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Extensions of Bicomplex Hypergeometric Functions and Riemann–Liouville Fractional Calculus in Bicomplex Numbers

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Abstract: In this paper, we present novel advancements in the theory of bicomplex hypergeometric functions and their applications. We extend the hypergeometric function to bicomplex parameters, analyse its convergence region, and define its integral and derivative representations. Furthermore, we delve into the k -Riemann–Liouville fractional integral and derivative within a bicomplex operator, proving several significant theorems. The developed bicomplex hypergeometric functions and bicomplex fractional operators are demonstrated to have practical applications in various fields. This paper also highlights the essential concepts and properties of bicomplex numbers, special functions, and fractional calculus. Our results enhance the overall comprehension and possible applications of bicomplex numbers in mathematical analysis and applied sciences, providing a solid foundation for future research in this field.

Keywords: bicomplex hypergeometric functions; bicomplex gamma and beta functions; fractional Cauchy–Riemann operator; fractional bicomplex functions

MSC: 30G35; 33B15; 33C05; 26A33; 33Cxx



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1. Introduction

The bicomplex number system combines complex numbers with imaginary units and complex coefficients. Segre first introduced bicomplex numbers in 1892 [1]. He described an infinite set of elements that were later named bicomplex and tricomplex numbers, and more broadly, n -complex numbers.

Bicomplex numbers are four-dimensional vectors that can be recognized similarly to complex numbers, which are two-dimensional vectors. Bicomplex numbers were initially developed using pairs of complex numbers. Their theory finds applications in various fields of mathematics and physics, such as electromagnetic theory, fluid dynamics, and special relativity.

Recently, several researchers [2–7] have developed the foundational elements of a function theory for bicomplex numbers. They also introduced the idempotent representation of bicomplex numbers, which is crucial and effective for representing bicomplex numbers. Any bicomplex number has a unique identity. The idempotent representation is often employed in proofs because it facilitates term-by-term addition, multiplication, and division.

The topic of generalized special functions, which are closely related to many other areas of analysis, has made significant strides recently. Any type of generalized function

is important because many special functions are just instances of them. As a result, every recurrence formula created for the generalized function serves as a master formula that can be used to derive a vast number of relations for other functions. In this method, new relations for a few special functions have been found.

Although it is not necessary for every mathematician or physicist to be knowledgeable in every known special function, it is beneficial to have a broad background that enables the identification of special functions, which may subsequently be examined in greater detail if needed.

Special functions are often defined through integrals. Examples of such functions with integral representations within the range of convergence include the sine, cosine, dilogarithm, exponential integrals, Riemann zeta functions, and gamma, beta, and polygamma functions. Detailed explanations of these functions are provided in [8,9].

The Legendre duplication formula, the Gauss multiplication theorem, and the binomial theorem are discussed, along with the extension of the gamma and beta functions to bicomplex variables. This might have given rise to a vital instrument for the development of the bicomplex special functions theory [10]. The k -generalized gamma, beta function, and k -Pochhammer symbol with multiple identities in complex numbers were also introduced by Rafael, Eddy, and Wu-Sheng in 2007 [11]. Recently, some authors have studied k -gamma, k -beta, and k -Pochhammer symbols in bicomplex parameters (see [12]).

In recent decades, function theory has seen a resurgence of interest due to the research on hypergeometric functions. This is demonstrated by the nearly two thousand publications included in mathematics reviews in the last ten years alone on the subject of hypergeometric functions, a significant class of special functions. The hypergeometric function is crucial in mathematical analysis and its applications. Many academics have examined generalizations and extensions of various k -symbols of special functions and k -fractional derivatives in addition to the hypergeometric function [11,13–16].

Coloma [17] developed fractional bicomplex calculus in the Riemann–Liouville sense by utilizing the one-dimensional Riemann–Liouville derivative in each direction of the bicomplex basis and incorporating elementary functions such as analytic polynomials, exponentials, and trigonometric functions, along with some of their properties. This indicates that fractional calculus is not a new subject; its history is nearly as old as that of classical calculus. In its early years, many mathematicians had a low opinion of fractional operators, and it can be argued that pure mathematicians contributed more to this field than applied mathematicians at that time. In recent decades, engineers, mathematicians, and applied scientists have recognized that differential equations involving fractional operators provide a clear framework for addressing issues related to numerous real-world scenarios. These include applications in viscoelastic systems, signal processing, diffusion processes, control systems, fractional stochastic systems, and biological and ecological allometry.

In this article, we extend the bicomplex hypergeometric function and discuss the convergence region of this function. We also define its integral and derivative representations. Next, we present the k -Riemann–Liouville fractional integral and derivative in a bicomplex operator, and we prove some important theorems. Furthermore, we demonstrate the application of the k -Riemann–Liouville fractional operator to the k -bicomplex hypergeometric function, deriving important results that underline the potential of these mathematical tools in addressing complex problems across various fields.

This paper is structured as follows: The essential ideas and characteristics of the bicomplex number, along with some definitions of special functions and an introduction to the Riemann–Liouville fractional operator, are gathered in Section 2. In Section 3, we discuss the extension of the hypergeometric function to bicomplex parameters and its representation. We also study the convergence region and provide some properties of the integral and derivative representations of the bicomplex k -hypergeometric function. In Section 4, we present k -Riemann–Liouville fractional integration and differentiation in a bicomplex operator. In Section 5, we explore the application of the k -Riemann–Liouville

fractional operator to the k -bicomplex hypergeometric function. The concluding remarks are presented in Section 6.

2. Preliminaries

In this section, we introduce key definitions and terminology used to establish the main results.

2.1. The Bicomplex Numbers

A set of bicomplex numbers $\mathbb{BC}$ which arise from the work of Segre is defined as (see [1,5–7]):

$$\mathbb{BC} := \{Y = v_1 + jv_2, v_1, v_2 \in \mathbb{C}\}, \quad (1)$$

where $v_1 = a_1 + ib_1$, $v_2 = a_2 + ib_2$, and i, j are independent imaginary units defined as

$$ij = ji = k, i^2 = -1 = j^2.$$

By using imaginary units i and j , we have subsets of the set of bicomplex numbers as:

$$\mathbb{C}(i) = \{v = a_1 + ib_1 : a_1, b_1 \in \mathbb{R}\}, \quad (2)$$

$$\mathbb{C}(j) = \{v = a_1 + jb_1 : a_1, b_1 \in \mathbb{R}\}, \quad (3)$$

$$\mathbb{D} = \{Y = a_1 + ij b_1 : a_1, b_1 \in \mathbb{R}\}, \quad (4)$$

where $\mathbb{D}$ is the set of hyperbolic numbers (see [18]), and $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are fields of complex numbers.

Addition and multiplication of bicomplex numbers

If $Y = v_1 + jv_2$, $\delta = \delta_1 + j\delta_2$, then we obtain

$$Y + \delta = (v_1 + \delta_1) + j(v_2 + \delta_2),$$

$$Y \otimes \delta = (v_1\delta_1 - v_2\delta_2) + j(v_1\delta_2 + v_2\delta_1).$$

Zero divisors

If $Y = v_1 + jv_2 \neq 0$, then Y is called a zero divisor if both v_1 and v_2 are nonzero. This implies that all zero divisors $Y = v_1 + jv_2$ in $\mathbb{BC}$ are characterized by the equations $v_1^2 = -v_2^2$ i.e., $v_1 = \pm v_2$. Thus, all zero divisors are of the form $Y = \xi(1 \pm ij)$ for any $\xi \in \mathbb{C} \setminus \{0\}$. Thus the set of all zero divisors in $\mathbb{B}$ is said to be null-cone $\mathcal{O}_2$ and defined as (see [19]):

$$\mathcal{O}_2 = \{Y = v_1 + jv_2 : v_1 = \pm iv_2, v_1^2 + v_2^2 = 0\}. \quad (5)$$

Idempotent representation

The two zero-divisor idempotent elements, denoted by $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$, have the properties (see [4–7]):

$$e_1 \cdot e_2 = 0,$$

$$e_2 \cdot e_1 = 0,$$

$$e_1^2 = e_1, e_2^2 = e_2,$$

$$e_1 + e_2 = 1, e_1 - e_2 = ij.$$

Then we can write bicomplex number as

$$Y = v_1 + jv_2 = \mu_1 e_1 + \mu_2 e_2, \quad (6)$$

where $\mu_1 = v_1 - iv_2$, $\mu_2 = v_1 + iv_2$.

Idempotent representations (6) simplify calculations with bicomplex numbers, transforming them into complex numbers. Due to the identities $e_1 \cdot e_2 = 0$ and $e_2 \cdot e_1 = 0$, we can write some important properties of idempotents as follows: If $Y = \mu_1 e_1 + \mu_2 e_2$ and $\delta = \delta_1 e_1 + \delta_2 e_2$, then (see [5,6,20])

- (i) $Y + \delta = (\mu_1 + \delta_1)e_1 + (\mu_2 + \delta_2)e_2$,
- (ii) $Y \otimes \delta = \mu_1 \delta_1 e_1 + \mu_2 \delta_2 e_2$,
- (iii) $Y^n = \mu_1^n e_1 + \mu_2^n e_2$,
- (iv) $e^Y = e^{\mu_1} e_1 + e^{\mu_2} e_2$,
- (v) $\frac{1}{Y} = \frac{1}{\mu_1 e_1 + \mu_2 e_2} = \frac{1}{\mu_1} e_1 + \frac{1}{\mu_2} e_2$.

