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The Ulam Stability of High-Order Variable-Order φ -Hilfer Fractional Implicit Integro-Differential Equations

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Abstract: This study investigates the initial value problem of high-order variable-order φ -Hilfer fractional implicit integro-differential equations. Due to the lack of the semigroup property in variable-order fractional integrals, solving these equations presents significant challenges. We introduce a novel approach that approximates variable-order fractional derivatives using a piecewise constant approximation method. This method facilitates an equivalent integral representation of the equations and establishes the Ulam stability criterion. In addition, we explore higher-order forms of fractional-order equations, thereby enriching the qualitative and stability results of their solutions.

Keywords: implicit integro-differential equations; high-order variable-order φ -Hilfer fractional derivatives; Ulam–Hyers stability; Ulam–Hyers–Rassias stability

MSC: 34A08; 34A09; 34D20



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1. Introduction

In recent years, interest in the qualitative and stability analysis of fractional differential equations has increased. These equations, which incorporate fractional derivatives and integrals, have applications in diverse fields, such as biomechanics, electrical engineering, and medical ultrasound detection (see [1–3]). Notably, although most fractional derivatives are of constant order, practical applications in physics, signal processing, and control systems often require variable-order fractional derivatives (see [4,5]). However, the lack of semigroup properties in variable-order fractional integrals complicates the analysis of the solution characteristics. Scholars have responded by proposing the piecewise constant approximation method, which simplifies variable-order fractional differential equations to constant order for analysis. For example, Refice et al. [6] discussed specific implicit variable-order fractional differential equations:

$$\begin{cases} D_{0+}^{u(t)} y(t) = \Phi(t, y(t), D_{0+}^{u(t)} y(t)), \\ y(0) = 0, y(b) = 0, \end{cases}$$

where $u(t) \in (1, 2]$, $t \in J = [0, T]$, and $\Phi(t, y(t), D_{0+}^{u(t)} y(t)) \in C(J, \mathbb{R}, \mathbb{R})$. $D_{0+}^{u(t)} y(t)$ denotes the variable-order Caputo fractional derivative of the function $y(t)$. Further results can be found in the literature [7–9].

The φ -Hilfer fractional differential equations, which are widely applied in practical problems [10], have yielded fundamental results concerning stability (see [11,12]). For instance, Sousa et al. [13] achieved notable findings on the Ulam stability of certain φ -Hilfer fractional integro-differential equations:

$$\begin{cases} {}^H D_{0+}^{\alpha, \beta; \varphi} u(t) = f(t, u(t)) + \int_0^t k(t, s, u(s)) ds, \\ I_{0+}^{1-\gamma} u(0) = \sigma, \end{cases}$$

where $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $t \in J = [0, T]$, $f(t, u(t)) \in C(J, \mathbb{R})$, and $k(t, s, u(s)) \in C(J, \mathbb{R}, \mathbb{R})$. ${}^H D_{0+}^{\alpha, \beta; \varphi} u(t)$ denotes the φ -Hilfer fractional derivative of the function $u(t)$. $I_{0+}^{1-\gamma}(\cdot)$ denotes the φ -Hilfer fractional integrals.

Due to the complexity of high-order φ -Hilfer fractional differential equations, there has been relatively little discussion about them in the literature. This paper addresses the initial value problem for a class of high-order variable-order φ -Hilfer fractional implicit integro-differential equations:

$$\begin{cases} {}^H D_{0+}^{\alpha(t), \beta; \varphi} v(t) = g(t, v(t), {}^H D_{0+}^{\alpha(t), \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau, \\ {}^H D_{0+}^{\alpha(t)-k, \beta; \varphi} v(0) = 0, k = 1, 2, \dots, n, \end{cases} \quad (1)$$

where $\alpha(t) \in (n-1, n]$, $\beta \in [0, 1]$, $t \in J = [0, T]$, $g(t, v(t), {}^H D_{0+}^{\alpha(t), \beta; \varphi} v(t)) \in C(J, \mathbb{R}, \mathbb{R})$, and $h(t, \tau, v(\tau)) \in C(J, \mathbb{R}, \mathbb{R})$. The function $\varphi(t)$ is a positively increasing function with a continuous derivative. To facilitate the results, we assume that if $\sigma(t) \equiv 1$, then $I^{\gamma(t)-k; \varphi} \sigma(t) = c_k$, $k = 1, \dots, n$, where $\gamma(t) = \alpha(t) + \beta(n - \alpha(t))$.

This paper first employs the method of piecewise constant approximation to transform variable-order φ -Hilfer fractional derivatives into constant-order fractional derivatives. It then uses the relationship between high-order φ -Hilfer fractional integrals and derivatives to derive the equivalent integral expression of high-order φ -Hilfer fractional implicit integro-differential equations. By applying the fixed-point theorem in Banach space, we establish criteria for Ulam–Hyers stability and Ulam–Hyers–Rassias stability for these equations. This study extends the φ -Hilfer fractional differential equations framework and discusses higher-order variable-order φ -Hilfer fractional differential equations.

