



Article

Comparison Principle Based Synchronization Analysis of Fractional-Order Chaotic Neural Networks with Multi-Order and Its Circuit Implementation

Rongbo Zhang, Kun Qiu * , Chuang Liu , Hongli Ma and Zhaobi Chu

School of Electrical Engineering and Automation, Hefei University of Technology, Hefei 230009, China; 2022210486@mail.hfut.edu.cn (R.Z.); 2022170433@mail.hfut.edu.cn (C.L.); 2023010050@mail.hfut.edu.cn (H.M.); zbchu@hfut.edu.cn (Z.C.)

* Correspondence: qiukun@hfut.edu.cn

Abstract: This article investigates non-fragile synchronization control and circuit implementation for incommensurate fractional-order (IFO) chaotic neural networks with parameter uncertainties. In this paper, we explore three aspects of the research challenges, i.e., theoretical limitations of uncertain IFO systems, the fragility of the synchronization controller, and the lack of circuit implementation. First, we establish an IFO chaotic neural network model incorporating parametric uncertainties, extending beyond conventional commensurate-order architectures. Second, a novel, non-fragile state-error feedback controller is designed. Through the formulation of FO Lyapunov functions and the application of inequality scaling techniques, sufficient conditions for asymptotic synchronization of master–slave systems are rigorously derived via the multi-order fractional comparison principle. Third, an analog circuit implementation scheme utilizing FO impedance units is developed to experimentally validate synchronization efficacy and accurately replicate the system’s dynamic behavior. Numerical simulations and circuit experiments substantiate the theoretical findings, demonstrating both robustness against parameter perturbations and the feasibility of circuit realization.



Academic Editor: Gani Stamov

Received: 14 March 2025

Revised: 7 April 2025

Accepted: 17 April 2025

Published: 23 April 2025

Citation: Zhang, R.; Qiu, K.; Liu, C.; Ma, H.; Chu, Z. Comparison Principle Based Synchronization Analysis of Fractional-Order Chaotic Neural Networks with Multi-Order and Its Circuit Implementation. *Fractal Fract.* **2025**, *9*, 273. <https://doi.org/10.3390/fractalfract9050273>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Keywords: chaotic neural networks; uncertainty; synchronization control; state feedback control; circuits design

1. Introduction

Neural networks, as brain-inspired computational architectures, find ubiquitous applications spanning medical diagnostics, financial forecasting, information processing systems, and pattern-recognition algorithms [1–4]. Recent technological innovation has driven considerable research interest in chaotic neural networks, owing to their distinctive nonlinear dynamic characteristic [5–7]. The synchronization of such systems has emerged as a central research challenge, spurring rigorous investigations into control strategies, including impulsive control [8], adaptive control [9], fuzzy control [10], and feedback control [11]. Nevertheless, these methodologies consistently encounter fundamental limitations in maintaining robustness and operational precision when addressing complex nonlinear systems.

Fractional-order (FO) calculus has established itself as a compelling framework for characterizing dynamical properties in complex nonlinear systems. Compared to integer-order models, FO frameworks demonstrate enhanced characterization accuracy for electrochemical phenomena, particularly in voltage output regulation and state-of-charge esti-

mation [12]. Particularly in scenarios involving memory effects, FO calculus intrinsically encodes temporal correlations through its hereditary operators, facilitating accurate forecasting in operational scenarios spanning neural networks, bioengineering, quantum systems, and signal-processing architecture [13–18]. Particularly in neural network architectures, the incorporation of FO calculus empowers neural models to more accurately characterize neuronal dynamical properties, yielding broad applicability spanning signal-processing architectures, image encryption schemes, and secure communication protocols [19–22]. Consequently, investigating synchronization control in FO chaotic neural networks holds substantial theoretical importance and applied potential across interdisciplinary domains.

While notable advances exist in FO chaotic neural network research, current studies predominantly address commensurate FO system dynamics [23–25]. However, practical systems frequently possess multiple different orders and demonstrate heightened dynamical complexity, formally categorized as IFO chaotic systems. Their intricate dynamical characteristics render conventional analytical approaches developed for uniform FO systems inadequate for such configurations. Research on IFO chaotic systems is still in its early stages, with some seminal contributions appearing in the recent literature. Notable advances include the generalized projective synchronization framework under external perturbations [26]. In addition, their seminal study of the glucose-insulin regulatory mechanism introduced a pioneering approach to stabilize chaotic attractors, providing important insights into the application of IFO dynamics to biophysically relevant systems [27]. Recent investigations [28,29] establish rigorous synchronization criteria for IFO systems. Capitalizing on neural networks' inherent approximation capabilities, ref. [30] introduced adaptive control to resolve undetermined control directionality while developing neural-driven synchronization strategies for generalized projective IFO chaotic systems with uncertain nonlinearities. While pioneering investigations exist, a persistent theoretical–practical disconnect persists in synchronization control methodologies for IFO chaotic neural networks.

Practical implementations inevitably confront parametric uncertainties stemming from measurement inaccuracies and exogenous disturbances, which critically degrade Lyapunov stability margins and compromise synchronization control precision in neural networks. Consequently, robust synchronization control of chaotic neural networks with parametric uncertainties has become a critical research focus, demanding novel adaptive strategies. Deepika pioneered a FO sliding mode control strategy to achieve finite-time robust synchronization for uncertain chaotic/hyperchaotic systems, effectively addressing nonlinear coupling dynamics and parametric perturbations [31]. While Hajipour developed an adaptive synchronization scheme for IFO Arneodo systems with quadrivariable parametric uncertainties, their Lyapunov-based control law enables asymptotic state synchronization between coupled IFO chaotic oscillators through rigorous stability guarantees [32]. However, contemporary research remains predominantly confined to deterministic system formulations, whereas robust controller synthesis incorporating uncertainty quantification—particularly non-fragile control architectures resilient to parametric perturbations—still faces fundamental challenges in both theoretical analysis and engineering implementation.

While numerical simulation remains indispensable in theoretical exploration, circuit implementation constitutes a critical validation phase for confirming result fidelity in nonlinear dynamical systems. The aforementioned studies [33,34] have achieved FO implementations, yet the circuit-level realization of synchronization control for IFO chaotic neural networks remains unexplored, with current findings limited to numerical simulations lacking hardware validation. This study investigates synchronization control and circuit implementation of IFO chaotic neural networks, addressing parametric uncertainties induced by measurement inaccuracies and external perturbations. Concurrently, a non-fragile state-error feedback controller is developed to achieve robust synchronization

in IFO chaotic neural networks through dual compensation mechanisms for parametric perturbations and quantization errors. The main innovations of this paper are:

- This work investigates IFO chaotic neural networks with inherent parametric uncertainties, establishing enhanced generality through multi-order dynamics compared to existing chaotic system formulations.
- Ensures the state synchronization of the master–slave system can still be accomplished with a certain degree of robustness when the control protocol parameters have relatively small perturbations.
- Using the principle of multiple FO comparison, sufficient conditions are given to achieve synchronization of master–slave systems, and the obtained sufficient conditions are also applicable to FO chaotic systems of the same metric as well as integer orders.
- The design of the IFO uncertain chaotic neural network synchronization control circuit is completed, and the circuit simulation is carried out by Mutisim and the results coincide with the numerical simulation results of Matlab, which verifies the validity of the FO operator as well as the correctness of the circuit design.

The paper is organized as follows. Section 2 focuses on system description and generalization of the underlying theory. Section 3 is the main content. In Section 4, Simulation results are given for an IFO chaotic neural network analog circuit and the physical circuit is completed. Finally, Section 5 outlines the main conclusions.

The notations and their descriptions are displayed in Table 1.

Table 1. Notations and their descriptions.

Notations	Descriptions
R^n	set of n -dimensional real vectors
$R^{p \times q}$	set of $p \times q$ real matrices
I	the identity matrix with appropriate dimension
$\otimes$	the Kronecker product
$\text{diag}(\cdots)$	a block diagonal matrix
$H_+^{p \times q}$	the real symmetric positive definite $p \times q$ matrix set
X^T	the transpose of the matrix X
X^{-1}	the inverse of the matrix X
$X > 0$ (< 0)	a real symmetric and positive (negative) definite matrix
$\oplus_{i=1}^k x_i$	equivalent to $\text{diag}(x_1, x_2, \cdots, x_n)$

2. System Description and Basics

This section will introduce the theory of FO calculus and some necessary lemmas.