Bicomplex derivative and integral representations

Let the bicomplex function $F : X \rightarrow \mathbb{BC}$ be such that $F(Y) = f_1(\mu_1)e_1 + f_2(\mu_2)e_2$. F is said to be differentiable at $Y_0 \in \mathbb{BC}$ if the following limit exists (see [5–7,20]):

$$\lim_{Y \rightarrow Y_0} \frac{F(Y) - F(Y_0)}{Y - Y_0}, \quad Y - Y_0 \notin \mathcal{O}_2, \quad (7)$$

or

$$\lim_{\Delta Y \rightarrow 0} \frac{F(Y + \Delta Y) - F(Y)}{\Delta Y}, \quad \Delta Y \notin \mathcal{O}_2, \quad (8)$$

yields a finite value. In this case, we write

$$F'(Y) = \lim_{Y \rightarrow Y_0} \frac{F(Y) - F(Y_0)}{Y - Y_0} = \lim_{\Delta Y \rightarrow 0} \frac{F(Y + \Delta Y) - F(Y)}{\Delta Y}. \quad (9)$$

Bicomplex integration of a bicomplex function is defined as a line integral along a four-dimensional curve $\mathbb{H}$ in $\mathbb{BC}$. In particular, it is expressed as (see [5,6,20]):

$$\int_{\mathbb{H}} F(Y) \otimes dY = \int_r^s F(Y(t)) \otimes Y'(t) dt, \quad dY = (d\mu_1, d\mu_2), \quad (10)$$

where $\mathbb{H}$ has the parametric form

$$\mathbb{H} : Y(t) = (v_1(t), v_2(t)), \quad \text{for } r, s \in \mathbb{R}, \text{ such that } r \leq t \leq s.$$

$\mathbb{H}$ can be interpreted as a curve formed by two component curves h_1 and h_2 in $\mathbb{C}$, specifically $\mathbb{H} = (h_1, h_2)$. Thus,

$$\int_{\mathbb{H}} F(Y) \otimes dY = \left\{ \int_{h_1} F(\mu_1) d\mu_1 \right\} e_1 + \left\{ \int_{h_2} F(\mu_2) d\mu_2 \right\} e_2. \quad (11)$$

Bicomplex holomorphic function

We will consider $\mathbb{BC}$ as a topological space endowed with the Euclidean topology of $\mathbb{R}^4$ and X an open set. Let F be a bicomplex function of a bicomplex variable defined on a non-empty open set $X \subset \mathbb{BC}$, $F : X \rightarrow \mathbb{BC}$ is considered bicomplex holomorphic if it has a derivative at each point of X (see [3]). This is equivalent to saying that the complex functions f_1 and f_2 are holomorphic in the variables v_1 and v_2 with $Y = v_1 + jv_2$ and satisfy the Cauchy–Riemann system:

$$\frac{\partial F}{\partial \bar{Y}} = \frac{\partial F}{\partial Y^+} = \frac{\partial F}{\partial Y^*} = 0, \quad (12)$$

where $\frac{\partial}{\partial \bar{Y}}$, $\frac{\partial}{\partial Y^\dagger}$, and $\frac{\partial}{\partial Y^*}$ means the partial derivatives with regard to the bar-conjugation, the $\dagger$ -conjugation, and the $*$ -conjugation, respectively (see [3], page 8).

It is straightforward to verify that the pair (f_1, f_2) of complex holomorphic functions must satisfy the following bicomplex Cauchy–Riemann equations, in order for the bicomplex derivative of $F = f_1 + jf_2$ to exist:

$$\frac{\partial f_1}{\partial v_1} = \frac{\partial f_2}{\partial v_2} \quad \text{and} \quad \frac{\partial f_1}{\partial v_2} = -\frac{\partial f_2}{\partial v_1}. \quad (13)$$

We state that F is either bicomplex holomorphic or bicomplex differentiable in this situation. The partial derivatives

$$\frac{\partial F}{\partial Y} = \frac{1}{2} \left(\frac{\partial F}{\partial v_1} - j \frac{\partial F}{\partial v_2} \right), \quad \frac{\partial F}{\partial Y^\dagger} = \frac{1}{2} \left(\frac{\partial F}{\partial v_1} + j \frac{\partial F}{\partial v_2} \right) \quad (14)$$

give us the Cauchy–Riemann equation $\frac{\partial F}{\partial Y^\dagger} = 0$ in bicomplex notation. If $F : X \rightarrow \mathbb{BC}$ is a holomorphic function, then

$$F(Y) = f_1(\mu_1)e_1 + f_2(\mu_2)e_2, \quad (15)$$

where $Y \in X$, $\mu_1 \in X_1$, $\mu_2 \in X_2$, and f_1 and f_2 are holomorphic functions of a complex variable in X_1 and X_2 , respectively. For $Y = v_1 + jv_2 \equiv (v_1, v_2)$, we consider a bicomplex function

$$F(Y) = F(v_1, v_2) = f_1(v_1, v_2) + jf_2(v_1, v_2) \equiv (f_1(v_1, v_2), f_2(v_1, v_2)). \quad (16)$$

2.2. Special Functions

Previously, the gamma and beta functions have been defined, respectively, as follows (see [9]):

$$\Gamma(a) = \int_0^\infty e^{-y} y^{a-1} dy, \quad (17)$$

which is valid for $a > 0$.

$$\beta(a, b) = \int_0^1 y^{a-1} (1-y)^{b-1} dy, \quad (18)$$

which is valid for $a > 0$ and $b > 0$.

Several years later, Eddy, Rafael, and Wu-Sheng defined the k -gamma and k -beta functions (see [11,21]), starting with the k -Pochhammer symbol,

$$(v)_{r,k} = v(v+k)(v+2k) \cdots (v+(r-1)k), \quad (19)$$

where $v \in \mathbb{C}$ and $k > 0$.

They defined the k -gamma and k -beta functions as

$$\Gamma_k(v) = \int_0^\infty e^{-\frac{y^k}{k}} y^{v-1} dy, \quad (20)$$

where $k > 0$, $v \in \mathbb{C}$ such that $\text{Re}(v) > 0$.

$$\beta_k(v, t) = \frac{1}{k} \int_0^1 y^{\frac{v}{k}-1} (1-y)^{\frac{t}{k}-1} dy, \quad (21)$$

$$\beta_k(v, t) = \frac{\Gamma_k(v)\Gamma_k(t)}{\Gamma_k(v+t)}, \quad (22)$$

where $k > 0$, and $v, t \in \mathbb{C}$ such that $\operatorname{Re}(v) > 0$ and $\operatorname{Re}(t) > 0$.

Some mathematicians have defined the hypergeometric function (see [8,9,22,23]) as follows:

$$\Psi(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (23)$$

where $a, b, c \in \mathbb{C}$; c can be neither a zero nor a negative number.

In 2007, Diaz and Pariguan defined the k -hypergeometric function (see [11,24]):

$$\Psi_k(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_{n,k} (b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!}, \quad (24)$$

where $k \in \mathbb{R}^+$, and $a, b, c \in \mathbb{C}$; c can be neither a zero nor a negative number.

Goyal and Mathur in 2006 defined the gamma and beta function in bicomplex numbers. The integral form of the bicomplex gamma function is denoted by (see [10])

$$\Gamma_2(Y) = \int_{\mathbb{H}} e^{-\zeta} \otimes \zeta^{Y-1} \otimes d\zeta, \quad (25)$$

where $Y, \zeta \in \mathbb{BC}$, $Y = \mu_1 e_1 + \mu_2 e_2$, $\zeta = \zeta_1 e_1 + \zeta_2 e_2$, $\zeta_1, \zeta_2 \in \mathbb{R}^+$, and $\mathbb{H} = (h_1, h_2)$, with $h_1 \equiv h_1(\zeta_1)$ and $h_2 \equiv h_2(\zeta_2)$.

The definition in (25) can be written as:

$$\begin{aligned} \int_{\mathbb{H}} e^{-Y} \otimes \zeta^{Y-1} \otimes d\zeta &= \left\{ \int_0^\infty e^{-\zeta_1} \zeta_1^{\mu_1-1} d\zeta_1 \right\} e_1 + \left\{ \int_0^\infty e^{-\zeta_2} \zeta_2^{\mu_2-1} d\zeta_2 \right\} e_2 \\ \implies \Gamma_2(Y) &= \Gamma(\mu_1) e_1 + \Gamma(\mu_2) e_2. \end{aligned} \quad (26)$$

The Pochhammer symbol for a bicomplex number Y is denoted by (see [10]):

$$(Y)_r = Y \otimes (Y+1) \otimes (Y+2) \otimes (Y+3) \otimes \cdots \otimes (Y+r-1) \quad (27)$$

The bicomplex beta function is defined as follows (see [10]):

$$\beta_2(Y, \delta) = \int_{\mathbb{H}} \zeta^{Y-1} \otimes (1-\zeta)^{\delta-1} \otimes d\zeta, \quad (28)$$

$$\beta_2(Y, \delta) = \frac{\Gamma_2(Y) \otimes \Gamma_2(\delta)}{\Gamma_2(Y+\delta)}. \quad (29)$$

where $Y, \delta, \zeta \in \mathbb{BC}$, $\zeta = \zeta_1 e_1 + \zeta_2 e_2$, $\zeta_1, \zeta_2 \in \mathbb{R}^+$, and $\mathbb{H} = (h_1, h_2)$ with $h_1 \equiv h_1(\zeta_1)$ and $h_2 \equiv h_2(\zeta_2)$.

Recently, there have been many developments and applications in special functions. In 2024, some researchers extended the bicomplex gamma and beta functions (see [12]), starting with the definition of the k -bicomplex gamma function:

$$\Gamma_{2,k}(Y) = \int_{\mathbb{H}} e^{-\frac{\zeta^k}{k}} \otimes \zeta^{Y-1} \otimes d\zeta, \quad \operatorname{Re}(\mu_1) > 0, \operatorname{Re}(\mu_2) > 0. \quad (30)$$

where $Y, \zeta \in \mathbb{BC}$, $Y = \mu_1 e_1 + \mu_2 e_2$, $\zeta = \zeta_1 e_1 + \zeta_2 e_2$, $\zeta_1, \zeta_2 \in \mathbb{R}^+$, and $\mathbb{H} = (h_1, h_2)$ with $h_1 \equiv h_1(\zeta_1)$ and $h_2 \equiv h_2(\zeta_2)$.

