k = 1, \ldots, n_{a}, \\
u_{h}^{1,0} = u_{0}^{1}, v_{h}^{1,0} = v_{0}^{1}, \quad (x, y) \in \Omega,
\end{align*}
\]
where
\[
\omega_{h}^{k} = (1 - \sigma) \omega_{0}^{k} + \sigma \omega_{h}^{k}, \quad k = 0, 1, 2, \ldots, n_{a}, \quad \omega_{h}^{k} = \omega_{h}^{1, n_{a}}, \quad n_{a} = \left[ \frac{1}{\alpha} + \frac{1}{2} \right].
\]

Now, we prove the stability of the fully discrete scheme. For this purpose, we introduce some useful lemmas.

**Lemma 1** ([21]). The bilinear form \((\cdot, I_{h}^{\sigma}, \cdot)\) satisfies the following properties
\[
(z_{h}, I_{h}^{\sigma} w_{h}) = (w_{h}, I_{h}^{\sigma} z_{h}), \quad \forall z_{h}, w_{h} \in Z_{h},
\]
and there exist constants \( \mu_{1} > 0 \) and \( \mu_{2} > 0 \) independent of \( h \), such that
\[
\begin{align*}
(z_{h}, I_{h}^{\sigma} z_{h}) &\geq \mu_{1}\|z_{h}\|^{2}, \quad \forall z_{h} \in Z_{h}, \\
(z_{h}, I_{h}^{\sigma} w_{h}) &\leq \mu_{2}\|z_{h}\||w_{h}|, \quad \forall z_{h}, w_{h} \in Z_{h}.
\end{align*}
\]

**Lemma 2** ([6]). For any grid function \( z^{n} \in Z \), we have
\[
\begin{align*}
Im \Delta(z^{n+\sigma}, I_{h}^{\sigma} z^{n+\sigma}) &= 0, \quad n = 0, 1, \ldots, M - 1, \\
Re(D_{h}^{\alpha} z^{n+\sigma}, I_{h}^{\sigma} z^{n+\sigma}) &\geq \frac{1}{2} D_{h}^{\alpha} \|z^{n+\sigma}\|^{2}, \quad n = 0, 1, \ldots, M - 1,
\end{align*}
\]
where Im(\(\mu\)) and Re(\(\mu\)) denote the imaginary part and the real part of \(\mu\), respectively.

**Lemma 3** ([16,20,22]). Suppose that the nonnegative sequences \(\{z^n, \kappa^n\}_{n=0,1,\cdots}\) satisfy
\[
\begin{align*}
D^\alpha_0 z^n &\leq \lambda_1 z^n + \lambda_2 z^0 + \kappa^0, \\
D^\alpha_0 z^{n+\sigma} &\leq \lambda_1 z^{n+1} + \lambda_2 z^n + \lambda_3 z^{n-1} + \kappa^n,
\end{align*}
\]
where \(\lambda_1, \lambda_2, \lambda_3 > 0\) are the given constants independent of \(\tau\). Then, there exists a positive constant \(\tau^*\) such that, when \(0 < \tau \leq \tau^*\), we have
\[
z^n \leq \left(6\tau^0 + \frac{12\tau^\alpha}{\Gamma(1+\alpha)} \max_{0 \leq \xi \leq n} \kappa^{\xi}\right)E_\alpha(2\lambda t^n), \quad n \geq 1,
\]
where \(E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+\alpha)}\) is the Mittag–Leffler function, and \(\lambda\) can be represented by \(\lambda_1, \lambda_2, \lambda_3\).

For the sake of simplicity, we define \((u,v;p,q) := (u,p) + (v,q)\). According to the above definition, (9) can also be written as
\[
(iD^\alpha_0 u^n_0, iD^\alpha_0 v^n_0; I^n_h q_h, I^n_h p_h) + \left(f(|u^n_0|^2 + |v^n_0|^2)\tilde{u}^{n+\sigma}_h, f(|\tilde{u}^{n+\sigma}_h|^2 + |\tilde{v}^{n+\sigma}_h|^2)\tilde{v}^{n+\sigma}_h, I^n_h q_h, I^n_h p_h\right)
\]
\[
+ a(u^n_0, I^n_h q_h) + a(v^n_0, I^n_h p_h) = (g^n_1, g^n_2; I^n_h q_h, I^n_h p_h).
\]

**Theorem 1.** Suppose \(u^n_0\) and \(v^n_0\) are the solution pair of the fully discrete scheme (9), then we have the following unconditionally stable result
\[
\|u^n_0\| + \|v^n_0\| \leq C\left(\|u^n_0\| + \|v^n_0\| + \sup_{t \in [0,T]} \|g_1(t)\| + \sup_{t \in [0,T]} \|g_2(t)\|\right), \quad n = 1,2,\cdots,M,
\]
where \(C\) is a constant independent of \(\tau\) and \(h\).