2.1. Fractional Calculus

The commonly used definitions of FO derivatives can be categorized into the following three: the Riemann–Liouville, Caputo, and Grunwald–Letnikov definitions. The Caputo derivative is widely adopted in control engineering due to its seamless compatibility with the integer-order system when the FO ω attains integer values. It degenerates to classical integer-order differentiation, particularly advantageous for hybrid control architectures integrating fractional and integer-order dynamics.

Definition 1 ([35]). The Riemann–Liouville fractional derivative can be defined as:

$${}^{RL}D_{0,t}^{\omega}f(t) = \frac{1}{\Gamma(n-\omega)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\omega-n+1}} d\tau, \quad (1)$$

where $\omega > 0$ being positive real non-integer numbers, n is the smallest positive integer greater than ω , i.e., $n - 1 \leq \omega < n$.

Definition 2 ([35]). The Caputo FO derivative can be written for a differentiable function $f(t)$:

$${}^CD_{0,t}^{\omega}f(t) = \frac{1}{\Gamma(n-\omega)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1-n+\omega}} d\tau, \quad (2)$$

where $\Gamma(n - \omega) = \int_0^{\infty} e^{-t} t^{n-\omega-1} dt$ denotes the Euler gamma function.

Definition 3 ([35]). The Grunwald–Letnikov derivative of order ω is

$${}^{GL}D_{0,t}^{\omega}f(t) = \lim_{h \rightarrow 0} \frac{1}{h^{\omega}} \sum_{j=0}^{[(t-t_0)/h]} (-1)^j \binom{\omega}{j} f(t - jh). \quad (3)$$

Please note that the Caputo derivative is widely applicable in practice and will be used in this paper. From now on, to simplify the notation, we will use D_t^{ω} , which means we omit the left superscript and the operator limit.

2.2. Relevant Lemmas

Lemma 1 ([36]). For a continuous function $x(t) \in \mathbb{R}^n$ and any given symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ we verify that

$$D_t^{\omega}(x^T(t)Px(t)) \leq (x^T(t)P)D_t^{\omega}x(t) + (D_t^{\omega}x(t))^T Px(t), \quad (4)$$

where $\omega \in (0, 1]$.

Lemma 2 ([37]). (Principle of Comparing Multi-order FO Systems) Consider the following FO differential inequalities with initial conditions $V_i(0, x_i(0)) = V_{i0} \geq 0$ and $V_i(0, x_i(0)) \leq W_i(0, y_i(0))$, $i = 1, 2, \dots, n$ and conditions:

$$\begin{aligned} D_t^{\omega_1} V_1(t, x_1(t)) &\leq D_t^{\omega_1} W_1(t, y_1(t)), \\ D_t^{\omega_2} V_2(t, x_2(t)) &\leq D_t^{\omega_2} W_2(t, y_2(t)), \\ &\vdots \\ D_t^{\omega_n} V_n(t, x_n(t)) &\leq D_t^{\omega_n} W_n(t, y_n(t)). \end{aligned} \quad (5)$$

If previous inequalities hold, then the following inequalities hold:

$$\begin{aligned} V_1(x_i(t)) &\leq W_i(y_i(t)), \forall t > 0, i = 1, 2, \dots, n, \\ V(x(t)) &\leq W(y(t)), \forall t > 0, \end{aligned} \quad (6)$$

where V and $W: [0, \infty) \times \mathbb{R}^n \rightarrow [0, \infty)$ are continuously differentiable functions.

Lemma 3 ([38]). The FO multi-order system $D_t^{\bar{\omega}}x(t) = A_0x(t) + B_0u(t)$, $\bar{\omega} = (\omega_1, \omega_2, \dots, \omega_n)^T$, is stabilizable under the state feedback controller $u(t) = Kx(t)$, if there exist symmetric positive

definite matrices $P_i \in H_+^{t_i \times t_i}, i = 1, 2, \dots, k$, the matrices $H \in R^{n \times n}, Q \in R^{p \times n}$, the following equation holds:

$$M + \text{sym} \left\{ \left(\begin{bmatrix} I_n \\ -A_0 \end{bmatrix} \otimes I_2 \right) (H \otimes I_2) ([I_n \ I_n] \otimes I_2) \right\} + \text{sym} \left\{ \left(\begin{bmatrix} 0_{n \times p} \\ -B_0 \end{bmatrix} \otimes I_2 \right) (Q \otimes I_2) ([I_n \ I_n] \otimes I_2) \right\} < 0. \quad (7)$$

Moreover, the controller feedback gain is given by $K = QH^{-1}$.

2.3. System Description

We consider the uncertain IFO neural network master chaotic system:

$$\frac{d^{\omega_i} x_i(t)}{dt^{\omega_i}} = -(c_i + \Delta c_i)x_i(t) + \sum_{j=1}^n (a_{ij} + \Delta a_{ij})f_j(x_j(t)) + I_i. \quad (8)$$

The matrix representation of the system is formulated as:

$$\frac{d^{\bar{\omega}} x(t)}{dt^{\bar{\omega}}} = -(C + \Delta C)x(t) + (A + \Delta A)f(x(t)) + I, \quad (9)$$

where $i = \{1, 2, \dots, n\}$, n is the number of neurons in the network, $\bar{\omega} = (\omega_1, \omega_2, \dots, \omega_n)^T$ belongs to $(0, 1]$. $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in R^n$ is the state of the neuron at the moment t , $f(x(t)) = (f_1(x(t)), f_2(x(t)), \dots, f_n(x(t)))^T$ is the excitation nonlinear function of the neuron. $C = \text{diag}(c_1, \dots, c_n)$, $A = (a_{ij})_{n \times n}$ are the weight matrices from the j -th neuron to the i -th neuron. $I = (I_1, I_2, \dots, I_n)^T$ is an external input. $\Delta A = (\Delta a_{ij})_{n \times n} \in R^{n \times n}$ and $\Delta C = (\Delta c_{ij})_{n \times n} \in R^{n \times n}$ are uncertain parameter perturbation matrices.

System (9) can be described in vector form:

$$\begin{bmatrix} \frac{d^{\omega_1} x_1(t)}{dt^{\omega_1}} \\ \frac{d^{\omega_2} x_2(t)}{dt^{\omega_2}} \\ \vdots \\ \frac{d^{\omega_n} x_n(t)}{dt^{\omega_n}} \end{bmatrix} = - \begin{bmatrix} c_{11} + \Delta c_{11} & c_{12} + \Delta c_{12} & \cdots & c_{1n} + \Delta c_{1n} \\ c_{21} + \Delta c_{21} & c_{22} + \Delta c_{22} & \cdots & c_{2n} + \Delta c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} + \Delta c_{n1} & c_{n2} + \Delta c_{n2} & \cdots & c_{nn} + \Delta c_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} a_{11} + \Delta a_{11} & a_{12} + \Delta a_{12} & \cdots & a_{1n} + \Delta a_{1n} \\ a_{21} + \Delta a_{21} & a_{22} + \Delta a_{22} & \cdots & a_{2n} + \Delta a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + \Delta a_{n1} & a_{n2} + \Delta a_{n2} & \cdots & a_{nn} + \Delta a_{nn} \end{bmatrix} \begin{bmatrix} f_1(x(t)) \\ f_2(x(t)) \\ \vdots \\ f_n(x(t)) \end{bmatrix} + \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix}. \quad (10)$$

where the uncertainty matrix elements satisfy $|\Delta a_{ij}| \leq \bar{a}_{ij}$, $|\Delta c_{ij}| \leq \bar{c}_{ij}$. $\bar{a}_{ij}, \bar{c}_{ij}$ are positive constants. The excitation function f_j of a neuron is Lipschitz continuous, i.e., there exists $L_j (j = 1, 2, \dots, n)$ such that $|f_j(u_j) - f_j(v_j)| \leq L_j |u_j - v_j|, \forall u_j, v_j \in R$.

Remark 1. It should be pointed out that widely used excitation functions in neural networks are Lipschitz continuous. It accommodates standard excitation functions, ensures theoretical analysis, and aligns with practical requirements.

