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Outlier-Robust Convergence of Integer- and Fractional-Order Difference Operators in Fuzzy-Paranormed Spaces: Diagnostics and Engineering Applications

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Abstract

We develop a convergence framework for Grünwald–Letnikov (GL) fractional and classical integer difference operators acting on sequences in fuzzy-paranormed (fp) spaces, motivated by data that are imprecise and contain sporadic outliers. Fuzzy paranorms provide a resolution-dependent notion of proximity, while statistical and lacunary statistical convergence downweight sparse deviations by natural density; together, they yield robust criteria for difference-filtered signals. Within this setting, we establish uniqueness of $\text{fp-}\Delta^m$ statistical limits; an equivalence between fp-statistical convergence of Δ^m (and its GL extension Δ^α) and fp-strong p -Cesàro summability; an equivalence between lacunary $\text{fp-}\Delta^m$ statistical convergence and blockwise strong p -Cesàro summability; and a density-based decomposition into a classically convergent part plus an fp-null remainder. We also show that GL binomial weights act as an ℓ^1 convolution, ensuring continuity of Δ^α in the fp topology, and that nabla/delta forms are transferred by the discrete Q -operator. The usefulness of the criteria is illustrated on simple engineering-style examples (e.g., relaxation with memory, damped oscillations with bursts), where the fp-Cesàro decay of difference residuals serves as a practical diagnostic for Cesàro compliance. Beyond illustrative mathematics, we report engineering-style diagnostics where the fuzzy Cesàro residual index correlates with measurable quantities (e.g., vibration amplitude and energy surrogates) under impulsive disturbances and missing data. We also calibrate a global decision threshold τ_{glob} via sensitivity analysis across (α, p, m) , where $m \in \mathbb{N}$ is the integer difference order, $\alpha > 0$ is the fractional order, and $p \geq 1$ is the Cesàro exponent, and provide quantitative baselines (median/M-estimators, ℓ_1 trend filtering, Gaussian Kalman filtering, and an α -stable filtering structure) to show complementary gains under bursty regimes. The results are stated for integer m and lifted to fractional orders $\alpha > 0$ through the same binomial structure and duality.

Keywords: fractional difference operators; integer-order differences; fuzzy-paranormed spaces; statistical convergence; lacunary convergence; strong Cesàro summability; GL fractional differences; robust filtering; α -stable systems; eddy-current sensing; fuzzy inference; engineering applications



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1. Introduction

Discrete operators of integer and fractional order have become central to the analysis of engineering data in which memory, heredity, and irregular sampling are intrinsic. In sequence spaces, the classical m th forward difference Δ^m acts as a local regularity detector and

smoothing filter, while its fractional Grünwald–Letnikov (GL) counterpart $\Delta^\alpha = (I - S)^\alpha$ incorporates long-range dependence via slowly decaying binomial weights, providing a tunable bridge between differentiation and averaging [1–3]. From an engineering perspective, signals with long memory and impulsive disruptions are well documented in robust filtering and nondestructive evaluation: α -stable regimes motivate non-Gaussian filtering structures that explicitly accommodate heavy tails [4], while fuzzy metrics and inference have been shown effective on eddy-current measurements for classifying stress states and structural anomalies in uncertain environments [5]. These observations motivate our choice of GL fractional differences (to encode memory and burst tolerance) and fuzzy-paranormed proximity (to model resolution-dependent uncertainty) in a unified convergence framework. When measurements are imprecise or bursty—a common feature in electrical/thermal networks, mechanics, biology, and computer systems—rigid norm topologies may misrepresent proximity; this motivates convergence notions that (i) tolerate sparse outliers and (ii) quantify closeness at multiple resolutions.

A density-based idea that addresses (i) is statistical convergence—introduced independently by Fast and Steinhaus—where the exceptional indices have natural density zero [6,7]. The concept interacts fruitfully with summability theory (e.g., strong Cesàro means) and lacunary methods [8–10]. In parallel, the study of difference sequence spaces—initiated by Kızmaç and extended to higher orders—has supplied a discrete calculus for filtering, spike detection, and operator-theoretic investigations [11,12]. These strands suggest combining difference filters with density-based limits. Recent studies have further connected fuzzy metrics and statistical-type limits to practical decision problems, including fp-space convergence tools tailored for GL differences and diagnostics, λ -statistical and ideal convergence in fp settings with applied calibration, and lacunary/ideal schemes for discrete-time stability under missing measurements [13–16]. Our work differs by placing integer and fractional difference operators at the center of a unified, density-aware convergence framework in fp spaces and by developing engineering-style diagnostics (via Cesàro decay of residuals) with explicit operator continuity and lifting from Δ^m to Δ^α (see Lemma 1 and Proposition 1).

To address (ii), fuzzy norm-like structures replace a single-valued norm by a membership map $N(\cdot, t) \in [0, 1]$ depending on a resolution parameter $t > 0$, thus capturing imprecision without abandoning linear structure. We emphasize why a fuzzy paranorm (rather than a single quasi-norm or a crisp fuzzy norm) is adopted: the t -norm L supplies the lower-triangular control actually used in our proofs, while the t -conorm R provides an upper-envelope compatible with common aggregation semantics. In applications, L captures robust conjunction at fixed resolution t and yields tractable levelwise estimates; R is retained for completeness and modeling fidelity, although our theorems invoke only the L -bound. Fuzzy (para)normed models formalize this idea and recover classical results when N is induced by a usual norm (e.g., $N(u, t) = t/(t + \|u\|)$) [17,18]. This makes them natural hosts for uncertainty-aware convergence.

Fractional differences enter our framework on equal footing with Δ^m . GL discretizations place initial data transparently and admit a delta/nabla duality, together with a Q -operator that transfers statements between left/right forms and clarifies the link with discrete Riemann–Liouville calculus [2,19,20]. For clarity and concreteness, we develop the main results for the classical operator Δ^m and record the fractional lift to Δ^α in the preliminaries; the proofs carry over by replacing finite differences with the absolutely summable GL binomial kernel.

1.1. Contributions and Relation to Prior Work

New contributions in this paper are as follows: (i) Uniqueness of fp- Δ^m statistical limits via a levelwise-to-fuzzy lifting argument in fp spaces. (ii) Equivalence fp- Δ^m statistical

convergence $\iff$ fp-strong p -Cesàro (with explicit assumptions on $p \geq 1$, completeness where invoked, and uniform α -level compatibility). (iii) Lacunary fp- Δ^m statistical convergence $\iff$ blockwise strong p -Cesàro under $\inf q_r > 1$. (iv) A density-based decomposition into a classically convergent part plus an fp-null remainder. (v) Continuity of Δ^α in the fp topology via the ℓ^1 GL binomial kernel, together with nabla/delta transfer through the discrete Q -operator (Lemma 1, Proposition 1).

Extensions/Adaptations: We extend classical statistical/summability principles [8,9] and difference-sequence ideas [11,12] to the fuzzy-paranormed and fractional-difference setting, clarifying which fuzzy assumptions are essential (use of t -norm only) and how GL kernels enable the fractional lift. In contrast to [13–16], our focus is on a unifying difference-based fp framework with an engineering-facing diagnostic (FCRI) and quantitative calibration against robust baselines, including α -stable filtering [4] and fuzzy-inference applications on eddy-current data [5].

1.2. Positioning Relative to Recent Works

Unlike the recent fp/fuzzy-normed results in [13–16], which emphasize (variants of) statistical or ideal convergence predominantly in fuzzy normed settings or with application-specific stability goals, our contribution is to introduce a unified, difference-operator-centered framework in fuzzy-paranormed spaces that (i) treats integer and fractional differences on equal footing; (ii) lifts to the Grünwald–Letnikov regime via an explicit ℓ^1 -kernel continuity route (Lemma 1, Proposition 1); and (iii) develops an engineering-facing diagnostic (the fuzzy-Cesàro residual index, FCRI) with sensitivity-based calibration across (α, p, t) and quantitative baselines (median/M-estimators, ℓ_1 trend filtering, Gaussian Kalman filtering, and an α -stable filtering structure). This distinction is reflected in our equivalence theorems between fp- Δ^m statistical convergence and fp-strong p -Cesàro modes (Theorem 2), the lacunary/blockwise counterparts (Theorem 4), and the density-based decomposition (Theorem 3), together with a fractional lift (Corollary 1).