2. Preliminaries

This section begins by introducing the fundamental definitions of φ -Hilfer fractional derivatives and integrals.

Definition 1. Let $\alpha(t) \in (n-1, n]$; $v(t), \varphi(t) \in C^n(J, \mathbb{R})$; and $\varphi(t)$ is a positively increasing function on J , with a continuous derivative on the same interval. The φ -Hilfer integral of variable-order is defined as follows:

$$I_{0+}^{\alpha(t); \varphi} v(t) = \frac{1}{\Gamma(\alpha(t))} \int_0^t \varphi'(s) (\varphi(t) - \varphi(s))^{\alpha(t)-1} v(s) ds, \quad t \in J.$$

Definition 2. Let $\alpha(t) \in (n-1, n]$; $\beta \in [0, 1]$; $v(t), \varphi(t) \in C^n(J, \mathbb{R})$; and $\varphi(t)$ is a positively increasing function on J , with a continuous derivative on the same interval and $\varphi'(t) \neq 0$. The φ -Hilfer derivative of variable-order is defined as follows:

$${}^H D_{0+}^{\alpha(t), \beta; \varphi} v(t) = I_{0+}^{\beta(n-\alpha(t)); \varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n I_{0+}^{(1-\beta)(n-\alpha(t)); \varphi} v(t).$$

Regarding the relationship between fractional derivatives and integrals of constant order, they possess the following properties, as documented in the literature [14].

Lemma 1. Let $\alpha, \beta > 0$, there is

$$I_{0+}^{\alpha; \varphi} I_{0+}^{\beta; \varphi} v(t) = I_{0+}^{\alpha+\beta; \varphi} v(t).$$

Lemma 2. Let $v(t) \in C^n(J, \mathbb{R})$, $\alpha \in (n-1, n)$, and $\beta \in [0, 1]$. There is

$$I_{0+}^{\alpha; \varphi} {}^H D_{0+}^{\alpha, \beta; \varphi} v(t) = v(t) - \sum_{k=1}^n \frac{(\varphi(t) - \varphi(0))^{\gamma-k}}{\Gamma(\gamma-k+1)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^{n-k} I_{0+}^{(1-\beta)(n-\alpha); \varphi} v(0),$$

where $\gamma = \alpha + \beta(n - \alpha)$.

Lemma 3. Let $k(t) = I_{0+}^{(1-\beta)(n-\alpha);\varphi} v(t)$. It follows that

$${}^H D_{0+}^{\alpha,\beta;\varphi} v(t) = I_{0+}^{n-\mu;\varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n k(t),$$

where $\mu = n(1 - \beta) + \beta\alpha$.

Lemma 4. Let $\alpha > 0$, $0 \leq \beta \leq 1$. We obtain

$${}^H D_{0+}^{\alpha,\beta;\varphi} I_{0+}^{\alpha;\varphi} v(t) = v(t).$$

Theorem 1 ([13]). Let (Δ, d) be a generalized complete metric space. Assume that $\mathcal{F} : \Delta \rightarrow \Delta$ is a strictly contractive operator with Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\mathcal{F}^{k+1}x, \mathcal{F}^k x) < \infty$, the following results hold:

- (i) The sequence $\{\mathcal{F}^n x\}$ converges to a fixed point x^* of $\mathcal{F}$.
- (ii) x^* is the unique fixed point of $\mathcal{F}$ in $\Delta^* = \{y \in \Delta \mid d(\mathcal{F}^n x, y) < \infty\}$.
- (iii) If $y \in \Delta^*$, then $d(y, \Delta^*) \leq \frac{1}{1-L} d(\mathcal{F}y, y)$.

3. Variable-Order φ -Hilfer Derivatives and Integrals

Through the following example, it can be observed that the φ -Hilfer fractional integrals of variable order do not satisfy semigroup properties.

Example 1. Let $\alpha(t) = 0.75t + 0.15$, $\beta(t) = 0.25t + 0.25$, $\varphi(t) = t$, and $v(t) = 1$; then,

$$\begin{aligned} I_{0+}^{\alpha(t)+\beta(t);\varphi} v(t) \Big|_{t=1} &= \frac{1}{\Gamma(\alpha(t) + \beta(t))} \int_0^t \varphi'(s) (\varphi(t) - \varphi(s))^{\alpha(t)+\beta(t)-1} u(s) ds \\ &= \frac{1}{\Gamma(t+1)} \int_0^t (t-s)^t v(s) ds \\ &\approx 0.805043. \end{aligned}$$

$$\begin{aligned} I_{0+}^{\alpha(t);\varphi} (I_{0+}^{\beta(t);\varphi} v(t)) \Big|_{t=1} &= \frac{1}{\Gamma(\alpha(t))} \int_0^t \varphi'(s) (\varphi(t) - \varphi(s))^{\alpha(t)-1} (I_{0+}^{\beta(s);\varphi} v(s)) ds \\ &= \frac{1}{\Gamma(0.75t + 0.15)} \int_0^t \frac{(t-s)^{0.25t-0.75} s^{0.25s+0.25}}{\Gamma(0.25s + 1.25)} ds \\ &\approx 1.09107. \end{aligned}$$