**Proof.** When \(n = 0\), we approximate \(\tilde{u}^0_h = \bar{u}^0_h\) and \(\tilde{v}^0_h = v^0_h\). Taking \(q = u^n_0\) and \(p = v^n_0\) in (16) and considering only the imaginary part of the result, we have
\[
Re(D^\alpha_0 u^n_0, D^\alpha_0 v^n_0; I^n_h q_h, I^n_h p_h) = Im(g^n_1, g^n_2; I^n_h q_h, I^n_h p_h)
\]
\[
- Im(f(|u^n_0|^2 + |v^n_0|^2)u^n_0, f(|\tilde{u}^{n+\sigma}_h|^2 + |\tilde{v}^{n+\sigma}_h|^2)\tilde{v}^{n+\sigma}_h, I^n_h q_h, I^n_h p_h).
\]

Make use of Lemmas 1 and 2, and apply Young’s inequality as well as triangle inequality to obtain
\[
D^\alpha_0 \|u^n_0\|^2 + D^\alpha_0 \|v^n_0\|^2 \leq 2(\|u^n_0\|^2 + \|v^n_0\|^2) + \|f(|u^n_0|^2 + |v^n_0|^2)\|^2 \|u^n_0\|^2 + \|\tilde{v}^{n+\sigma}_h\|^2 \|g^{n+\sigma}_1\| + \sup_{t \in [0,T]} \|g_1(t)\|^2 + \|g_2(t)\|^2 \|g_2(t)\|^2.
\]

When \(n \geq 1\), taking \(q = u^n_0\) and \(p = v^n_0\) in (16) and considering only the imaginary part of the result, we have
\[
Re(D^\alpha_0 u^n_0, D^\alpha_0 v^n_0; I^n_h u^n_h, I^n_h v^n_h) = Im(g^n_1, g^n_2; I^n_h u^n_h, I^n_h v^n_h)
\]
\[
- Im(f(|\tilde{u}^{n+\sigma}_h|^2 + |\tilde{v}^{n+\sigma}_h|^2)\tilde{u}^{n+\sigma}_h, f(|\tilde{u}^{n+\sigma}_h|^2 + |\tilde{v}^{n+\sigma}_h|^2)\tilde{v}^{n+\sigma}_h, I^n_h u^n_h, I^n_h v^n_h).
\]

Make use of Lemmas 1 and 2, and apply Young’s inequality as well as triangle inequality to obtain
\[ D_w^n \| u_h^n + \sigma \|^2 + D_v^n \| v_h^n + \sigma \|^2 \leq \| s_{11}^{n+\sigma} \|^2 + \| s_{22}^{n+\sigma} \|^2 + 2(\| u_h^{n+1} \|^2 + \| v_h^{n+1} \|^2) \\
+ \left\| f \left( (\tilde{u}_h^{n+\sigma})^2 + |\tilde{v}_h^{n+\sigma}|^2 \right) \right\| + \left\| f \left( (\tilde{u}_h^{n+\sigma})^2 + |\tilde{v}_h^{n+\sigma}|^2 \right) \right\| \\
\leq \sup_{t \in [0,T]} \| s_1(t) \|^2 + \sup_{t \in [0,T]} \| s_2(t) \|^2 + \lambda_1 \left( \| u_h^{n+1} \|^2 + \| v_h^{n+1} \|^2 \right) \\
+ \lambda_2 (\| u_h^n \|^2 + \| v_h^n \|^2) + \lambda_3 (\| u_h^{n-1} \|^2 + \| v_h^{n-1} \|^2), \] (21)

where \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are constants independent of \( \tau \) and \( h \). Use Lemma 3 to obtain

\[ \| u_h^n \| + \| v_h^n \| \leq C \left( \| u_0 \| + \| v_0 \| \right) + \sup_{t \in [0,T]} \| s_1(t) \| + \sup_{t \in [0,T]} \| s_2(t) \|), \] (22)

which imply that the stability holds. \( \Box \)