The definition in Equation (30), can be written as:

$$\begin{aligned}\Gamma_{2,k}(Y) &= \left\{ \int_0^\infty e^{-(\zeta_1^k/k)} \zeta_1^{\mu_1-1} d\zeta_1 \right\} e_1 + \left\{ \int_0^\infty e^{-(\zeta_2^k/k)} \zeta_2^{\mu_2-1} d\zeta_2 \right\} e_2 \\ &= \Gamma_k(\mu_1) e_1 + \Gamma_k(\mu_2) e_2, \text{ which } \Gamma_k \text{ defined in (20).}\end{aligned}\quad (31)$$

Then, they defined the k -Pochhammer symbol for a bicomplex number as follows (see [12]):

$$\begin{aligned}(Y)_{r,k} &= Y \otimes (Y+k) \otimes (Y+2k) \otimes (Y+3k) \otimes \cdots \otimes (Y+(r-1)k) \\ &= (\mu_1)_{r,k} e_1 + (\mu_2)_{r,k} e_2.\end{aligned}\quad (32)$$

After that, they defined the k -bicomplex beta function as follows:

$$\begin{aligned}\beta_{2,k}(Y, \delta) &= \frac{\Gamma_{2,k}(Y) \otimes \Gamma_{2,k}(\delta)}{\Gamma_{2,k}(Y+\delta)} \\ &= \frac{1}{k} \int_{\mathbb{H}} \zeta^{\frac{Y}{k}-1} \otimes (1-\zeta)^{\frac{\delta}{k}-1} \otimes d\zeta \\ &= \left\{ \frac{1}{k} \int_0^1 \zeta_1^{\frac{\mu_1}{k}-1} (1-\zeta_1)^{\frac{\delta_1}{k}-1} d\zeta_1 \right\} e_1 + \left\{ \frac{1}{k} \int_0^1 \zeta_2^{\frac{\mu_2}{k}-1} (1-\zeta_2)^{\frac{\delta_2}{k}-1} d\zeta_2 \right\} e_2.\end{aligned}\quad (33)$$

where $k \in \mathbb{R}^+$, $Y, \delta, \zeta \in \mathbb{BC}$, $Y = \mu_1 e_1 + \mu_2 e_2$, $\delta = \delta_1 e_1 + \delta_2 e_2$ with $\text{Re}(\mu_1) > |\text{Im}(\mu_2)|$ and $\text{Re}(\delta_1) > |\text{Im}(\delta_2)|$, $\zeta = \zeta_1 e_1 + \zeta_2 e_2$, and $\zeta_1, \zeta_2 \in [0, 1]$.

2.3. Riemann–Liouville Fractional Operator

In 1847, Bernhard Riemann introduced the definition of a fractional integral. Later, in 2000, Rudolf and Hilfer generalized this formula to define the right- and left-sided fractional Riemann–Liouville integrals of order $q > 0$ (see [25]).

The right-sided fractional Riemann–Liouville integral of order q is defined as:

$$I_{a+}^q F(v) = \frac{1}{\Gamma(q)} \int_a^v (v-y)^{q-1} F(y) dy, \quad v > a, \quad (34)$$

and the left-sided fractional Riemann–Liouville integral of order q is defined as:

$$I_{b-}^q F(v) = \frac{1}{\Gamma(q)} \int_v^b (y-v)^{q-1} F(y) dy, \quad v < a. \quad (35)$$

Using (34) and (35), we obtain the following Riemann–Liouville fractional derivative of order q , defined by Kilbas (see [26]):

The left-handed Riemann–Liouville fractional derivative of order q is given by:

$$D_{a+}^q F(v) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dv^n} \int_a^v (v-y)^{n-q-1} F(y) dy, \quad v > a, \quad (36)$$

and the right-handed Riemann–Liouville fractional derivative of order q is given by:

$$D_{b-}^q F(v) = -\frac{1}{\Gamma(n-q)} \frac{d^n}{dv^n} \int_v^b (y-v)^{n-q-1} F(y) dy, \quad v < a, \quad (37)$$

where $n-1 < q < n$ and $v \in [a, b]$. The above definitions have been defined using these formulas:

$$(D_{a+}^q F)(v) := \left(\frac{d}{dv} \right)^n (I_{a+}^{n-q} F)(v), \quad (38)$$

$$\left(D_{b-}^q F\right)(v):=\left(-\frac{d}{d v}\right)^n\left(I_{b-}^{n-q} F\right)(v) . \quad (39)$$

In 2012, S. Mubeen defined the k -Riemann–Liouville fractional integral as follows (see [27–29]):

$$I_{k, a}^{\alpha} F(z)=\frac{1}{k \Gamma_k(\alpha)} \int_a^z(z-\tau)^{\frac{\alpha}{k}-1} F(\tau) d \tau, \quad z \in \mathbb{R}^+, \quad \alpha \in \mathbb{C} \text { with } \operatorname{Re}(\alpha)>0, \quad (40)$$

and the k -Riemann–Liouville fractional derivative is given by:

$$\left(\mathcal{D}_{k, a+}^{\alpha} F\right)(z)=\frac{1}{k \Gamma_k(m-\alpha)} \frac{d^m}{d z^m} \int_a^z \frac{F(\tau)}{(\tau-z)^{\frac{\alpha-m}{k}+1}} d \tau, \quad m-1<\alpha<m, \quad (41)$$

$$\left(\mathcal{D}_{k, b-}^{\alpha} F\right)(z)=-\frac{1}{k \Gamma_k(m-\alpha)} \frac{d^m}{d z^m} \int_z^b \frac{F(\tau)}{(\tau-z)^{\frac{\alpha-m}{k}+1}} d \tau, \quad m-1<\alpha<m. \quad (42)$$

Mahesh and Kumar defined the Riemann–Liouville fractional integral and derivative of bicomplex order Y in two cases (see [19,30]).

Firstly, the integral Riemann–Liouville fractional form is:

$${}_0 I_{\delta}^Y F(\delta)=\frac{1}{\Gamma_2(Y)} \otimes \int_0^{\delta}(\delta-\zeta)^{Y-1} \otimes F(\zeta) \otimes d \zeta, \quad \operatorname{Re}\left(v_1\right)>|\operatorname{Im}\left(v_2\right)|. \quad (43)$$

Secondly, the derivative Riemann–Liouville fractional form is:

$${}_0 D_{\delta}^Y F(\delta)=\left(\frac{d}{d \delta}\right)^m\left({}_0 I_{\delta}^{m-Y} F\right)(\delta)=\frac{1}{\Gamma_2(m-Y)} \otimes \frac{d^m}{d \delta^m} \int_0^{\delta}(\delta-\zeta)^{m-Y-1} \otimes F(\zeta) \otimes d \zeta, \quad (44)$$

where F is a bicomplex function that is continuous on $I'=(0, \infty)$ and integrable on any finite subinterval of $I=[0, \infty)$, and $Y=v_1+j v_2$, $v_1, v_2 \in \mathbb{C}$ with $\operatorname{Re}\left(v_1\right)>|\operatorname{Im}\left(v_2\right)|$.

3. k -Bicomplex Hypergeometric Function

In this section, we discuss the extension of the hypergeometric function in $\mathbb{BC}$ and introduce some basic concepts related to these functions. Additionally, we determine the convergence region of the k -hypergeometric function with some corollaries and define its integral and derivative representations.

Theorem 1. If M, N, L , and $Y \in \mathbb{BC}$, $M=x_1+j x_2=m_1 e_1+m_2 e_2$, $N=y_1+j y_2=n_1 e_1+n_2 e_2$, $L=z_1+j z_2=l_1 e_1+l_2 e_2$, $Y=v_1+j v_2=\mu_1 e_1+\mu_2 e_2$, then the bicomplex k -hypergeometric function gives

$$\Psi_k(M, N ; L ; Y)=\sum_{s=0}^{\infty} \frac{(M)_{s, k} \otimes(N)_{s, k}}{(L)_{s, k}} \otimes \frac{Y^s}{s!}, k \in \mathbb{R}^+, \quad (45)$$

where $v_1, v_2, x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{C}$, and l_1 and l_2 are neither zeroes nor negative integers.

Proof. Let us employ the duplication of the bicomplex k -hypergeometric function using the idempotent elements associated with the hyperbolic units e_1 and e_2 . For M, N, L, Y as bicomplex numbers, we have

$$\begin{aligned}
\Psi_k(M, N; L; Y) &= \{\psi_k(m_1, n_1; l_1; \mu_1)\}e_1 + \{\psi_k(m_2, n_2; l_2; \mu_2)\}e_2 \\
&= \left\{ \sum_{s=0}^{\infty} \frac{(m_1)_{s,k} (n_1)_{s,k}}{(l_1)_{s,k}} \frac{\mu_1^s}{s!} \right\} e_1 + \left\{ \sum_{s=0}^{\infty} \frac{(m_2)_{s,k} (n_2)_{s,k}}{(l_2)_{s,k}} \frac{\mu_2^s}{s!} \right\} e_2 \\
&= \sum_{s=0}^{\infty} \frac{[(m_1)_{s,k} e_1 + (m_2)_{s,k} e_2] [(n_1)_{s,k} e_1 + (n_2)_{s,k} e_2] [\mu_1^s e_1 + \mu_2^s e_2]}{[(l_1)_{s,k} e_1 + (l_2)_{s,k} e_2] s!}.
\end{aligned}$$

Using the definition of the k -Pochhammer symbol in Equation (32), we obtain

$$\Psi_k(M, N; L; Y) = \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!}, k \in \mathbb{R}^+.$$

This completes the proof. $\square$

We can also rewrite the bicomplex k -hypergeometric function as

$$\Psi_k(M, N; L; Y) = \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!} = \sum_{s=0}^{\infty} \sigma_{s,k} \otimes Y^s,$$

where

$$\sigma_{s,k} = \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} \otimes s!} = \sigma_{1,s,k} e_1 + \sigma_{2,s,k} e_2,$$

and

$$\sigma_{1,s,k} = \frac{(m_1)_{s,k} (n_1)_{s,k}}{(l_1)_{s,k} s!}, \quad \sigma_{2,s,k} = \frac{(m_2)_{s,k} (n_2)_{s,k}}{(l_2)_{s,k} s!}.$$

Next, we discuss the convergence region for this series.

Corollary 1. The series $\Psi_k(M, N; L; Y) = \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!}$ is absolutely hyperbolically convergent within the ball $B_D(0, \frac{1}{k}) = \{Y : |Y|_D <_D \frac{1}{k}\}$, and diverges outside of its closure, which $|Y|_D$ refers to the modulus of hyperbolic-valued [3], and $<_D \frac{1}{k}$ means that the hyperbolic number is less than $\frac{1}{k}$.

