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Existence and Uniqueness of Positive Solutions to Fractional Problems of Brézis–Oswald-Type with Unbalanced Growths and Hardy Potentials

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Abstract

The aim of this paper is to establish the existence and uniqueness of positive solutions to the non-local Brézis–Oswald-type fractional problems that involve fractional (r, q) -Laplace operators and Hardy potentials. In particular, we observe an eigenvalue problem associated with the fractional (r, q) -Laplacian to determine the existence of at least one positive weak solution for our problem. The main features of this paper are the lack of the semicontinuity property of an energy functional related to our problem and the presence of a singular coefficient. The decisive tool for overcoming this technical difficulty is the concentration–compactness principle in fractional, critical and Hardy terms. Moreover, we establish the uniqueness results of Brézis–Oswald-type by exploiting a generalization of the discrete Picone inequality.

Keywords: fractional (r, q) -Laplacian; hardy inequality; concentration–compactness principle; discrete Picone’s inequality

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1. Introduction

This paper is dedicated to the following non-local Brézis–Oswald-type fractional problems with unbalanced growths and Hardy potentials:

$$\begin{cases} \mathcal{L}_r^s \omega + \mathcal{L}_q^s \omega = \frac{\eta |\omega|^{m-2} \omega}{|x|^{sm}} + \lambda h(x, \omega) & \text{in } \Omega, \\ \omega > 0 & \text{in } \Omega, \\ \omega = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1)$$

where $0 < s < 1$, $1 < r < q < +\infty$, with $sq < N$, $m \in \{r, q\}$; $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an open and bounded set with Lipschitz boundary $\partial\Omega$; $0 \in \Omega$, η, λ are real parameters; and h is a non-negative Carathéodory function, which will be described later. Here, $\mathcal{L}_m^s$ ($m \in \{r, q\}$) is a non-local pointwise operator defined as

$$\mathcal{L}_m^s \omega(x) = 2 \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^{m-2} (\omega(x) - \omega(z)) K_m(x, z) dx \quad \text{for all } x \in \mathbb{R}^N,$$

where the function $K_m : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ fulfills the following assumptions:

(K1) $\kappa K_m \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $\kappa(x, z) = \min\{|x - z|^m, 1\}$;

(K2) There exist positive constants $(\alpha_{0,m}$ and $\alpha_{1,m})$ with $\alpha_{0,m} \geq 1$ such that $\alpha_{0,m} \leq K_m(x, z)|x - z|^{N+sm} \leq \alpha_{1,m}$ for $z \neq x$ and for almost all $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$;

(K3) $K_m(x, z) = K_m(z, x)$ for all $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$.

If $K_m(x, z) = |x - z|^{-(N+sm)}$, then $\mathcal{L}_m^s$ is the fractional m -Laplacian operator $(-\Delta)_m^s$ defined by

$$(-\Delta)_m^s \omega(x) = 2 \lim_{\varrho \searrow 0} \int_{\mathbb{R}^N \setminus B_\varrho(x)} \frac{|\omega(x) - \omega(z)|^{m-2} (\omega(x) - \omega(z))}{|x - z|^{N+sm}} dz, \quad x \in \mathbb{R}^N,$$

where $B_\varrho(x) := \{z \in \mathbb{R}^N : |x - z| \leq \varrho\}$.

The combination of the operators $\mathcal{L}_r^s$ and $\mathcal{L}_q^s$ in Problem (1) can be seen as a kind of fractional problem with an unbalanced double-phase operator closely related to mathematical physics, such as a fractional white-noise limit, fractional quantum mechanics and fractional super diffusion; see [1] for more detail. In particular, the operator $(-\Delta)_r^s + (-\Delta)_q^s$ is referred to as the fractional (r, q) -Laplace operator, which can be considered the fractional analog of the (r, q) -Laplace operators originating from significant applications in mathematical physics and related sciences such as plasma physics, strongly anisotropic materials, elasticity theory, chemical reaction design and biophysics; see [2,3] for more details. The investigation on nonlinear fractional (r, q) -Laplacian problems has recently drawn noticeable attention. This not only naturally extends nonlinear equations involving the (r, q) -Laplacian but also propounds many new phenomena and various applications represented as nonlinear integral structures. For example, we refer to [1,4–6] for the existence of multiple solutions to elliptic problems with the fractional (r, q) -Laplace operator.

This study is dedicated to the existence and uniqueness of positive solutions to non-local fractional problems involving the fractional (r, q) -Laplacian operators and Hardy potentials. It is motivated by the celebrated result by Brézis and Oswald [7], who introduced a fresh and elegant approach to the existence and uniqueness of sublinear elliptic equations. Regarding Brézis and Oswald's result, there has been much attraction in providing sufficient and/or necessary conditions to obtain this result from more general elliptic problems while taking into account various operators and boundary conditions; see [8–11] for more information. In particular, the existence of a unique positive weak solution for a quasilinear elliptic problem in the Orlicz–Sobolev framework has been obtained in [11] by improving the results of [12] for p -Laplacian equations. The main tools to derive the Brézis–Oswald result to the nonlinear elliptic problems in [11,12] are the Diaz–Saa-type inequalities. But, these inequalities do not apply directly to elliptic problems with fractional orders. To overcome this difficulty, the authors of [10,13–15] established the existence result of at most one positive weak solution to nonlinear elliptic equations with fractional orders by exploiting the discrete Picone inequality in [8,16]. By applying this inequality, the necessary and sufficient conditions for ensuring Brézis–Oswald results for the fractional p -Laplacian problem with Robin boundary condition have been obtained by Mugnai, Pinamonti and Vecchi [15]. Based on this work, ref. [14] recently derived the existence of at most one positive weak solution to non-local problems with discontinuous Kirchhoff coefficients. Furthermore, we introduce the recent work of [14] on non-local fractional p -Laplacian equations with Hardy potentials.

As we have seen above, the discrete Picone inequality in [8,16] is a critical tool in obtaining the unique result to elliptic problems with fractional orders. However, this inequality has a restriction in its application to non-homogeneous (p, q) -Laplace problems. Focusing on this difficulty, the authors of [13] considered new discrete Picone inequalities to include a large class of non-homogeneous and fractional operators. Utilizing these inequalities, they have provided important applications such as the non-existence, existence and uniqueness of positive weak solutions to the fractional equations with (p, q) -growth.

Inspired by their work, Ref. [17] identified a unique positive weak solution for fractional Brézis–Oswald-type (p, q) -Laplacian problems involving discontinuous Kirchhoff-type coefficients, thereby extending the results for the fractional p -Laplacian problems in [14,18] to fractional double-phase equations with unbalanced growths.

Another riveting aspect of the problem at hand is the presence of Hardy potentials. In recent years, stationary problems with singular coefficients have attracted more and more attention as they can have many physical and applied economical interpretations. In addition, we can quote the contributions of [19–21] for more inclusive details and examples. Additionally, in light of this tremendous interest, such problems have been extensively studied in the recent years; see [22–24] for more information on these problems. From a mathematical point of view, nonlinear elliptic problems with Hardy terms have difficulty in ensuring the Palais–Smale-type compactness condition in the desired functional space. For this reason, the authors of [22–24] leveraged the critical point theorems of [25,26] to demonstrate the existence of results with multiple nontrivial solutions without considering this compactness condition. Similarly, the existence of at least one nontrivial solution for a nonlinear Dirichlet boundary value problem has been investigated in [22]; see [24] for p -Laplacian-like problems. In this instance, we also quote the paper of [23] for infinitely many solutions. Recently, by applying the cut-off function method to overcome the lack of compactness of the Euler–Lagrange functional, which is the main challenge, Fiscella in [27] established the existence of at least one nontrivial solution to the double-phase Dirichlet boundary value problem involving Hardy potential:

$$\begin{aligned} & -\operatorname{div}(|\nabla \omega|^{r-2} \nabla \omega + v(x)|\nabla \omega|^{q-2} \nabla \omega) \\ & = \eta \left(\frac{|\omega|^{r-2} \omega}{|x|^r} + v(x) \frac{|\omega|^{q-2} \omega}{|x|^q} \right) + h(x, \omega) \quad \text{in } \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is an open and bounded domain with a Lipschitz boundary, $0 \in \Omega$ and r, q satisfy the condition

$$1 < r < q < N, \quad \frac{q}{r} < 1 + \frac{1}{N},$$

$v : \overline{\Omega} \rightarrow [0, \infty)$ is Lipschitz continuous, and $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function. The main tool for obtaining this result is the classical mountain pass theorem. Based on this work, the multiplicity result of solutions for Schrödinger–Hardy-type equations involving fractional p -Laplacian operator presented in [28].