Obviously,

$$I_{0+}^{\alpha(t)+\beta(t);\varphi} v(t) \neq I_{0+}^{\alpha(t);\varphi} (I_{0+}^{\beta(t);\varphi} v(t)).$$

In order to address such issues, this paper introduces a method using piecewise constant approximation, approximating variable-order φ -Hilfer fractional integrals as constant-order equivalents. This approach ensures semigroup properties of fractional integrals and subsequently derives an equivalent integral expression form of the equations.

Lemma 5 ([9]). Let $\theta(t) \in C(J, (0, 1))$. For any $\varepsilon > 0$, there exists a natural number $n = n(\varepsilon)$ and a series of points $T_k = T_k(\varepsilon) : 0 = T_0 < T_1 < T_2 < \dots < T_{n-1} < T_n = T$, satisfying the condition

$$|\theta(t) - \alpha(t)| \leq \varepsilon, \quad t \in J,$$

where $\alpha(t) = \sum_{k=1}^n \theta(T_{k-1})J_k(t)$, $J_k(t)$ represents the indicator of the interval $(T_{k-1}, T_k]$, $k = 1, 2, \dots, n$, i.e.,

$$J_k(t) = \begin{cases} 1, & \text{if } t \in (T_{k-1}, T_k], \\ 0, & \text{otherwise.} \end{cases}$$

According to Lemma 5, for the given $\varepsilon > 0$, it can be established that there is a partition $P = (T_{k-1}, T_k]$, $k = 1, 2, \dots, n$ of the interval $[0, T]$ and an ε -approximation $\theta(t)$ of $\alpha(t)$ such that $|\theta(t) - \alpha(t)| \leq \varepsilon$, $t \in [0, T]$. In this paper, the case of a fixed partition $0 = T_0 < T_1 < T_2 < \dots < T_n = T$ is considered.

Therefore, for $t \in (T_{k-1}, T_k]$, there is $\alpha(t) = \theta(T_{k-1}) = q_k$ such that

$${}^H D_{0+}^{\alpha(t), \beta; \varphi} v(t) = {}^H D_{0+}^{q_k, \beta; \varphi} v(t).$$

In conclusion, for Equation (1), we only need to consider the following equations:

$$\begin{cases} {}^H D_{0+}^{q_k, \beta; \varphi} v(t) = g(t, v(t), {}^H D_{0+}^{q_k, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau, & t \in (T_{k-1}, T_k], \\ {}^H D_{0+}^{q_k - k, \beta; \varphi} v(0) = 0, & k = 1, 2, \dots, n, \end{cases} \quad (2)$$

where $k = 1, 2, 3, \dots, n$. $q_k \in (n-1, n]$, $\beta \in [0, 1]$, $t \in J$, $g(t, v(t), {}^H D_{0+}^{q_k, \beta; \varphi} v(t)) \in C(J, \mathbb{R}, \mathbb{R})$, $h(t, \tau, v(\tau)) \in C(J, \mathbb{R}, \mathbb{R})$, and if $\sigma(t) \equiv 1$, then $I^{\gamma_k - k; \varphi} \sigma(t) = c_k$, $k = 1, \dots, n$, $\gamma_k = q_k + \beta(n - q_k)$.

Theorem 2. The equivalent integral form of Equation (2) is

$$v(t) = I^{q_k; \varphi} f(t) + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i I^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik; \varphi} f(T_{k-1}), \quad t \in (T_{k-1}, T_k], \quad (3)$$

where $f(t) = g(t, v(t), {}^H D_{0+}^{q_k, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau$, $\mu_k = n(1 - \beta) + \beta q_k$.