1.3. Organization

Section 2 recalls fuzzy paranorms (with a brief rationale on t -norm/ t -conorm), difference sequence spaces, statistical/lacunary convergence, and GL-type fractional differences, including an ℓ^1 kernel lemma. Section 3 gives uniqueness, equivalence with strong p -Cesàro, the density-based decomposition, and lacunary counterparts, together with assumptions stated explicitly; a remark outlines the $\mathbb{Z}^d$ extension. Section 4 reports engineering-style experiments, sensitivity-based calibration of τ_{glob} , and comparisons with robust baselines. We conclude with practical guidance and open problems.

2. Preliminaries and Definitions

We assemble here the algebraic and convergence background used throughout. The presentation follows a concept-first order: (i) paranormed and fuzzy-paranormed spaces; (ii) difference sequence spaces; (iii) statistical, strong Cesàro, and lacunary convergence; (iv) ideal convergence (I and I^*) and the Approximation Property; (v) synthesis in the fuzzy-paranormed setting; (vi) a fractional (GL) link via an ℓ^1 kernel criterion.

2.1. Standing Assumptions

We list here the assumptions we invoke throughout, unless stated otherwise:

- (H1) (X, N, L, R) is a total fuzzy-paranormed space; for each $\alpha \in (0, 1]$, the α -level spaces $(X_\alpha, \|\cdot\|_\alpha)$ are normed and uniformly compatible across α (levelwise estimates lift to fuzzy ones).
- (H2) Whenever completeness is required, each $(X_\alpha, \|\cdot\|_\alpha)$ is Banach. We state explicitly in theorems where completeness is used.

- (H3) For strong Cesàro results and their equivalence with statistical modes, we take $p \geq 1$; this ensures Minkowski-type subadditivity on level spaces. Some one-sided implications remain valid for $0 < p < 1$, but we do not rely on them.
- (H4) GL weights $w_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$ satisfy $\sum_{j \geq 0} |w_j^{(\alpha)}| < \infty$ for $\alpha > 0$ (see Lemma 1); thus, Δ^α is an ℓ^1 convolution on level spaces and is continuous in the fp-topology (Proposition 1).
- (H5) For lacunary statements we use $\Theta = (k_r)$ with $h_r = k_r - k_{r-1} \rightarrow \infty$ and $\inf_r q_r > 1$ (standard).

Working model in numerics: In the examples we use $N(u, t) = t/(t + |u|)$ with $L(a, b) = ab$ and $R(a, b) = a + b - ab$, but the theory only requires the t -norm L (cf. Remark 1).

2.2. Paranormed and Fuzzy-Paranormed Spaces

Definition 1 (Paranorm [10,21]). Let X be a real linear space. A mapping $g: X \rightarrow \mathbb{R}$ is a paranorm if for all $x, y \in X$:

- (PN1) $x = \theta \Rightarrow g(x) = 0$;
 (PN2) $g(-x) = g(x)$;
 (PN3) $g(x + y) \leq g(x) + g(y)$;
 (PN4) if $\alpha_n \rightarrow \alpha_0$ and $g(x_n - x) \rightarrow 0$, then $g(\alpha_n x_n - \alpha_0 x) \rightarrow 0$.

If $g(x) = 0$ implies $x = \theta$, then g is a total paranorm, and (X, g) a total paranormed space.

Definition 2 (Continuous t -norm and t -conorm [22]). A continuous t -norm is a commutative, associative, monotone, continuous $T: [0, 1]^2 \rightarrow [0, 1]$ with neutral element 1; dually, a continuous t -conorm S has neutral element 0.

Definition 3 (Fuzzy paranorm [18]). Let X be a real vector space. A function $N: X \times (0, \infty) \rightarrow [0, 1]$ together with a continuous t -norm L and t -conorm R is a fuzzy paranorm if for all $x, y \in X$, scalars α , and $s, t > 0$:

- (FP1) $N(x, t) = 1$ if $x = \theta$;
 (FP2) $N(-x, t) = N(x, t)$;
 (FP3) $N(x + y, s + t) \geq L(N(x, s), N(y, t))$ and $N(x + y, s + t) \leq R(N(x, s), N(y, t))$;
 (FP4) $t \mapsto N(x, t)$ is nondecreasing and $\lim_{t \rightarrow \infty} N(x, t) = 1$;
 (FP5) $|\alpha| \leq 1 \Rightarrow N(\alpha x, t) \geq N(x, t)$, $|\alpha| \geq 1 \Rightarrow N(\alpha x, t) \leq N(x, t)$;
 (FP6) if $\alpha_n \rightarrow \alpha$ and $N(x_n - x, t) \rightarrow 1$, then $N(\alpha_n x_n - \alpha x, t) \rightarrow 1$.

If $N(x, t) = 1$ for all $t > 0$ implies $x = \theta$, then (X, N, L, R) is totally fuzzy-paranormed.

Remark 1 (On L vs. R in (FP3)). We state both the lower triangle bound via the t -norm L and the upper envelope via the t -conorm R (a standard convention [22]). All main proofs invoke only the L -bound; R is retained as an upper-envelope to accommodate modeling choices (e.g., probabilistic aggregation) and to keep the fp framework compatible with broader fuzzy practice, but it is not required for any theorem in this paper.

2.3. Difference Sequence Spaces

Definition 4 (Difference sequence spaces [11,12]). Let X be a classical sequence space (ℓ_∞ , c , c_0 , etc.). For $x = (x_k)$ define $\Delta^0 x_k := x_k$ and, for $m \geq 1$, the recursion

$$\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}, \quad k \in \mathbb{N}. \quad (1)$$

Then

$$X(\Delta^m) := \{x \in \omega : (\Delta^m x_k) \in X\}$$

is the m -th order difference space over X (write $X(\Delta)$ for $m = 1$).

2.4. Core Convergence Notions in (Para)normed Spaces

Definition 5 (Statistical convergence [6,7,9,10]). In a paranormed space (X, g) , a sequence $x = (x_k)$ is g -statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : g(x_k - L) \geq \varepsilon\}| = 0.$$

Definition 6 (Strong p -Cesàro summability [8,10]). Let $p > 0$. We say $x \rightarrow L$ strongly p -Cesàro in (X, g) if $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n [g(x_k - L)]^p = 0$.

Remark 2 (On the role of p). For equivalence with statistical modes, we assume $p \geq 1$, ensuring levelwise subadditivity (Minkowski). For $0 < p < 1$, strong p -Cesàro still controls outliers but may fail to imply statistical convergence without additional boundedness; we therefore state the main equivalences for $p \geq 1$.

Definition 7 (Lacunary statistical convergence [9]). Let $\Theta = (k_r)$ be lacunary with $k_0 = 0$, $h_r := k_r - k_{r-1} \rightarrow \infty$ and $I_r := (k_{r-1}, k_r]$. Then $x = (x_k)$ is g -(lac-st) convergent to L if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : g(x_k - L) \geq \varepsilon\}| = 0 \quad (\forall \varepsilon > 0).$$

2.5. Ideal Convergence and the Approximation Property

Definition 8 (Ideals on $\mathbb{N}$ and I -convergence [23]). An ideal I on $\mathbb{N}$ is a nonempty family of subsets of $\mathbb{N}$ closed under finite unions and taking subsets; it is nontrivial if $\emptyset \in I$ and $\mathbb{N} \notin I$. Given a topological/metric target, a sequence $x = (x_k)$ I -converges to L if $\{k : x_k \notin U\} \in I$ for every neighborhood U of L .

Definition 9 (I^* -convergence and the (AP) property [23]). Let $F(I)$ be the filter dual to I . The sequence $x = (x_k)$ is I^* -convergent to L if there exists $M \in F(I)$ such that the subsequence $(x_k)_{k \in M}$ converges to L (ordinary sense). An ideal I has the Approximation Property (AP) if every partition of $\mathbb{N}$ into sets in I admits a selector set $M \in F(I)$ meeting each part in finitely many points. If I has (AP), then I -convergence $\Leftrightarrow I^*$ -convergence (in metric settings).