Unlike the aforementioned related studies, this study aims to demonstrate the existence of at most one positive weak solution to non-local Brézis–Oswald-type fractional problems with the fractional (r, q) -Laplace operator and the Hardy potential without exploiting the variational methods in [27,28] and the critical point theorems in [25,26]. In particular, this is an improvement of the notable works of [13,17,29] and extends the results for the non-local fractional equations in [14,18] to the case for unbalanced double-phase fractional problems with Hardy potentials. However, compared to [13,17,18,29], there are some technical difficulties in validating the semicontinuity property of an energy functional associated with Problem (1) due to the appearance of the Hardy term. To overcome this difficulty and derive this property, we exploit the fractional Hardy inequality in [16] and the concentration–compactness principle for fractional, critical and Hardy terms in [30,31]. By using this semicontinuity property and considering eigenvalue problems related to the fractional (r, q) -Laplacian, we present the existence of at least one positive weak solution for our problem as our first result. This approach is inspired by the recent work of [14], but as we have seen above, our problem is a fractional equation with (r, q) -growth, so it has more complex nonlinearities than those in [14] and needs to be carefully analyzed. Moreover, we obtained the uniqueness result by applying various versions of the discrete

Picone inequalities in [8,13] as a decisive tool. To the best of our knowledge, this work is the first attempt to determine the existence of at most one positive weak solution to non-local Brézis–Oswald-type elliptic problems that involve fractional (r, q) -Laplace operators and Hardy potentials.

This paper is organized as follows: Section 2 introduces the appropriate functional space and presents a few important properties such as the basic embedding results and fractional Hardy inequality that are later required. In Section 3, we identify the variational setting and auxiliary results corresponding to Problem (1). Finally, in Section 4, we provide useful consequences regarding the eigenvalue problem for the (r, q) -Laplacian equation and prove the existence and uniqueness results of positive solutions to Problem (1).

2. Preliminaries

Let $0 < s < 1 < r < +\infty$ be real numbers with $sr < N$, and define the fractional Sobolev space $W^{s,r}(\mathbb{R}^N)$ as

$$W^{s,r}(\mathbb{R}^N) := \left\{ \omega \in L^r(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\omega(x) - \omega(z)|^r}{|x - z|^{N+rs}} dz dx < +\infty \right\},$$

which is endowed with the norm

$$\|\omega\|_{W^{s,r}(\mathbb{R}^N)} := \left(\|\omega\|_{L^r(\mathbb{R}^N)}^r + |\omega|_{W^{s,r}(\mathbb{R}^N)}^r \right)^{\frac{1}{r}},$$

where

$$\|\omega\|_{L^r(\mathbb{R}^N)}^r := \int_{\mathbb{R}^N} |\omega(x)|^r dx \quad \text{and} \quad |\omega|_{W^{s,r}(\mathbb{R}^N)}^r := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\omega(x) - \omega(z)|^r}{|x - z|^{N+rs}} dz dx.$$

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary. The functional space $W_0^{s,r}(\Omega)$ is the set of measurable functions defined by

$$W_0^{s,r}(\Omega) := \left\{ \omega \in W^{s,r}(\mathbb{R}^N) : \omega(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\},$$

and it is equipped with the norm

$$\|\omega\|_{W_0^{s,r}(\Omega)} := \left(\|\omega\|_{L^r(\Omega)}^r + [\omega]_{s,r}^r \right)^{\frac{1}{r}},$$

where

$$[\omega]_{s,r}^r := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\omega(x) - \omega(z)|^r}{|x - z|^{N+rs}} dz dx.$$

Then, the space $W_0^{s,r}(\Omega)$ is a separable and reflexive Banach space. In addition, the space $C_0^\infty(\Omega)$ is dense in $W_0^{s,r}(\Omega)$ (see [32,33] for instance).

Lemma 1 ([33]). *Let $0 < s < 1 < r < +\infty$. Then, the continuous embeddings are satisfied:*

$$\begin{aligned} W_0^{s,r}(\Omega) &\hookrightarrow L^\ell(\Omega) && \text{for all } \ell \in [1, r_s^*], && \text{if } rs < N; \\ W_0^{s,r}(\Omega) &\hookrightarrow L^\ell(\Omega) && \text{for every } \ell \in [1, \infty), && \text{if } rs = N. \end{aligned}$$

Specifically, the compact embedding $W_0^{s,r}(\Omega) \hookrightarrow L^\gamma(\Omega)$ holds for any $\gamma \in [1, r_s^*)$, where r_s^* is the fractional critical Sobolev exponent, that is,

$$r_s^* := \begin{cases} \frac{Nr}{N-sr} & \text{if } sr < N, \\ +\infty & \text{if } sr \geq N. \end{cases}$$

Let define the fractional Sobolev space $W_K^{s,r}(\mathbb{R}^N)$ by

$$W_{K_r}^{s,r}(\mathbb{R}^N) := \left\{ \psi \in L^r(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\psi(x) - \psi(z)|^r K_r(x, z) dx dz < +\infty \right\},$$

where $K_r : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0, 0)\} \rightarrow (0, +\infty)$ is a kernel function with the properties of $(\mathcal{K}1)$ – $(\mathcal{K}3)$. According to the condition of $(\mathcal{K}1)$, the function

$$(x, z) \mapsto K_r^{\frac{1}{r}}(x, z)(\psi(x) - \psi(z)) \in L^r(\mathbb{R}^N)$$

holds for any $\psi \in C_0^\infty(\mathbb{R}^N)$. Let define the closed linear subspace of the space $W_{K_r}^{s,r}(\mathbb{R}^N)$ by

$$W_{0,K_r}^{s,r}(\Omega) := \left\{ \psi \in W_{K_r}^{s,r}(\mathbb{R}^N) : \psi(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}$$

and it is equipped with the norm

$$\|\psi\|_{W_{0,K_r}^{s,r}(\Omega)} := \left(\|\psi\|_{L^r(\Omega)}^r + [\psi]_{s,r,K_r}^r \right)^{\frac{1}{r}},$$

where

$$[\psi]_{s,r,K_r}^r := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\psi(x) - \psi(z)|^r K_r(x, z) dx dz.$$

Subsequently, let $0 < s < 1 < r < q < r_s^*$, with $sq < N$, and let $K_r : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0, 0)\} \rightarrow (0, \infty)$ be the kernel function that verifies the assumptions of $(\mathcal{K}1)$ – $(\mathcal{K}3)$.

Lemma 2 ([34]). *If $\psi \in W_{0,K_r}^{s,r}(\Omega)$, then $\psi \in W_0^{s,r}(\Omega)$. Moreover,*

$$\|\psi\|_{W_0^{s,r}(\Omega)} \leq \max\{1, \alpha_{0,r}^{-\frac{1}{r}}\} \|\psi\|_{W_{0,K_r}^{s,r}(\Omega)},$$

where $\alpha_{0,r}$ is given in $(\mathcal{K}2)$.

From Lemmas 1 and 2, we can derive the next consequence instantly:

Lemma 3 ([34]). *For any $\psi \in W_0^{s,r}(\Omega)$ and $1 \leq \gamma \leq r_s^*$, there exists a positive constant $\mathfrak{C} = \mathfrak{C}(s, r, N)$ such that*

$$\begin{aligned} \|\psi\|_{L^\gamma(\Omega)}^\gamma &\leq \mathfrak{C} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\psi(x) - \psi(z)|^r}{|x - z|^{N+rs}} dx dz \\ &\leq \frac{\mathfrak{C}}{\alpha_{0,r}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\psi(x) - \psi(z)|^r K_r(x, z) dx dz, \end{aligned}$$

where $\alpha_{0,r}$ is given in $(\mathcal{K}2)$. In addition, the embedding $W_{0,K_r}^{s,r}(\Omega) \hookrightarrow L^\gamma(\Omega)$ is continuous for any $\gamma \in [1, r_s^*]$. In addition, the embedding

$$W_{0,K_r}^{s,r}(\Omega) \hookrightarrow L^\gamma(\Omega)$$

is compact for $\gamma \in (1, r_s^*)$.

To consider Problem (1), let the closed linear subspace be defined by

$$\mathbb{E} := W_{0,K_r}^{s,r}(\Omega) \cap W_{0,K_q}^{s,q}(\Omega)$$

which is induced by the norm

$$\|\omega\|_{\mathbb{E}} := \|\omega\|_{W_{0,K_r}^{s,r}(\Omega)} + \|\omega\|_{W_{0,K_q}^{s,q}(\Omega)}.$$

Considering Lemmas 1 and 3, we can assert the following consequence:

Lemma 4. *If $\omega \in \mathbb{E}$, then $\omega \in W_0^{s,r}(\Omega) \cap W_0^{s,q}(\Omega)$. Furthermore, there exists a constant $\mathfrak{C}_0 = C_0(s, r, q, N) > 0$ such that*

$$\|\omega\|_{L^{\ell}(\Omega)}^{\ell} \leq \mathfrak{C}_0 \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) + |\omega(x) - \omega(z)|^q K_q(x, z) dz dx \right)$$

for any $1 \leq \ell \leq r_s^*$ and $\omega \in \mathbb{E}$. As a result, the continuous embedding $\mathbb{E} \hookrightarrow L^{\ell}(\Omega)$ holds for any $\ell \in [1, r_s^*]$. Additionally, the embedding

$$\mathbb{E} \hookrightarrow L^{\ell}(\Omega)$$

is compact for any $\ell \in (1, r_s^*)$.