Proof. For $t \in [0, T_1]$, integrating Equation (2) for $q_1 \in (n-1, n]$ order on both sides, according to Lemma 2, we obtain

$$\begin{aligned} I_{0+}^{q_1; \varphi} {}^H D_{0+}^{q_1, \beta; \varphi} v(t) &= v(t) - \sum_{k=1}^n \frac{(\varphi(t) - \varphi(0))^{\gamma_k - k}}{\Gamma(\gamma_k - k + 1)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^{n-k} I_{0+}^{(1-\beta)(n-q_1); \varphi} v(0) \\ &= I_{0+}^{q_1; \varphi} \left(g(t, v(t), {}^H D_{0+}^{q_1, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right). \end{aligned}$$

Thus,

$$\begin{aligned} v(t) &= I_{0+}^{q_1; \varphi} \left(g(t, v(t), {}^H D_{0+}^{q_1, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right) \\ &\quad + \sum_{k=1}^n \frac{(\varphi(t) - \varphi(0))^{\gamma_k - k}}{\Gamma(\gamma_k - k + 1)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^{n-k} I_{0+}^{(1-\beta)(n-q_1); \varphi} v(0) \\ &= I_{0+}^{q_1; \varphi} \left(g(t, v(t), {}^H D_{0+}^{q_1, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right) \\ &\quad + \sum_{k=1}^n I^{\gamma_k - k; \varphi} \sigma(t) \cdot \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^{n-k} I_{0+}^{(1-\beta)(n-q_1); \varphi} v(0). \end{aligned} \quad (4)$$

where $\sigma(t) \equiv 1$. Then, by integrating $n - k - \mu_1$, $\mu_1 = n(1 - \beta) + \beta q_1$ order on both sides of Equation (4), based on Lemmas 1 and 3, and the assumptions $I^{\gamma_k - k; \varphi} 1 = c_k$, $k = 1, \dots, n$, we have

$$I_{0+}^{n-k-\mu_1;\varphi} v(t) = I_{0+}^{n-k-\mu_1+q_1;\varphi} \left(g(t, v(t), {}^H D_{0+}^{q_1, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right) + \sum_{k=1}^n c_k D_{0+}^{q_1-k, \beta; \varphi} v(0).$$

From the initial value conditions ${}^H D_{0+}^{q_1-k, \beta; \varphi} v(0) = 0, k = 1, \dots, n$, it follows that

$$I_{0+}^{n-k-\mu_1;\varphi} v(t) = I_{0+}^{n-k-\mu_1+q_1;\varphi} \left(g(t, v(t), {}^H D_{0+}^{q_1, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right). \quad (5)$$

Further, by differentiating $n - k - \mu_1$ order on both sides of Equation (5), based on Lemma 4, there is

$$v(t) = I_{0+}^{q_1;\varphi} \left(g(t, v(t), {}^H D_{0+}^{q_1, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right), t \in [0, T_1].$$

For $t \in (T_1, T_2]$, we have

$$I^{n-k-\mu_2;\varphi} v(t) = I^{n-k-\mu_2+q_2;\varphi} \left(g(t, v(t), {}^H D^{q_2, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right) + \sum_{k=1}^n c_k \sum_{k=1}^n c_k D^{q_2-k, \beta; \varphi} v(T_1), \quad (6)$$

where $\mu_2 = n(1 - \beta) + \beta q_2$. Then, by differentiating $n - k - \mu_2$ order on both sides of Equation (6), there is

$$v(t) = I_{0+}^{q_2;\varphi} \left(g(t, v(t), {}^H D_{0+}^{q_2, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau \right) + \sum_{k=1}^n c_k \sum_{k=1}^n c_k I^{q_1-q_2-n-\mu_2+2k;\varphi} \left(g(T_1, v(T_1), {}^H D^{q_1, \beta; \varphi} v(T_1)) + \int_0^{T_1} h(T_1, \tau, v(\tau)) d\tau \right).$$

Repeat the steps above, for $t \in (T_{k-1}, T_k]$, let

$$f(t) = g(t, v(t), {}^H D^{q_k, \beta; \varphi} v(t)) + \int_0^t h(t, \tau, v(\tau)) d\tau,$$

we obtain

$$v(t) = I^{q_k;\varphi} f(t) + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i I^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik;\varphi} f(T_{k-1}).$$

□

4. Ulam Stability

To commence, we present the definitions of Ulam–Hyers stability and Ulam–Hyers–Rassias stability of Equation (2).

Definition 3. Equation (2) is Ulam–Hyers stable if there exists a positive number $f_k > 0$ such that for any $\varepsilon > 0$ and for each solution $v(t)$ of the inequality

$$\left| {}^H D_{0+}^{q_k, \beta; \varphi} v(t) - g(t, v(t), {}^H D_{0+}^{q_k, \beta; \varphi} v(t)) - \int_0^t h(t, \tau, v(\tau)) d\tau \right| \leq \varepsilon, t \in (T_{k-1}, T_k], \quad (7)$$

there exists a solution $v_0(t)$ of Equation (2) with

$$|v(t) - v_0(t)| \leq f_k \varepsilon.$$

Definition 4. Equation (2) is Ulam–Hyers–Rassias stable with respect to $\xi_k(t) \in C((T_{k-1}, T_k], \mathbb{R})$, if there exists a positive number $f_k > 0$ such that for any $\varepsilon > 0$ and for each solution $v(t)$ of the inequality