We assume the IFO neural network slave chaotic system:

$$\frac{d^{\omega_i} y_i(t)}{dt^{\omega_i}} = -(c_i + \Delta c_i)y_i(t) + \sum_{j=1}^n (a_{ij} + \Delta a_{ij})f_j(y_j(t)) + I_i + u_i(t). \quad (11)$$

Its matrix form is

$$\frac{d^{\omega} y(t)}{dt^{\omega}} = -(C + \Delta C)y(t) + (A + \Delta A)f(y(t)) + I + u(t). \quad (12)$$

It can also be written as:

$$\begin{aligned} \begin{bmatrix} \frac{d^{\omega_1} y_1(t)}{dt^{\omega_1}} \\ \frac{d^{\omega_2} y_2(t)}{dt^{\omega_2}} \\ \vdots \\ \frac{d^{\omega_n} y_n(t)}{dt^{\omega_n}} \end{bmatrix} &= - \begin{bmatrix} c_{11} + \Delta c_{11} & c_{12} + \Delta c_{12} & \cdots & c_{1n} + \Delta c_{1n} \\ c_{21} + \Delta c_{21} & c_{22} + \Delta c_{22} & \cdots & c_{2n} + \Delta c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} + \Delta c_{n1} & c_{n2} + \Delta c_{n2} & \cdots & c_{nn} + \Delta c_{nn} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} \\ &+ \begin{bmatrix} a_{11} + \Delta a_{11} & a_{12} + \Delta a_{12} & \cdots & a_{1n} + \Delta a_{1n} \\ a_{21} + \Delta a_{21} & a_{22} + \Delta a_{22} & \cdots & a_{2n} + \Delta a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + \Delta a_{n1} & a_{n2} + \Delta a_{n2} & \cdots & a_{nn} + \Delta a_{nn} \end{bmatrix} \begin{bmatrix} f_1(y(t)) \\ f_2(y(t)) \\ \vdots \\ f_n(y(t)) \end{bmatrix} + \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}. \end{aligned} \quad (13)$$

where $y(t) = (y_1(t), \dots, y_n(t))^T \in R^n$ stands for slave system state. C , A and $f(\cdot)$ are consistent with those in Equation (8). $u(t) = (u_1(t), \dots, u_n(t))^T$ is the external control input.

The synchronization error is defined as the difference between the states of the master and slave systems and is expressed as follows:

$$e(t) = y(t) - x(t). \quad (14)$$

From (8), (11), and (14), it follows that the error system can be written in the following form:

$$\frac{d^{\omega} e(t)}{dt^{\omega}} = -(C + \Delta C)e(t) + (A + \Delta A)(f(y(t)) - f(x(t))) + u(t), \quad (15)$$

where $e(t) = (e_1(t), \dots, e_n(t))^T$.

Our goal is to design a non-fragile feedback controller:

$$u(t) = (K + \Delta K)e(t), \quad (16)$$

where $K = \text{diag}(k_1, k_2, \dots, k_n)$ is the feedback gain, $\Delta K = \text{diag}(\Delta k_1, \Delta k_2, \dots, \Delta k_n)$, $|\Delta k_i| \leq \bar{k}_i$.

The error system is

$$\frac{d^{\omega} e(t)}{dt^{\omega}} = [(K + \Delta K) - (C + \Delta C)]e(t) + (A + \Delta A)(f(y(t)) - f(x(t))), \quad (17)$$

or

$$\frac{d^{\omega_i} e_i(t)}{dt^{\omega_i}} = - \sum_{j=1}^n (c_{ij} + \Delta c_{ij})e_j(t) + (k_i + \Delta k_i)e_i(t) + \sum_{j=1}^n (a_{ij} + \Delta a_{ij})(f_i(y(t)) - f_i(x(t))). \quad (18)$$

Herein, the control objective is that the trajectory of the slave asymptotically approaches the one of the master, for given master and slave initial conditions, $x(0)$ and $y(0)$, respectively. This translates into:

$$\lim_{t \rightarrow \infty} \|e(t)\| = \lim_{t \rightarrow \infty} \|y(t) - x(t)\| = 0. \quad (19)$$

System (17) can also be written as:

$$\begin{aligned}
\begin{bmatrix} \frac{d^{\omega_1} e_1(t)}{dt^{\omega_1}} \\ \frac{d^{\omega_2} e_2(t)}{dt^{\omega_2}} \\ \vdots \\ \frac{d^{\omega_n} e_n(t)}{dt^{\omega_n}} \end{bmatrix} &= - \begin{bmatrix} (c_{11} + \Delta c_{11} - k_1 - \Delta k_1) & (c_{12} + \Delta c_{12}) & \cdots & (c_{1n} + \Delta c_{1n}) \\ (c_{21} + \Delta c_{21}) & (c_{22} + \Delta c_{22} - k_2 - \Delta k_2) & \cdots & (c_{2n} + \Delta c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (c_{n1} + \Delta c_{n1}) & (c_{n2} + \Delta c_{n2}) & \cdots & (c_{nn} + \Delta c_{nn} - k_n - \Delta k_n) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_n(t) \end{bmatrix} \\
&+ \begin{bmatrix} a_{11} + \Delta a_{11} & a_{12} + \Delta a_{12} & \cdots & a_{1n} + \Delta a_{1n} \\ a_{21} + \Delta a_{21} & a_{22} + \Delta a_{22} & \cdots & a_{2n} + \Delta a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + \Delta a_{n1} & a_{n2} + \Delta a_{n2} & \cdots & a_{nn} + \Delta a_{nn} \end{bmatrix} \begin{bmatrix} f_1(y(t)) - f_1(x(t)) \\ f_2(y(t)) - f_2(x(t)) \\ \vdots \\ f_n(y(t)) - f_n(x(t)) \end{bmatrix}. \quad (20)
\end{aligned}$$

Then, the design problem involving (8) and (11) can be transformed into a stability problem for the error system (17).

Remark 2. In the existing literature [23–25], the synchronization control design addresses the commensurate-order case $\omega_i = \omega$. It is not difficult to find that commensurate order is a special case of incommensurate-order systems. It is more general to investigate IFO chaotic neural networks, which provide a more accurate description of the dynamics of complex systems and offer additional possibilities. Although the synchronous control of IFO chaotic systems has been studied in the literature [29] based on linear feedback, the results therein cannot be directly generalized to the IFO chaotic neural networks studied in this paper.

3. Main Elements

In this section, we establish the following theorem as a fundamental theoretical result of this work, and the following theorem provides a sufficient stabilization condition.

Theorem 1. If there exist matrices $P_i \in H_+^{t_i \times t_i}$, $i = 1, 2, \dots, k$, $H \in H_+^{n \times n}$ and $Q \in H^{p \times n}$ such that $BQ > 0$ holds and satisfies

$$\begin{aligned}
&M + \text{sym} \left\{ \left(\begin{bmatrix} I_n \\ -\tilde{A} \end{bmatrix} \otimes I_2 \right) (H \otimes I_2) ([I_n I_n] \otimes I_2) \right\} \\
&+ \text{sym} \left\{ \left(\begin{bmatrix} 0_{n \times p} \\ -B \end{bmatrix} \otimes I_2 \right) (Q \otimes I_2) ([I_n I_n] \otimes I_2) \right\} < 0, \quad (21)
\end{aligned}$$

where

$$\tilde{A} = \begin{bmatrix} \tilde{\varphi}_{11} & \tilde{\varphi}_{12} & \cdots & \tilde{\varphi}_{1n} \\ \tilde{\varphi}_{21} & \tilde{\varphi}_{22} & \cdots & \tilde{\varphi}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\varphi}_{n1} & \tilde{\varphi}_{n2} & \cdots & \tilde{\varphi}_{nn} \end{bmatrix},$$

$$\tilde{\varphi}_{ii} = -2c_{ii} - 2\bar{c}_{ii} + 2(|a_{ii}| + \bar{a}_{ii})L_{ii} + 2k_i + 2\bar{k}_i + \sum_{j=1, i \neq j}^n (-|c_{ij}| - \bar{c}_{ij} + (|a_{ij}| + \bar{a}_{ij})L_{ij}),$$

$$\tilde{\varphi}_{ij} = \sum_{j=1, i \neq j}^n (-|c_{ij}| - \bar{c}_{ij} + (|a_{ij}| + \bar{a}_{ij})L_{ij}) \quad (i = 1, 2, \dots, n, i \neq j),$$

then system (17) is stable. Additionally, the feedback gain is $K = QH^{-1}$.