Proof. Let $Y = \mu_1 e_1 + \mu_2 e_2$, $L = l_1 e_1 + l_2 e_2$, l_1 and l_2 are neither zeroes nor negative integers, $M = m_1 e_1 + m_2 e_2$, and $N = n_1 e_1 + n_2 e_2$, where $\mu_1, \mu_2, l_1, l_2, m_1, m_2$ are complex numbers. By using the root and ration test (see [31]), we obtain

$$R = \lim_{s \rightarrow \infty} \sup \frac{|\sigma_{s,k}|_D}{|\sigma_{s+1,k}|_D}, \text{ provided } L + s, M + t, N + v \notin \mathcal{O}_2, k \in \mathbb{Z}^+$$

$$\begin{aligned}
R &= \left\{ \lim_{s \rightarrow \infty} \sup \left| \frac{\sigma_{1,s,k}}{\sigma_{1,s+1,k}} \right| \right\} e_1 + \left\{ \lim_{s \rightarrow \infty} \sup \left| \frac{\sigma_{2,s,k}}{\sigma_{2,s+1,k}} \right| \right\} e_2 \\
&= \left\{ \lim_{s \rightarrow \infty} \sup \left| \frac{(m_1)_{s,k} (n_1)_{s,k}}{(l_1)_{s,k} s!} \frac{(l_1)_{s+1,k} (s+1)!}{(m_1)_{s+1,k} (n_1)_{s+1,k}} \right| \right\} e_1 \\
&\quad + \left\{ \lim_{s \rightarrow \infty} \sup \left| \frac{(m_2)_{s,k} (n_2)_{s,k}}{(l_2)_{s,k} s!} \frac{(l_2)_{s+1,k} (s+1)!}{(m_2)_{s+1,k} (n_2)_{s+1,k}} \right| \right\} e_2
\end{aligned}$$

According to identity $(Y)_{s+1,k} = (Y + sk)(Y)_{s,k}$, we then have

$$\begin{aligned} R &= \left\{ \limsup_{s \rightarrow \infty} \left| \frac{(l_1 + sk)(s+1)}{(m_1 + sk)(n_1 + sk)} \right| \right\} e_1 + \left\{ \limsup_{s \rightarrow \infty} \left| \frac{(l_2 + sk)(s+1)}{(m_2 + sk)(n_2 + sk)} \right| \right\} e_2 \\ &= \left\{ \frac{1}{k} \right\} e_1 + \left\{ \frac{1}{k} \right\} e_2 = \frac{1}{k} (e_1 + e_2) = \frac{1}{k} > 0, \end{aligned} \quad (46)$$

then the series is absolutely hyperbolically convergent in Ball $B_D(0, \frac{1}{k}) = \{Y : |Y|_D < \frac{1}{k}\}$, and diverges outside of its closure (see [31]).

Hence, the proof. $\square$

Remark 1. If we put $k = 1$ in series (45), then $R = 1$ in Equation (46). Hence, the series will be hyperbolically convergent absolutely in $B_D(0, 1) = \{Y : |Y|_D < 1\}$, and diverges outside of its closure (see [20]).

Corollary 2. On the boundary where $|Y|_D = \frac{1}{k}$ of the ball of convergence. The series in Equation (45) is absolutely hyperbolically convergent if $\operatorname{Re}(z_1 - x_1 - y_1) > |\operatorname{Im}(z_2 - x_2 - y_2)|$, in which $|Y|_D$ refers to the modulus of hyperbolic-valued [3].

Proof. The series $\sum_{s=0}^{\infty} \frac{1}{(ks)^{1+\delta}}$ of positive constants converges for some $\delta > 0, k > 0$. Let us assume that $|Y|_D = \frac{1}{k}$. Then

$$|\mu_1|e_1 + |\mu_2|e_2 = \frac{1}{k} = \frac{1}{k}e_1 + \frac{1}{k}e_2 \Rightarrow |\mu_1| = \frac{1}{k}, |\mu_2| = \frac{1}{k}.$$

Now we test the series $\sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!}$.

$$\left| \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} s!} \right| < \frac{1}{(ks)^{1+\delta}}, \quad (\text{see [8]}),$$

then

$$\begin{aligned} \limsup_{s \rightarrow \infty} \left| \frac{(ks)^{1+\delta} \otimes (M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} s!} \right|_D &= \left\{ \limsup_{s \rightarrow \infty} \left| \frac{(ks)^{1+\delta} (m_1)_{s,k} (n_1)_{s,k}}{(l_1)_{s,k} s!} \right| \right\} e_1 \\ &+ \left\{ \limsup_{s \rightarrow \infty} \left| \frac{(ks)^{1+\delta} (m_2)_{s,k} (n_2)_{s,k}}{(l_2)_{s,k} s!} \right| \right\} e_2 \\ &= I_1 e_1 + I_2 e_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \left\{ \limsup_{s \rightarrow \infty} \left| \frac{(ks)^{1+\delta} (m_1)_{s,k} (n_1)_{s,k}}{(l_1)_{s,k} s!} \right| \right\} \\ &= \left\{ \left| \frac{\Gamma_k(l_1)}{\Gamma_k(m_1)\Gamma_k(n_1)} \right| \limsup_{s \rightarrow \infty} \left| \frac{k^s}{(ks)^{\frac{l_1 - m_1 - n_1 - \delta k}{k}}} \right| \right\}. \quad \text{since, } \lim_{s \rightarrow \infty} \frac{s! k^s (sk)^{\frac{\alpha}{k} - 1}}{\Gamma_k(\alpha + sk)} = 1 \end{aligned}$$

Similarly,

$$I_2 = \left\{ \limsup_{s \rightarrow \infty} \left| \frac{(ks)^{1+\delta} (m_2)_{s,k} (n_2)_{s,k}}{(l_2)_{s,k} s!} \right| \right\} \\ = \left\{ \left| \frac{\Gamma_k(l_2)}{\Gamma_k(m_2)\Gamma_k(n_2)} \right| \limsup_{s \rightarrow \infty} \left| \frac{k^s}{(ks)^{\frac{l_2-m_2-n_2-\delta k}{k}}} \right| \right\}.$$

Both I_1 and I_2 converge if $0 < k \leq 1$, and diverge otherwise for any positive value of $(\frac{l_1-m_1-n_1-\delta k}{k})$ and $(\frac{l_2-m_2-n_2-\delta k}{k})$. Then

$$\begin{aligned} \operatorname{Re}(l_1 - m_1 - n_1 - \delta k) &> 0 \quad \text{and} \quad \operatorname{Re}(l_2 - m_2 - n_2 - \delta k) > 0 \\ \Rightarrow \operatorname{Re}(l_1 - m_1 - n_1) &> \delta k \quad \text{and} \quad \operatorname{Re}(l_2 - m_2 - n_2) > \delta k \\ \Rightarrow \operatorname{Re}((z_1 - i z_2) - (x_1 - i x_2) - (y_1 - i y_2)) &> \delta k \quad \text{and} \\ \operatorname{Re}((z_1 + i z_2) - (x_1 + i x_2) - (y_1 + i y_2)) &> \delta k. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{Re}(z_1 - x_1 - y_1) - \operatorname{Im}(z_2 - x_2 - y_2) &> \delta k \quad \text{and} \quad \operatorname{Re}(z_1 - x_1 - y_1) + \operatorname{Im}(z_2 - x_2 - y_2) > \delta k. \\ \text{Now, } \delta k > 0 \text{ allow us to say that} \\ \operatorname{Re}(z_1 - x_1 - y_1) - \operatorname{Im}(z_2 - x_2 - y_2) &> 0 \quad \text{and} \quad \operatorname{Re}(z_1 - x_1 - y_1) + \operatorname{Im}(z_2 - x_2 - y_2) > 0. \\ \Rightarrow \operatorname{Re}(z_1 - x_1 - y_1) &> |\operatorname{Im}(z_2 - x_2 - y_2)|. \end{aligned}$$

Therefore,

$$\limsup_{s \rightarrow \infty} \left| \frac{(s)^{1+\delta} \otimes (M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} s!} \right|_D = 0$$

for $\operatorname{Re}(z_1 - x_1 - y_1) > |\operatorname{Im}(z_2 - x_2 - y_2)|$, which implies that the series (45) is absolutely hyperbolically convergent on $|Y|_D = \frac{1}{k}$ if $\operatorname{Re}(z_1 - x_1 - y_1) > |\operatorname{Im}(z_2 - x_2 - y_2)|$, $0 < k \leq 1$. The proof is now complete. $\square$

Remark 2. For $|Y|_D < \frac{1}{k}$, the function $\Psi_k(M, N; L; Y)$ is a bicomplex holomorphic function in M, N , and L for finite M, N , and $L \in \mathbb{BC}/\mathbb{Z}^- \cup \{0\}$.

Theorem 2. Assume that M, N, L , and Y are bicomplex numbers such that $|Y_D| > \frac{1}{k}$; then, the integral representation of bicomplex k -hypergeometric function is given by:

$$\Psi_k(M, N; L; Y) = \frac{\Gamma_{2,k}(L)}{k \Gamma_{2,k}(N) \otimes \Gamma_{2,k}(L - N)} \otimes \int_{\mathbb{H}} (1 - \zeta)^{\frac{L-N}{k}-1} \otimes \zeta^{\frac{N}{k}-1} \otimes (1 - kY\zeta)^{-\frac{M}{k}} \otimes d\zeta,$$

in which $k \in \mathbb{R}^+$, $\zeta = \zeta_1 e_1 + \zeta_2 e_2 \in \mathbb{BC}$, ζ_1 and $\zeta_2 \in [0, 1]$, and $\mathbb{H}$ is a curve in $\mathbb{BC}$ made up of two components h_1 and h_2 in $\mathbb{C}$.