$$\left| {}^H D_{0+}^{q_k, \beta; \varphi} v(t) - g(t, v(t), {}^H D_{0+}^{q_k, \beta; \varphi} v(t)) - \int_0^t h(t, \tau, \tau(s)) d\tau \right| \leq \varepsilon \xi_k(t), t \in (T_{k-1}, T_k], \quad (8)$$

there exists a solution $v_0(t)$ of Equation (2) with

$$|v(t) - v_0(t)| \leq f_k \varepsilon \xi_k(s).$$

To establish the Ulam stability of Equation (2), it is conventionally required to verify the fulfillment of the specified conditions.

(H₁) The functions $g \in C(J \times \mathbb{R} \times \mathbb{R})$ and $h \in C(J \times \mathbb{R} \times \mathbb{R})$ are continuous.

(H₂) For any $t \in J$, there exists a nonnegative constant L_g ; there is

$$|g(t, v_1, v_2) - g(t, w_1, w_2)| \leq L_g(|v_1 - w_1| + |v_2 - w_2|).$$

(H₃) For any $t \in J$, there exists a nonnegative constant L_h ; there is

$$|h(t, s, v_1) - h(t, s, w_1)| \leq L_h|v_1 - w_1|.$$

Theorem 3. Assume that the conditions (H₁), (H₂), and (H₃) are satisfied, and $\varphi(t)$ is an increasing function with $\varphi'(t) \neq 0$. If there exists a continuously differentiable function $v_k(t)$ such that the inequality (7) holds, and

$$\begin{aligned} M_k = & \left[\left(L_g + \frac{L_g}{1 - L_g} \right) + \left(L_h + \frac{L_h}{1 - L_g} \right) T_k \right] \left[\frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)} \right. \\ & \left. + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik}}{\Gamma(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik + 1)} \right] < 1, \end{aligned} \quad (9)$$

then a unique continuous solution $v_{k0}(t)$ exists such that

$$|v_k(t) - v_{k0}(t)| \leq \varepsilon \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{(1 - M_k)\Gamma(q_k + 1)},$$

i.e., Equation (2) is Ulam–Hyers stable.

Proof. Let Δ denote the collection of all continuous real-valued functions defined on the interval J . Define a generalized metric on Δ as follows:

$$d(a, b) = \inf\{\omega \in [0, \infty) \mid |a(t) - b(t)| \leq \omega, t \in (T_{k-1}, T_k]\}.$$

Then, (Δ, d) is considered to be a generalized complete metric space.

Define the operator $\mathcal{F} : \Delta \rightarrow \Delta$ as

$$\mathcal{F}v(t) = I^{q_k, \varphi} f(t) + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i I^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik; \varphi} f(T_{k-1}). \quad (10)$$

We consider the case where k is even.

Firstly, we prove that $\mathcal{F}$ is a strictly compact operator of the generalized metric space Δ . For any $a, b \in \Delta$, there exists $\omega_{kab} \in [0, \infty)$ such that $d(a, b) = |a(t) - b(t)| \leq \omega_{kab}$; then,

$$\begin{aligned}
 & |\mathcal{F}a(t) - \mathcal{F}b(t)| \\
 & \leq \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} \left| g(z, a(z), {}^H D^{q_k, \beta; \varphi} a(z)) - g(z, b(z), {}^H D^{q_k, \beta; \varphi} b(z)) \right| dz \\
 & \quad + \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} \int_0^z |h(z, \tau, a(\tau)) - h(z, \tau, b(\tau))| d\tau dz \\
 & \quad + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \left\{ \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \right. \\
 & \quad \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik-1} \\
 & \quad \times \left| g(T_{k-1}, a(T_{k-1}), {}^H D^{q_k, \beta; \varphi} a(T_{k-1})) - g(T_{k-1}, b(T_{k-1}), {}^H D^{q_k, \beta; \varphi} b(T_{k-1})) \right| dz \\
 & \quad + \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \\
 & \quad \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik-1} \\
 & \quad \times \int_0^{T_{k-1}} |h(T_{k-1}, \tau, a(\tau)) - h(T_{k-1}, \tau, b(\tau))| d\tau dz \Big\}
 \end{aligned} \tag{11}$$

According to the conditions $(H_2), (H_3)$, we obtain

$$\begin{aligned}
 & \left| {}^H D^{q_k, \beta; \varphi} a(z) - {}^H D^{q_k, \beta; \varphi} b(z) \right| \leq \left| g(z, a(z), {}^H D^{q_k, \beta; \varphi} a(z)) - g(z, b(z), {}^H D^{q_k, \beta; \varphi} b(z)) \right| \\
 & \quad + \int_{T_{k-1}}^t |h(t, \tau, a(\tau)) - h(t, \tau, b(\tau))| d\tau \\
 & \leq L_f \left(|a(z) - b(z)| + \left| {}^H D^{q_k, \beta; \varphi} a(z) - {}^H D^{q_k, \beta; \varphi} b(z) \right| \right) \\
 & \quad + L_h \int_0^t |a(\tau) - b(\tau)| d\tau.
 \end{aligned}$$