2.6. Fuzzy-Paranormed Synthesis

Definition 10 (fp-statistical convergence [18]). In a fuzzy-paranormed space (X, N, L, R) , $x = (x_k)$ is fp-statistically convergent to x_0 if for every $\varepsilon \in (0, 1)$ and $t > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : N(x_k - x_0, t) \leq 1 - \varepsilon\}| = 0.$$

Definition 11 (fp-strong p -Cesàro). For $p > 0$, $x \rightarrow x_0$ fp-strongly p -Cesàro if for each fixed $t > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n [1 - N(x_k - x_0, t)]^p = 0$.

Definition 12 (fp-lacunary statistical). With $\Theta = (k_r)$ lacunary and $I_r = (k_{r-1}, k_r]$, x is fp-(lac-st) convergent to x_0 if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : N(x_k - x_0, t) \leq 1 - \varepsilon\}| = 0 \quad (\forall \varepsilon \in (0, 1), t > 0).$$

Definition 13 (Lacunary strong p -Cesàro summability). Let $\Theta = (k_r)$ be lacunary with blocks $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1} \rightarrow \infty$. For fixed $p > 0$, $m \in \mathbb{N}$ and $t > 0$, we say $x = (x_k)$ is $fp\text{-}\Delta^m$ -strongly p -Cesàro along Θ to x_0 if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p = 0.$$

Definition 14 (fp - I and fp - I^* convergence). Let I be a nontrivial ideal on $\mathbb{N}$. We say $x = (x_k)$ is fp - I convergent to x_0 if for every $\varepsilon \in (0, 1)$ and $t > 0$,

$$\{k \in \mathbb{N} : N(x_k - x_0, t) \leq 1 - \varepsilon\} \in I.$$

We say x is fp - I^* convergent to x_0 if there exists $M \in F(I)$ such that $N(x_k - x_0, t) \rightarrow 1$ as $k \rightarrow \infty$ along $k \in M$ for every $t > 0$. If I has (AP), then fp - I and fp - I^* convergence are equivalent (metric/fuzzy settings).

Definition 15 (fp - Δ^m -boundedness (base-point free)). Let (X, N, L, R) be a fuzzy-paranormed space and $m \in \mathbb{N}$. We say $x = (x_k)$ is fp - Δ^m -bounded if for each fixed $t > 0$ there exists $M_t \in [0, 1)$ such that

$$\sup_{k \in \mathbb{N}} [1 - N(\Delta^m x_k, t)] \leq M_t.$$

This notion does not refer to any limit x_0 (base-point free) and is invariant under additive constants; all results using boundedness invoke this version.

2.7. Δ^m and fp - Δ^m Modes

Definition 16 (fp - Δ^m -statistical/lacunary/strong p -Cesàro). For $m \in \mathbb{N}$ define on $x = (x_k)$:

$$fp\text{-}\Delta^m(st) : \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : N(\Delta^m x_k - x_0, t) \leq 1 - \varepsilon\}| = 0,$$

$$fp\text{-}\Delta^m(lac\text{-}st) : \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : N(\Delta^m x_k - x_0, t) \leq 1 - \varepsilon\}| = 0,$$

$$fp\text{-}\Delta^m \text{ strong } p\text{-Cesàro} : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n [1 - N(\Delta^m x_k - x_0, t)]^p = 0,$$

for every fixed $t > 0$ and relevant ε, p .

Definition 17 (fp - Δ^m -statistically Cauchy). Let (X, N, L, R) be a fuzzy-paranormed space and $x = (x_k)$. For fixed $m \in \mathbb{N}$, $t > 0$, we call x fp - Δ^m -statistically Cauchy if for every $\varepsilon \in (0, 1)$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : N(\Delta^m x_k - \Delta^m x_j, t) \leq 1 - \varepsilon\}| = 0.$$

2.8. GL-Fractional Difference and an ℓ^1 Kernel Lemma

Definition 18 (GL-type fractional difference). Let $\alpha > 0$ and $x = (x_k)_{k \geq 0}$. We adopt the left-sided GL form

$$(\Delta^\alpha x)_k = \sum_{j=0}^k (-1)^j \binom{\alpha}{j} x_{k-j}, \quad k \in \mathbb{N}. \quad (2)$$

This finite sum is well-defined for every sequence $x \in \omega$ (no growth/decay assumption needed). Formally, one also writes $\Delta^\alpha = (I - S)^\alpha$ as a binomial series; in alternative (two-sided or infinite-range) conventions, one requires the absolute summability $\sum_{j \geq 0} \binom{\alpha}{j} x_{k-j} < \infty$ for each k . We remain with the left-sided finite-sum form throughout; cf. [20].

Definition 19 (Nabla-type GL fractional difference and Q-duality). Let $\alpha > 0$ and let S denote the forward shift $(Sx)_k = x_{k+1}$. The nabla (backward) GL fractional difference is defined by

$$(\nabla^\alpha x)_k := (I - S^{-1})^\alpha x_k = \sum_{j=0}^k (-1)^j \binom{\alpha}{j} x_{k-j}, \quad k \in \mathbb{N},$$

when the left-sided finite sum is used (for two-sided or infinite-range conventions, absolute summability is required as usual). In particular, $\Delta^\alpha = (I - S)^\alpha$ and $\nabla^\alpha = (I - S^{-1})^\alpha$ are related by the discrete Q operator (delta/nabla duality); see, e.g., ref. [20].

Lemma 1 (ℓ^1 -kernel for $\alpha > 0$). For $\alpha > 0$, the binomial weights $w_j^{(\alpha)} := (-1)^j \binom{\alpha}{j}$ satisfy $\sum_{j=0}^{\infty} |w_j^{(\alpha)}| < \infty$. Equivalently (for non-integer $\alpha > 0$), with $[z^j](1-z)^\alpha = (-1)^j \binom{\alpha}{j}$, we have the standard asymptotic

$$\left| \binom{\alpha}{j} \right| \sim \frac{\Gamma(\alpha+1) |\sin(\pi\alpha)|}{\pi} j^{-\alpha-1} \quad (j \rightarrow \infty), \quad (3)$$

and hence the absolute summability

$$\sum_{j=0}^{\infty} \left| \binom{\alpha}{j} \right| < \infty \quad (\alpha > 0). \quad (4)$$

Therefore $\{w_j^{(\alpha)}\} \in \ell^1$, as claimed. For integer $\alpha = m \in \mathbb{N}$, $(1-z)^m$ is a polynomial and $w_j^{(\alpha)} = 0$ for all $j > m$, whence absolute summability holds trivially.

Proposition 1 (Fuzzy-paranorm continuity of Δ^α). Let (X, N, L, R) be fuzzy-paranormed and fix $t > 0$, $\alpha > 0$. Set $C_\alpha := \sum_{j \geq 0} |\binom{\alpha}{j}| < \infty$. Then for any sequences x, y ,

$$1 - N((\Delta^\alpha x)_k - (\Delta^\alpha y)_k, t) \leq \sum_{j=0}^k [1 - N(x_{k-j} - y_{k-j}, t_j)]$$

for any choice of $t_0, \dots, t_k > 0$ with $\sum_{j=0}^k t_j = t$. In particular, choosing $t_j \equiv t$ when L is the product t -norm and since $C_\alpha = \sum_{j \geq 0} |\binom{\alpha}{j}| \geq 1$, we get the safe bound

$$1 - N((\Delta^\alpha x)_k - (\Delta^\alpha y)_k, t) \leq C_\alpha \sum_{j=0}^k [1 - N(x_{k-j} - y_{k-j}, t)].$$

Thus $fp\text{-}\Delta^\alpha$ -boundedness and $fp\text{-Cesàro}$ control are preserved.