3. Error Analysis

Here, for the need of error analysis, we introduce an orthogonal projection \( H \) from \( H_0^1(\Omega) \cap H^2(\Omega) \) to \( Z_h \), which satisfies

\[ a(z - Hz, I_h^2q) = 0, \quad \forall q \in Z_h. \] (23)

Lemma 4 ([11,21]). There exists a constant \( C^* > 0 \) independent of \( h \) and \( \tau \), such that

\[ \| z - Hz \| \leq C^* h^2 \| z \|_{3,\tau}, \quad \forall z \in H_0^1(\Omega) \cap W^{3,\tau}(\Omega), s > 1. \] (24)

For implementing the error analysis, we also need to provide the local truncation error.

Lemma 5 ([17]). For a function \( z \in C^3([0,T]) \), the local truncation error between \( D_w^n z(t_{n+\sigma}) \) and \( C_0 D^w_\sigma z(t_{n+\sigma}) \) satisfies

\[ |D_w^\sigma z(t_{n+\sigma}) - C_0 D^w_\sigma z(t_{n+\sigma})| = O(\tau^{3-s}), \] (25)

where \( t_{n+\sigma} = (n + \sigma)\tau \).

Theorem 2. Suppose \( u(t_n) \) and \( v(t_n) \) are the exact solution pair, and \( u_h^n \) and \( v_h^n \) are the numerical solution pair at \( t_n \), respectively. There exists a constant \( C \) independent of mesh size \( h \) and time step length size \( \tau \), such that

\[ \| u(t_n) - u_h^n \| + \| v(t_n) - v_h^n \| \leq C(\tau^2 + h^2), \quad n = 1, 2, \ldots, M. \] (26)

Proof. For convenience, we let

\[ u(t_n) - u_h^n = u(t_n) - Hu^n + Hu^n - u_h^n, \]
\[ v(t_n) - v_h^n = v(t_n) - Hv^n + Hv^n - v_h^n. \] (27)

We assume that \( \| u^n \|_{\infty} \) and \( \| v^n \|_{\infty} \) of the time semi-discrete solution pair are bounded. We now define \( P_1 \) and \( P_2 \) as the local truncation errors

\[ P_1^{n+\sigma} = i \left( D_w^n u^{n+\sigma} - C_0 D^w_\sigma u(t_{n+\sigma}) \right) + \Delta \left( u^{n+\sigma} - u(t_{n+\sigma}) \right) \]
\[ + f \left( |u^{n+\sigma}|^2 + |v^{n+\sigma}|^2 \right) u^{n+\sigma} - f \left( |u(t_{n+\sigma})|^2 + |v(t_{n+\sigma})|^2 \right) u(t_{n+\sigma}), \]
\[ P_2^{n+\sigma} = i \left( D_v^n v^{n+\sigma} - C_0 D^v_\sigma v(t_{n+\sigma}) \right) + \Delta \left( v^{n+\sigma} - v(t_{n+\sigma}) \right) \]
\[ + f \left( |u^{n+\sigma}|^2 + |v^{n+\sigma}|^2 \right) v^{n+\sigma} - f \left( |u(t_{n+\sigma})|^2 + |v(t_{n+\sigma})|^2 \right) v(t_{n+\sigma}). \] (28)
From the Taylor expansion and Lemma 5, we know
\[
|D_0^\alpha z^{n+\sigma} - D_0^\alpha z(t_{n+\sigma})| = O(\tau^2),
\]
\[
|\Delta z^{n+\sigma} - z(t_{n+\sigma})| = O(\tau^2),
\]
\[
|f((\bar{u}^{n+\sigma})^2 + |\bar{\sigma}^{n+\sigma}|^2)\bar{\sigma}^{n+\sigma} - f((u(t_{n+\sigma}))^2 + |\sigma(t_{n+\sigma})|^2)z(t_{n+\sigma})| = O(\tau^2).
\]