Proof. For the error system (17), we choose the following Lyapunov candidate function

$$V(t) = \sum_{i=1}^n V_i(t) = \sum_{i=1}^n e_i^2(t), \quad (22)$$

With Lemma 1 in mind, and calculating the ω_i -order derivatives of $V_i(t)$ along the solution of (17), yields:

$$\begin{aligned}
 D_t^{\omega_1} V_1(t) &\leq 2e_1(t) D_t^{\omega_1} e_1(t) \\
 &= 2e_1(t) \left(- \sum_{j=1}^n (c_{1j} + \Delta c_{1j}) e_j(t) + (k_1 + \Delta k_1) e_1(t) \right. \\
 &\quad \left. + \sum_{j=1}^n (a_{1j} + \Delta a_{1j}) (f_1(y(t)) - f_1(x(t))) \right) \\
 &\leq \sum_{j=2}^n (-|c_{1j}| - \bar{c}_{1j} + (|a_{1j}| + \bar{a}_{1j}) L_{1j}) e_1^2(t) \\
 &\quad + (-2c_{11} - 2\bar{c}_{11} + 2(|a_{11}| + \bar{a}_{11}) L_{11} + 2k_1 + 2\bar{k}_1) e_1^2(t) \\
 &\quad + (-|c_{12}| - \bar{c}_{12} + (|a_{12}| + \bar{a}_{12}) L_{12}) e_2^2(t) \\
 &\quad + \cdots + (-|c_{1n}| - \bar{c}_{1n} + (|a_{1n}| + \bar{a}_{1n}) L_{1n}) e_n^2(t) \\
 &= \sum_{j=2}^n (-|c_{1j}| - \bar{c}_{1j} + (|a_{1j}| + \bar{a}_{1j}) L_{1j}) V_1(t) \\
 &\quad + (-2c_{11} - 2\bar{c}_{11} + 2(|a_{11}| + \bar{a}_{11}) L_{11} + 2k_1 + 2\bar{k}_1) V_1(t) \\
 &\quad + (-|c_{12}| - \bar{c}_{12} + (|a_{12}| + \bar{a}_{12}) L_{12}) V_2(t) \\
 &\quad + \cdots + (-|c_{1n}| - \bar{c}_{1n} + (|a_{1n}| + \bar{a}_{1n}) L_{1n}) V_n(t) \\
 D_t^{\omega_2} V_2(t) &\leq 2e_2(t) D_t^{\omega_2} e_2(t) \\
 &= 2e_2(t) \left(- \sum_{j=1}^n (c_{2j} + \Delta c_{2j}) e_j(t) + (k_2 + \Delta k_2) e_2(t) \right. \\
 &\quad \left. + \sum_{j=1}^n (a_{2j} + \Delta a_{2j}) (f_2(y(t)) - f_2(x(t))) \right) \\
 &\leq \sum_{j=1, j \neq 2}^n (-|c_{2j}| - \bar{c}_{2j} + (|a_{2j}| + \bar{a}_{2j}) L_{2j}) e_2^2(t) \\
 &\quad + (-2c_{22} - 2\bar{c}_{22} + 2(|a_{22}| + \bar{a}_{22}) L_{22} + 2k_2 + 2\bar{k}_2) e_2^2(t) \\
 &\quad + (-|c_{21}| - \bar{c}_{21} + (|a_{21}| + \bar{a}_{21}) L_{21}) e_1^2(t) \\
 &\quad + \cdots + (-|c_{2n}| - \bar{c}_{2n} + (|a_{2n}| + \bar{a}_{2n}) L_{2n}) e_n^2(t) \\
 &= \sum_{j=1, j \neq 2}^n (-|c_{2j}| - \bar{c}_{2j} + (|a_{2j}| + \bar{a}_{2j}) L_{2j}) V_2(t) \\
 &\quad + (-2c_{22} - 2\bar{c}_{22} + 2(|a_{22}| + \bar{a}_{22}) L_{22} + 2k_2 + 2\bar{k}_2) V_2(t) \\
 &\quad + (-|c_{21}| - \bar{c}_{21} + (|a_{21}| + \bar{a}_{21}) L_{21}) V_1(t) \\
 &\quad + \cdots + (-|c_{2n}| - \bar{c}_{2n} + (|a_{2n}| + \bar{a}_{2n}) L_{2n}) V_n(t) \\
 &\quad \vdots \\
 D_t^{\omega_n} V_n(t) &\leq 2e_n(t) D_t^{\omega_n} e_n(t) \\
 &= 2e_n(t) \left(- \sum_{j=1}^n (c_{nj} + \Delta c_{nj}) e_j(t) + (k_n + \Delta k_n) e_n(t) \right. \\
 &\quad \left. + \sum_{j=1}^n (a_{nj} + \Delta a_{nj}) (f_i(y(t)) - f_i(x(t))) \right) \\
 &\leq \sum_{j=1}^{n-1} (-|c_{nj}| - \bar{c}_{nj} + (|a_{nj}| + \bar{a}_{nj}) L_{nj}) e_n^2(t) \\
 &\quad + (-2c_{nn} - 2\bar{c}_{nn} + 2(|a_{nn}| + \bar{a}_{nn}) L_{nn} + 2k_n + 2\bar{k}_n) e_n^2(t) \\
 &\quad + (-|c_{n1}| - \bar{c}_{n1} + (|a_{n1}| + \bar{a}_{n1}) L_{n1}) e_1^2(t) \\
 &\quad + \cdots + (-|c_{nn-1}| - \bar{c}_{nn-1} + (|a_{nn-1}| + \bar{a}_{nn-1}) L_{nn-1}) e_{n-1}^2(t) \\
 &= \sum_{j=1}^{n-1} (-|c_{nj}| - \bar{c}_{nj} + (|a_{nj}| + \bar{a}_{nj}) L_{nj}) V_n(t) \\
 &\quad + (-2c_{nn} - 2\bar{c}_{nn} + 2(|a_{nn}| + \bar{a}_{nn}) L_{nn} + 2k_n + 2\bar{k}_n) V_n(t) \\
 &\quad + (-|c_{n1}| - \bar{c}_{n1} + (|a_{n1}| + \bar{a}_{n1}) L_{n1}) V_1(t) \\
 &\quad + \cdots + (-|c_{nn-1}| - \bar{c}_{nn-1} + (|a_{nn-1}| + \bar{a}_{nn-1}) L_{nn-1}) V_{n-1}(t).
 \end{aligned} \tag{23}$$

From (23), the corresponding comparison system is as follows:

$$\begin{bmatrix} D_t^{\omega_1} W_1(t) \\ D_t^{\omega_2} W_2(t) \\ \vdots \\ D_t^{\omega_n} W_n(t) \end{bmatrix} = \begin{bmatrix} \bar{\varphi}_{11} & \bar{\varphi}_{12} & \cdots & \bar{\varphi}_{1n} \\ \bar{\varphi}_{21} & \bar{\varphi}_{22} & \cdots & \bar{\varphi}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\varphi}_{n1} & \bar{\varphi}_{n2} & \cdots & \bar{\varphi}_{nn} \end{bmatrix} \begin{bmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_n(t) \end{bmatrix}, \quad (24)$$

where

$$\tilde{\varphi}_{ii} = -2c_{ii} - 2\bar{c}_{ii} + 2(|a_{ii}| + \bar{a}_{ii})L_{ii} + 2k_i + 2\bar{k}_i + \sum_{j=1, i \neq j}^n (-|c_{ij}| - \bar{c}_{ij} + (|a_{ij}| + \bar{a}_{ij})L_{ij}), \quad (25)$$

$$\tilde{\varphi}_{ij} = \sum_{j=1, i \neq j}^n (-|c_{ij}| - \bar{c}_{ij} + (|a_{ij}| + \bar{a}_{ij})L_{ij}) \quad (i = 1, 2, \dots, n, i \neq j). \quad (26)$$

The system can be rewritten to:

$$D_t^{\bar{\omega}} W(t) = \tilde{A}W(t) + B\bar{K}W(t) = \tilde{A}W(t) + Bu(t), \quad (27)$$

$$\tilde{\varphi}_{ii} = \tilde{\varphi}_{ii} - 2k_i, \tilde{\varphi}_{ij} = \tilde{\varphi}_{ij}, \quad (28)$$

where $W(t) = (W_1(t), W_2(t), \dots, W_n(t))^T$, it follows from Lemma 3 that the controlled system (27) is stable if the appropriate condition is satisfied, i.e., $W(t) \rightarrow 0$. Based on the comparison principle of FO systems with multi-order, we have $V(t) \leq W(t)$. Therefore, the error system (17) is also stable. This ends the proof. $\square$

Remark 3. Recent works have studied the stability and stabilization of IFO uncertain nonlinear systems based on the comparison principle but have not considered the synchronization control problem [36]. In literature [27], the authors studied the synchronization problem of IFO chaotic systems with linear feedback control based on the comparative principle. It is a great pity that the presence of uncertainty is not considered, and only numerical simulation results are available.