Thus, there is

$$\left| {}^H D^{q_k, \beta; \varphi} a(z) - {}^H D^{q_k, \beta; \varphi} b(z) \right| \leq \frac{L_g}{1 - L_g} |a(z) - b(z)| + \frac{L_h}{1 - L_g} \int_0^t |a(\tau) - b(\tau)| d\tau. \tag{12}$$

Substituting (12) into (11), we obtain

$$\begin{aligned}
 & |\mathcal{F}a(t) - \mathcal{F}b(t)| \\
 & \leq \left(L_g + \frac{L_g}{1 - L_g} \right) \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} |a(z) - b(z)| dz \\
 & \quad + \left(L_g + \frac{L_h}{1 - L_g} \right) \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} \int_0^z |a(\tau) - b(\tau)| d\tau dz \\
 & \quad + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \left\{ \left(L_g + \frac{L_g}{1 - L_g} \right) \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \right.
 \end{aligned}$$

$$\begin{aligned}
& \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik - 1} |a(T_{k-1}) - b(T_{k-1})| dz \\
& + \left(L_g + \frac{L_h}{1 - L_g} \right) \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \\
& \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik - 1} \int_0^{T_{k-1}} |a(\tau) - b(\tau)| d\tau dz \Big\} \\
& \leq \omega_{kab} \left[\left(L_g + \frac{L_g}{1 - L_g} \right) + \left(L_h + \frac{L_h}{1 - L_g} \right) T_k \right] \left[\frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)} \right. \\
& \left. + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik}}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik + 1\right)} \right].
\end{aligned}$$

Denoting

$$\begin{aligned}
M_k = & \left[\left(L_g + \frac{L_g}{1 - L_g} \right) + \left(L_h + \frac{L_h}{1 - L_g} \right) T_k \right] \left[\frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)} \right. \\
& \left. + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik}}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik + 1\right)} \right].
\end{aligned}$$

Therefore, we have

$$d(\mathcal{F}a, \mathcal{F}b) \leq M_k d(a, b), \quad t \in (T_{k-1}, T_k].$$

For any $a \in \Delta$, since g, h, a are bounded, there is $d(\mathcal{F}a, a) < \infty$. Based on (9), we can conclude that $\mathcal{F}$ is a strictly compact operator. According to Theorem 1(i), a continuous function v_{k0} exists, which satisfies that $\mathcal{F}^n a \rightarrow v_{k0}$ when $n \rightarrow \infty$, and $\mathcal{F}v_{k0} = v_{k0}$.

Since a, v_{k0} are bounded, there is a constant $\omega_{ka v_{k0}} \in [0, \infty)$ such that $d(a, v_{k0}) < \omega_{ka v_{k0}} < \infty$. Then, $\{a \in \Delta | d(a, v_{k0}) < \infty\}$ is equivalent to Δ . Then, by Theorem 1(ii), v_{k0} is the only fixed point.

According to $v_k(t)$ satisfying the inequality (7), it follows that

$$\left| v_k(t) - I^{q_k; \varphi} \left(g(t, v_k(t), {}^H D_{0+}^{q_k, \beta; \varphi} v_k(t)) + \int_0^t h(t, \tau, v_k(\tau)) ds \right) \right| \leq \varepsilon \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)},$$

i.e.,

$$d(v_k, I^{q_k; \varphi} v_k) \leq \varepsilon \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)}.$$

Then,

$$d(v_k, \mathcal{F}v_k) \leq d(v_k, I^{q_k; \varphi} v_k) \leq \varepsilon \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)}.$$

By Theorem 1(iii), we obtain

$$|v_k(t) - v_{k0}(t)| \leq \frac{1}{1 - M_k} d(\mathcal{F}v_k, v_k) \leq \varepsilon \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{(1 - M_k) \Gamma(q_k + 1)}.$$

Thus, Equation (2) is Ulam–Hyers stable. $\square$

In summary, when $t \in (T_{k-1}, T_k]$, the solution of the inequality (7) is v_k and the unique solution of Equation (2) is v_{k0} . Then, the solution of inequality (7) on $t \in J$ can be expressed as follows:

$$v(t) = \begin{cases} v_1, t \in (T_0, T_1], \\ v_2, t \in (T_1, T_2], \\ \dots, \\ v_n, t \in (T_{n-1}, T_n]. \end{cases}$$