From the above we obtain that the constant $C$ exists, that is independent of time step $\tau$ and mesh size $h$, such that
\[
\|P_1^{n+\sigma}\| + \|P_2^{n+\sigma}\| \leq C\tau^2.
\]

Let $\eta_h^{n+\sigma} = H\sigma^{n+\sigma} - q^{n+\sigma}$, $\eta_h^{n+\sigma} = H\sigma^{n+\sigma} - q^{n+\sigma}$, $n = 0, 1, \cdots, M$. Using the finite volume element scheme we created and definitions of $P_1^{n+\sigma}$ and $P_2^{n+\sigma}$, we can obtain the following equation
\[
\begin{align*}
&\mathcal{i}(D_0^\alpha \xi_h^{n+\sigma}, D_0^\alpha \eta_h^{n+\sigma}; I_h^0 q, I_h^0 p) + a(\xi_h^{n+\sigma}, I_h^0 \eta) + a(\eta_h^{n+\sigma}, I_h^0 p) + (H_1^1 + H_2^1; I_h^0 q, I_h^0 p) \\
&= \mathcal{i}(D_0^\alpha (H\sigma^{n+\sigma} - u^{n+\sigma}), D_0^\alpha (H\sigma^{n+\sigma} - q^{n+\sigma}); I_h^0 q, I_h^0 p) + (P_1^{n+\sigma}, P_2^{n+\sigma}; I_h^0 q, I_h^0 p),
\end{align*}
\]
(31)
where $q, p \in Z_h$, $n = 1, 2, \cdots, M$, and
\[
H_1^{n+\sigma} = f(|\bar{u}^{n+\sigma}|^2 + |\bar{\sigma}^{n+\sigma}|^2)\bar{\eta}^{n+\sigma} - f(|\eta_h^{n+\sigma}|^2 + |\sigma_h^{n+\sigma}|^2)\eta_h^{n+\sigma},
\]
\[
H_2^{n+\sigma} = f(|\bar{u}^{n+\sigma}|^2 + |\bar{\sigma}^{n+\sigma}|^2)\bar{\eta}^{n+\sigma} - f(|\eta_h^{n+\sigma}|^2 + |\sigma_h^{n+\sigma}|^2)\eta_h^{n+\sigma}.
\]

Using the proven conclusion, triangle inequality and Hölder inequality, we have
\[
\begin{align*}
\|H_1^{n+\sigma}\|^2 + \|H_2^{n+\sigma}\|^2 \\
\leq \|f'(\xi_1)(|\bar{u}^{n+\sigma}|^2 - |\bar{\eta}^{n+\sigma}|^2 + |\bar{\sigma}^{n+\sigma}|^2 - |\bar{\eta}_h^{n+\sigma}|^2)\bar{\eta}_h^{n+\sigma} + f(|\bar{u}^{n+\sigma}|^2 + |\bar{\sigma}^{n+\sigma}|^2)\eta_h^{n+\sigma} \|^2 \\
+ \|f'(\xi_2)(|\bar{u}^{n+\sigma}|^2 - |\bar{\eta}^{n+\sigma}|^2 + |\bar{\sigma}^{n+\sigma}|^2 - |\bar{\eta}_h^{n+\sigma}|^2)\bar{\eta}_h^{n+\sigma} + f(|\bar{u}^{n+\sigma}|^2 + |\bar{\sigma}^{n+\sigma}|^2)\eta_h^{n+\sigma} \|^2 \\
\leq C(\|\xi_h^{n+\sigma}\|^2 + \|\eta_h^{n+\sigma}\|^2 + h^2),
\end{align*}
\]
(32)
where $\xi_1$ is a function between $\bar{u}^{n+\sigma}$ and $\bar{\eta}_h^{n+\sigma}$, $\xi_2$ is a function between $\bar{\sigma}^{n+\sigma}$ and $\bar{\eta}_h^{n+\sigma}$, and the constant $C$ is independent of $\tau, h$.