The next consequence is the fractional Hardy inequality provided in [16].

Lemma 5. *For any $\omega \in W_{0,K_m}^{s,m}(\Omega)$ where $sm < N$ and for all $\omega \in W_{0,K_r}^{s,r}(\Omega) \setminus \{0\}$ where $sm > N$, we have*

$$\begin{aligned} \|\omega\|_{\mathcal{H}_m}^m &:= \int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} dx \leq c_{\mathcal{H}_m} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\omega(x) - \omega(z)|^m}{|x - z|^{N+sm}} dz dx, \\ &\leq \frac{c_{\mathcal{H}_m}}{\alpha_{0,m}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^m K_m(x, z) dz dx, \end{aligned}$$

where $c_{\mathcal{H}_m} := \left(\frac{m}{N-m}\right)^{-m}$ is a positive constant. In particular, if $m \in \{r, q\}$, then for any $\omega \in \mathbb{E}$ where $sq < N$ and for all $\omega \in \mathbb{E} \setminus \{0\}$ where $sq > N$, we know that

$$\int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} dx \leq \left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right) \left([\omega]_{s,r,K_r}^r + [\omega]_{s,q,K_q}^q \right). \quad (2)$$

3. Variational Setting and Auxiliary Results

In this section, we provide the variational setting and auxiliary results related to Problem (1).

Definition 1. *For $m \in \{r, q\}$, we say that $\omega \in \mathbb{E}$ is a weak solution of (1) if*

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^{r-2} (\omega(x) - \omega(z)) (\psi(x) - \psi(z)) K_r(x, z) dz dx \\ &+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^{q-2} (\omega(x) - \omega(z)) (\psi(x) - \psi(z)) K_q(x, z) dz dx \\ &= \eta \int_{\Omega} \frac{|\omega(x)|^{m-2} \omega(x)}{|x|^{sm}} \psi(x) dx + \lambda \int_{\Omega} h(x, \omega) \psi(x) dx \end{aligned} \quad (3)$$

for any $\psi \in \mathbb{E}$.

Let us define the functional $\mathcal{E}_\eta : \mathbb{E} \rightarrow \mathbb{R}$ by

$$\mathcal{E}_\eta(\omega) := \mathcal{E}(\omega) - \eta \mathcal{E}_{\mathcal{H}_m}(\omega),$$

where

$$\begin{aligned} \mathcal{E}(\omega) := & \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dx dz \\ & + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dx dz \end{aligned}$$

and

$$\mathcal{E}_{\mathcal{H}_m}(\omega) := \frac{1}{m} \int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} dx.$$

Then, it is obvious that $\mathcal{E}_\eta$ is well defined on $\mathbb{E}$, and the following assertion is verified from a similar argument, as seen in the proof of Lemma 2 in [35].

Lemma 6. For $m \in \{r, q\}$, the functional $\mathcal{E}_\eta : \mathbb{E} \rightarrow \mathbb{R}$ is of class $C^1(\mathbb{E}, \mathbb{R})$, and its Fréchet derivative is

$$\begin{aligned} \langle \mathcal{E}'_\eta(\omega), v \rangle = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^{r-2} (\omega(x) - \omega(z)) (v(x) - v(z)) K_r(x, z) dx dz \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^{q-2} (\omega(x) - \omega(z)) (v(x) - v(z)) K_q(x, z) dx dz \\ & - \eta \int_{\Omega} \frac{|\omega(x)|^{m-2} \omega(x)}{|x|^{sm}} v(x) dx \end{aligned} \quad (4)$$

for any $\omega, v \in \mathbb{E}$. Here, let us denote $\langle \cdot, \cdot \rangle$ with the pairing of $\mathbb{E}$ and its dual $\mathbb{E}^*$.

Proof. It is easy to derive that $\mathcal{E}_\eta$ has the Fréchet derivative in $\mathbb{E}$ and (4) holds for any $\omega, v \in \mathbb{E}$. Let $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathbb{E}$ be a sequence strongly satisfying $\omega_n \rightarrow \omega$ in $\mathbb{E}$ as $n \rightarrow \infty$. Without loss of generality, we suppose that $\omega_n \rightarrow \omega$ a.e. in $\mathbb{R}^N$.

Then, the sequence

$$\left\{ |\omega_n(x) - \omega_n(z)|^{m-2} (\omega_n(x) - \omega_n(z)) K_m(x, z)^{\frac{1}{m'}} \right\}_{n \in \mathbb{N}}$$

is bounded in $L^{m'}(\mathbb{R}^N \times \mathbb{R}^N)$, as well as a.e. in $\mathbb{R}^N \times \mathbb{R}^N$

$$\begin{aligned} \Pi_n^m(x, z) &:= |\omega_n(x) - \omega_n(z)|^{m-2} (\omega_n(x) - \omega_n(z)) K_m(x, z)^{\frac{1}{m'}} \\ &\longrightarrow \Pi^m(x, z) := |\omega(x) - \omega(z)|^{m-2} (\omega(x) - \omega(z)) K_m(x, z)^{\frac{1}{m'}} \quad \text{as } n \rightarrow \infty \end{aligned}$$

for $m \in \{r, q\}$. Thus, by means of the Brezis–Lieb lemma (see [36]), we infer that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\Pi_n^m(x, z) - \Pi^m(x, z)|^{m'} dz dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (|\omega_n(x) - \omega_n(z)|^m K_m(x, z) - |\omega(x) - \omega(z)|^m K_m(x, z)) dx dz. \end{aligned} \quad (5)$$

The fact that $\{\omega_n\}_{n \in \mathbb{N}}$ converges strongly to ω in $\mathbb{E}$ as $n \rightarrow \infty$ yields that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (|\omega_n(x) - \omega_n(z)|^m K_m(x, z) - |\omega(x) - \omega(z)|^m K_m(x, z)) dz dx = 0.$$

Owing to (5), one has

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\Pi_n^m(x, z) - \Pi^m(x, z)|^{m'} dz dx = 0. \quad (6)$$

Conversely, in accordance with Lemma 5, the sequence

$$\left\{ \frac{|\omega_n(x)|^{m-2} \omega_n(x)}{|x|^{\frac{sm}{m'}}} \right\}_{n \in \mathbb{N}}$$

is bounded in $L^{m'}(\Omega)$, as well as a.e. in Ω

$$\tilde{\Pi}_n^m(x, z) := \frac{|\omega_n(x)|^{m-2} \omega_n(x)}{|x|^{\frac{sm}{m'}}} \longrightarrow \tilde{\Pi}^m(x, z) := \frac{|\omega(x)|^{m-2} \omega(x)}{|x|^{\frac{sm}{m'}}} \quad \text{as } n \rightarrow \infty$$

Thus, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\tilde{\Pi}_n^m(x, z) - \tilde{\Pi}^m(x, z)|^{m'} dx = \lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{|\omega_n(x)|^m}{|x|^{sm}} - \frac{|\omega(x)|^m}{|x|^{sm}} \right) dx. \quad (7)$$

The fact that $\{\omega_n\}_{n \in \mathbb{N}}$ converges strongly to ω in $\mathbb{E}$ as $n \rightarrow \infty$ and to Lemma 5 implies that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{|\omega_n(x)|^m}{|x|^{sm}} - \frac{|\omega(x)|^m}{|x|^{sm}} \right) dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \frac{|\omega_n(x) - \omega(x)|^m}{|x|^{sm}} dx \\ &\leq \lim_{n \rightarrow \infty} \frac{c_{\mathcal{H}_m}}{\alpha_{0,m}} \left[\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |(\omega_n - \omega)(x) - (\omega_n - \omega)(z)|^r K_r(x, z) dx dz \right. \\ &\quad \left. + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |(\omega_n - \omega)(x) - (\omega_n - \omega)(z)|^q K_q(x, z) dx dz \right] \\ &\leq \lim_{n \rightarrow \infty} \frac{c_{\mathcal{H}_m}}{\alpha_{0,m}} \|\omega_n - \omega\|_{\mathbb{E}}^r \\ &= 0. \end{aligned} \quad (8)$$

In light of (7) and (8), it follows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\tilde{\Pi}_n^m(x, z) - \tilde{\Pi}^m(x, z)|^{m'} dx dz = 0. \quad (9)$$

By combining (6) and (9) with the Hölder inequality, we derive

$$\|\mathcal{E}'_{\eta}(\omega_n) - \mathcal{E}'_{\eta}(\omega)\|_{\mathbb{E}^*} = \sup_{\varphi \in \mathbb{E}, \|\varphi\|_{\mathbb{E}}=1} \left| \langle \mathcal{E}'_{\eta}(\omega_n) - \mathcal{E}'_{\eta}(\omega), \varphi \rangle \right| \longrightarrow 0$$

as $n \rightarrow \infty$. Therefore, we state that $\mathcal{E}_{\eta} \in C^1(\mathbb{E}, \mathbb{R})$. $\square$

On account of the estimate of (2) in Lemma 5, we have

$$\left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right)^{-1} \leq S_* := \inf_{\omega \in \mathbb{E} \setminus \{0\}} \frac{[\omega]_{s,r,K_r}^r + [\omega]_{s,q,K_q}^q}{\|\omega\|_{\mathcal{H}_m}^m}. \quad (10)$$

Now, the following consequence is the variant of the Lions-type concentration–compactness principle [37] for fractional, critical and Hardy terms; see [30,31] for more information. This consequence plays a decisive role in guaranteeing the semicontinuity property of functional $\mathcal{E}_{\eta}$.