The unique solution of Equation (1) on $t \in J$ can be expressed as follows:

$$v_0(t) = \begin{cases} v_{10}, t \in (T_0, T_1], \\ v_{20}, t \in (T_1, T_2], \\ \dots, \\ v_{n0}, t \in (T_{n-1}, T_n]. \end{cases}$$

Let $\lambda = \min \left\{ \frac{(\varphi(T_1) - \varphi(0))^{q_1}}{(1-M_1)\Gamma(q_1+1)}, \dots, \frac{(\varphi(T_n) - \varphi(T_{n-1}))^{q_n}}{(1-M_n)\Gamma(q_n+1)} \right\}$; then, we can obtain

$$|v(t) - v_0(t)| \leq \lambda \varepsilon.$$

That is, Equation (1) is Ulam–Hyers stable.

In order to obtain the Ulam–Hyers–Rassias stability of Equation (2), the following condition is assumed to hold.

(H₄) There exist the constant ρ_k and monotonically nondecreasing function $\xi_k(t) \in C((T_{k-1}, T_k], \mathbb{R})$ satisfying

$$I_{0+}^{q_k; \varphi} \xi_k(t) \leq \rho_k \xi_k(t), t \in (T_{k-1}, T_k].$$

Theorem 4. Assume that the conditions (H₁), (H₂), (H₃), and (H₄) are satisfied, and $\varphi(t)$ is an increasing function with $\varphi'(t) \neq 0$. If there exists a continuously differentiable function $v_k(t)$ such that the inequality (8) holds, and

$$\begin{aligned} N_k = & \rho_k \xi_k(t) \left[\left(L_g + \frac{L_g}{1 - L_g} \right) + \left(L_h + \frac{L_h}{1 - L_g} \right) T_k \right] \left[\frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)} \right. \\ & \left. + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{j-k-i}}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik + 1\right)} \right] < 1, \end{aligned} \quad (13)$$

then a unique continuous solution $v_{k0}(t)$ exists such that

$$|v_k(t) - v_{k0}(t)| \leq \varepsilon \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{(1 - N_k)\Gamma(q_k + 1)} \xi(t),$$

i.e., Equation (2) is Ulam–Hyers–Rassias stable.

Proof. Similar to Theorem 3, we define a generalized metric on Δ as follows:

$$d(a, b) = \inf \{ \omega \in [0, \infty) \mid |a(t) - b(t)| \leq \omega \xi_k(t), t \in (T_{k-1}, T_k] \}.$$

Then, (Δ, d) is considered to be a generalized complete metric space.

For any $a, b \in \Delta$, there exists $\omega_{kab} \in [0, \infty)$ such that $d(a, b) = |a(t) - b(t)| \leq \omega_{kab} \xi_k(t)$; then, when k is a even number, we have

$$|\mathcal{F}a(t) - \mathcal{F}b(t)|$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} \left| g(z, a(z), {}^H D_{0+}^{q_k, \beta; \varphi} a(z)) - g(z, b(z), {}^H D_{0+}^{q_k, \beta; \varphi} b(z)) \right| dz \\
&\quad + \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} \int_0^z |h(t, \tau, a(\tau)) - h(t, \tau, b(\tau))| d\tau dz \\
&\quad + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \left\{ \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \right. \\
&\quad \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik - 1} \\
&\quad \times \left| g(T_{k-1}, a(T_{k-1}), {}^H D^{q_k, \beta; \varphi} a(T_{k-1})) - g(T_{k-1}, b(T_{k-1}), {}^H D^{q_k, \beta; \varphi} b(T_{k-1})) \right| dz \\
&\quad + \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \\
&\quad \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik - 1} \\
&\quad \times \int_0^{T_{k-1}} |h(T_{k-1}, \tau, a(\tau)) - h(z, \tau, b(\tau))| d\tau dz \Big\} \\
&\leq \left(L_g + \frac{L_g}{1 - L_g} \right) \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} |a(z) - b(z)| dz \\
&\quad + \left(L_g + \frac{L_h}{1 - L_g} \right) \frac{1}{\Gamma(q_k)} \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{q_k-1} \int_0^z |a(\tau) - b(\tau)| d\tau dz \\
&\quad + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \left\{ \left(L_g + \frac{L_g}{1 - L_g} \right) \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \right. \\
&\quad \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik - 1} |a(T_{k-1}) - b(T_{k-1})| dz \\
&\quad + \left(L_g + \frac{L_h}{1 - L_g} \right) \frac{1}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik\right)} \\
&\quad \times \int_{T_{k-1}}^t \varphi'(z)(\varphi(t) - \varphi(z))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik - 1} \int_0^{T_{k-1}} |a(\tau) - b(\tau)| d\tau dz \Big\}.
\end{aligned}$$