By setting $q = \eta_h^{n+\sigma}$ and $p = \eta_h^{n+\sigma}$ in (31), taking the imaginary part and using use of Lemma 1, we obtain
\[
\begin{align*}
&\text{Re}\left(D^\alpha \xi_h^{n+\sigma}, D^\alpha \eta_h^{n+\sigma}; I_h^0 q, I_h^0 p\right) \\
&= \text{Re}\left(D^\alpha (H\sigma^{n+\sigma} - u^{n+\sigma}), D^\alpha (H\sigma^{n+\sigma} - q^{n+\sigma}); I_h^0 q, I_h^0 p\right) \\
&\quad + \text{Im}\left(P_1^{n+\sigma}, P_2^{n+\sigma}; I_h^0 \eta, I_h^0 \eta\right) \\
&\quad - \text{Im}\left(H_1^{n+\sigma}, H_2^{n+\sigma}, I_h^0 \eta, I_h^0 \eta\right) \\
&\leq \frac{3}{2}\|\xi_h^{n+\sigma}\|^2 + \frac{3}{2}\|\eta_h^{n+\sigma}\|^2 + \frac{1}{2}\|D^\alpha (H\sigma^{n+\sigma} - q^{n+\sigma})\|^2 + \frac{1}{2}\|P_1^{n+\sigma}\|^2 + \frac{1}{2}\|P_2^{n+\sigma}\|^2.
\end{align*}
\]
(33)

Using Lemma 2, we can find the constant $C$ independent of $\tau$ and $h$, such that
\[
\begin{align*}
&\|D^\alpha \xi_h^{n+\sigma}\|^2 + \|D^\alpha \eta_h^{n+\sigma}\|^2 \\
&\leq C(\|\xi_h^{n+1}\|^2 + \|\eta_h^{n+1}\|^2 + \|\xi_h^{n-1}\|^2 + \|\eta_h^{n-1}\|^2 + \|\xi_h^{n}\|^2 + \|\eta_h^{n}\|^2 + \tau^2 + h^2).
\end{align*}
\]
(34)
Using Lemma 3, we can obtain

$$
\| \xi^n_h \| + \| \eta^n_h \| \leq C (\tau^2 + h^2).
$$

(35)

Make use of Lemma 4 and (27) to obtain

$$
\|
\begin{aligned}
&u(t_n) - u^n_h \| + \| v(t_n) - v^n_h \| \\
&\leq \|
\begin{aligned}
&u(t_n) - Hu^n \| + \| v(t_n) - Hv^n \| \\
&\leq C(\tau^2 + h^2), \quad n = 1, 2, \ldots, M.
\end{aligned}
\end{aligned}
\right.
$$

(36)

The proof ends. \( \square \)

4. Numerical Tests

In this section, for verifying the theoretical results, we implement some numerical tests. Firstly, the algorithm process is introduced. Using the definition of \( D^n_m \) and taking \( q = p = \phi_j \), \( j = 1, 2, \ldots, N \), we can rewrite (16) in the following matrix form

$$
\left( \begin{array}{c}
M_1 \ O \\
O \ M_1
\end{array} \right)
\left( \begin{array}{c}
U^{n+1}_h \\
V^{n+1}_h
\end{array} \right) =
\left( \begin{array}{c}
C^{n+\sigma}_1 A^{n+\sigma} \ O \\
\begin{array}{c}
O \\
C^{n+\sigma}_2 A^{n+\sigma}
\end{array}
\end{array} \right)
\left( \begin{array}{c}
U^n_h \\
V^n_h
\end{array} \right) +
\left( \begin{array}{c}
F^n \ O \\
O \ F^n
\end{array} \right)
\left( \begin{array}{c}
U^n_h \\
V^n_h
\end{array} \right),
$$

(37)

where

\[
A = ((\phi_j, I^n_h \phi_k))_{j, k=1, \ldots, N}, \quad U^n_h = (u^n_1, u^n_2, \ldots, u^n_N)^T, \quad M_1 = \frac{iC^{(n)}_0}{\mu} A + \sigma B,
\]

\[
B = (a(\phi_j, I^n_h \phi_k))_{j, k=1, \ldots, N}, \quad V^n_h = (v^n_1, v^n_2, \ldots, v^n_N)^T, \quad M_2 = \frac{iA}{\mu} \sum^n_{j=1} (C^{(n-1)}_{n-j} - C^{(n)}_{n-j+1}),
\]

\[
C^{n+\sigma}_1 = ((s^{n+\sigma}_1, I^n_h \phi_1), (s^{n+\sigma}_1, I^n_h \phi_2), \ldots, (s^{n+\sigma}_1, I^n_h \phi_N))^T,
\]