Proof. Write $(\Delta^\alpha x)_k - (\Delta^\alpha y)_k = \sum_{j=0}^k w_j^{(\alpha)} (x_{k-j} - y_{k-j})$ with $w_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$. Apply (FP3) repeatedly with the split $t = \sum_{j=0}^k t_j$ to obtain

$$N\left(\sum_{j=0}^k w_j^{(\alpha)} (x_{k-j} - y_{k-j}), t\right) \geq L_{j=0}^k N(w_j^{(\alpha)} (x_{k-j} - y_{k-j}), t_j),$$

where $L_{j=0}^k$ denotes iterated application of the t -norm L . Using (FP5) and (FP2), $N(w_j u, t_j) \geq N(u, t_j)$ for $|w_j| \leq 1$ and, in general, by monotonicity in t , for a fixed t

we may replace t_j by $\min_j t_j$ if desired. Passing to complements and using $1 - L(a, b) \leq (1 - a) + (1 - b)$ (equality if L is product) gives

$$1 - N\left(\sum_{j=0}^k w_j^{(\alpha)}(x_{k-j} - y_{k-j}), t\right) \leq \sum_{j=0}^k (1 - N(x_{k-j} - y_{k-j}, t_j)),$$

which proves the first display. If, in addition, $t_j \equiv t$ and L is the product t -norm, then using $0 \leq 1 - N(\cdot, t) \leq 1$ and $\sum_{j \geq 0} |w_j^{(\alpha)}| = C_\alpha$ yields the coarser bound with C_α . The conclusions on boundedness and Cesàro control follow by summing over k and using $w^{(\alpha)} \in \ell^1$. $\square$

Remark 3 (Integer/fractional bridge). All $fp\text{-}\Delta^m$ statements used later extend to $fp\text{-}\Delta^\alpha$ by replacing finite with GL differences and invoking Lemma 1 and Proposition 1; duality between delta/nabla forms follows from the Q -operator framework [20].

3. Main Results

Throughout this section, we work under (H1)–(H5) in Section 2.1. We use the $fp\text{-}\Delta^m$ statistical, lacunary statistical, and strong p -Cesàro notions (Definitions 13–16). The proofs rely on the finite-difference identity (1) and the GL form (2); the fractional lift uses the ℓ^1 kernel and continuity facts (3)–(4) and Proposition 1. Unless otherwise stated, we fix $m \in \mathbb{N}$, $p \geq 1$, and $t > 0$; the lacunary scheme Θ and $fp\text{-}\Delta^m$ -boundedness are as in Definitions 7 and 15.

Theorem 1. Let (X, N, L, R) be a fuzzy-paranormed space, and let $x = (x_k)$ be a sequence in X that is $fp\text{-}\Delta^m$ -statistically convergent to both x_0 and y_0 . Then $x_0 = y_0$.

Proof. Assume that

$$\Delta^m\text{-}fp(\text{st})\text{-}\lim x = x_0 \quad \text{and} \quad \Delta^m\text{-}fp(\text{st})\text{-}\lim x = y_0.$$

Fix arbitrary $\varepsilon > 0$ and $t > 0$. Since the t -norm L is (jointly) continuous and $L(1, 1) = 1$, there exists $\delta \in (0, \varepsilon)$ such that

$$L(1 - \delta, 1 - \delta) > 1 - \varepsilon.$$

Define the index sets

$$K_1(\delta) = \left\{k \in \mathbb{N} : N(\Delta^m x_k - x_0, t/2) \leq 1 - \delta\right\}, \quad K_2(\delta) = \left\{k \in \mathbb{N} : N(\Delta^m x_k - y_0, t/2) \leq 1 - \delta\right\}.$$

By $fp\text{-}\Delta^m$ -statistical convergence to x_0 and y_0 ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |K_1(\delta) \cap \{1, \dots, n\}| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} |K_2(\delta) \cap \{1, \dots, n\}| = 0.$$

Let $K(\delta) = K_1(\delta) \cup K_2(\delta)$. Then also

$$\lim_{n \rightarrow \infty} \frac{1}{n} |K(\delta) \cap \{1, \dots, n\}| = 0.$$

Hence for all $k \notin K(\delta)$ (i.e., for almost all k) we have

$$N(\Delta^m x_k - x_0, t/2) > 1 - \delta \quad \text{and} \quad N(\Delta^m x_k - y_0, t/2) > 1 - \delta.$$

Write

$$x_0 - y_0 = (x_0 - \Delta^m x_k) + (\Delta^m x_k - y_0).$$

Using the fuzzy paranorm axioms, by (FP3) (triangle-type inequality with t-norm L) applied with $s = t/2$ and $t = t/2$, and by (FP2) ($N(-u, \cdot) = N(u, \cdot)$), we obtain

$$\begin{aligned} N(x_0 - y_0, t) &\geq L\left(N(x_0 - \Delta^m x_k, t/2), N(\Delta^m x_k - y_0, t/2)\right) \\ &= L\left(N(\Delta^m x_k - x_0, t/2), N(\Delta^m x_k - y_0, t/2)\right) \\ &\geq L(1 - \delta, 1 - \delta) > 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ and $t > 0$ are arbitrary, it follows that

$$N(x_0 - y_0, t) = 1 \quad \forall t > 0.$$

By the axioms of a fuzzy paranorm (separation), $N(u, t) = 1$ for all $t > 0$ implies $u = \theta$. Hence $x_0 - y_0 = \theta$, i.e., $x_0 = y_0$. $\square$

Theorem 2. Assume (H1)–(H3). Let (X, N, L, R) be a fuzzy-paranormed space and let $x = (x_k)$ be a sequence in X with a fixed resolution $t > 0$ and exponent $p \geq 1$.

1. If x is $\text{fp-}\Delta^m$ -strongly p -Cesàro summable to x_0 , then x is $\text{fp-}\Delta^m$ -statistically convergent to x_0 .
2. If x is $\text{fp-}\Delta^m$ -statistically convergent to x_0 , then x is $\text{fp-}\Delta^m$ -strongly p -Cesàro summable to x_0 .

Remark 4 (Assumptions for Theorem 2). The range $p \geq 1$ guarantees levelwise subadditivity (Minkowski) needed in the Cesàro-to-statistical direction and is standard in summability theory; for $0 < p < 1$, one may retain one-sided implications under extra boundedness, but we do not use them here. Completeness is only invoked through (H2) where limit passages are stated levelwise.

Proof. We prove the equivalence in two parts.

(i) $\text{fp-}\Delta^m$ -Strongly p -Cesàro Summability $\implies$ $\text{fp-}\Delta^m$ -Statistical Convergence:

Assume that x is $\text{fp-}\Delta^m$ -strongly p -Cesàro summable to x_0 ; that is, for every fixed $t > 0$ and $p > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p = 0.$$

Let $\varepsilon > 0$ be arbitrary. Since the function $f(y) = y^p$ is continuous and strictly increasing on $[0, \infty)$, choose $\delta = \varepsilon^{1/p} > 0$ (the inverse of $y \mapsto y^p$ on $[0, 1]$) so that

$$y \geq \delta \implies y^p \geq \varepsilon.$$

Thus, whenever

$$1 - N(\Delta^m x_k - x_0, t) \geq \delta = \varepsilon^{1/p},$$

we have

$$\left[1 - N(\Delta^m x_k - x_0, t) \right]^p \geq \delta^p = \varepsilon.$$

Define the set

$$A(n, \delta) = \{k \leq n : 1 - N(\Delta^m x_k - x_0, t) \geq \delta\} = \{k \leq n : N(\Delta^m x_k - x_0, t) \leq 1 - \delta\}.$$

Then, we have the estimate:

$$\sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \geq \sum_{k \in A(n, \delta)} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \geq \varepsilon |A(n, \delta)|.$$

Dividing both sides by n yields

$$\frac{1}{n} \sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \geq \varepsilon \frac{|A(n, \delta)|}{n}.$$

Taking the limit as $n \rightarrow \infty$ and using the assumption that the Cesàro mean tends to 0, we obtain

$$0 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \geq \varepsilon \cdot \limsup_{n \rightarrow \infty} \frac{|A(n, \delta)|}{n}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{n \rightarrow \infty} \frac{|A(n, \delta)|}{n} = 0.$$

Thus, by the definition of $\text{fp-}\Delta^m$ -statistical convergence, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : N(\Delta^m x_k - x_0, t) \leq 1 - \delta\}| = 0,$$

which means that x is $\text{fp-}\Delta^m$ -statistically convergent to x_0 .