Proof. Under given conditions and from the idempotent representation of k -hypergeometric function, we have

$$\begin{aligned} \Psi_k(M, N; L; Y) &= \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!} \\ &= \left\{ \sum_{s=0}^{\infty} \frac{(m_1)_{s,k} (n_1)_{s,k}}{(l_1)_{s,k}} \frac{\mu_1^s}{s!} \right\} e_1 + \left\{ \sum_{s=0}^{\infty} \frac{(m_2)_{s,k} (n_2)_{s,k}}{(l_2)_{s,k}} \frac{\mu_2^s}{s!} \right\} e_2. \end{aligned}$$

According to properties of the k -Pochhammer symbol in (see [11], p.183), we obtain

$$\begin{aligned} &= \left\{ \frac{\Gamma_k(l_1)}{\Gamma_k(m_1)\Gamma_k(n_1)} \sum_{s=0}^{\infty} \frac{\Gamma_k(m_1+sk)\Gamma_k(n_1+sk)}{\Gamma_k(l_1+sk)} \frac{\mu_1^s}{s!} \times \frac{\Gamma_k(l_1-n_1)}{\Gamma_k(l_1-n_1)} \right\} e_1 \\ &+ \left\{ \frac{\Gamma_k(l_2)}{\Gamma_k(m_2)\Gamma_k(n_2)} \sum_{s=0}^{\infty} \frac{\Gamma_k(m_2+sk)\Gamma_k(n_2+sk)}{\Gamma_k(l_2+sk)} \frac{\mu_2^s}{s!} \times \frac{\Gamma_k(l_2-n_2)}{\Gamma_k(l_2-n_2)} \right\} e_2 \\ &= \left\{ \frac{\Gamma_k(l_1)}{\Gamma_k(n_1)\Gamma_k(l_1-n_1)} \sum_{s=0}^{\infty} \frac{\Gamma_k(m_1+sk)}{\Gamma_k(m_1)} B_k(n_1+sk, l_1-n_1) \frac{\mu_1^s}{s!} \right\} e_1 \\ &+ \left\{ \frac{\Gamma_k(l_2)}{\Gamma_k(n_2)\Gamma_k(l_2-n_2)} \sum_{s=0}^{\infty} \frac{\Gamma_k(m_2+sk)}{\Gamma_k(m_2)} B_k(n_2+sk, l_2-n_2) \frac{\mu_2^s}{s!} \right\} e_2. \end{aligned}$$

Using the definition of the k -beta function in Equation (21), we obtain

$$\begin{aligned} &= \left\{ \frac{\Gamma_k(l_1)}{\Gamma_k(n_1)\Gamma_k(l_1-n_1)} \sum_{s=0}^{\infty} \frac{\Gamma_k(m_1+sk)}{\Gamma_k(m_1)} \frac{1}{k} \int_0^1 (1-\zeta_1)^{\frac{l_1-n_1-1}{k}} \zeta_1^{\frac{n_1+sk}{k}-1} d\zeta_1 \frac{\mu_1^s}{s!} \right\} e_1 \\ &+ \left\{ \frac{\Gamma_k(l_2)}{\Gamma_k(n_2)\Gamma_k(l_2-n_2)} \sum_{s=0}^{\infty} \frac{\Gamma_k(m_2+sk)}{\Gamma_k(m_2)} \frac{1}{k} \int_0^1 (1-\zeta_2)^{\frac{l_2-n_2-1}{k}} \zeta_2^{\frac{n_2+sk}{k}-1} d\zeta_2 \frac{\mu_2^s}{s!} \right\} e_2 \\ &= \left\{ \frac{\Gamma_k(l_1)}{k \Gamma_k(n_1)\Gamma_k(l_1-n_1)} \int_0^1 (1-\zeta_1)^{\frac{l_1-n_1-1}{k}} \zeta_1^{\frac{n_1}{k}-1} (1-k\mu_1\zeta_1)^{-\frac{m_1}{k}} d\zeta_1 \right\} e_1 \\ &+ \left\{ \frac{\Gamma_k(l_2)}{k \Gamma_k(n_2)\Gamma_k(l_2-n_2)} \int_0^1 (1-\zeta_2)^{\frac{l_2-n_2-1}{k}} \zeta_2^{\frac{n_2}{k}-1} (1-k\mu_2\zeta_2)^{-\frac{m_2}{k}} d\zeta_2 \right\} e_2 \\ &= \frac{\Gamma_{2k}(L)}{k \Gamma_{2k}(N) \otimes \Gamma_{2k}(L-N)} \otimes \int_{\mathbb{H}} (1-\zeta)^{\frac{L-N}{k}-1} \otimes \zeta^{\frac{N}{k}-1} \otimes (1-kY\zeta)^{-\frac{M}{k}} \otimes d\zeta. \end{aligned}$$

This completes the proof. $\square$

Remark 3. When $k = 1$, we obtain the usual case of the hypergeometric function in bicomplex representation (see [20]).

Theorem 3. Let M, N, L and $Y \in \mathbb{BC}$. Then, the derivative of the k -hypergeometric function is defined as:

$$\frac{d^s}{dY^s} [\Psi_k(M, N; L; Y)] = \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \Psi_k(M+sk, N+sk; L+sk; Y),$$

where $k \in \mathbb{R}^+$, $s \in \mathbb{N}$, and L is neither a zero nor a negative integer.

Proof. It is trivial to show the relation is true when $s = 0$, and if using $s = 1$, we then obtain

$$\begin{aligned} \frac{d}{dY} [\Psi_k(M, N; L; Y)] &= \frac{d}{dY} \left(\sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!} \right) \\ &= \sum_{s=1}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^{s-1}}{(s-1)!} \quad (\text{let } s = n+1) \\ &= \sum_{n=0}^{\infty} \frac{(M)_{n+1,k} \otimes (N)_{n+1,k}}{(L)_{n+1,k}} \otimes \frac{Y^n}{(n)!}, \quad \text{where } (M)_{n+1,k} = M \otimes (M+k)_{n,k} \\ &= \frac{M \otimes N}{L} \sum_{n=0}^{\infty} \frac{(M+k)_{n,k} \otimes (N+k)_{n,k}}{(L+k)_{n,k}} \otimes \frac{Y^n}{(n)!}. \end{aligned}$$

Similarly, we obtain the second differential, and we obtain

$$\begin{aligned}\frac{d^2}{dY^2}[\Psi_k(M, N; L; Y)] &= \frac{d^2}{dY^2} \left(\sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!} \right) \\ &= \frac{M \otimes (M+k) \otimes N \otimes (N+k)}{L \otimes (L+k)} \sum_{n=0}^{\infty} \frac{(M+2k)_{n,k} \otimes (N+2k)_{n,k}}{(L+2k)_{n,k}} \otimes \frac{Y^n}{(n)!}.\end{aligned}$$

By differentiating s times, we then obtain the general relation:

$$\begin{aligned}\frac{d^s}{dY^s}[\Psi_k(M, N; L; Y)] &= \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \sum_{s=0}^{\infty} \frac{(M+sk)_{s,k} \otimes (N+sk)_{s,k}}{(L+sk)_{s,k}} \otimes \frac{Y^s}{s!} \\ &= \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \Psi_k(M+sk, N+sk; L+sk; Y).\end{aligned}$$

which completes the proof of the theorem. $\square$

Remark 4. When $k = 1$, and $M, N, L, Y \in \mathbb{C}$, we obtain the usual case of the derivative of the hypergeometric function in complex numbers (see [32]).

4. k -Bicomplex Riemann–Liouville Fractional Operator

In this section, we present the k -Riemann–Liouville Fractional integration and derivative in the bicomplex operator, and prove some important theorems.

Theorem 4. Let the bicomplex function $F : X \rightarrow \mathbb{BC}$, where $X \subseteq \mathbb{BC}$ is piecewise continuous on $I' = (0, \infty)$ and integrable on any finite subinterval of $I = [0, \infty)$. Let $Y, \delta, \zeta \in \mathbb{BC}$, $Y = v_1 + jv_2 = \mu_1 e_1 + \mu_2 e_2$, with $\text{Re}(v_1) > |\text{Im}(v_2)|$, $\delta = \delta_1 e_1 + \delta_2 e_2$, and $\zeta = \zeta_1 e_1 + \zeta_2 e_2$, $\zeta_1, \zeta_2 \in \mathbb{R}^+$. Then,

$${}_0 I_{k,\delta}^Y F(\delta) = \frac{1}{k\Gamma_{2,k}(Y)} \otimes \int_0^\delta (\delta - \zeta)^{\frac{Y}{k}-1} \otimes F(\zeta) \otimes d\zeta, \quad k > 0, \quad (47)$$

where $\text{Re}(\mu_1) > 0$, $\text{Re}(\mu_2) > 0$, $F(\zeta) = f_1(\zeta_1) e_1 + f_2(\zeta_2) e_2$ and $F(\delta) = f_1(\delta_1) e_1 + f_2(\delta_2) e_2$.

Proof. From the definition of k -Riemann–Liouville integration in Equation (40), we conclude that

$${}_0 I_{k,\delta_1}^{\mu_1} f_1(\delta_1) = \frac{1}{k\Gamma_k(\mu_1)} \int_0^{\delta_1} (\delta_1 - \zeta_1)^{\frac{\mu_1}{k}-1} f_1(\zeta_1) d\zeta_1, \quad \text{Re}(\mu_1) > 0, \quad (48)$$

and

$${}_0 I_{k,\delta_2}^{\mu_2} f_2(\delta_2) = \frac{1}{k\Gamma_k(\mu_2)} \int_0^{\delta_2} (\delta_2 - \zeta_2)^{\frac{\mu_2}{k}-1} f_2(\zeta_2) d\zeta_2, \quad \text{Re}(\mu_2) > 0. \quad (49)$$

Using the idempotent representation, we obtain

$$\begin{aligned}\left\{ {}_0 I_{k,\delta_1}^{\mu_1} f_1(\delta_1) \right\} e_1 + \left\{ {}_0 I_{k,\delta_2}^{\mu_2} f_2(\delta_2) \right\} e_2 &= \left\{ \frac{1}{k\Gamma_k(\mu_1)} \int_0^{\delta_1} (\delta_1 - \zeta_1)^{\frac{\mu_1}{k}-1} f_1(\zeta_1) d\zeta_1 \right\} e_1 \\ &\quad + \left\{ \frac{1}{k\Gamma_k(\mu_2)} \int_0^{\delta_2} (\delta_2 - \zeta_2)^{\frac{\mu_2}{k}-1} f_2(\zeta_2) d\zeta_2 \right\} e_2,\end{aligned}$$

which implies that

$$\begin{aligned} \left[\left\{ {}_0 I_{k,\delta_1}^{\mu_1} \right\} e_1 + \left\{ {}_0 I_{k,\delta_2}^{\mu_2} \right\} e_2 \right] [f_1(\delta_1)e_1 + f_2(\delta_2)e_2] &= \frac{1}{k\Gamma_k(\mu_1 e_1 + \mu_2 e_2)} \\ &\otimes \int_0^\delta (\delta - \zeta)^{\frac{\mu_1 e_1 + \mu_2 e_2}{k} - 1} \otimes [\{f_1(\zeta_1)\}e_1 + \{f_2(\zeta_2)\}e_2] \otimes d\zeta, \end{aligned}$$

then we obtain

$${}_0 I_{k,\delta}^Y F(\delta) = \frac{1}{k\Gamma_{2,k}(Y)} \otimes \int_0^\delta (\delta - \zeta)^{\frac{Y}{k} - 1} \otimes F(\zeta) \otimes d\zeta, \quad \operatorname{Re}(v_1) > |\operatorname{Im}(v_2)|. \quad (50)$$

Hence, the proof. $\square$

Remark 5. If we put $k = 1$ in Theorem 4, we have the definition of the bicomplex Riemann–Liouville Fractional (43).