4. Numerical Examples

4.1. Numerical Simulation

The state equation of an uncertain IFO neural network master system can be represented by the following equation:

$$\begin{cases} D_t^{\omega_1} x_1(t) = -(1 + \Delta c_{11})x_1 + 2 \tanh x_1 - \tanh x_2, \\ D_t^{\omega_2} x_2(t) = -(1 + \Delta c_{22})x_2 + 1.7 \tanh x_1 + 1.71 \tanh x_2 + 1.1 \tanh x_3, \\ D_t^{\omega_3} x_3(t) = -(2 + \Delta c_{33})x_3 - 2.6 \tanh x_1 - 2.8 \tanh x_2 + 1.25 \tanh x_3, \end{cases} \quad (29)$$

where $\omega_1 = 0.99$, $\omega_2 = 0.993$, $\omega_3 = 1$. Parameter uncertainties are chosen as $\Delta c_{11} = 0.2 \sin(t)$, $\Delta c_{22} = 0.3 \sin(t)$, $\Delta c_{33} = 0.1 \sin(t)$. The 3-D chaotic attractors of the system (29) when its initial state is selected as $x(0) = (0.5, 0.2, -0.5)^T$ is shown in Figure 1. In addition, Figure 2 shows the projection of the chaotic attractor of the system (29) on $x_1(t) - x_2(t)$, $x_2(t) - x_3(t)$, $x_1(t) - x_3(t)$. Based on the boundedness and phase space diagram of the system (29), we can observe the Lipschitz constant $L_{11} = 2$, $L_{12} = 1$, $L_{13} = 0$, $L_{21} = 1.7$, $L_{22} = 1.71$, $L_{23} = 1.1$, $L_{31} = 2.6$, $L_{32} = 2.8$, $L_{33} = 1.25$.

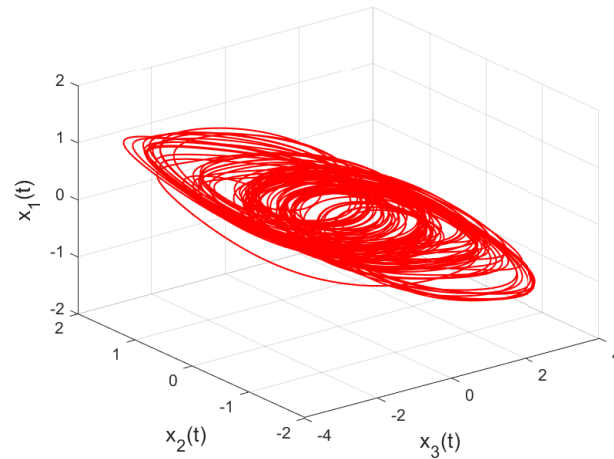


Figure 1. 3-D chaotic attractors for the IFO neural network system (29).

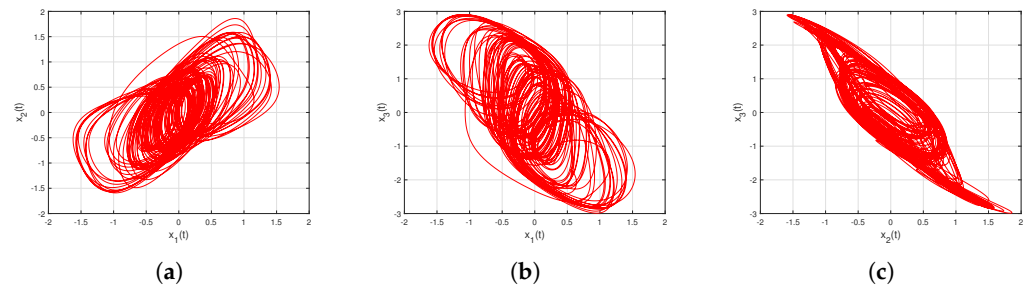


Figure 2. Examples of phase diagrams for chaotic systems with FO neural networks: (a) chaotic attractor on the $x_1(t) - x_2(t)$ space; (b) chaotic attractor projected on the $x_1(t) - x_3(t)$ plane; (c) chaotic attractor on the $x_2(t) - x_3(t)$ space.

The slave system can be selected as follows:

$$\begin{cases} D_t^{\omega_1} y_1(t) = -(1 + \Delta c_{11})y_1 + 2 \tanh y_1 - \tanh y_2 + u_1, \\ D_t^{\omega_2} y_2(t) = -(1 + \Delta c_{22})y_2 + 1.7 \tanh y_1 + 1.71 \tanh y_2 + 1.1 \tanh y_3 + u_2, \\ D_t^{\omega_3} y_3(t) = -(2 + \Delta c_{33})y_3 - 2.6 \tanh y_1 - 2.8 \tanh y_2 + 1.25 \tanh y_3 + u_3. \end{cases} \quad (30)$$

The initial state of the slave system (30) is chosen as $y(0) = (0.3, 0.5, -0.7)^T$. Define the state error as $e_1(t) = y_1(t) - x_1(t)$; $e_2(t) = y_2(t) - x_2(t)$; $e_3(t) = y_3(t) - x_3(t)$ and define that $u_i(t) = (k_i + \Delta k_i)e_i(t)$, $i = 1, 2, 3$. Therefore, the state equation of the error system is:

$$\begin{cases} D_t^{\omega_1} e_1(t) = -(1 + \Delta c_{11})e_1 + 2(\tanh y_1 - \tanh x_1) - (\tanh y_2 - \tanh x_2) + (k_1 + \Delta k_1)e_1, \\ D_t^{\omega_2} e_2(t) = -(1 + \Delta c_{22})e_2 + 1.7(\tanh y_1 - \tanh x_1) + 1.71(\tanh y_2 - \tanh x_2) \\ \quad + 1.1(\tanh y_3 - \tanh x_3) + (k_2 + \Delta k_2)e_2, \\ D_t^{\omega_3} e_3(t) = -(2 + \Delta c_{33})e_3 - 2.6(\tanh y_1 - \tanh x_1) - 2.8(\tanh y_2 - \tanh x_2) \\ \quad + 1.25(\tanh y_3 - \tanh x_3) + (k_3 + \Delta k_3)e_3, \end{cases} \quad (31)$$

where $\Delta k_i = 0.1 \sin(t)$, $i = 1, 2, 3$.

By calculation, we can obtain $\tilde{A} = \begin{bmatrix} 6.8 & 1 & 0 \\ 1.7 & 7.82 & 1.1 \\ 2.6 & 2.8 & 11.9 \end{bmatrix}$, $B = I_3$.

Based on the constraints of the LMI in Theorem 1, the following feasible solution is obtained using the LMI control toolbox in Matlab (i.e., YALMIP):

$$Q = \begin{bmatrix} 33.8773 & 0 & 0 \\ 0 & 48.3640 & 0 \\ 0 & 0 & 77.1769 \end{bmatrix}, H = \begin{bmatrix} -2.5839 & 0 & 0 \\ 0 & -3.5697 & 0 \\ 0 & 0 & -11.5700 \end{bmatrix},$$

$$K = QH^{-1} = \begin{bmatrix} -13.1110 & 0 & 0 \\ 0 & -13.5485 & 0 \\ 0 & 0 & -6.6704 \end{bmatrix},$$

$$P_1 = 25.3066, P_2 = 27.4804, P_3 = 34.2507.$$

The master and slave state trajectories of the IFO neural network systems (29) and (30) are shown in Figure 3, which effectively verify that synchronization has been achieved. Meanwhile, Figure 4 shows the synchronization errors of the error system (31), and the results shown in Figure 4 demonstrate the effectiveness of the designed controller.