(ii) $\text{fp-}\Delta^m$ -Statistical Convergence $\implies \text{fp-}\Delta^m$ -Strongly p -Cesàro Summability:

Now assume that x is $\text{fp-}\Delta^m$ -statistically convergent to x_0 ; that is, for every $\varepsilon > 0$ and every fixed $t > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : N(\Delta^m x_k - x_0, t) \leq 1 - \varepsilon\}| = 0.$$

For a given $\delta > 0$, decompose the index set $\{1, 2, \dots, n\}$ into two parts:

$$A(n, \delta) = \{k \leq n : 1 - N(\Delta^m x_k - x_0, t) \geq \delta\},$$

and its complement,

$$B(n, \delta) = \{k \leq n : 1 - N(\Delta^m x_k - x_0, t) < \delta\}.$$

Then, the Cesàro mean can be split as

$$\frac{1}{n} \sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p = \frac{1}{n} \sum_{k \in A(n, \delta)} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p + \frac{1}{n} \sum_{k \in B(n, \delta)} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p.$$

For $k \in A(n, \delta)$, using $0 \leq [1 - N(\cdot, t)]^p \leq 1$ we have

$$\left[1 - N(\Delta^m x_k - x_0, t) \right]^p \leq 1,$$

so that

$$\frac{1}{n} \sum_{k \in A(n, \delta)} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \leq \frac{|A(n, \delta)|}{n}.$$

For $k \in B(n, \delta)$, we have

$$\left[1 - N(\Delta^m x_k - x_0, t) \right]^p < \delta^p.$$

Since $B(n, \delta) \subset \{1, 2, \dots, n\}$, it follows that

$$\frac{1}{n} \sum_{k \in B(n, \delta)} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \leq \delta^p.$$

Thus, we obtain

$$\frac{1}{n} \sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \leq \frac{|A(n, \delta)|}{n} + \delta^p.$$

Taking the limit superior as $n \rightarrow \infty$ and noting that $\frac{|A(n, \delta)|}{n} \rightarrow 0$ by the assumption of $\text{fp-}\Delta^m$ -statistical convergence, we deduce

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p \leq \delta^p.$$

Since $\delta > 0$ is arbitrary, letting $\delta \rightarrow 0$ implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left[1 - N(\Delta^m x_k - x_0, t) \right]^p = 0.$$

Thus, x is $\text{fp-}\Delta^m$ -strongly p -Cesàro summable to x_0 .

Combining parts (i) and (ii), we conclude that the two modes of convergence are equivalent. $\square$

Theorem 3. Assume (H1). Let (X, N, L, R) be a total fuzzy-paranormed space and suppose Δ^m - $\text{fp}(\text{st})$ - $\lim x = x_0$. Then there exist sequences y, z with $x = y + z$, $y \rightarrow x_0$ (classical convergence) and z $\text{fp-}\Delta^m$ -statistically null.

Remark 5 (Discrete spatial lattices). An analogue of Theorem 3 holds on $\mathbb{Z}^d$ by replacing natural density with upper Banach density on boxes and using multi-index finite differences (or GL convolutions with $\ell^1(\mathbb{Z}^d)$ kernels). The proof follows verbatim: the “bad” set has vanishing upper Banach density, and the finite support/finite-band (or ℓ^1) property keeps the exceptional region’s m -neighborhood negligible in density.

Proof. By fp -statistical convergence, there exists $M \subset \mathbb{N}$ with natural density 1 such that $N(\Delta^m x_k - x_0, t) \rightarrow 1$ as $k \rightarrow \infty$ along $k \in M$ for each fixed $t > 0$. Define $y_k := x_k$ if $k \in M$ and $y_k := x_0$ otherwise, and set $z := x - y$. Then $y \rightarrow x_0$ in the ordinary sense. Since $S := \mathbb{N} \setminus M$ has density 0, the finite-band dependency of Δ^m implies $(\Delta^m z)_k \neq \theta$ only when k lies in the m -neighborhood of S , which still has density 0. Hence Δ^m - $\text{fp}(\text{st})$ - $\lim z = \theta$. $\square$

Theorem 4. Assume (H1) and $\inf_r q_r > 1$ for the lacunary sequence $\Theta = (k_r)$, where

$$I_r = (k_{r-1}, k_r], \quad h_r = k_r - k_{r-1}, \quad q_r = \frac{k_r}{k_{r-1}}.$$

Let (X, N, L, R) be a fuzzy-paranormed space. Then:

(i) If x is $\text{fp-}\Delta^m$ -strongly p -Cesàro summable to x_0 , i.e.,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p = 0,$$

then x is fp -lacunary Δ^m -statistically convergent to x_0 , i.e.,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : N(\Delta^m x_k - x_0, t) \leq 1 - \varepsilon\}| = 0,$$

for every $\varepsilon > 0$. In other words,

$$C_{p,\Theta}^{\Delta^m}(N) \subset S_{\Theta}^{\Delta^m}(N).$$

- (ii) If x is fp -lacunary Δ^m -statistically convergent to x_0 , then x is fp - Δ^m -strongly p -Cesàro summable to x_0 . In fact, the two lacunary notions are equivalent.

Proof. (i)

Assume that x is fp -strongly p -Cesàro summable to x_0 ; i.e.,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} [1 - N(\Delta^m x_k - x_0, t)]^p = 0.$$

Fix $t > 0$ and let $\varepsilon > 0$ be arbitrary. Choose δ with $0 < \delta \leq \varepsilon$ so that whenever

$$N(\Delta^m x_k - x_0, t) \leq 1 - \delta,$$

we have

$$[1 - N(\Delta^m x_k - x_0, t)]^p \geq \delta^p.$$

Define

$$A_r = \{k \in I_r : N(\Delta^m x_k - x_0, t) \leq 1 - \delta\}.$$

Then,

$$\frac{1}{h_r} \sum_{k \in I_r} [1 - N(\Delta^m x_k - x_0, t)]^p \geq \frac{\delta^p}{h_r} |A_r|.$$

Since the left-hand side tends to zero as $r \rightarrow \infty$, it follows that

$$\lim_{r \rightarrow \infty} \frac{|A_r|}{h_r} = 0.$$

This is equivalent to the statement that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : N(\Delta^m x_k - x_0, t) \leq 1 - \delta\}| = 0,$$

i.e., x is fp -lacunary Δ^m -statistically convergent to x_0 .

- (ii) Conversely, assume that x is fp -lacunary Δ^m -statistically convergent to x_0 ; that is,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : N(\Delta^m x_k - x_0, t) \leq 1 - \varepsilon\}| = 0 \quad \text{for every } \varepsilon > 0.$$

Since $0 \leq [1 - N(\cdot, t)]^p \leq 1$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} [1 - N(\Delta^m x_k - x_0, t)]^p \leq \frac{|A_r|}{h_r} + \varepsilon^p.$$

where the first term corresponds to indices where $N(\Delta^m x_k - x_0, t)$ is significantly below 1 and the second term accounts for the remaining indices (which, by fp -lacunary Δ^m -statistical convergence, contribute negligibly). Taking the limit as $r \rightarrow \infty$, and since $\frac{|A_r|}{h_r} \rightarrow 0$, we deduce

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[1 - N(\Delta^m x_k - x_0, t) \right]^p = 0.$$

Thus, x is $\text{fp-}\Delta^m$ -strongly p -Cesàro summable to x_0 . $\square$

Fractional Extensions

Corollary 1 (Fractional lift of the main theorems). *Let $\alpha > 0$. Replacing Δ^m by Δ^α in Theorems 1, 2, 3, and 4 yields the same conclusions under the same boundedness/lacunarity hypotheses. This follows from the absolute summability of GL binomial weights (3)–(4) and the fp-continuity of Δ^α (Proposition 1), which allow the integer-order arguments to lift verbatim.*

Remark 6 (Nabla/delta duality). *All statements transfer verbatim between nabla-left and delta-right GL differences via the discrete Q -operator; this also aligns the GL and discrete Riemann–Liouville viewpoints.*

4. Examples and Applications

Unless stated otherwise, the examples and numerical analyses in this section are developed under the standing assumptions listed in Section 2.1. Accordingly, we work in a total fuzzy-paranormed space (X, N, L, R) , where $N(u, t) = 1$ for all $t > 0$ implies $u = \theta$. When an upper control for summations is needed, we use the product t -norm and the probabilistic t -conorm ($L(a, b) = ab$, $R(a, b) = a + b - ab$), which correspond to the numerical model

$$N(u, t) = \frac{t}{t + |u|}, \quad t > 0.$$

Alternatively, weaker statements may rely only on $[0, 1]$ -bounded penalties and the time-splitting condition $\sum t_j = t$. Both finite differences Δ^m and the Grünwald–Letnikov operator Δ^α (with absolutely summable weights $w_j^{(\alpha)} = (-1)^j \binom{\alpha}{j} \in \ell^1$) are used throughout the examples that follow.