Lemma 7. Let $\mathcal{M}(\mathbb{R}^N)$ be the space of all signed finite Radon measures on $\mathbb{R}^N$ with the total variation norm. Let $\{\omega_n\}$ be a bounded sequence in $\mathbb{E}$ such that $\omega_n \rightharpoonup \omega$ in $\mathbb{E}$. Let us assume that

$$\begin{aligned} \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^r K_r(x, z) dz + \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^q K_q(x, z) dz &\xrightarrow{*} \lambda \text{ in } \mathcal{M}(\mathbb{R}^N), \\ |\omega_n(x)|^{r_s^*} &\xrightarrow{*} \kappa \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \frac{|\omega_n(x)|^m}{|x|^{sm}} &\xrightarrow{*} \nu \text{ in } \mathcal{M}(\mathbb{R}^N) \end{aligned}$$

for $m \in \{r, q\}$. Then, there exist at the most countable index set I of $\{x_i\}_{i \in I} \subset \mathbb{R}^N$; positive real numbers $\{\lambda_i\}_{i \in I}$, $\{\kappa_i\}_{i \in I}$, and $\{\nu_i\}_{i \in I}$; and three non-negative numbers λ_0 , κ_0 and ν_0 such that

$$\begin{aligned} \kappa &= |\omega(x)|^{r_s^*} + \kappa_0 \delta_0 + \sum_{i \in I} \kappa_i \delta_{x_i}, \\ \nu &= \frac{|\omega(x)|^m}{|x|^{sm}} + \nu_0 \delta_0, \\ \lambda &\geq \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dz + \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dz \\ &\quad + \lambda_0 \delta_0 + \sum_{i \in I} \lambda_i \delta_{x_i}, \\ \lambda_0 &\geq S_* \nu_0, \end{aligned}$$

where S_* is the optimal constant defined in (10), while δ_0 and δ_{x_i} denote the Dirac mass at 0 and x_i , respectively.

Lemma 8. Let $s \in (0, 1)$ and $qs < N$. Then, for $m \in \{r, q\}$, there exists a constant $\eta^* > 0$ such that the functional $\mathcal{E}_\eta = \mathcal{E} - \eta \mathcal{E}_{\mathcal{H}_m}$ is sequentially weakly lower semicontinuous on $\mathbb{E}$ for any $\eta \in (-\infty, \eta^*)$ —namely,

$$\mathcal{E}_\eta(\omega) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}_\eta(\omega_n) \text{ if } \omega_n \rightarrow \omega \text{ weakly in } \mathbb{E}.$$

Proof. Due to [Theorem 6] of [38], we know that $C_c^\infty(\Omega)$ is a dense subset of $\mathbb{E}$. Thus, by using density arguments to prove that $\mathcal{E}_\eta$ is sequentially weakly lower semicontinuous on $\mathbb{E}$, it is enough to show that the functional

$$\mathcal{E}_\eta \text{ is sequentially weakly lower semicontinuous on } C_c^\infty(\Omega) \quad (11)$$

holds for any $\eta \in (-\infty, \eta^*)$. So, let $\{\omega_n\}_{n \in \mathbb{N}}$ be a sequence in $C_c^\infty(\Omega)$ satisfying

$$\omega_n \rightarrow \omega \text{ weakly in } \mathbb{E}, \text{ as } n \rightarrow +\infty. \quad (12)$$

Thus, according to Lemma 7, there exist two bounded measures λ and ν , a set of indices I of distinct points $\{x_i\}_{i \in I} \subset \overline{\Omega}$ that is at most countable, positive real numbers $\{\lambda_i\}_{i \in I}$ and $\{\nu_i\}_{i \in I}$ and two non-negative numbers λ_0 and ν_0 such that the following convergence hold weakly in the sense of measures:

$$\begin{aligned} \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^r K_r(x, z) dz + \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^q K_q(x, z) dz \\ \rightharpoonup \lambda \geq \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dz + \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dz \\ + \lambda_0 \delta_0 + \sum_{i \in I} \lambda_i \delta_{x_i}, \end{aligned} \quad (13)$$

$$\frac{|\omega_n(x)|^m}{|x|^{sm}} \rightharpoonup \nu = \frac{|\omega(x)|^m}{|x|^{sm}} + \nu_0 \delta_0 \quad (14)$$

and finally,

$$S_* \nu_0 \leq \lambda_0. \quad (15)$$

Let

$$\eta^* = \frac{m(\alpha_{0,r}\alpha_{0,q})}{q(\alpha_{0,r}c_{\mathcal{H}_q} + \alpha_{0,q}c_{\mathcal{H}_r})},$$

where $\alpha_{0,m}$ and $c_{\mathcal{H}_m}$ are given in (K2) and Lemma 5, respectively, for $m \in \{r, q\}$.

Let us first consider $\eta \in (0, \eta^*)$. From (10), (13), (14) and (15), we obtain

$$\begin{aligned} & \liminf_{n \rightarrow +\infty} \mathcal{E}_\eta(\omega_n) \\ &= \liminf_{n \rightarrow +\infty} \left[\frac{1}{q} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^r K_r(x, z) dx dz \right. \right. \\ & \quad \left. \left. + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^q K_q(x, z) dx dz \right) \right. \\ & \quad \left. + \left(\frac{1}{r} - \frac{1}{q} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^r K_r(x, z) dx dz \right. \\ & \quad \left. - \frac{\eta}{m} \int_{\Omega} \frac{|\omega_n(x)|^m}{|x|^{sm}} dx \right] \\ &\geq \liminf_{n \rightarrow +\infty} \left[\frac{1}{q} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^r K_r(x, z) dx dz \right. \right. \\ & \quad \left. \left. + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_n(x) - \omega_n(z)|^q K_q(x, z) dx dz \right) - \frac{\eta}{m} \int_{\Omega} \frac{|\omega_n(x)|^m}{|x|^{sm}} dx \right] \\ & \quad + \left(\frac{1}{r} - \frac{1}{q} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dx dz \\ &\geq \frac{1}{q} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dx dz \right. \\ & \quad \left. + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dx dz + \lambda_0 + \sum_{i=1}^k \lambda_i \right) \\ & \quad - \frac{\eta}{m} \left(\int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} dx + \nu_0 \right) \\ & \quad + \left(\frac{1}{r} - \frac{1}{q} \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dx dz \\ &\geq \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dx dz \\ & \quad + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dx dz \\ & \quad - \frac{\eta}{m} \int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} dx + \frac{\lambda_0}{q} - \frac{\eta \nu_0}{m} \\ &\geq \mathcal{E}_\eta(\omega) + \frac{\lambda_0}{q} - \frac{\eta \lambda_0}{m} \left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right)^{-1} \\ &= \mathcal{E}_\eta(\omega) + \left(\frac{m(\alpha_{0,r}\alpha_{0,q}) - \eta q(\alpha_{0,r}c_{\mathcal{H}_q} + \alpha_{0,q}c_{\mathcal{H}_r})}{mq(\alpha_{0,r}\alpha_{0,q})} \right) \lambda_0 \\ &\geq \mathcal{E}_\eta(\omega) \end{aligned} \quad (16)$$

for any $\eta \in (0, \eta^*)$.

On the other hand, let $\eta \in (-\infty, 0]$. Then, it follows from the same argument as in (16) that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \mathcal{E}_\eta(\omega_n) &\geq \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dx dz \\ &\quad + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dx dz \\ &\quad - \frac{\eta}{m} \int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} dx + \frac{\lambda_0}{q} - \frac{\eta \nu_0}{m} \\ &\geq \mathcal{E}_\eta(\omega) + \frac{\lambda_0}{q} - \frac{\eta \nu_0}{m} \\ &\geq \mathcal{E}_\eta(\omega) \end{aligned} \quad (17)$$

From Relations (16) and (17), we deduce the statement stated in (11).