\[
C^{n+\sigma}_2 = ((s^{n+\sigma}_2, I^n_h \phi_1), (s^{n+\sigma}_2, I^n_h \phi_2), \ldots, (s^{n+\sigma}_2, I^n_h \phi_N))^T,
\]

\[
F^n = (f(|\sum^n_{i=1} \tilde{u}^{n+\sigma}_i \phi_i|^2 + |\sum^n_{i=1} \tilde{v}^{n+\sigma}_i \phi_i|^2) \phi_j, I^n_h \phi_k))_{j, k=1, \ldots, N}.
\]

Making use of the initial values of \( U^n_h, V^n_h, U^{n+1}_h \), and \( V^{n+1}_h \) obtained by (10), we can compute (37).

Here, the test function space \( \mathcal{Q}_h \) is taken to be the space of the piecewise constant functions corresponding to \( \Gamma^n_h \). The basis function is

$$
\phi_{i^n_0}(P) = \begin{cases} 
1, & P \in \mathcal{K}^e_{i^n_0} \\
0, & \text{elsewhere.}
\end{cases}
$$

(38)

In what follows, we will carry out numerical computations by taking several examples.

Example 1. Consider the following one-dimensional nonlinear coupled time-fractional Schrödinger equations

\[
\begin{aligned}
&i \frac{D^n_0 u + \partial_{xx} u + (|u|^2 + |v|^2) u = g_1, \quad x \in (-10, 10), \quad t \in (0, 1],}
&i \frac{D^n_0 v + \partial_{xx} v + (|u|^2 + |v|^2) v = g_2, \quad x \in (-10, 10), \quad t \in (0, 1],}
&u(-10, t) = u(10, t) = 0, \quad v(-10, t) = v(10, t) = 0, \quad t \in (0, 1],
&u(x, 0) = 0, \quad v(x, 0) = 0, \quad x \in [-10, 10],
\end{aligned}
\]

where \( g_1 \) and \( g_2 \) are determined by the exact solution pair \( u = it^2 \cos(\frac{2x}{10}) \) and \( v = it^3 \sin(\frac{2x}{10}) \).
For reducing the computational cost and memory, we investigate the temporal convergence and spatial convergence together by refining $\tau$ and $h$ simultaneously. In Table 1, by taking $N = M = 40, 80, 160, 320$, we arrive at errors in $L^2$-norm of $u$ and $v$, as well as its space-time convergence orders, when $h$ takes different values. Based on the computed results, one can find that the space-time convergence order is close to 2, regardless of the different $h \in (0, 1)$.

In order to test the influence of the space parameter $N$, we draw the comparison images between the numerical solution and the exact solution with different $N$ in Figures 1 and 2. To check the influence of the changed fractional parameter $\alpha$, we draw the errors of $|u - u_h|$ and $|v - v_h|$ with different $\alpha$ in Figures 3 and 4, from which one can see that the variation of the fractional parameter has a slight impact on errors.

Table 1. $L^2$-errors and convergence rates of the proposed scheme with different $\alpha$.