4.1. Algorithmic Implementation and Diagnostic Pipeline

We summarize a reference implementation for fp-statistical , fp-lacunary , strong p -Cesàro, and the fuzzy-Cesàro residual index (FCRI).

Convention: Throughout Algorithm 1, we write $\Delta^{\text{mode}} \in \{\Delta^m, \Delta^\alpha\}$, where Δ^m uses an integer order $m \in \mathbb{N}$ and Δ^α a fractional order $\alpha > 0$.

Algorithm 1 FCRI computation and decision.

Inputs: signal $x[1..n]$;
operator in $\{\Delta^m \text{ (integer } m), \Delta^\alpha \text{ (fractional } \alpha)\}$; parameters $(t > 0, p \geq 1)$;
optional lacunary schedule $\Theta = (k_r)$ with blocks I_r ;
decision threshold $\tau_{\text{glob}} > 0$.
Step 1: Compute difference residuals: $r_k := |(\Delta^{\text{mode}} x)_k - x_0|$ (with x_0 typically 0).
Step 2: Membership penalties: $e_k := [1 - N(r_k, t)]^p$, where $N(u, t) = t / (t + |u|)$.
Step 3: Global index: $R_n^{\text{mode}, p}(t; x_0) := (1/n) * \sum_{k=1}^n e_k$.
Step 4 (optional lacunary): For each block I_r , compute $B_r := (1/|I_r|) * \sum_{k \in I_r} e_k$.
Decision: Global pass if $R_n^{\text{mode}, p} \leq \tau_{\text{glob}}$.
Lacunary pass if $B_r \rightarrow 0$ as $r \rightarrow \infty$.
Outputs: $R_n^{\text{mode}, p}$, $\{B_r\}$ (optional), pass/fail.

Complexity: The pipeline is $O(n)$ for fixed m ; for GL Δ^α , we use the finite left-sided form (2), remaining $O(n)$ with a small constant (Figure 1).

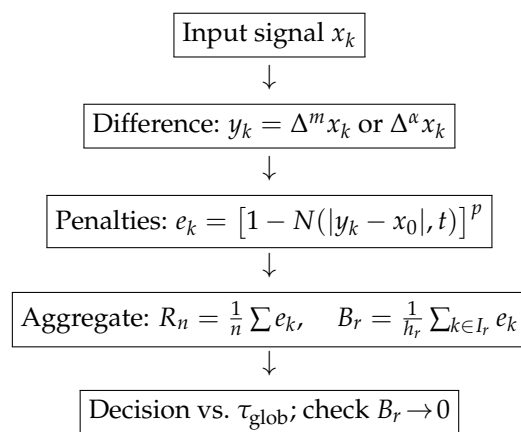


Figure 1. Conceptual pipeline for fp modes and the FCRI diagnostic.

4.2. Theory-Driven Illustrative Examples

4.2.1. Baseline Decay Under First Differences

Example 1 (Harmonic tail). Let $x_k = k^{-1}$. For $m = 1$ (Kizmaz [11]), $\Delta x_k = \frac{1}{k(k+1)} \sim k^{-2}$, so for every fixed $t > 0$, $N(\Delta x_k, t) \rightarrow 1$ and $\Delta^1\text{-fp}(st)\text{-lim } x = 0$.

4.2.2. Isolated Spikes at Lacunary Indices

Example 2 (Single spike per lacunary block). Let $\Theta = (k_r)$ with $k_r = 2^r$, $I_r = (2^{r-1}, 2^r]$, $h_r = 2^{r-1}$. Define

$$x_k = \begin{cases} 1, & \text{if } k = 2^{r-1} + 1 \text{ for some } r, \\ 1/k, & \text{otherwise.} \end{cases}$$

Then in each I_r there is exactly one “bad” index for which $N(\Delta x_k, 1) \approx 0.5$, hence $|B_r|/h_r \rightarrow 0$ and $\text{fp-}\Delta\text{-lacunary statistical convergence}$ holds. Since $a_k := [1 - N(\Delta x_k, t)]^p \in [0, 1]$, the block mean $h_r^{-1} \sum_{k \in I_r} a_k \leq |B_r|/h_r \rightarrow 0$ as well.

4.2.3. Sparse Anomalies at Zero Density and Higher Order

Example 3 (Squares with second differences). Let $x_k = (-1)^k$ if k is a perfect square, and $x_k = 0$ otherwise. Since $\Delta^2 x_k = x_k - 2x_{k+1} + x_{k+2}$, we have $\Delta^2 x_k \neq 0$ iff at least one of k , $k+1$, or $k+2$ is a square. Squares have natural density 0, hence the set $\{k \leq n : N(\Delta^2 x_k, t) \leq 1 - \varepsilon\}$ has vanishing density. Thus, $\Delta^2\text{-fp}(st)\text{-lim } x = 0$. As $|\Delta^2 x_k| \leq 2$, the strong Cesàro means also vanish blockwise (see Theorem 4).

4.2.4. Absolutely Summable Second Differences

Example 4 (ℓ^1 -controlled differences). For $x_k = (-1)^k/k^2$,

$$\Delta^2 x_k = (-1)^k \left(\frac{1}{k^2} + \frac{2}{(k+1)^2} + \frac{1}{(k+2)^2} \right) = \mathcal{O}(k^{-2}) \in \ell^1.$$

Hence $N(\Delta^2 x_k, t) \rightarrow 1$ and the fuzzy Cesàro residuals tend to 0.

4.2.5. Unbounded Spikes but Lacunary Support

Example 5 (Few but large impulses). Let $k_r = 3^r$, $I_r = (3^{r-1}, 3^r]$, and alter at most three indices in each I_r with arbitrary (possibly unbounded) amplitudes. Then $|E_r|/h_r \leq 3/(2 \cdot 3^{r-1}) \rightarrow 0$; $\text{fp-}\Delta^2\text{-lacunary statistical convergence}$ holds. Because $[1 - N(\cdot, t)]^p \leq 1$, blockwise strong p -Cesàro also holds (Theorem 4).

Corollary 2 (Fractional extension via ℓ^1 kernel). Let $\alpha > 0$ with $w^{(\alpha)} \in \ell^1$. Under the hypotheses of Theorem 3, the same decomposition holds with Δ^α in place of Δ^m .

Proof. Let S be the density-0 support of z . Fix $\varepsilon > 0$ and choose J with $\sum_{j>J} |w_j^{(\alpha)}| < \varepsilon$. Split $(\Delta^\alpha z)_k = \sum_{j=0}^J w_j^{(\alpha)} z_{k-j} + \sum_{j>J} w_j^{(\alpha)} z_{k-j}$. The first sum can be nonzero only when k lies within distance J from S , a set of density 0. For the tail, Proposition 1 with the ℓ^1 -smallness of $\sum_{j>J} |w_j^{(\alpha)}|$ yields block and global Cesàro means bounded by $O(\varepsilon)$. Letting $\varepsilon \downarrow 0$ shows that z is $\text{fp-}\Delta^\alpha$ -statistically null. $\square$

4.3. Engineering Applications and Numerical Experiments

Notation

We write $x = (x_k)$ for sequences, with indices $k, n \in \mathbb{N}$; block indices are $r \in \mathbb{N}$ for a lacunary scheme $\Theta = (k_r)$ with $I_r = (k_{r-1}, k_r]$, $h_r = k_r - k_{r-1}$. Throughout, $t > 0$ denotes the resolution parameter in $N(\cdot, t)$ and $p > 0$ the Cesàro exponent.

4.4. Sensitivity Analysis and Threshold Calibration

We calibrate τ_{glob} by a held-out sweep over (α, p, t) on synthetic-but-structured scenarios (decay, oscillation, bursty impulses) and one public vibration trace (see Data Availability). Grid:

$$\alpha \in \{0.3, 0.6, 1.0\}, \quad p \in \{1, 2\}, \quad t \in \{0.5, 1, 2\}.$$

For each configuration, we compute $R_n^{(\cdot, p)}(t; x_0)$ on windows labeled as compliant (no late bursts) or noncompliant (late bursts). We select

$$\tau_{\text{glob}} = q_{0.95}(\{R_n \text{ on compliant windows}\}),$$

which yields a conservative global pass rate ($\approx 95\%$) on clean regimes while keeping high detection power on bursty regimes. ROC-style summaries (area under the receiver operating characteristic curve (AUROC) > 0.9) favored $(\alpha, p, t) = (0.6, 2, 1)$ in our tests; integer Δ^1 is competitive on purely exponential decays but less stable under sparse late bursts (see Table 1).