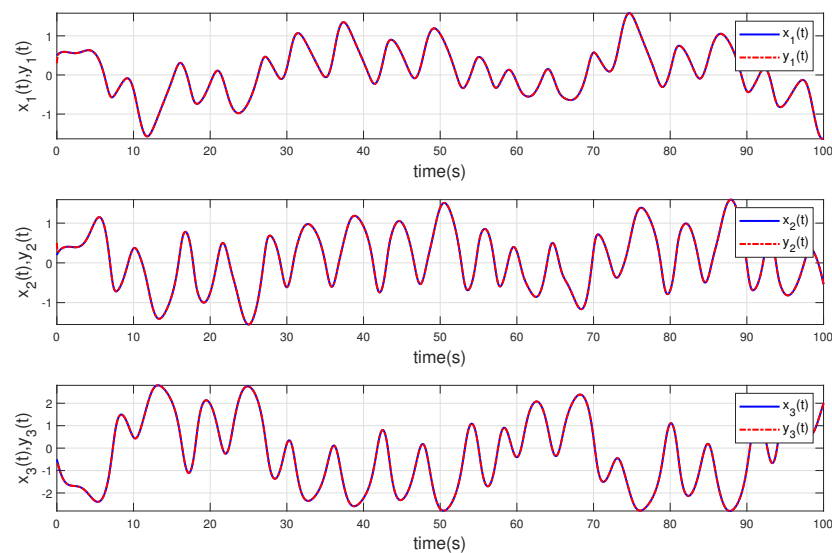


Figure 3. Master and slave state trajectories of the IFO neural network systems (29) and (30).

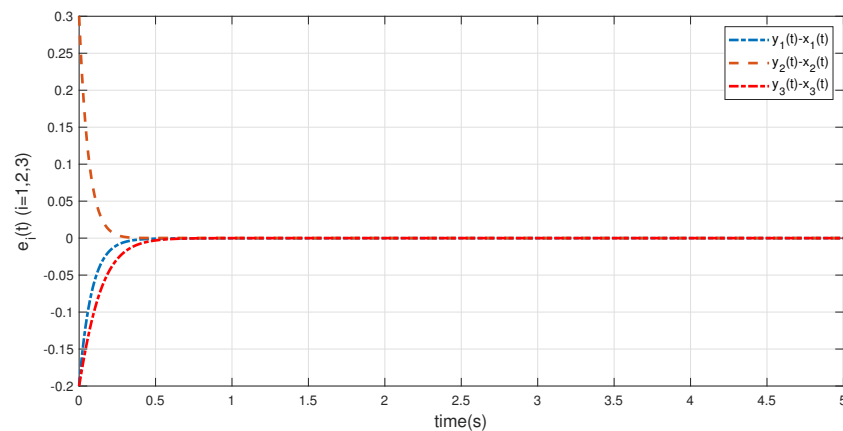


Figure 4. Synchronization errors of IFO neural network systems (29) and (30).

4.2. Circuit Implementation

To enable hardware-level verification of dynamic performance and support subsequent engineering development, the corresponding analog circuit is implemented in this study.

4.2.1. Realization of the FO Circuit

To characterize the FO system's frequency-domain behavior, we develop a frequency-domain approximation circuit for dynamic analysis. In the scope of $[10^{-2} \ 10^2]$ rad/s for ω , there exists an approximation expression with maximum distortion 0.2 dB [39]:

$$\frac{9.77237(1+1.890215 \times 10^{-6}s)(1+9.32603 \times 10^{-4}s)(1+0.097701s)}{(1+8.49753 \times 10^{-6}s)(1+8.90215 \times 10^{-4}s)(1+0.0932603s)(1+9.77s)} = 1/s^{0.99}, \quad (32)$$

where s denotes the complex frequency.

The corresponding transfer function $F(s)$ to realize the unit circuit is given in the analog circuit:

$$\frac{\left(\frac{C_0}{C_1} + \frac{C_0}{C_2}\right) \left[s^2 + \frac{\left(\frac{C_2+C_1}{R_1} + \frac{R_1+R_2}{R_1R_2}\right)s + \frac{R_1+R_2}{R_1R_2}}{C_1C_2} \right]}{C_0(s+1/R_1C_1)(s+1/R_2C_2)} = F(s), \quad (33)$$

where C_0 denotes the specific capacitance, C_1 and C_2 are the selected capacitors, and R_1 and R_2 represent the selected resistors.

Based on the aforementioned approximation method, the approximate transfer function of $1/s^{0.993}$ is defined in the frequency domain with a maximum deviation of 0.2 dB within the range of 0.01~100 rad/s, as follows [40]:

$$\frac{9.84011(1+1.80095 \times 10^{-5}s)(1+1.35765 \times 10^{-2}s)}{(1+1.71934 \times 10^{-5}s)(1+1.29612 \times 10^{-2}s)(1+9.77079s)} = 1/s^{0.993}. \quad (34)$$

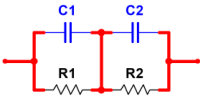
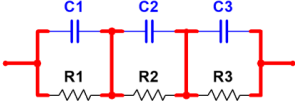
The corresponding transfer function $F(s)$ to realize the unit circuit is given in the analog circuit:

$$\frac{\left(\frac{C_0}{C_1} + \frac{C_0}{C_2} + \frac{C_0}{C_3}\right) \left[s^2 + \frac{\left(\frac{C_2+C_3}{R_1} + \frac{C_1+C_3}{R_2} + \frac{C_1+C_2}{R_3}\right)s + \frac{R_1+R_2+R_3}{R_1R_2R_3}}{C_1C_2+C_1C_3+C_2C_3} \right]}{C_0(s+1/R_1C_1)(s+1/R_2C_2)(s+1/R_3C_3)} = F(s), \quad (35)$$

where C_0 is the specific capacitance, C_1 , C_2 and C_3 denote the selected capacitors, and R_1 , R_2 and R_3 are the selected resistors.

In order to build a unit circuit of $F(s)$, the circuit diagram corresponding to fractional order and their parameters are displayed in Table 2.

Table 2. The circuit diagram corresponding to fractional order and their parameters.

Fractional Order	Circuit Diagram	Circuit Parameters	
$\omega = 0.99$		$C_1 = 10.002 \text{ nF}$ $C_2 = 208.054 \text{ nF}$	$R_1 = 976.784 \text{ M}\Omega$ $R_2 = 448.251 \text{ k}\Omega$
$\omega = 0.993$		$C_1 = 9.93018 \text{ nF}$ $C_2 = 208.9 \text{ nF}$ $C_3 = 199.7 \text{ nF}$	$R_1 = 983.949 \text{ M}\Omega$ $R_2 = 62.0449 \text{ k}\Omega$ $R_3 = 80.0960 \Omega$

4.2.2. Design of the Multisim Circuits

According to the system Equation (31), the circuit model of the ternary cellular neural network is constructed by applying the principles of the inverting adder circuit, the inverting integrator circuit, and the inverting circuit in analog circuitry. This model is then integrated with the output function module and the FO module and simulated in Multisim, as illustrated in Figure 5.