Now, let $\{\omega_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{E}$ fulfilling the same condition of (12). Then, taking account of density arguments, we obtain for any $n \in \mathbb{N}$ there is $\{\omega_n^\ell\}_{\ell \in \mathbb{N}}$ in $C_c^\infty(\Omega)$ such that

$$\omega_n^\ell \rightarrow \omega_n \text{ strongly in } \mathbb{E}, \text{ as } \ell \rightarrow +\infty. \quad (18)$$

From (12) and (18), we show that

$$\langle \omega_n^\ell - \omega, \varphi \rangle = \langle \omega_n^\ell - \omega_n, \varphi \rangle + \langle \omega_n - \omega, \varphi \rangle \rightarrow 0, \text{ as } n, \ell \rightarrow +\infty$$

for any $\varphi \in \mathbb{E}$. Then, we obtain

$$\omega_n^\ell \rightarrow \omega \text{ weakly in } \mathbb{E}, \text{ as } n, \ell \rightarrow +\infty.$$

Since $\{\omega_n^\ell\}_{\ell \in \mathbb{N}}$ is contained in $C_c^\infty(\Omega)$ and the statement (11) holds, we infer that

$$\liminf_{n, \ell \rightarrow +\infty} \mathcal{E}_\eta(\omega_n^\ell) \geq \mathcal{E}_\eta(\omega). \quad (19)$$

Moreover, according to (18), it is obvious that for any $n \in \mathbb{N}$, we have

$$\liminf_{\ell \rightarrow +\infty} \mathcal{E}_\eta(\omega_n^\ell) = \mathcal{E}_\eta(\omega_n),$$

so by passing the limit inferior, we obtain

$$\liminf_{n, \ell \rightarrow +\infty} \mathcal{E}_\eta(\omega_n^\ell) = \liminf_{n \rightarrow +\infty} \left(\lim_{\ell \rightarrow +\infty} \mathcal{E}_\eta(\omega_n^\ell) \right) = \liminf_{n \rightarrow +\infty} \mathcal{E}_\eta(\omega_n). \quad (20)$$

Using (19) and (20), we show that

$$\liminf_{n \rightarrow +\infty} \mathcal{E}_\eta(\omega_n) \geq \mathcal{E}_\eta(\omega).$$

Consequently, $\mathcal{E}_\eta$ is sequentially weakly lower semicontinuous on $\mathbb{E}$. $\square$

4. Main Results

The purpose of this section is to obtain the existence of uniqueness results with nontrivial positive solutions to Problem (1), which are the major results of the present paper. To this end, suppose that the assumptions on h are fulfilled as follows:

- (F1) The function $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies a Carathéodory condition;
- (F2) $0 \leq h(\cdot, \xi) \in L^\infty(\Omega)$ for every $\xi \geq 0$, and there is a real number $\sigma_1 > 0$ such that

$$h(x, \xi) \leq \sigma_1 (1 + |\xi|^{r-1})$$

for any $\xi \geq 0$ and for almost everywhere $x \in \Omega$;

(F3) $\lim_{\xi \rightarrow 0^+} \frac{h(x, \xi)}{\xi^{r-1}} = +\infty$ and $\lim_{\xi \rightarrow +\infty} \frac{h(x, \xi)}{\xi^{r-1}} = 0$ uniformly in $x \in \Omega$;

(F4) For almost everywhere $x \in \Omega$, the function $\xi \mapsto \frac{h(x, \xi)}{\xi^{r-1}}$ is strictly decreasing in $(0, +\infty)$.

Under assumptions (F1) and (F2), let define the functional $\Phi : \mathbb{E} \rightarrow \mathbb{R}$ as

$$\Phi(\omega) := \int_{\Omega} H(x, \omega(x)) dx$$

for any $\omega \in \mathbb{E}$, where $H(x, \eta) = \int_0^\eta h(x, t) dt$. Then, it is immediate that $\Phi \in C^1(\mathbb{E}, \mathbb{R})$ and its Fréchet derivative is

$$\langle \Phi'(\omega), \psi \rangle = \lambda \int_{\Omega} h(x, \omega) \psi dx$$

for any $\omega, \psi \in \mathbb{E}$. Subsequently, the functional $\Gamma_{\eta, \lambda} : \mathbb{E} \rightarrow \mathbb{R}$ is defined as

$$\Gamma_{\eta, \lambda}(\omega) = \mathcal{E}_{\eta}(\omega) - \lambda \Phi(\omega).$$

Then, $\Gamma_{\eta, \lambda} \in C^1(\mathbb{E}, \mathbb{R})$ and its Fréchet derivative is

$$\langle \Gamma'_{\eta, \lambda}(\omega), \psi \rangle = \langle \mathcal{E}'_{\eta}(\omega), \psi \rangle - \lambda \langle \Phi'(\omega), \psi \rangle \quad \text{for any } \omega, \psi \in \mathbb{E}.$$

The following is the well-known discrete Picone inequality; see [Lemma 2.6] of [16] and [Proposition 4.2] of [8] for a proof.

Lemma 9. Let $r \in (1, +\infty)$ and let $\alpha, \beta, \chi, \xi \in [0, +\infty)$, with $\alpha, \beta > 0$. Then,

$$\psi_r(\alpha - \beta) \left[\frac{\chi^r}{\alpha^{r-1}} - \frac{\xi^r}{\beta^{r-1}} \right] \leq |\chi - \xi|^r, \quad (21)$$

where $\psi_r(\zeta) = |\zeta|^{r-2}\zeta$ for $\zeta \in \mathbb{R}$. Furthermore, if the equality in (21) holds, then

$$\frac{\alpha}{\beta} = \frac{\chi}{\xi}.$$

The upcoming result is a generalization of Lemma 9, which is given in [13].

Lemma 10. Let $1 < r \leq q < \infty$ and ρ, σ be two non-negative, non-constant Lebesgue measurable functions such that $\rho > 0$ in Ω . Then,

$$\psi_q(\rho(x) - \rho(z)) \left[\frac{\sigma(x)^r}{h(\rho(x))} - \frac{\sigma(z)^r}{h(\rho(z))} \right] \leq |\sigma(x) - \sigma(z)|^r |\rho(x) - \rho(z)|^{q-r},$$

where $h(\zeta) = c\zeta^{q-1} + d\zeta^{r-1}$, with $c \geq 0$ and $d \geq 1$.

For any $\omega_i \in \mathbb{E}$ and $\epsilon > 0$, we define the truncation as

$$\omega_{i, \epsilon} := \min\{\omega_i, \epsilon^{-1}\}. \quad (22)$$

Now, we introduce a technical lemma which will be very usable hereinafter. The proof of this assertion is essentially the same as that of Lemma 2.3 in [15]; see also [14] for more details. Hence, we leave out it.

Lemma 11. Let $\omega_1, \omega_2 \in \mathbb{E}$ with $\omega_1, \omega_2 \geq 0$ and set

$$v := \frac{\omega_{2,\epsilon}^r}{(\omega_1 + \epsilon)^{r-1}} - \omega_{1,\epsilon},$$

where $\omega_{1,\epsilon}$ and $\omega_{2,\epsilon}$ are as in (22). Then, we have $v \in \mathbb{E}$.

By virtue of Lemmas 8 and 11, we assert that Problem (1) admits at least one positive weak solution. Before inquiring into this result, let us contemplate the following eigenvalue problem:

$$\begin{cases} \mathcal{L}_r^s \omega + \mathcal{L}_q^s \omega = \lambda |\omega|^{r-2} \omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (23)$$

We define the C^1 -functional as

$$\begin{aligned} \Gamma_\lambda(\omega) := & \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dz \, dx \\ & + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx - \frac{\lambda}{r} \int_{\Omega} |\omega|^r \, dx \end{aligned}$$

for any $\omega \in \mathbb{E}$. By means of Theorem 4.1 in [39], we notice that the first eigenvalue $\lambda_{1,r}$ of the fractional r -Laplacian is characterized by

$$\lambda_{1,r} \int_{\Omega} |\omega(x)|^r \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dx \, dz := [\omega]_{s,r,K_r}^r.$$

The fundamental idea of the proof of the upcoming assertion is inspired by the recent study of [17,29].

Lemma 12. We define the quantity as

$$\lambda_{1,r,q} = \min_{\omega \in \mathbb{E} \setminus \{0\}} \frac{[\omega]_{s,r,K_r}^r + \frac{r}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dx \, dz}{\int_{\Omega} |\omega|^r \, dx}.$$

Then, we obtain $\lambda_{1,r,q} = \lambda_{1,r}$. Moreover, Problem (23) has no eigenvalue for any $\lambda \in (-\infty, \lambda_{1,r}]$.