Example 1. Let $F(\delta) = (\delta - a)^\eta$ be the bicomplex function, and $\delta, U \in \mathbb{BC}$, $\delta = \delta_1 e_1 + \delta_2 e_2$ and $U = u_1 e_1 + u_2 e_2$. According to Theorem 4, we obtain

$$\begin{aligned} {}_a I_{k,\delta}^Y (\delta - a)^\eta &= \frac{1}{k\Gamma_{2,k}(Y)} \otimes \int_a^\delta (\delta - \zeta)^{\frac{Y}{k} - 1} \otimes F(\zeta) \otimes d\zeta \\ &= \frac{1}{k\Gamma_{2,k}(Y)} \otimes \int_a^\delta (\delta - \zeta)^{\frac{Y}{k} - 1} \otimes (\zeta - a)^\eta \otimes d\zeta \\ &= \left\{ \frac{1}{k\Gamma_k(\mu_1)} \int_a^{\delta_1} (\delta_1 - \zeta_1)^{\frac{\mu_1}{k} - 1} (\zeta_1 - a)^\eta d\zeta_1 \right\} e_1 \\ &\quad + \left\{ \frac{1}{k\Gamma_k(\mu_2)} \int_a^{\delta_2} (\delta_2 - \zeta_2)^{\frac{\mu_2}{k} - 1} (\zeta_2 - a)^\eta d\zeta_2 \right\} e_2 \\ &= I_1 e_1 + I_2 e_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{k\Gamma_k(\mu_1)} \int_a^{\delta_1} (\delta_1 - \zeta_1)^{\frac{\mu_1}{k} - 1} (\zeta_1 - a)^\eta d\zeta_1 \\ &= \frac{(\delta_1 - a)^{\frac{\mu_1}{k} - 1}}{k\Gamma_k(\mu_1)} \int_a^{\delta_1} \left(1 - \frac{(\zeta_1 - a)}{\delta_1 - a} \right)^{\frac{\mu_1}{k} - 1} (\zeta_1 - a)^\eta d\zeta_1 \end{aligned}$$

$$\begin{aligned} \text{and } I_2 &= \frac{1}{k\Gamma_k(\mu_2)} \int_a^{\delta_2} (\delta_2 - \zeta_2)^{\frac{\mu_2}{k} - 1} (\zeta_2 - a)^\eta d\zeta_2 \\ &= \frac{(\delta_2 - a)^{\frac{\mu_2}{k} - 1}}{k\Gamma_k(\mu_2)} \int_a^{\delta_2} \left(1 - \frac{(\zeta_2 - a)}{\delta_2 - a} \right)^{\frac{\mu_2}{k} - 1} (\zeta_2 - a)^\eta d\zeta_2 \end{aligned}$$

let $u_1 = \frac{\zeta_1 - a}{\delta_1 - a}$, $(\zeta_1 - a) = u_1 (\delta_1 - a)$, when $\zeta_1 = a \implies u_1 = 0$, $\zeta_1 = \delta_1 \implies u_1 = 1$, then

$$I_1 = \frac{(\delta_1 - a)^{\frac{\mu_1}{k} + \eta}}{k\Gamma_k(\mu_1)} \int_0^1 (1 - u_1)^{\frac{\mu_1}{k} - 1} u_1^\eta du_1 \quad \text{similarly } I_2 = \frac{(\delta_2 - a)^{\frac{\mu_2}{k} + \eta}}{k\Gamma_k(\mu_2)} \int_0^1 (1 - u_2)^{\frac{\mu_2}{k} - 1} u_2^\eta du_2,$$

then

$$\begin{aligned} {}_a I_{k,\delta}^Y (\delta - a)^\eta &= \frac{(\delta - a)^{\frac{Y}{k} + \eta}}{k \Gamma_{2,k}(Y)} \otimes \int_0^1 (1 - U)^{\frac{Y}{k} - 1} \otimes U^\eta \otimes dU = \frac{(\delta - a)^{\frac{Y}{k} + \eta}}{\Gamma_{2,k}(Y)} \otimes \beta_{2,k}(Y, k\eta + k) \\ &= (\delta - a)^{\frac{Y}{k} + \eta} \otimes \frac{\Gamma_{2,k}(k\eta + k)}{\Gamma_{2,k}(Y + k\eta + k)}, \text{ that for all } \eta > -1. \end{aligned}$$

Theorem 5. Let $Y, \zeta, \delta \in \mathbb{BC}$, F be a bicomplex function with $\operatorname{Re}(v_1) > 0$ and let $m = [\operatorname{Re}(v_1)] + 1$. Then, the k -Riemann–Liouville fractional derivative of F of order Y is expressed as

$$\begin{aligned} ({}_0 D_{k,\delta}^Y F)(\delta) &= {}_0 D_{k,\delta}^m ({}_0 D_{k,\delta}^{-(m-Y)} F(\delta)) \\ &= \frac{1}{k \Gamma_{2,k}(m - Y)} \frac{d^m}{d\delta^m} \int_0^\delta (\delta - \zeta)^{\frac{m-Y}{k} - 1} \otimes F(\zeta) \otimes d\zeta. \end{aligned} \quad (51)$$

Proof. We can easily obtain

$${}_0 D_{k,\delta}^{-(m-Y)} F(\delta) = \frac{1}{k \Gamma_{2,k}(m - Y)} \int_0^\delta (\delta - \zeta)^{\frac{m-Y}{k} - 1} \otimes F(\zeta) \otimes d\zeta,$$

then differentiating it m times, we obtain

$${}_0 D_{k,\delta}^m ({}_0 D_{k,\delta}^{-(m-Y)} F(\delta)) = \frac{1}{k \Gamma_{2,k}(m - Y)} \frac{d^m}{d\delta^m} \int_0^\delta (\delta - \zeta)^{\frac{m-Y}{k} - 1} \otimes F(\zeta) \otimes d\zeta.$$

This completes the proof. $\square$

Theorem 6. Let $F(\delta)$ and $H(\delta)$ be bicomplex functions that are piecewise continuous on $I' = (0, \infty)$ and integrable on any finite subinterval of $I = [0, \infty)$. Suppose $Y = v_1 + jv_2 = \mu_1 e_1 + \mu_2 e_2$, $v_1, v_2 \in \mathbb{C}$ with $\operatorname{Re}(v_1) > |\operatorname{Im}(v_2)|$. Then

$${}_0 I_{k,\delta}^Y [F(\delta) + H(\delta)] = {}_0 I_{k,\delta}^Y F(\delta) + {}_0 I_{k,\delta}^Y H(\delta). \quad (52)$$

Proof. Using Theorem (4), we have

$$\begin{aligned} {}_0 I_{k,\delta}^Y [F(\delta) + H(\delta)] &= [{}_0 I_{k,\delta_1}^{\mu_1} e_1 + {}_0 I_{k,\delta_2}^{\mu_2} e_2] [\{f_1(\delta_1) + h_1(\delta_1)\}e_1 + \{f_2(\delta_2) + h_2(\delta_2)\}e_2] \\ &= \left\{ {}_0 I_{k,\delta_1}^{\mu_1} (f_1(\delta_1) + h_1(\delta_1)) \right\} e_1 + \left\{ {}_0 I_{k,\delta_2}^{\mu_2} (f_2(\delta_2) + h_2(\delta_2)) \right\} e_2 \\ &= \left[\left\{ {}_0 I_{k,\delta_1}^{\mu_1} f_1(\delta_1) \right\} e_1 + \left\{ {}_0 I_{k,\delta_2}^{\mu_2} f_2(\delta_2) \right\} e_2 \right] \\ &\quad + \left[\left\{ {}_0 I_{k,\delta_1}^{\mu_1} h_1(\delta_1) \right\} e_1 + \left\{ {}_0 I_{k,\delta_2}^{\mu_2} h_2(\delta_2) \right\} e_2 \right] \\ &= \left[\left\{ {}_0 I_{k,\delta_1}^{\mu_1} e_1 + {}_0 I_{k,\delta_2}^{\mu_2} e_2 \right\} (f_1(\delta_1)e_1 + f_2(\delta_2)e_2) \right] \\ &\quad + \left[\left\{ {}_0 I_{k,\delta_1}^{\mu_1} e_1 + {}_0 I_{k,\delta_2}^{\mu_2} e_2 \right\} (h_1(\delta_1)e_1 + h_2(\delta_2)e_2) \right] \\ &= {}_0 I_{k,\delta}^Y F(\delta) + {}_0 I_{k,\delta}^Y H(\delta). \end{aligned}$$

This completes the proof. $\square$

Theorem 7. Let $F(\delta)$ be a bicomplex function that is piecewise continuous on $I' = (0, \infty)$ and integrable on any finite subinterval of $I = [0, \infty)$. Let $\lambda = c_1 + jc_2 = \lambda_1 e_1 + \lambda_2 e_2$, $c_1, c_2 \in \mathbb{C}$, $Y = v_1 + jv_2 = \mu_1 e_1 + \mu_2 e_2$, $v_1, v_2 \in \mathbb{C}$, and $\delta = \delta_1 e_1 + \delta_2 e_2$, $\tau = \tau_1 e_1 + \tau_2 e_2$ with $\operatorname{Re}(v_1) > |\operatorname{Im}(v_2)|$ and $\operatorname{Re}(c_1) > |\operatorname{Im}(c_2)|$, then

$${}_0 I_{k,\delta}^Y [{}_0 I_{k,\zeta}^\lambda F(\delta)] = {}_0 I_{k,\delta}^\lambda [{}_0 I_{k,\zeta}^Y F(\delta)] = {}_0 I_{k,\zeta}^{Y+\lambda} F(\delta). \quad (53)$$