<table>
<thead>
<tr>
<th>$N = M$</th>
<th>$\alpha = 0.25$</th>
<th>Order</th>
<th>$|u - u_h|$</th>
<th>Order</th>
<th>$|u - u_h|$</th>
<th>Order</th>
<th>$|u - u_h|$</th>
<th>Order</th>
<th>$|u - u_h|$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>$7.2311 \times 10^{-3}$</td>
<td>1.9953</td>
<td>$8.0798 \times 10^{-3}$</td>
<td>1.9950</td>
<td>$6.0127 \times 10^{-3}$</td>
<td>1.9949</td>
<td>$3.8495 \times 10^{-3}$</td>
<td>1.9948</td>
<td>$4.8110 \times 10^{-3}$</td>
<td>1.9947</td>
</tr>
<tr>
<td>80</td>
<td>$1.8392 \times 10^{-3}$</td>
<td>1.9950</td>
<td>$2.0164 \times 10^{-3}$</td>
<td>1.9948</td>
<td>$2.0026 \times 10^{-3}$</td>
<td>1.9947</td>
<td>$1.5035 \times 10^{-3}$</td>
<td>1.9946</td>
<td>$6.9428 \times 10^{-4}$</td>
<td>1.9945</td>
</tr>
<tr>
<td>160</td>
<td>$4.6246 \times 10^{-4}$</td>
<td>1.9977</td>
<td>$5.0456 \times 10^{-4}$</td>
<td>1.9975</td>
<td>$1.9987 \times 10^{-4}$</td>
<td>1.9974</td>
<td>$3.7574 \times 10^{-4}$</td>
<td>1.9973</td>
<td>$2.4129 \times 10^{-4}$</td>
<td>1.9972</td>
</tr>
<tr>
<td>320</td>
<td>$1.1586 \times 10^{-4}$</td>
<td>1.9990</td>
<td>$1.2623 \times 10^{-4}$</td>
<td>1.9988</td>
<td>$1.9990 \times 10^{-5}$</td>
<td>1.9987</td>
<td>$9.3854 \times 10^{-5}$</td>
<td>1.9986</td>
<td>$2.0012 \times 10^{-5}$</td>
<td>1.9985</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N = M$</th>
<th>$|v - v_h|$</th>
<th>Order</th>
<th>$|v - v_h|$</th>
<th>Order</th>
<th>$|v - v_h|$</th>
<th>Order</th>
<th>$|v - v_h|$</th>
<th>Order</th>
<th>$|v - v_h|$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
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<td>40</td>
<td>$1.1670 \times 10^{-2}$</td>
<td>1.9977</td>
<td>$1.1110 \times 10^{-2}$</td>
<td>1.9975</td>
<td>$7.6529 \times 10^{-3}$</td>
<td>1.9974</td>
<td>$4.8110 \times 10^{-3}$</td>
<td>1.9973</td>
<td>$1.2071 \times 10^{-3}$</td>
<td>1.9972</td>
</tr>
<tr>
<td>80</td>
<td>$2.9829 \times 10^{-3}$</td>
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<td>$2.7871 \times 10^{-3}$</td>
<td>1.9988</td>
<td>$1.9990 \times 10^{-3}$</td>
<td>1.9987</td>
<td>$1.9200 \times 10^{-3}$</td>
<td>1.9986</td>
<td>$1.2071 \times 10^{-3}$</td>
<td>1.9985</td>
</tr>
<tr>
<td>160</td>
<td>$7.5194 \times 10^{-4}$</td>
<td>1.9994</td>
<td>$6.9905 \times 10^{-4}$</td>
<td>1.9993</td>
<td>$4.8049 \times 10^{-4}$</td>
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<td>$1.2071 \times 10^{-3}$</td>
<td>1.9990</td>
</tr>
<tr>
<td>320</td>
<td>$1.8866 \times 10^{-4}$</td>
<td>1.9994</td>
<td>$1.7508 \times 10^{-4}$</td>
<td>1.9993</td>
<td>$1.9990 \times 10^{-5}$</td>
<td>1.9992</td>
<td>$1.2009 \times 10^{-5}$</td>
<td>1.9991</td>
<td>$1.2071 \times 10^{-3}$</td>
<td>1.9990</td>
</tr>
</tbody>
</table>

Example 2. Consider the following two-dimensional nonlinear coupled time-fractional Schrödinger equations

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\alpha}{1} D_t^\alpha u + \Delta u + f(|u|^2 + |v|^2)u = g_1, \ (x, y) \in \Omega, \ 0 < t \leq 1, \\
\frac{\alpha}{1} D_t^\alpha v + \Delta v + f(|u|^2 + |v|^2)v = g_2, \ (x, y) \in \Omega, \ 0 < t \leq 1, \\
u(x, y, t) = 0, v(x, y, t) = 0, \ (x, y) \in \partial \Omega, \ 0 < t \leq 1, \\
u(x, y, 0) = 0, v(x, y, 0) = 0, \ (x, y) \in \Omega,
\end{array} \right. \\
\end{align*}
\]
where \( \bar{\Omega} = [0, 1]^2 \), \( g_1 \) and \( g_2 \) are determined by the exact solution pair \( u = (i + 2)t^2 \sin(2\pi x) \sin(2\pi y) \) and \( v = it^2 \sin(\pi x) \sin(\pi y) \). The nonlinear term \( f(s) \) is selected as the following three cases [20]:

Case 1. \( f(s) = s \),
Case 2. \( f(s) = \ln(1 + s) \),
Case 3. \( f(s) = 1 - e^{-s} \).