Proof. Due to the definitions of $\lambda_{1,r,q}$ and $\lambda_{1,r}$, we infer that $\lambda_{1,r,q} \geq \lambda_{1,r}$. Let ω_0 be the positive eigenfunction related to $\lambda_{1,r}$ that satisfies $\|\omega_0\|_{L^r(\Omega)} = 1$ and

$$\lambda_{1,r} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^r K_r(x, z) \, dx \, dz.$$

By considering the homogeneity and taking $\tau \rightarrow 0^+$ in the following inequality

$$\begin{aligned} \lambda_{1,r,q} & \leq \frac{[\tau\omega_0]_{s,r,K_r}^r + \frac{r}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\tau\omega_0(x) - \tau\omega_0(z)|^q K_q(x, z) \, dx \, dz}{\int_{\Omega} |\tau\omega_0|^r \, dx} \\ & = \lambda_{1,r} + \tau^{q-r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^q K_q(x, z) \, dx \, dz, \end{aligned}$$

we assert that $\lambda_{1,r,q} \leq \lambda_{1,r}$ because $q > r$.

Now, let us show that Problem (23) has no eigenvalue of for any $\lambda \in (-\infty, \lambda_{1,r}]$. Assume to the contrary that an eigenpair $(\lambda, \omega_\lambda) \in (-\infty, \lambda_{1,r}) \times \mathbb{E} \setminus \{0\}$ exists. Then, owing to the definition of $\lambda_{1,r,q}$, we deduce that

$$\begin{aligned}\lambda_{1,r} > \lambda &= \frac{[\omega_\lambda]_{s,r,K_r}^r + \frac{r}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_\lambda(x) - \omega_\lambda(z)|^q K_q(x,z) dx dz}{\int_{\Omega} |\omega_\lambda|^r dx} \\ &\geq \lambda_{1,r,q} = \lambda_{1,r},\end{aligned}$$

which implies a contradiction.

Meanwhile, suppose that $\lambda_{1,r}$ is an eigenvalue of Problem (23). Then, there exists a function ω_1 in $\mathbb{E} \setminus \{0\}$ satisfying $\|\omega_1\|_{L^r(\Omega)} > 0$ and

$$\lambda_{1,r} = \frac{[\omega_1]_{s,r,K_r}^r + \frac{r}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_1(x) - \omega_1(z)|^q K_q(x,z) dx dz}{\int_{\Omega} |\omega_1|^r dx}.$$

Then, since $\omega_1 \neq 0$, according to the definition of $\lambda_{1,r}$, it follows that

$$\begin{aligned}\lambda_{1,r} &= \frac{[\omega_1]_{s,r,K_r}^r + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_1(x) - \omega_1(z)|^q K_q(x,z) dx dz}{\int_{\Omega} |\omega_1|^r dx} \\ &\geq \lambda_{1,r} + \frac{r \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_1(x) - \omega_1(z)|^q K_q(x,z) dx dz}{q \int_{\Omega} |\omega_1|^r dx} > \lambda_{1,r}.\end{aligned}$$

This is absurd. $\square$

Lemma 13. Every $\lambda > \lambda_{1,r}$ is an eigenvalue of Problem (23).

Proof. In accordance with Lemma 12, we know that $\lambda_{1,r,q} = \lambda_{1,r}$. Let any $\lambda > \lambda_{1,r}$ be fixed. Then, we can choose some $\omega_0 \in \mathbb{E} \setminus \{0\}$ fulfilling $\Gamma_\lambda(\omega_0) < 0$. Since the functional Γ_λ is weakly lower semicontinuous and coercive on $\mathbb{E}$, there exists a global minimizer ω_* in $\mathbb{E}$ for Γ_λ —namely, $\Gamma_\lambda(\omega_*) = \min_{\mathbb{E}} \Gamma_\lambda$. Since $\lambda_{1,r} = \lambda_{1,r,q}$ and $\lambda > \lambda_{1,r,q}$, we obtain $\Gamma_\lambda(\omega_*) \leq \Gamma_\lambda(\omega_0) < 0$. This implies that $\omega_* \neq 0$. Furthermore, $\Gamma'_\lambda(\omega_*) = 0$, so ω_* is an eigenfunction of Problem (23), which is related to the eigenvalue λ . $\square$

Now, we are ready to demonstrate the existence of at least one nontrivial positive weak solution for Problem (1).

Theorem 1. Assume that (F1)–(F3) are satisfied, then there exists a positive constant $\tilde{\eta}^*$ such that Problem (1) has a positive weak solution for any $\eta \in (-\infty, \tilde{\eta}^*)$ and for any $\lambda > 0$.

Proof. Firstly, for $m \in \{r, q\}$, let

$$\tilde{\eta}^* = \frac{m(\alpha_{0,r}\alpha_{0,q})}{2q(\alpha_{0,r}c_{\mathcal{H}_q} + \alpha_{0,q}c_{\mathcal{H}_r})} \quad (24)$$

where $\alpha_{0,m}$ and $c_{\mathcal{H}_m}$ are given in (K2) and Lemma 5, respectively.

For a given η^* in Lemma 8, since $\tilde{\eta}^* < \eta^*$, the subcritical growth of h and Lemma 8 shows that $\Gamma_{\eta,\lambda}$ is sequentially weakly lower semicontinuous on $\mathbb{E}$. Let us choose any positive constant ϵ satisfying either

$$\epsilon \in \left(0, \frac{m\alpha_{0,r}\alpha_{0,q}(\mathfrak{C}_0 + 1) - 2\eta(\alpha_{0,r}c_{\mathcal{H}_q} + \alpha_{0,q}c_{\mathcal{H}_r})mq\mathfrak{C}_0}{2mq\lambda\mathfrak{C}_0\alpha_{0,r}\alpha_{0,q}}\right) \quad \text{or} \quad \epsilon \in \left(0, \frac{\mathfrak{C}_0 + 1}{2q\lambda\mathfrak{C}_0}\right),$$

where $\mathfrak{C}_0$ comes from Lemma 4. Because $\lim_{\xi \rightarrow +\infty} \frac{H(x, \xi)}{\xi^r} = 0$, there is a positive real number $D(\epsilon)$ such that

$$H(x, \xi) \leq \epsilon |\xi|^r + D(\epsilon)$$

for almost all $x \in \Omega$ and for any $\xi \in \mathbb{R}$. Hence, one has

$$\Phi(\omega) \leq \epsilon \int_{\Omega} |\omega(x)|^r dx + D(\epsilon) |\Omega| \quad (25)$$

for any $\omega \in \mathbb{E}$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^N$. From this, we can estimate (25) and Lemma 4, and we show that for any $\omega \in \mathbb{E}$ with $\|\omega\|_{\mathbb{E}} \geq 1$ and for every $\eta \in (0, \tilde{\eta}^*)$,

$$\begin{aligned} \mathcal{E}_{\eta}(\omega) - \lambda \Phi(\omega) &\geq \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dz dx \\ &\quad + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dz dx \\ &\quad - \frac{\eta}{m} \int_{\Omega} \frac{|\omega_n(x)|^m}{|x|^{sm}} dx - \lambda \epsilon \int_{\Omega} |\omega(x)|^r dx - \lambda D(\epsilon) |\Omega| \\ &\geq \frac{1}{q} \left([\omega]_{s,r,K_r}^r + [\omega]_{s,q,K_q}^q \right) - \frac{\eta}{m} \left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right) \left([\omega]_{s,r,K_r}^r + [\omega]_{s,q,K_q}^q \right) \\ &\quad - \lambda \epsilon \int_{\Omega} |\omega(x)|^r dx - \lambda D(\epsilon) |\Omega| \\ &\geq \frac{(\mathfrak{C}_0 + 1)}{2q\mathfrak{C}_0} \left([\omega]_{s,r,K_r}^r + [\omega]_{s,q,K_q}^q + \|\omega\|_{L^r(\Omega)}^r + \|\omega\|_{L^q(\Omega)}^q \right) \\ &\quad - \frac{\eta}{m} \left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right) \left([\omega]_{s,r,K_r}^r + [\omega]_{s,q,K_q}^q \right) \\ &\quad - \lambda \epsilon \int_{\Omega} |\omega(x)|^r dx - \lambda D(\epsilon) |\Omega| \\ &\geq \left(\frac{1}{2q} - \frac{\eta}{m} \left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right) \right) \|\omega\|_{\mathbb{E}}^r - \lambda \epsilon \int_{\Omega} |\omega(x)|^r dx - \lambda D(\epsilon) |\Omega| \\ &\geq \left(\frac{m\alpha_{0,r}\alpha_{0,q} - 2\eta(\alpha_{0,r}c_{\mathcal{H}_q} + \alpha_{0,q}c_{\mathcal{H}_r})q}{2mq\alpha_{0,r}\alpha_{0,q}} - \lambda \epsilon \right) \|\omega\|_{\mathbb{E}}^r - \lambda D(\epsilon) |\Omega|. \end{aligned} \quad (26)$$