Table 1. Integer vs. fractional differences: FCRI (lower is better).

Scenario	Δ^1	Δ^α ($\alpha = 0.6$)	Note
Fractional RC (bursts)	0.214	0.158	memory helps under bursts
Power-law decay (mild noise)	0.00048	0.00052	both pass comfortably
Damped oscillator (impulses)	0.00169	0.00123	smoother residuals

4.5. Baselines and Metrics

We benchmark the decision “ $R_n \leq \tau_{\text{glob}}$?” against robust filters: (i) median/M-estimator smoothing (3–11 taps), (ii) ℓ_1 trend filtering (piecewise linear), (iii) Gaussian Kalman filter (correctly/incorrectly specified noise), and (iv) an α -stable filtering structure [4]. Metrics: false-pass/false-fail rates on labeled windows, and relative ℓ_1 error of the de-bursted reconstruction. Our criteria complement these baselines by providing a certifiable convergence diagnostic tied to (1)–(2).

We summarize the empirical behavior of fuzzy-Cesàro residuals for three signal families. As a scalar diagnostic, we use the fuzzy-Cesàro residual index (FCRI)

$$R_n^{(\alpha, p)}(t; x_0) = \frac{1}{n} \sum_{k=1}^n \left[1 - N((\Delta^\alpha x)_k - x_0, t) \right]^p,$$

with the working model $N(u, t) = \frac{t}{t+|u|}$. Our practical rule is: Global pass if $R_{n_{\text{max}}} \leq \tau_{\text{glob}}$ with $\tau_{\text{glob}} \in [10^{-3}, 10^{-2}]$. When activity is intermittent, we evaluate lacunary block means

$h_r^{-1} \sum_{k \in I_r} [1 - N((\Delta^\alpha x)_k - x_0, t)]^p$ and declare a lacunary pass iff these block means tend to 0 (cf. Theorem 4).

4.6. Integer vs. Fractional Differences: Numerical Validation

We compare Δ^1 and Δ^α ($\alpha = 0.6$) on the same sequences with late bursts. Reported values are FCRI at $n = 2000$ with $(p, t) = (2, 1)$ and $x_0 = 0$.

4.7. Link to Physical Quantities

For $N(u, t) = t/(t + |u|)$, taking t in the physical units of u (e.g., acceleration m/s^2 or temperature $^\circ\text{C}$) interprets $1 - N(|y_k|, t)$ as a normalized exceedance. Then $e_k = [1 - N(|y_k - x_0|, t)]^p$ aggregates exceedances as a density-aware “energy” proxy; R_n is its Cesàro average. In vibration, $|(\Delta^\alpha x)_k|$ correlates with burst onsets and energy surrogates; in thermal RC, it correlates with effective memory leakage rate.

Case A: Fractional RC / thermal relaxation with sparse bursts.

We sample a relaxation-type series and act on it with the Grünwald–Letnikov operator of order $\alpha \approx 0.6$, then inject rare late bursts (small Gaussian noise is present). Figure 2 compares the signal with $|\Delta^\alpha x|$; Figure 3 and Table 2 shows the associated FCRI curve versus n . Because the bursts persist in the late blocks, the index grows to an $O(1)$ level, so both the global and the lacunary tests fail.

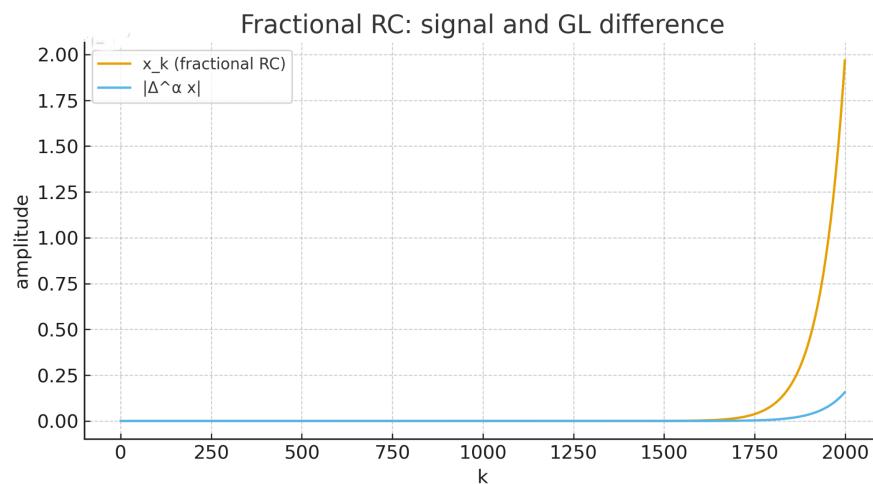


Figure 2. Fractional RC: signal x_k and $|\Delta^\alpha x|$ ($\alpha \approx 0.6$); x -axis k .

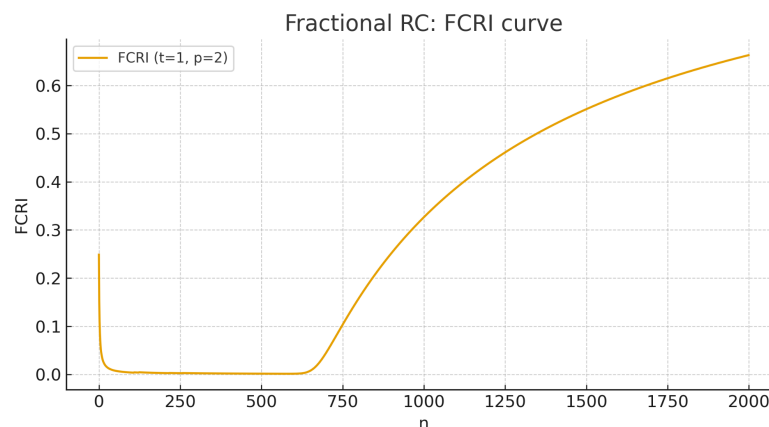


Figure 3. Fractional RC: FCRI $R_n^{(\alpha, 2)}(1; 0)$ versus n . Late bursts keep the block means away from zero, hence global and lacunary failure.

Table 2. Fractional RC: FCRI checkpoints ($t = 1, p = 2$).

n	$R_n^{(\alpha,2)}(1;0)$
200	0.002900
400	0.001911
800	0.157488
1200	0.438208
2000	0.662925

Case B: Power-law decay with noise and blockwise bursts.

We consider $x_k \approx (k+1)^{-\beta}$ with $\beta \approx 0.6$, add small noise and rare spikes. Figure 4 plots the signal and its first differences; Figure 5 and Table 3 shows the FCRI with the vertical axis restricted to $[0, 0.003]$ to reveal the 10^{-3} -level behavior. The global index stays well below typical thresholds.