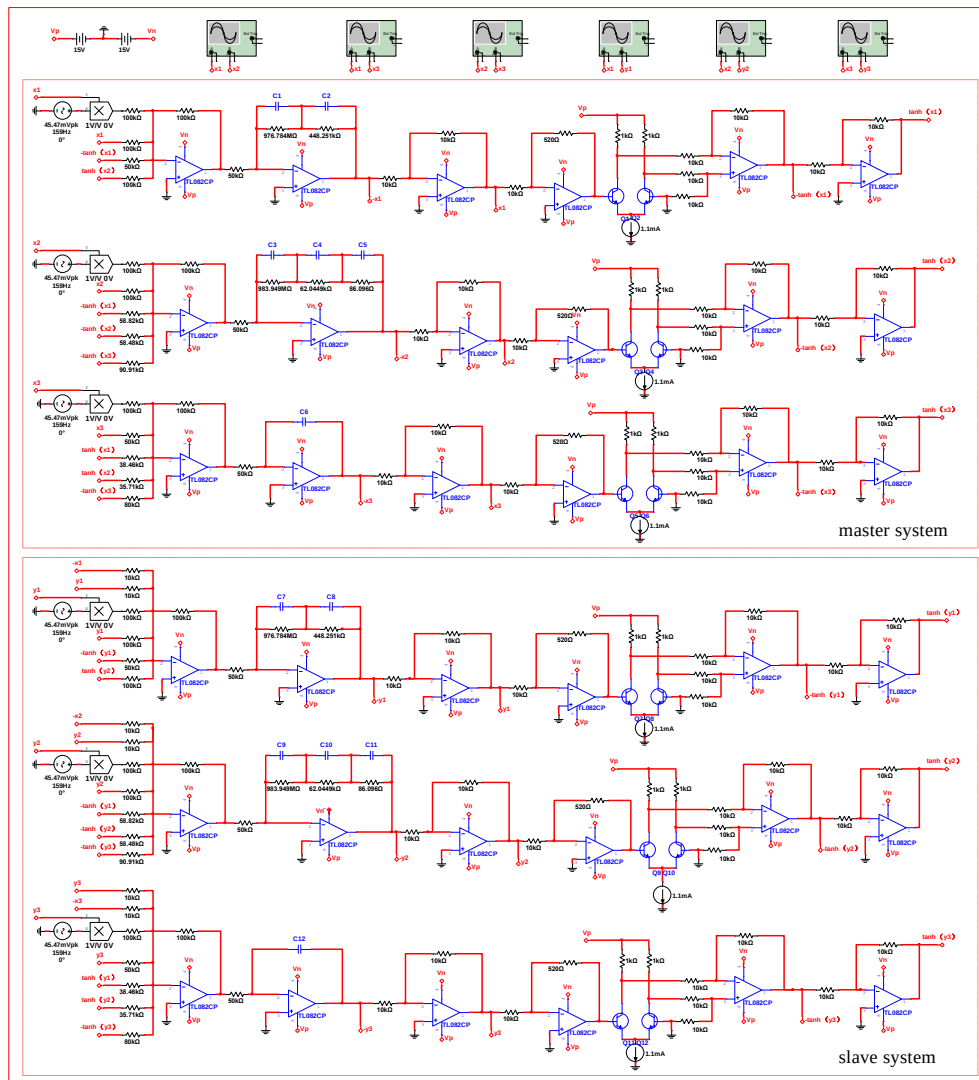


Figure 5. Overall circuit synchronization diagram.

The phase portrait of the chaotic attractor generated by the IFO neural network system is depicted in Figure 6, while Figure 7 illustrates the relationship between the corresponding synchronization state vectors. These figures demonstrate that the master–slave system achieves synchronization through the regulation of the linear feedback controller. Furthermore, the results align with both the Matlab simulation and the circuit simulation outcomes, thereby validating the correctness of the circuit design presented in this section.

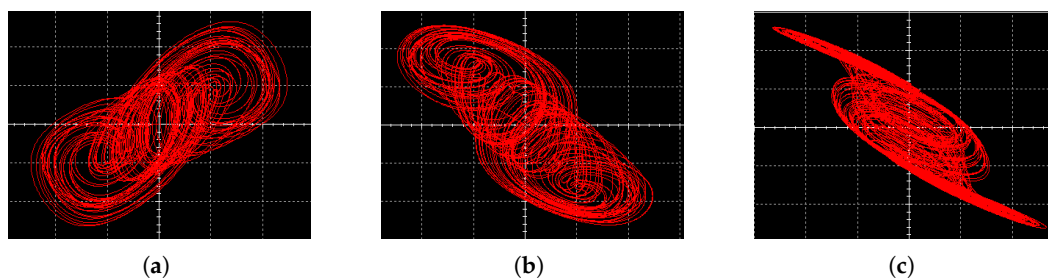


Figure 6. IFO Phase trajectory diagram of a chaotic neural network system: (a) $x_1(t) - x_2(t)$ phase diagram (b) $x_1(t) - x_3(t)$ phase diagram (c) $x_2(t) - x_3(t)$ phase diagram.

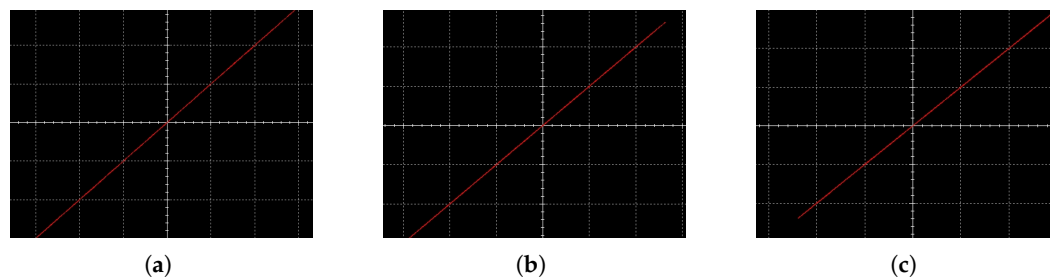


Figure 7. Synchronization curves for master–slave IFO chaotic neural network systems: (a) $x_1(t) - x_2(t)$ phase diagram (b) $x_1(t) - x_3(t)$ phase diagram (c) $x_2(t) - x_3(t)$ phase diagram.

5. Conclusions

This paper addressed the synchronization control and circuit implementation of IFO chaotic neural networks with parameter uncertainties. By establishing a dynamic model for IFO systems, the study breaks through the limitations of traditional commensurate-order frameworks, offering a more generalized approach for analyzing complex neural networks. A novel non-fragile state-error feedback controller was designed to ensure robust synchronization under parameter perturbations, supported by FO Lyapunov functions and the multi-order fractional comparison principle. Sufficient conditions for asymptotic synchronization were derived, applicable to both commensurate and integer-order systems. Furthermore, the proposed synchronization strategy was validated through numerical simulations and circuit experiments, demonstrating consistency between theoretical predictions and circuit implementation. Future work may focus on extending the framework to higher-dimensional systems and exploring real-world applications under more complex environmental conditions.

Author Contributions: Methodology, writing—original draft preparation, R.Z.; software, C.L.; validation, Z.C.; data curation, H.M.; writing—review and editing, supervision and project administration, K.Q. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Funds of China (No. 62202138), Anhui Provincial Key Research and Development Project (202304a05020060) and Anhui Provincial Natural Science Outstanding Youth Foundation (2408085J038) and 2024 University Student Innovative Experiment Program at Hefei University of Technology (X2024103591).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Che, H.; Wang, J. A Two-Timescale Duplex Neurodynamic Approach to Biconvex Optimization. *IEEE Trans. Neural Netw. Learn. Syst.* **2019**, *30*, 2503–2514. [[CrossRef](#)] [[PubMed](#)]
2. Chen, W.; Ding, D.; Dong, H.; Wei, G. Distributed Resilient Filtering for Power Systems Subject to Denial-of-Service Attacks. *IEEE Trans. Syst. Man Cybern. Syst.* **2019**, *49*, 1688–1697. [[CrossRef](#)]
3. Ding, D.; Wang, Z.; Han, Q.L. A Set-Membership Approach to Event-Triggered Filtering for General Nonlinear Systems Over Sensor Networks. *IEEE Trans. Autom. Control* **2020**, *65*, 1792–1799. [[CrossRef](#)]
4. Zong, G.; Ren, H.; Karimi, H.R. Event-Triggered Communication and Annular Finite-Time Filtering for Networked Switched Systems. *IEEE Trans. Cybern.* **2021**, *51*, 309–317. [[CrossRef](#)] [[PubMed](#)]
5. Chen, M.; Shao, S.Y.; Shi, P.; Shi, Y. Disturbance-Observer-Based Robust Synchronization Control for a Class of Fractional-Order Chaotic Systems. *IEEE Trans. Circuits Syst. II Express Briefs* **2017**, *64*, 417–421. [[CrossRef](#)]
6. Heng, L.; Sheng-Gang, L.; Ye-Guo, S.; Hong-Xing, W. Prescribed performance synchronization for fractional-order chaotic systems. *Chin. Phys. B* **2015**, *24*, 157–164.