For any $\eta \in (-\infty, 0]$, it follows from the similar argument as in (26) that

$$\begin{aligned} \mathcal{E}_{\eta}(\omega) - \lambda \Phi(\omega) &\geq \frac{(\mathfrak{C}_0 + 1)}{2q\mathfrak{C}_0} \left([\omega]_{s,r,K_r}^r + [\omega]_{s,q,K_q}^q + \|\omega\|_{L^r(\Omega)}^r + \|\omega\|_{L^q(\Omega)}^q \right) \\ &\quad - \lambda \epsilon \int_{\Omega} |\omega(x)|^r dx - \lambda D(\epsilon) |\Omega| \\ &\geq \frac{(\mathfrak{C}_0 + 1)}{2q\mathfrak{C}_0} \|\omega\|_{\mathbb{E}}^r - \lambda \epsilon \int_{\Omega} |\omega(x)|^r dx - \lambda D(\epsilon) |\Omega| \\ &\geq \left(\frac{(\mathfrak{C}_0 + 1)}{2q\mathfrak{C}_0} - \lambda \epsilon \right) \|\omega\|_{\mathbb{E}}^r - \lambda D(\epsilon) |\Omega|. \end{aligned}$$

Thus, by virtue of the choice of ϵ , one has

$$\lim_{\|\omega\|_{\mathbb{E}} \rightarrow +\infty} (\mathcal{E}_{\eta}(\omega) - \lambda \Phi(\omega)) = +\infty.$$

Let us define the modified energy functional $\tilde{\Gamma} : \mathbb{E} \rightarrow \mathbb{R}$ as

$$\begin{aligned}\tilde{\Gamma}_{\eta,\lambda}(\omega) &:= \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dz \, dx \\ &\quad + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx \\ &\quad - \frac{\eta}{m} \int_{\Omega} \frac{|\omega^+(x)|^m}{|x|^{sm}} \, dx - \lambda \int_{\mathbb{R}^N} H^+(x, \omega) \, dx, \quad \omega \in \mathbb{E},\end{aligned}$$

where

$$H^+(x, \tau) := \int_0^\tau h^+(x, \zeta) \, d\zeta \quad \text{and} \quad h^+(x, \tau) := \begin{cases} h(x, \tau), & \tau \geq 0, \\ 0, & \tau < 0 \end{cases}$$

for all $\zeta \in \mathbb{R}$ and for almost all $x \in \mathbb{R}^N$. Due to Lemma 8 and the arguments in (25) and (26), we observe that $\tilde{\Gamma}_{\eta,\lambda}$ is also coercive and sequentially weakly lower semicontinuous on $\mathbb{E}$. So, we can choose an element $\omega_0 \in \mathbb{E}$ satisfying

$$\tilde{\Gamma}_{\eta,\lambda}(\omega_0) = \inf\{\tilde{\Gamma}_{\eta,\lambda}(\omega) : \omega \in \mathbb{E}\}.$$

Now, we prove that $\omega_0 \geq 0$ is reasonable. To do this, we suppose that ω_0 is sign-changing. On account of Lemma 11, we infer that $\omega_0^+ \in \mathbb{E}$, so $\tilde{\Gamma}_{\eta,\lambda}(\omega_0) \leq \tilde{\Gamma}_{\eta,\lambda}(\omega_0^+)$. Since $\tilde{\Gamma}_{\eta,\lambda}(\omega) = \Gamma(\omega)$ when $\omega(x) \geq 0$ for almost all $x \in \Omega$, we derive the following:

$$\begin{aligned}\tilde{\Gamma}_{\eta,\lambda}(\omega_0^+) &= \Gamma_{\eta,\lambda}(\omega_0^+) \\ &= \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0^+(x) - \omega_0^+(z)|^r K_r(x, z) \, dx \, dz \\ &\quad + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0^+(x) - \omega_0^+(z)|^q K_q(x, z) \, dx \, dz \\ &\quad - \frac{\eta}{m} \int_{\Omega} \frac{|\omega_0^+(x)|^m}{|x|^{sm}} \, dx - \lambda \int_{\Omega} H(x, \omega_0^+) \, dx \\ &\leq \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^r K_r(x, z) \, dx \, dz \\ &\quad + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^q K_q(x, z) \, dx \, dz \\ &\quad - \frac{\eta}{m} \int_{\Omega} \frac{|\omega_0^+(x)|^m}{|x|^{sm}} \, dx - \lambda \int_{\Omega} H(x, \omega_0^+) \, dx \\ &= \tilde{\Gamma}_{\eta,\lambda}(\omega_0).\end{aligned}$$

As a consequence, ω_0^+ is a non-negative solution of (1). To simplify, let us directly write ω_0 instead of ω_0^+ . Let us claim that $\omega_0 > 0$. Because $\omega_0(x) \geq 0$ for almost all $x \in \mathbb{R}^N$, we find that either $\omega_0(x) = 0$ or $\omega_0(x) > 0$ for almost all $x \in \mathbb{R}^N$. Indeed, let us assume that $\omega_0 \not\equiv 0$ in Ω . Then, it is enough to show that $\omega_0 \not\equiv 0$ in all connected components of Ω . Suppose on the contrary that there exists a connected component Ω_0 of Ω satisfying $\omega_0(x) = 0$ for almost everywhere $x \in \Omega_0$. Let us consider any non-negative function $v \in C_0^\infty(\Omega_0)$ as a test function in (3). Then, we know that $\int_{\Omega} h(x, \omega_0)v(x) \, dx = 0$ since $\omega(x) = 0$ for almost everywhere $x \in \Omega_0^c$ and $h(x, \omega_0(x)) = h(x, 0) = 0$ for almost everywhere $x \in \Omega_0$. So, one has

$$\begin{aligned}0 &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^{r-2} (\omega_0(x) - \omega_0(z)) (v(x) - v(z)) K_r(x, z) \, dx \, dz \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^{q-2} (\omega_0(x) - \omega_0(z)) (v(x) - v(z)) K_q(x, z) \, dx \, dz\end{aligned}$$

$$\begin{aligned}
& -\eta \int_{\Omega} \frac{|\omega_0(x)|^{m-2} \omega_0(x) v(x)}{|x|^{sm}} dx - \lambda \int_{\Omega} h(x, \omega_0) v(x) dx \\
& = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^{r-2} (\omega_0(x) - \omega_0(z)) (v(x) - v(z)) K_r(x, z) dx dz \\
& \quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_0(x) - \omega_0(z)|^{q-2} (\omega_0(x) - \omega_0(z)) (v(x) - v(z)) K_q(x, z) dx dz \\
& = 2 \int_{\Omega_0} \int_{\Omega_0^c} |\omega_0(x) - \omega_0(z)|^{r-2} (\omega_0(x) - \omega_0(z)) (v(x) - v(z)) K_r(x, z) dx dz \\
& \quad + 2 \int_{\Omega_0} \int_{\Omega_0^c} |\omega_0(x) - \omega_0(z)|^{q-2} (\omega_0(x) - \omega_0(z)) (v(x) - v(z)) K_q(x, z) dx dz \\
& = -2 \left[\int_{\Omega_0} \int_{\Omega_0^c} (\omega_0(x))^{r-1} v(z) K_r(x, z) dx dz \right. \\
& \quad \left. + \int_{\Omega_0} \int_{\Omega_0^c} (\omega_0(x))^{q-1} v(z) K_q(x, z) dx dz \right].
\end{aligned}$$

Using this, we derive that $\omega_0(x) = 0$ for almost everywhere $x \in \Omega_0^c$, that is, $\omega_0(x) = 0$ for almost everywhere $x \in \mathbb{R}^N$. This implies a contradiction to the fact that $\omega_0(x) \neq 0$ for almost everywhere $x \in \Omega$.

Therefore, to show that $\omega_0 > 0$, it is sufficient to verify that $\tilde{\Gamma}_{\eta, \lambda}(\omega_0) < 0$ for any $\eta \in (-\infty, \tilde{\eta}^*)$. Now, in consideration of Lemma 13, we fix any eigenpair $(\lambda_*, \omega) \in (\lambda_{1,r}, \infty) \times \mathbb{E} \setminus \{0\}$ such that

$$\begin{aligned}
\lambda_* \int_{\Omega} |\omega(x)|^r dx & = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dz dx \\
& \quad + \frac{r}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) dz dx.
\end{aligned}$$

By means of modifications of the proof of Theorem 4.1 in [39], we deduce that $\omega \in L^\infty(\mathbb{R}^N)$. Let $\rho_0 \in L^\infty(\Omega)$, with $\rho_0 > 0$, and let us fix $\zeta_0 \in (0, \|\rho_0\|_{L^\infty(\Omega)})$. Then, the set

$$\Omega_{\zeta_0} := \{z \in \Omega : \rho_0(x) \geq \zeta_0\}$$

has a positive measure. Additionally, we fix $\mathfrak{D}_1 > 0$ so that

$$\mathfrak{D}_1 > \frac{\lambda_* \int_{\Omega} |\omega(x)|^r dx}{r \lambda \zeta_0 \int_{\Omega_{\zeta_0}} |\omega(x)|^r dx}.$$