In Example 2, we test the convergence order of the proposed scheme by solving the two-dimensional nonlinear coupled fractional Schrödinger equations. For the convenience of calculation, we take the same space partition in both \( x \) and \( y \) directions. For the ease of description, we denote the numerical solution \( u_h \) in Cases 1, 2, and 3 as \( u_{h1} \), \( u_{h2} \), and \( u_{h3} \), respectively. The \( L^2 \)-errors and convergence rates with different \( \alpha \) and cases are listed in Table 2. We can observe again that the proposed scheme has an optimal convergence result with \( O(\tau^2 + h^2) \), and for three different nonlinear terms, all the convergence orders are close to 2. This shows that our scheme is effective for solving these nonlinear equations.

To observe the approximation effect for different spatial parameters \( N \) on \( |u_h| \) and \( |v_h| \), we separately plotted Figure 5 for \( |u_h| \) and Figure 6 for \( |v_h| \) of Case 1 with \( \alpha = 0.5, M = 20 \) and the exact solution pair \( |u| \) and \( |v| \) at \( t = 1 \). In order to test the influence of time \( t \), we draw the error graphs \( |u - u_h| \) and \( |v - v_h| \) of Case 2 with \( \alpha = 0.75, N = M = 40 \) at different time points in Figure 7.
Table 2. $L^2$-errors and convergence rates of the proposed scheme with different $\alpha$ for Cases 1–3.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$N = M$</th>
<th>$| u - u_{h1} |$</th>
<th>Order</th>
<th>$| u - u_{h2} |$</th>
<th>Order</th>
<th>$| u - u_{h3} |$</th>
<th>Order</th>
</tr>
</thead>
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<tr>
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<td>$6.8221 \times 10^{-2}$</td>
<td></td>
<td>$7.9125 \times 10^{-2}$</td>
<td></td>
<td>$8.1875 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>$1.4410 \times 10^{-2}$</td>
<td></td>
<td>$2.2432 \times 10^{-2}$</td>
<td></td>
<td>$2.2192 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>$3.6666 \times 10^{-3}$</td>
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<td>$1.9745 \times 10^{-3}$</td>
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<td>$1.9646 \times 10^{-3}$</td>
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</tr>
<tr>
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<td>$2.0371 \times 10^{-3}$</td>
<td></td>
<td>$2.0451 \times 10^{-3}$</td>
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<tr>
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<td>$2.2423 \times 10^{-2}$</td>
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<tr>
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<td></td>
<td>$2.0431 \times 10^{-3}$</td>
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</tr>
</tbody>
</table>

Figure 4. $|v - v_h|$ with $N = M = 100$ for different $\alpha$. 

Table 2. $L^2$-errors and convergence rates of the proposed scheme with different $\alpha$ for Cases 1–3.
Figure 5. \(|u|\) and \(|u_h|\) of Case 1 with \(\alpha = 0.5, M = 20\) for different \(N\) at \(t = 1\).

Figure 6. Cont.
5. Conclusion and Advancements

In this article, to solve the nonlinear coupled time-fractional Schrödinger equations, we establish a fully discrete finite volume element method based on the $L2 - 1_\sigma$ formula. We provide the theoretical proof of stability, and derive the optimal error estimate. Finally, choosing several numerical examples, we carry out the numerical tests and obtain the approximate second-order convergence results, which are consistent with the theoretical
optimal results, and indicate that the convergence is almost unaffected by the changed fractional parameter when the solution has good regularity. In future work, we will continue to develop the finite volume element method for the coupled fractional Klein-Gordon-Schrödinger equations.

Author Contributions: X.Z.: methodology, validation, software, and writing—original draft preparation; Y.Y.: conceptualization, formal analysis, validation, and writing—review and editing; H.L.: validation, formal analysis, and writing—review and editing, funding acquisition; Z.F.: validation, formal analysis, and writing—review and editing; Y.L.: conceptualization, methodology, validation, formal analysis, funding acquisition, and writing—review and editing. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: All the data were computed using our algorithm. Data sharing is not applicable to this article as no dataset was generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflicts of interest.

References


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