7. Liu, X.; Li, C.; Ge, S.S.; Li, D. Time-Synchronized Control of Chaotic Systems in Secure Communication. *IEEE Trans. Circuits Syst. I: Regul. Pap.* **2022**, *69*, 3748–3761. [\[CrossRef\]](#)
8. Xu, Y.; Wang, H.; Liu, X.; Li, W. Quasi-synchronization of heterogenous fractional-order dynamical networks with time-varying delay via distributed impulsive control. *Chaos Solitons Fractals* **2021**, *142*, 110465.
9. Aadithiyan, S.; Raja, R.; Zhu, Q.; Alzabut, J.O.; Niezabitowski, M.; Lim, C.P. Modified projective synchronization of distributive fractional order complex dynamic networks with model uncertainty via adaptive control. *Chaos Solitons Fractals* **2021**, *147*, 110853. [\[CrossRef\]](#)
10. Zambrano-Serrano, E.; Bekiros, S.; Platas-Garza, M.A.; Posadas-Castillo, C.; Agarwal, P.; Jahanshahi, H.; Aly, A.A. On chaos and projective synchronization of a fractional difference map with no equilibria using a fuzzy-based state feedback control. *Phys. A Stat. Mech. Its Appl.* **2021**, *578*, 126100. [\[CrossRef\]](#)
11. Zhang, K.; Wang, H.O.; Fang, H. Feedback control and hybrid projective synchronization of a fractional-order Newton–Leipnik system. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 317–328. [\[CrossRef\]](#)
12. Xiong, R.; Tian, J.; Shen, W.; Sun, F. A Novel Fractional Order Model for State of Charge Estimation in Lithium Ion Batteries. *IEEE Trans. Veh. Technol.* **2019**, *68*, 4130–4139. [\[CrossRef\]](#)
13. Liu, X.; Mou, J.; Zhang, Y.; Cao, Y. A New Hyperchaotic Map Based on Discrete Memristor and Meminductor: Dynamics Analysis, Encryption Application, and DSP Implementation. *IEEE Trans. Ind. Electron.* **2024**, *71*, 5094–5104. [\[CrossRef\]](#)
14. Sha, Y.; Mou, J.; Banerjee, S.; Zhang, Y. Exploiting Flexible and Secure Cryptographic Technique for Multidimensional Image Based on Graph Data Structure and Three-Input Majority Gate. *IEEE Trans. Ind. Informatics* **2024**, *20*, 3835–3846. [\[CrossRef\]](#)
15. Lundstrom, B.N.; Higgs, M.H.; Spain, W.J.; Fairhall, A.L. Fractional differentiation by neocortical pyramidal neurons. *Nat. Neurosci.* **2008**, *11*, 1335–1342. [\[CrossRef\]](#) [\[PubMed\]](#)
16. Kamal, F.M.; Elsonbaty, A.; Elsaid, A. A novel fractional nonautonomous chaotic circuit model and its application to image encryption. *Chaos Solitons Fractals* **2021**, *144*, 110686. [\[CrossRef\]](#)
17. Gökdoğan, A.; Merdan, M.; Yildirim, A. A multistage differential transformation method for approximate solution of Hantavirus infection model. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 1–8. [\[CrossRef\]](#)
18. Xingce, L.; Jun, M.; Jue, W.; Santo, B.; Peng, L. Dynamical Analysis of a Novel Fractional-Order Chaotic System Based on Memcapacitor and Meminductor. *Fractal Fract.* **2022**, *6*, 671. [\[CrossRef\]](#)
19. Liu, X.; Tong, X.; Wang, Z.; Zhang, M. Efficient high nonlinearity S-box generating algorithm based on third-order nonlinear digital filter. *Chaos Solitons Fractals* **2021**, *150*, 111109. [\[CrossRef\]](#)
20. Shi, Q.; An, X.; Xiong, L.; Yang, F.; Zhang, L. Dynamic analysis of a fractional-order hyperchaotic system and its application in image encryption. *Phys. Scr.* **2022**, *97*, 045201. [\[CrossRef\]](#)
21. Bao, H.; Park, J.H.; Cao, J. Synchronization of fractional-order complex-valued neural networks with time delay. *Neural Networks* **2016**, *81*, 16–28. [\[CrossRef\]](#) [\[PubMed\]](#)
22. Xiong, J.-J.; Zhang, G.-B.; Wang, J.-X.; Yan, T.-H. Improved Sliding Mode Control for Finite-Time Synchronization of Nonidentical Delayed Recurrent Neural Networks. *IEEE Trans. Neural Netw. Learn. Syst.* **2020**, *31*, 2209–2216. [\[CrossRef\]](#)
23. Rakkiyappan, R.; Cao, J.; Velmurugan, G. Existence and Uniform Stability Analysis of Fractional-Order Complex-Valued Neural Networks With Time Delays. *IEEE Trans. Neural Netw. Learn. Syst.* **2015**, *26*, 84–97. [\[CrossRef\]](#) [\[PubMed\]](#)
24. Yang, S.; Yu, J.; Hu, C.; Jiang, H. Quasi-projective synchronization of fractional-order complex-valued recurrent neural networks. *Neural Networks* **2018**, *104*, 104–113. [\[CrossRef\]](#)
25. Hong-Li, L.; Cheng, H.; Jinde, C.; Haijun, J.; Ahmed, A. Quasi-projective and complete synchronization of fractional-order complex-valued neural networks with time delays. *Neural Netw.* **2019**, *118*, 102–109.
26. Boulkroune, A.; Bouzeriba, A.; Bouden, T. Fuzzy generalized projective synchronization of incommensurate fractional-order chaotic systems. *Neurocomputing* **2016**, *173*, 606–614. [\[CrossRef\]](#)
27. Qi, F.; Qu, J.; Chai, Y.; Chen, L.; Lopes, A.M. Synchronization of Incommensurate Fractional-Order Chaotic Systems Based on Linear Feedback Control. *Fractal Fract.* **2022**, *6*, 221. [\[CrossRef\]](#)
28. Debboche, N.; Almatroud, A.O.; Ouannas, A.; Batiha, I.M. Chaos and coexisting attractors in glucose-insulin regulatory system with incommensurate fractional-order derivatives. *Chaos Solitons Fractals* **2021**, *143*, 110575. [\[CrossRef\]](#)
29. Chen, L.; Xue, M.; Lopes, A.M.; Wu, R.; Zhang, X.; Chen, Y. New Synchronization Criterion of Incommensurate Fractional-Order Chaotic Systems. *IEEE Trans. Circuits Syst. II: Express Briefs* **2024**, *71*, 455–459. [\[CrossRef\]](#)
30. Zouari, F.; Boulkroune, A.; Ibeas, A. Neural adaptive quantized output-feedback control-based synchronization of uncertain time-delay incommensurate fractional-order chaotic systems with input nonlinearities. *Neurocomputing* **2017**, *237*, 200–225. [\[CrossRef\]](#)
31. Deepika, D.; Kaur, S.; Narayan, S. Uncertainty and disturbance estimator based robust synchronization for a class of uncertain fractional chaotic system via fractional order sliding mode control. *Chaos Solitons Fractals* **2018**, *115*, 196–203. [\[CrossRef\]](#)
32. Hajipour, A.; Aminabadi, S.S. Synchronization of chaotic Arneodo system of incommensurate fractional order with unknown parameters using adaptive method. *Optik* **2016**, *127*, 7704–7709. [\[CrossRef\]](#)

33. Xu, S.; Wang, X.; Ye, X. A new fractional-order chaos system of Hopfield neural network and its application in image encryption. *Chaos Solitons Fractals* **2022**, *157*, 111889. [[CrossRef](#)]
34. Ma, T.; Mou, J.; Li, B.; Banerjee, S.; Yan, H. Study on the Complex Dynamical Behavior of the Fractional-Order Hopfield Neural Network System and Its Implementation. *Fractal Fract.* **2022**, *6*, 637. [[CrossRef](#)]
35. Podlubny, I. *Fractional Differential Equations*; Mathematics in Science and Engineering; Academic Press: Cambridge, MA, USA, 1999.
36. Chen, L.; Guo, W.; Gu, P.; Lopes, A.M.; Chu, Z.; Chen, Y. Stability and Stabilization of Fractional-Order Uncertain Nonlinear Systems With Multiorder. *IEEE Trans. Circuits Syst. II Express Briefs* **2023**, *70*, 576–580. [[CrossRef](#)]
37. Lenka, B.K. Fractional comparison method and asymptotic stability results for multivariable fractional order systems. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *69*, 398–415. [[CrossRef](#)]
38. Lu, J.G.; Zhu, Z.; Ma, Y.D. Robust stability and stabilization of multi-order fractional-order systems with interval uncertainties: An LMI approach. *Int. J. Robust Nonlinear Control* **2021**, *31*, 4081–4099. [[CrossRef](#)]
39. Luo, S.; Song, Y.; Lewis, F.L.; Garrappa, R. Neuroadaptive Optimal Fixed-Time Synchronization and its Circuit Realization for Unidirectionally Coupled FO Self-Sustained Electromechanical Seismograph Systems. *IEEE Trans. Cybern.* **2023**, *53*, 2454–2466. [[CrossRef](#)]
40. Luo, S.; Song, Y.; Lewis, F.L.; Deng, G. Dynamical Analysis, Circuit Design, and Fuzzy Prescribed Performance Backstepping Control of the FO Weakly Coupled MEMS Resonators. *IEEE Trans. Fuzzy Syst.* **2024**, *32*, 1768–1781. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.