According to the condition (H_4) , there is

$$\begin{aligned}
|\mathcal{F}a(t) - \mathcal{F}b(t)| &\leq \rho_k \omega_{kab} \xi_k(t) \left[\left(L_g + \frac{L_g}{1 - L_g} \right) + \left(L_h + \frac{L_h}{1 - L_g} \right) T_k \right] \left[\frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)} \right. \\
&\quad \left. + \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik}}{\Gamma\left(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik + 1\right)} \right].
\end{aligned}$$

Denoting

$$N_k = \rho_k \xi_k(t) \left[\left(L_g + \frac{L_g}{1 - L_g} \right) + \left(L_h + \frac{L_h}{1 - L_g} \right) T_k \right] \left[\frac{(\varphi(T_k) - \varphi(T_{k-1}))^{q_k}}{\Gamma(q_k + 1)} \right]$$

$$+ \sum_{i=1}^{k-1} \left(\sum_{k=1}^n c_k \right)^i \frac{(\varphi(T_k) - \varphi(T_{k-1}))^{\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik}}{\Gamma(\sum_{j=k-i}^{k-1} q_j - q_k + \sum_{j=k-i}^k \mu_j - in + 2ik + 1)} \Big].$$

Therefore, there is

$$d(\mathcal{F}a, \mathcal{F}b) \leq N_k d(a, b), t \in (T_{k-1}, T_k].$$

For any $a \in \Delta$, since g, h, a are bounded, there exists ω_{ka} such that

$$|\mathcal{F}a - a| \leq \omega_{ka} \zeta_k(t),$$

i.e., $d(\mathcal{F}a, a) < \infty$. Based on (13), we can conclude that $\mathcal{F}$ is a strictly compact operator. According to Theorem 1(i), a continuous function v_{k0} exists, which satisfies that $\mathcal{F}^n a \rightarrow v_{k0}$ when $n \rightarrow \infty$, and $\mathcal{F}v_{k0} = v_{k0}$.

Since a, v_{k0} are bounded, and $\min \zeta_k(t) > 0$, there is a constant $\omega_{kav_{k0}} \in [0, \infty)$ such that

$$|\mathcal{F}a - v_{k0}| \leq \omega_{kav_{k0}} \zeta_k(t),$$

i.e., $d(a, v_{k0}) < \omega_{kav_{k0}} < \infty$. Then, $\{a \in \Delta | d(a, v_{k0}) < \infty\}$ is equivalent to Δ . Then, by Theorem 1(ii), we know that v_{k0} is the only fixed point.

By $v_k(t)$ satisfying the inequality (8), it follows that

$$|v_k(t) - I_{0+}^{q_k; \varphi} \left(g(t, v_k(t), {}^H D_{0+}^{q_k; \beta; \varphi} v_k(t)) + \int_0^t h(t, \tau, v_k(\tau)) d\tau \right)| \leq \varepsilon I_{0+}^{q_k; \varphi} \zeta_k(t),$$

i.e.,

$$d(v_k, I_{0+}^{q_k; \varphi} v_k) \leq \varepsilon I_{0+}^{q_k; \varphi} \zeta_k(t).$$

Then,

$$d(v_k, \mathcal{F}v_k) \leq d(v_k, I_{0+}^{q_k; \varphi} v_k) \leq \varepsilon I_{0+}^{q_k; \varphi} \zeta_k(t).$$

According to Theorem 1(iii), we obtain

$$|v_k(t) - v_{k0}(t)| \leq \frac{1}{1 - N_k} d(\mathcal{F}v_k, v_k) \leq \varepsilon \frac{\rho_k}{1 - N_k} \zeta_k(t).$$

Then, Equation (2) is Ulam–Hyers–Rassias stable. $\square$

For Equation (1), let $\zeta(t) = \min \{ \zeta_1(t), \dots, \zeta_n(t) \}$, $\delta = \min \left\{ \frac{\rho_1}{1 - N_1}, \dots, \frac{\rho_n}{1 - N_n} \right\}$; then, there is

$$|v(t) - v_0(t)| \leq \delta \varepsilon \zeta(t).$$

That is, Equation (1) is Ulam–Hyers–Rassias stable.

5. Conclusions

The paper also examines the equivalent integral expressions of high-order variable-order φ -Hilfer fractional implicit integro-differential equations under certain conditions, extending the study to higher orders. Moreover, using the piecewise constant approximation method, it investigates the criteria for Ulam–Hyers and Ulam–Hyers–Rassias stability of variable-order φ -Hilfer fractional implicit integro-differential equations. This enhancement advances the framework of high-order φ -Hilfer fractional differential equations, enriching the qualitative and stability analyses of its solutions. In addition, it considers the existence and stability of solutions to multiterm Hilfer fractional differential equations.

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