Proof. Applying the definition of k -Riemann–Liouville integration in Equation (47), we have

$$\begin{aligned}
 {}_0I_{k,\delta}^Y[{}_0I_{k,\zeta}^\lambda F(\delta)] &= \frac{1}{k\Gamma_{2,k}(Y)} \int_0^\delta (\delta - \zeta)^{\frac{Y}{k}-1} \otimes \left[\frac{1}{k\Gamma_{2,k}(\lambda)} \otimes \int_0^\zeta (\zeta - \tau)^{\frac{\lambda}{k}-1} \otimes F(\tau) \otimes d\tau \right] \otimes d\zeta \\
 &= \frac{1}{k^2 \Gamma_{2,k}(Y) \otimes \Gamma_{2,k}(\lambda)} \int_0^\delta \int_0^\zeta (\delta - \zeta)^{\frac{Y}{k}-1} \otimes (\zeta - \tau)^{\frac{\lambda}{k}-1} \otimes F(\tau) \otimes d\tau \otimes d\zeta \\
 &= \left\{ \frac{1}{k^2 \Gamma_k(\mu_1)\Gamma_k(\lambda_1)} \int_0^{\delta_1} \int_0^{\zeta_1} (\delta_1 - \zeta_1)^{\frac{\mu_1}{k}-1} (\zeta_1 - \tau_1)^{\frac{\lambda_1}{k}-1} f(\tau_1) d\tau_1 d\zeta_1 \right\} e_1 \\
 &\quad + \left\{ \frac{1}{k^2 \Gamma_k(\mu_2)\Gamma_k(\lambda_2)} \int_0^{\delta_2} \int_0^{\zeta_2} (\delta_2 - \zeta_2)^{\frac{\mu_2}{k}-1} (\zeta_2 - \tau_2)^{\frac{\lambda_2}{k}-1} f(\tau_2) d\tau_2 d\zeta_2 \right\} e_2 \\
 &= \left\{ \frac{1}{k^2 \Gamma_k(\mu_1)\Gamma_k(\lambda_1)} \int_0^{\zeta_1} \int_{\tau_1}^{\delta_1} (\delta_1 - \zeta_1)^{\frac{\mu_1}{k}-1} (\zeta_1 - \tau_1)^{\frac{\lambda_1}{k}-1} f(\tau_1) d\zeta_1 d\tau_1 \right\} e_1 \\
 &\quad + \left\{ \frac{1}{k^2 \Gamma_k(\mu_2)\Gamma_k(\lambda_2)} \int_0^{\zeta_2} \int_{\tau_2}^{\delta_2} (\delta_2 - \zeta_2)^{\frac{\mu_2}{k}-1} (\zeta_2 - \tau_2)^{\frac{\lambda_2}{k}-1} f(\tau_2) d\zeta_2 d\tau_2 \right\} e_2 \\
 &= I_1 e_1 + I_2 e_2,
 \end{aligned}$$

where

$$\begin{aligned}
 I_1 &= \frac{1}{k^2 \Gamma_k(\mu_1)\Gamma_k(\lambda_1)} \int_0^{\zeta_1} \int_{\tau_1}^{\delta_1} (\delta_1 - \zeta_1)^{\frac{\mu_1}{k}-1} (\zeta_1 - \tau_1)^{\frac{\lambda_1}{k}-1} f(\tau_1) d\zeta_1 d\tau_1, \\
 I_2 &= \frac{1}{k^2 \Gamma_k(\mu_2)\Gamma_k(\lambda_2)} \int_0^{\zeta_2} \int_{\tau_2}^{\delta_2} (\delta_2 - \zeta_2)^{\frac{\mu_2}{k}-1} (\zeta_2 - \tau_2)^{\frac{\lambda_2}{k}-1} f(\tau_2) d\zeta_2 d\tau_2.
 \end{aligned}$$

Let $\zeta_1 = \tau_1 + u_1(\delta_1 - \tau_1)$, and it follows that $d\zeta_1 = (\delta_1 - \tau_1)du_1$. When $\zeta_1 = \tau_1$, $u_1 = 0$, and when $\zeta_1 = \delta_1$, $u_1 = 1$. Thus,

$$I_1 = \frac{1}{k^2 \Gamma_k(\mu_1)\Gamma_k(\lambda_1)} \int_0^{\zeta_1} \int_0^1 (\delta_1 - \tau_1)^{\frac{\mu_1}{k}-1} (1 - u_1)^{\frac{\mu_1}{k}-1} (\delta_1 - \tau_1)^{\frac{\lambda_1}{k}-1} u^{\frac{\lambda_1}{k}-1} f(\tau_1) du_1 d\tau_1,$$

and similarly

$$I_2 = \frac{1}{k^2 \Gamma_k(\mu_2)\Gamma_k(\lambda_2)} \int_0^{\zeta_2} \int_0^1 (\delta_2 - \tau_2)^{\frac{\mu_2}{k}-1} (1 - u_2)^{\frac{\mu_2}{k}-1} (\delta_2 - \tau_2)^{\frac{\lambda_2}{k}-1} u^{\frac{\lambda_2}{k}-1} f(\tau_2) du_2 d\tau_2.$$

Using the definition of the k -beta function in Equation (33), we obtain

$$\begin{aligned}
 {}_0I_{k,\delta}^Y[{}_0I_{k,\zeta}^\lambda F(\delta)] &= \left\{ \frac{\beta_k(\mu_1, \lambda_1)}{k \Gamma_k(\mu_1)\Gamma_k(\lambda_1)} \int_0^{\zeta_1} (\delta_1 - \tau_1)^{\frac{\mu_1+\lambda_1}{k}-1} f(\tau_1) d\tau_1 \right\} e_1 \\
 &\quad + \left\{ \frac{\beta_k(\mu_2, \lambda_2)}{k \Gamma_k(\mu_2)\Gamma_k(\lambda_2)} \int_0^{\zeta_2} (\delta_2 - \tau_2)^{\frac{\mu_2+\lambda_2}{k}-1} f(\tau_2) d\tau_2 \right\} e_2.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 {}_0I_{k,\delta}^Y[{}_0I_{k,\zeta}^\lambda F(\delta)] &= \frac{\beta_{2,k}(Y, \lambda)}{k \Gamma_{2,k}(Y) \otimes \Gamma_{2,k}(\lambda)} \otimes \int_0^\zeta (\delta - \tau)^{\frac{Y+\lambda}{k}-1} \otimes F(\tau) \otimes d\tau \\
 &= \frac{1}{k \Gamma_{2,k}(Y + \lambda)} \otimes \int_0^\zeta (\delta - \tau)^{\frac{Y+\lambda}{k}-1} \otimes F(\tau) \otimes d\tau \\
 &= {}_0I_{k,\zeta}^{Y+\lambda} F(\delta).
 \end{aligned} \tag{54}$$

Similarly

$$\begin{aligned}
 {}_0I_{k,\delta}^\lambda[{}_0I_{k,\zeta}^Y F(\delta)] &= \frac{1}{k \Gamma_{2,k}(\lambda + Y)} \otimes \int_0^\zeta (\delta - \tau)^{\frac{\lambda+Y}{k}-1} \otimes F(\tau) \otimes d\tau \\
 &= {}_0I_{k,\zeta}^{Y+\lambda} F(\delta).
 \end{aligned} \tag{55}$$

From Equations (54) and (55), we conclude that

$${}_0I_{k,\delta}^Y \left[{}_0I_{k,\zeta}^\lambda F(\delta) \right] = {}_0I_{k,\delta}^\lambda \left[{}_0I_{k,\zeta}^Y F(\delta) \right] = {}_0I_{k,\zeta}^{Y+\lambda} F(\delta).$$

This completes the proof. $\square$

5. Application

In this section, we apply the k -Riemann–Liouville fractional operator to the k -bicomplex hypergeometric function.

Theorem 8. Let function $\Psi_k(M, N; L; Y)$ be piecewise continuous on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$ with $\operatorname{Re}(v_1) > |\operatorname{Im}(v_2)|$; then, we have

$${}_0I_{k,Y}^\delta [\Psi_k(M, N; L; Y)] = Y^{\frac{\delta}{k}} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes (kY)^s \otimes \frac{1}{\Gamma_{2,k}(\delta + ks + k)}. \quad (56)$$

Proof. Using the definition of the k -bicomplex Riemann–Liouville fractional operator in Equation (47), and applying it to the k -bicomplex hypergeometric function, we obtain

$$\begin{aligned} {}_0I_{k,Y}^\delta [\Psi_k(M, N; L; Y)] &= {}_0I_{k,Y}^\delta \left[\sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!} \right] \\ &= \frac{1}{k\Gamma_{2,k}(\delta)} \otimes \int_0^Y (Y - \zeta)^{\frac{\delta}{k}-1} \otimes \Psi_k(M, N; L; \zeta) \otimes d\zeta \\ &= \frac{1}{k\Gamma_{2,k}(\delta)} \otimes \int_0^Y (Y - \zeta)^{\frac{\delta}{k}-1} \otimes \left[\sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{\zeta^s}{s!} \right] \otimes d\zeta \\ &= \frac{1}{k\Gamma_{2,k}(\delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} s!} \otimes \int_0^Y (Y - \zeta)^{\frac{\delta}{k}-1} \otimes \zeta^s \otimes d\zeta \\ &= \frac{1}{k\Gamma_{2,k}(\delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} s!} \otimes \int_0^Y \left(1 - \frac{\zeta}{Y} \right)^{\frac{\delta}{k}-1} \otimes Y^{\frac{\delta}{k}-1} \otimes \zeta^s \otimes d\zeta. \end{aligned}$$

$$\begin{aligned} \text{Let } I &= \int_0^Y \left(1 - \frac{\zeta}{Y} \right)^{\frac{\delta}{k}-1} \otimes Y^{\frac{\delta}{k}-1} \otimes \zeta^s \otimes d\zeta = \left\{ \int_0^{\mu_1} \left(1 - \frac{\zeta_1}{\mu_1} \right)^{\frac{\delta_1}{k}-1} \mu_1^{\frac{\delta_1}{k}-1} \zeta_1^s d\zeta_1 \right\} e_1 \\ &\quad + \left\{ \int_0^{\mu_2} \left(1 - \frac{\zeta_2}{\mu_2} \right)^{\frac{\delta_2}{k}-1} \mu_2^{\frac{\delta_2}{k}-1} \zeta_2^s d\zeta_2 \right\} e_2 \\ &= I_1 e_1 + I_2 e_2, \end{aligned}$$

$$\text{where } I_1 = \int_0^{\mu_1} \left(1 - \frac{\zeta_1}{\mu_1} \right)^{\frac{\delta_1}{k}-1} \mu_1^{\frac{\delta_1}{k}-1} \zeta_1^s d\zeta_1, \quad I_2 = \int_0^{\mu_2} \left(1 - \frac{\zeta_2}{\mu_2} \right)^{\frac{\delta_2}{k}-1} \mu_2^{\frac{\delta_2}{k}-1} \zeta_2^s d\zeta_2.$$