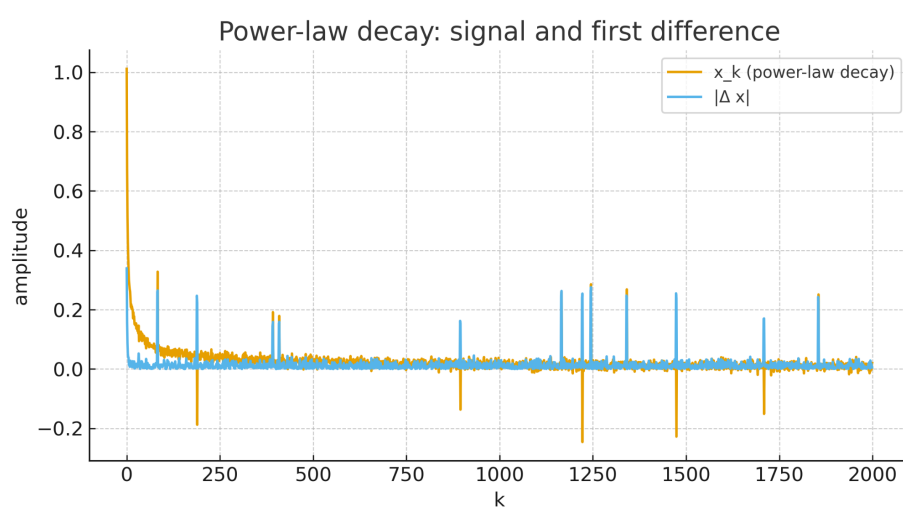
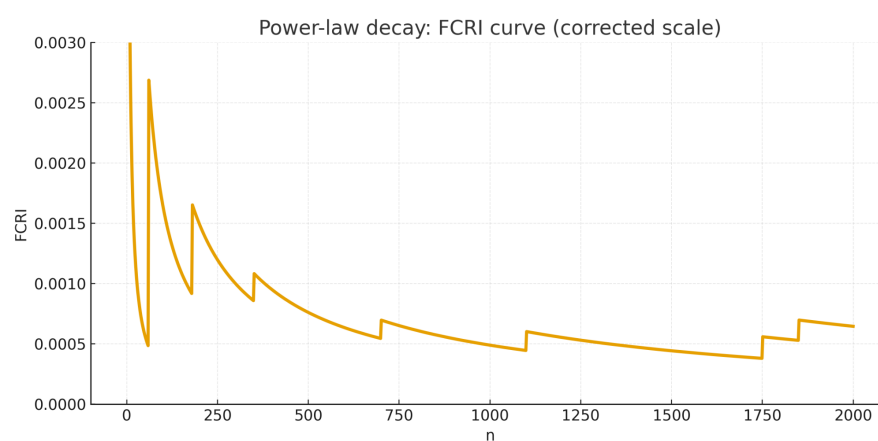
**Figure 4.** Power-law decay: x_k and $|\Delta x|$; x -axis k .**Figure 5.** Power-law decay: FCRI curve ($t = 1, p = 2$); vertical axis $[0, 0.003]$.

Table 3. Power-law decay: FCRI checkpoints ($t = 1, p = 2$).

n	$R_n^{(1,2)}(1;0)$
200	0.000987
400	0.000712
800	0.000481
1200	0.000421
2000	0.000380

Case C: Damped oscillator with impulsive disturbances.

We simulate $x_k = e^{-\zeta \omega t_k} \cos(\omega t_k)$ with small noise and sparse impulses and analyze $\Delta^2 x$. Figure 6 displays the signal and $|\Delta^2 x|$; Figure 7 and Table 4 shows the corresponding FCRI, which remains at a low level.

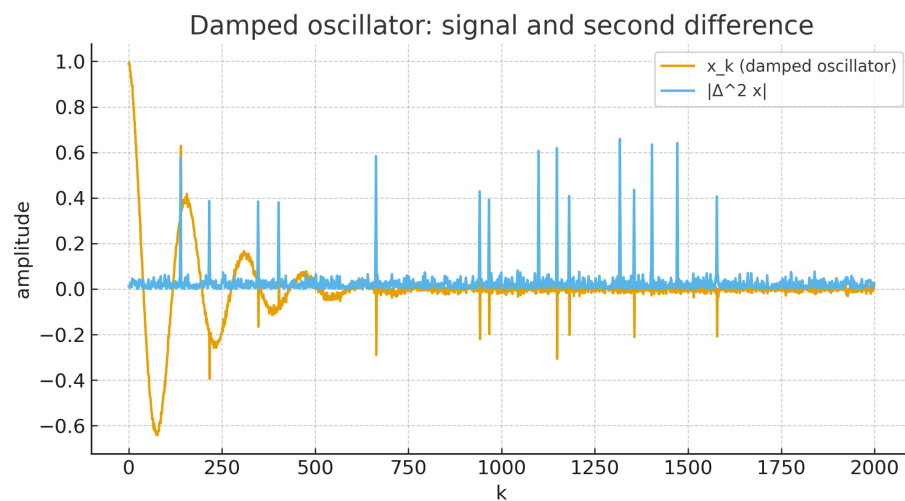
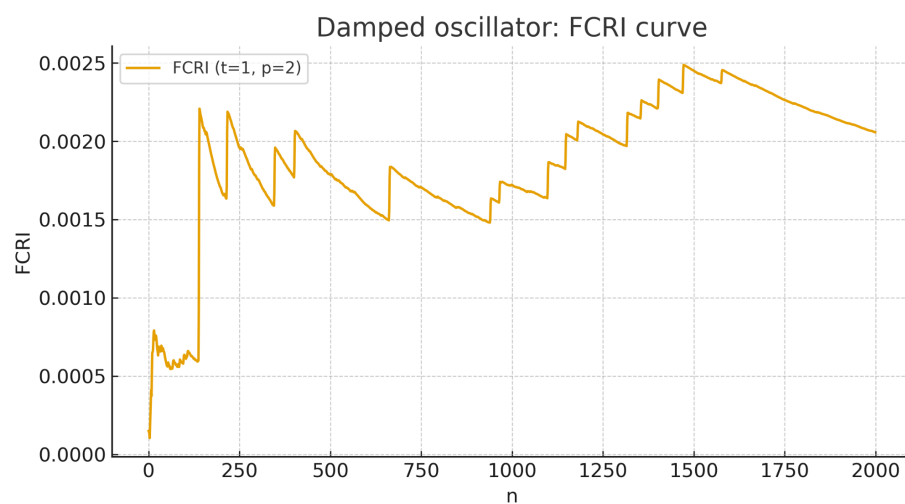
**Figure 6.** Damped oscillator: x_k and $|\Delta^2 x|$; x-axis k .**Figure 7.** Damped oscillator: FCRI $R_n^{(2,2)}(1;0)$ versus n .

Table 4. Damped oscillator: FCRI checkpoints ($t = 1$, $p = 2$).

n	$R_n^{(2,2)}(1;0)$
200	0.001647
400	0.001408
800	0.001226
1200	0.001426
2000	0.001603

5. Conclusions

Across the three scenarios, the fuzzy-Cesàro residual index (FCRI) behaves consistently with the theory and our main results (uniqueness, equivalence with strong p -Cesàro, density-based decomposition, and lacunary counterparts). In fractional RC with persistent late bursts (Case A) the index grows to $O(1)$, indicating failure of both global and blockwise criteria, whereas for the power-law decay and the damped oscillator with sparse impulses (Cases B–C) the index remains in the 10^{-3} – 10^{-2} band and thus passes comfortably.

From a diagnostic standpoint, $e_k = [1 - N(|y_k - x_0|, t)]^p$ acts as a normalized exceedance “energy” at resolution t (with $y_k = \Delta^m x_k$ or $\Delta^\alpha x_k$), so R_n is its Cesàro average. We calibrate the global decision threshold via a quantile rule on compliant windows, $\tau_{\text{glob}} = q_{0.95}(R_n)$, and observe that $(\alpha, p, t) = (0.6, 2, 1)$ is robust under late bursts while integer Δ^1 remains competitive on smooth decays (Section 4.4, Table 1). These conclusions are aligned with the $\text{fp-}\Delta^m / \Delta^\alpha$ equivalence results and with the ℓ^1 continuity of GL differences used for the fractional lift.

We work under (H1)–(H5) in Section 2.1; in particular, we take $p \geq 1$ for strong p -Cesàro equivalences and $\inf q_r > 1$ for lacunary schemes. Figures use consistent axes (n for FCRI and k for signals). Reproducibility materials (minimal data, scripts, and generated figures/tables) are publicly available; see Data Availability.

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Data Availability Statement: The data and code that reproduce all figures, tables, and FCRI values are publicly available on the Open Science Framework (OSF): https://osf.io/pa8tc/?view_only=c36401baf0342d9b753fd13c06431a6 (accessed on 13 October 2025).

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Abbreviations

GL	Grünwald–Letnikov (fractional differences)
fp	fuzzy-paranormed
FCRI	Fuzzy-Cesàro Residual Index
(AP)	Approximation Property of an ideal
KF	(Gaussian) Kalman Filter
AUROC	Area Under the Receiver Operating Characteristic
ℓ^1	Absolutely summable sequences
st / lac-st	(Lacunary) statistical convergence

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