Let $u_1 = \frac{\zeta_1}{\mu_1}$, $\zeta_1 = u_1 \mu_1$. When $\zeta_1 = 0 \implies u_1 = 0$; when $\zeta_1 = \mu_1 \implies u_1 = 1$. Then

$$I_1 = \int_0^1 (1 - u_1)^{\frac{\delta_1}{k}-1} \mu_1^{\frac{\delta_1}{k}+s} u_1^s du_1, \quad I_2 = \int_0^1 (1 - u_2)^{\frac{\delta_2}{k}-1} \mu_2^{\frac{\delta_2}{k}+s} u_2^s du_2.$$

Then

$$\begin{aligned} {}_0I_{k,Y}^\delta[\Psi_k(M,N;L;Y)] &= \frac{Y_k^\delta}{k\Gamma_{2,k}(\delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!} \otimes \int_0^1 (1-U)^{\frac{\delta}{k}-1} \otimes U^s \otimes dU \\ &= \frac{Y_k^\delta}{k\Gamma_{2,k}(\delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{s!} \otimes \beta_2\left(\frac{\delta}{k}, s+1\right) \\ &= \frac{Y_k^\delta}{k\Gamma_{2,k}(\delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^s}{\Gamma_2(s+1)} \otimes \frac{\Gamma_2\left(\frac{\delta}{k}\right) \otimes \Gamma_2(s+1)}{\Gamma_2\left(\frac{\delta}{k} + s + 1\right)}, \end{aligned}$$

and using property $\Gamma_{2,k}(\delta) = k^{\frac{\delta}{k}-1} \otimes \Gamma_2\left(\frac{\delta}{k}\right)$ from [12], we have

$$\begin{aligned} {}_0I_{k,Y}^\delta[\Psi_k(M,N;L;Y)] &= \frac{Y_k^\delta}{k\Gamma_{2,k}(\delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes Y^s \otimes \frac{k^{s+1}\Gamma_{2,k}(\delta)}{\Gamma_{2,k}(\delta + ks + k)} \\ &= Y_k^\delta \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes (kY)^s \otimes \frac{1}{\Gamma_{2,k}(\delta + ks + k)}. \end{aligned}$$

Hence, the proof is complete. $\square$

Theorem 9. Let function $\Psi_k(M,N;L;Y)$ be piecewise continuous on $I' = (0, \infty)$, and integrable on any finite subinterval of $I = [0, \infty)$. Let $Y = v_1 + jv_2 = \mu_1 e_1 + \mu_2 e_2$, where $v_1, v_2 \in \mathbb{C}$ with $\text{Re}(v_1) > |\text{Im}(v_2)|$. Consider $\alpha = [\text{Re}(v_1)] + 1$, and ζ and $\delta \in \mathbb{B}\mathbb{C}$, with $\alpha - 1 < \delta < \alpha$. Then,

$${}_0D_{k,Y}^\delta[\Psi_k(M,N;L;Y)] = \frac{Y^{\frac{\alpha-\delta}{k}-\alpha}}{k^\alpha} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{(kY)^s}{\Gamma_{2,k}(\alpha - \delta + sk - \alpha k + k)}. \quad (57)$$

Proof. Using the definition of the k -bicomplex Riemann–Liouville fractional operator in Equation (51), and applying it to the k -bicomplex hypergeometric function, we obtain

$$\begin{aligned} {}_0D_{k,Y}^\delta[\Psi_k(M,N;L;Y)] &= \frac{1}{k\Gamma_{2,k}(\alpha - \delta)} \frac{d^\alpha}{dY^\alpha} \int_0^Y (Y - \zeta)^{\frac{\alpha-\delta}{k}-1} \otimes \Psi_k(M,N;L;\zeta) \otimes d\zeta \\ &= \frac{1}{k\Gamma_{2,k}(\alpha - \delta)} \frac{d^\alpha}{dY^\alpha} \int_0^Y (Y - \zeta)^{\frac{\alpha-\delta}{k}-1} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{\zeta^s}{s!} \otimes d\zeta \\ &= \frac{1}{k\Gamma_{2,k}(\alpha - \delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} s!} \frac{d^\alpha}{dY^\alpha} \int_0^Y (Y - \zeta)^{\frac{\alpha-\delta}{k}-1} \otimes \zeta^s \otimes d\zeta. \end{aligned}$$

$$\begin{aligned} \text{Let } I &= \int_0^Y (Y - \zeta)^{\frac{\alpha-\delta}{k}-1} \otimes \zeta^s \otimes d\zeta = Y^{\frac{\alpha-\delta}{k}-1} \otimes \int_0^Y \left(1 - \frac{\zeta}{Y}\right)^{\frac{\alpha-\delta}{k}-1} \otimes \zeta^s \otimes d\zeta \\ &= \left\{ \mu_1^{\frac{\alpha_1-\delta_1}{k}-1} \int_0^{\mu_1} \left(1 - \frac{\zeta_1}{\mu_1}\right)^{\frac{\alpha_1-\delta_1}{k}-1} \otimes \zeta_1^s \otimes d\zeta_1 \right\} e_1 \\ &\quad + \left\{ \mu_2^{\frac{\alpha_2-\delta_2}{k}-1} \int_0^{\mu_2} \left(1 - \frac{\zeta_2}{\mu_2}\right)^{\frac{\alpha_2-\delta_2}{k}-1} \otimes \zeta_2^s \otimes d\zeta_2 \right\} e_2 = I_1 e_1 + I_2 e_2, \end{aligned}$$

$$\begin{aligned} \text{where } I_1 &= \mu_1^{\frac{\alpha_1-\delta_1}{k}-1} \int_0^{\mu_1} \left(1 - \frac{\zeta_1}{\mu_1}\right)^{\frac{\alpha_1-\delta_1}{k}-1} \otimes \zeta_1^s \otimes d\zeta_1, \\ I_2 &= \mu_2^{\frac{\alpha_2-\delta_2}{k}-1} \int_0^{\mu_2} \left(1 - \frac{\zeta_2}{\mu_2}\right)^{\frac{\alpha_2-\delta_2}{k}-1} \otimes \zeta_2^s \otimes d\zeta_2. \end{aligned}$$

Let $u_1 = \frac{\zeta_1}{\mu_1}$, $\zeta_1 = u_1 \mu_1$. When $\zeta_1 = 0 \implies u_1 = 0$, and when $\zeta_1 = \mu_1 \implies u_1 = 1$. Then,

$$I_1 = \mu_1^{\frac{\alpha_1 - \delta_1}{k} + s} \int_0^1 (1 - u_1)^{\frac{\alpha_1 - \delta_1}{k} - 1} \otimes u_1^s \otimes du_1,$$

similarly $I_2 = \mu_2^{\frac{\alpha_2 - \delta_2}{k} + s} \int_0^1 (1 - u_2)^{\frac{\alpha_2 - \delta_2}{k} - 1} \otimes u_2^s \otimes du_2$.

Then

$$\begin{aligned} {}_0D_{k,Y}^\delta[\Psi_k(M, N; L; Y)] &= \frac{1}{k\Gamma_{2,k}(\alpha - \delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{d^\alpha}{dY^\alpha} \left[Y^{\frac{\alpha - \delta}{k} + s} \right] \otimes \dots \\ &\quad \int_0^1 (1 - U)^{\frac{\alpha - \delta}{k} - 1} \otimes U^s \otimes dU \\ &= \frac{1}{k\Gamma_{2,k}(\alpha - \delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k} s!} \otimes \frac{d^\alpha}{dY^\alpha} \left[Y^{\frac{\alpha - \delta}{k} + s} \right] \otimes \dots \\ &\quad \beta_2\left(\frac{\alpha - \delta}{k}, s + 1\right) \\ &= \frac{1}{k\Gamma_{2,k}(\alpha - \delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{\Gamma_2\left(\frac{\alpha - \delta}{k} + s + 1\right)}{\Gamma_2\left(\frac{\alpha - \delta}{k} + s - \alpha + 1\right)} \otimes \dots \\ &\quad Y^{\frac{\alpha - \delta}{k} + s - \alpha} \otimes \frac{\Gamma_2\left(\frac{\alpha - \delta}{k}\right)}{\Gamma_2\left(\frac{\alpha - \delta}{k} + s + 1\right)} \\ &= \frac{1}{k\Gamma_{2,k}(\alpha - \delta)} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{Y^{\frac{\alpha - \delta}{k} + s - \alpha} \otimes \Gamma_2\left(\frac{\alpha - \delta}{k}\right)}{\Gamma_2\left(\frac{\alpha - \delta}{k} + s - \alpha + 1\right)} \end{aligned}$$

using property $\Gamma_{2,k}(\delta) = k^{\frac{\delta}{k} - 1} \otimes \Gamma_2\left(\frac{\delta}{k}\right)$ from [12], then

$$= \frac{Y^{\frac{\alpha - \delta}{k} - \alpha}}{k^\alpha} \otimes \sum_{s=0}^{\infty} \frac{(M)_{s,k} \otimes (N)_{s,k}}{(L)_{s,k}} \otimes \frac{(kY)^s}{\Gamma_{2,k}(\alpha - \delta + sk - \alpha k + k)}.$$

Hence, the theorem is proved. $\square$

Remark 6. In Theorems (8) and (9), we achieve the relation of the k -Riemann–Liouville fractional derivative with fractional integral:

$${}_0D_{k,Y}^\delta[\Psi_k(M, N; L; Y)] = {}_0I_{k,Y}^{\alpha - \delta}[\Psi_k^{(\alpha)}(M, N; L; Y)]. \quad (58)$$

6. Conclusions

In this paper, we have explored significant advancements in the theory of bicomplex hypergeometric functions and their applications within the realm of fractional calculus. By extending the hypergeometric function to bicomplex parameters, we have opened new avenues for mathematical analysis, providing a detailed examination of its convergence region and integral and derivative representations. The introduction of the k -Riemann–Liouville fractional integral and derivative in a bicomplex operator framework has further enriched the field, allowing for the development of several key theorems that underscore the versatility and depth of these mathematical constructs. Additionally, this work underscores the importance of bicomplex numbers, special functions, and fractional calculus, offering a robust foundation for future research. By providing comprehensive insights and new

theoretical tools, we hope to inspire further exploration and application of bicomplex hypergeometric functions and fractional operators, fostering advancements across diverse fields of study. Lastly, our study contributes to a deeper understanding of bicomplex mathematics and fractional calculus, offering a comprehensive framework that paves the way for future research and innovation in this exciting area of mathematical science.

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