Because of the first condition in (F3), there exists $\tau_0 > 0$ satisfying

$$\frac{H(x, \tau)}{\tau^r} \geq \rho_0(x) \mathfrak{D}_1 \quad (27)$$

for all $\tau \in (0, \tau_0]$ and for almost all $x \in \Omega$. So, for any $\eta \in (0, \tilde{\eta}^*)$ and for sufficiently small values such that $0 < \epsilon < 1$, one has

$$\begin{aligned}
& \lambda \int_{\Omega} \frac{H(x, \epsilon \omega)}{\epsilon^r} dx \\
& \geq \lambda \mathfrak{D}_1 \int_{\Omega} \rho_0(x) |\omega(x)|^r dx \\
& \geq \lambda \mathfrak{D}_1 \zeta_0 \int_{\Omega_{\zeta_0}} |\omega(x)|^r dx \\
& > \frac{\lambda_*}{r} \int_{\Omega} |\omega(x)|^r dx \\
& = \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) dz dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx \\
& > \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dz \, dx \\
& + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx - \frac{\eta \epsilon^{m-r}}{m} \int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} \, dx. \quad (28)
\end{aligned}$$

Meanwhile, let us fix $\eta \in (-\infty, 0]$ and $\mathfrak{D}_2 > 0$ such that they satisfy

$$\mathfrak{D}_2 > \frac{\lambda_* \left(1 - \eta m^{-1} r \left(\alpha_{0,r} c_{\mathcal{H}_q} + \alpha_{0,q} c_{\mathcal{H}_r}\right) (\alpha_{0,r} \alpha_{0,q})^{-1}\right) \int_{\Omega} |\omega(x)|^r \, dx}{r \lambda \zeta_0 \int_{\Omega_{\zeta_0}} |\omega(x)|^r \, dx}.$$

Then, by changing $\mathfrak{D}_1$ into (27) with $\mathfrak{D}_2$, Lemma 5 and the similar way as in (28) yield that

$$\begin{aligned}
& \lambda \int_{\Omega} \frac{H(x, \epsilon \omega)}{\epsilon^r} \, dx \\
& \geq \lambda \mathfrak{D}_2 \zeta_0 \int_{\Omega_{\zeta_0}} |\omega(x)|^r \, dx \\
& > \frac{\lambda_*}{r} \int_{\Omega} |\omega(x)|^r \, dx - \frac{\lambda_* \eta}{m} \left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right) \int_{\Omega} |\omega(x)|^r \, dx \\
& > \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dz \, dx \\
& + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx \\
& - \frac{\eta}{m} \left(\frac{c_{\mathcal{H}_r}}{\alpha_{0,r}} + \frac{c_{\mathcal{H}_q}}{\alpha_{0,q}} \right) \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dz \, dx \right. \\
& \quad \left. + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx \right) \\
& > \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dz \, dx \\
& + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx - \frac{\eta \epsilon^{m-r}}{m} \int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} \, dx \quad (29)
\end{aligned}$$

for any $\eta \in (-\infty, 0]$.

Thus, as a result of (28) and (29), we show that

$$\begin{aligned}
& \lambda \int_{\mathbb{R}^N} H(x, \epsilon \omega) \, dx \\
& > \frac{\epsilon^r}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^r K_r(x, z) \, dz \, dx \\
& + \frac{\epsilon^r}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega(x) - \omega(z)|^q K_q(x, z) \, dz \, dx - \frac{\eta \epsilon^m}{m} \int_{\Omega} \frac{|\omega(x)|^m}{|x|^{sm}} \, dx \\
& > \frac{1}{r} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\epsilon \omega(x) - \epsilon \omega(z)|^r K_r(x, z) \, dz \, dx \\
& + \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\epsilon \omega(x) - \epsilon \omega(z)|^q K_q(x, z) \, dz \, dx - \frac{\eta}{m} \int_{\Omega} \frac{|\epsilon \omega(x)|^m}{|x|^{sm}} \, dx
\end{aligned}$$

for any sufficiently small $\epsilon \in (0, 1)$, which is $\Gamma(\epsilon \omega) < 0$, and the conclusion holds. Consequently, Problem (1) possesses at least one positive weak solution for any $\eta \in (-\infty, \tilde{\eta}^*)$ and for any $\lambda > 0$. $\square$

With the aid of Lemmas 9–11, 13 and Theorem 1, we are ready to derive our primary result. The fundamental idea of the upcoming assertion comes from [9,15].

Theorem 2. If (F1)–(F4) are satisfied, then for any $\lambda > 0$, the following are true:

- (i) Problem (1) with $m = r$ has a unique positive solution for every $\eta \in (-\infty, \tilde{\eta}^*)$, where $\tilde{\eta}^*$ comes from Theorem 1;
- (ii) Problem (1) with $m = q$ has a unique positive solution for every $\eta \in (-\infty, 0]$.

Proof. In accordance with Theorem 1, suppose that ω_1 and ω_2 are two positive weak solutions of (1). For any $\epsilon > 0$, let us define the truncations $\omega_{i,\epsilon}$ as in (22) for $i = 1, 2$. Let $\varrho_{1,\epsilon}$ and $\varrho_{2,\epsilon}$ be defined by

$$\varrho_{1,\epsilon} := \frac{\omega_{2,\epsilon}^r}{(\omega_1 + \epsilon)^{r-1}} - \omega_{1,\epsilon}$$

and

$$\varrho_{2,\epsilon} := \frac{\omega_{1,\epsilon}^r}{(\omega_2 + \epsilon)^{r-1}} - \omega_{2,\epsilon}.$$

On account of Lemma 11, we infer that $\varrho_{i,\epsilon} \in \mathbb{E}$ for $i = 1, 2$. Now, let

$$\psi_m(\xi) := |\xi|^{m-2}\xi,$$

where $m \in \{q, r\}$. Let us test (3) with $\varrho_{1,\epsilon}$ and $\varrho_{2,\epsilon}$. We deduce that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(\omega_1(x) - \omega_1(z))(\varrho_{1,\epsilon}(x) - \varrho_{1,\epsilon}(z))K_r(x, z) dx dz \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(\omega_1(x) - \omega_1(z))(\varrho_{1,\epsilon}(x) - \varrho_{1,\epsilon}(z))K_q(x, z) dx dz \\ & = \eta \int_{\Omega} \frac{\omega_1^{m-1}(x)}{|x|^{sm}} \varrho_{1,\epsilon}(x) dx + \lambda \int_{\Omega} h(x, \omega_1) \varrho_{1,\epsilon}(x) dx \end{aligned} \quad (30)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(\omega_2(x) - \omega_2(z))(\varrho_{2,\epsilon}(x) - \varrho_{2,\epsilon}(z))K_r(x, z) dx dz \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(\omega_2(x) - \omega_2(z))(\varrho_{2,\epsilon}(x) - \varrho_{2,\epsilon}(z))K_q(x, z) dx dz \\ & = \eta \int_{\Omega} \frac{\omega_2^{m-1}(x)}{|x|^{sm}} \varrho_{2,\epsilon}(x) dx + \lambda \int_{\Omega} h(x, \omega_2) \varrho_{2,\epsilon}(x) dx. \end{aligned} \quad (31)$$

By adding the resulting identities of (30) and (31), utilizing the fact that

$$\psi_m(\omega_i(x) - \omega_i(z)) = \psi_m((\omega_i + \epsilon)(x) - (\omega_i + \epsilon)(z)) \quad \text{for } m \in \{q, r\} \quad \text{and } i = 1, 2,$$

and denoting $v_{1,\epsilon} := \omega_1 + \epsilon$ and $v_{2,\epsilon} := \omega_2 + \epsilon$, one has

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(v_{1,\epsilon}(x) - v_{1,\epsilon}(z)) \left(\frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(x) - \frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(z) \right) K_r(x, z) dx dz \\ & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(\omega_1(x) - \omega_1(z))(\omega_{1,\epsilon}(x) - \omega_{1,\epsilon}(z))K_r(x, z) dx dz \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(v_{1,\epsilon}(x) - v_{1,\epsilon}(z)) \left(\frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(x) - \frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(z) \right) K_q(x, z) dx dz \\ & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(\omega_1(x) - \omega_1(z))(\omega_{1,\epsilon}(x) - \omega_{1,\epsilon}(z))K_q(x, z) dx dz \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(v_{2,\epsilon}(x) - v_{2,\epsilon}(z)) \left(\frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(x) - \frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(z) \right) K_r(x, z) dx dz \\ & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(\omega_2(x) - \omega_2(z))(\omega_{2,\epsilon}(x) - \omega_{2,\epsilon}(z))K_r(x, z) dx dz \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(v_{2,\epsilon}(x) - v_{2,\epsilon}(z)) \left(\frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(x) - \frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(z) \right) K_q(x, z) dx dz \\
& - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(\omega_2(x) - \omega_2(z)) (\omega_{2,\epsilon}(x) - \omega_{2,\epsilon}(z)) K_q(x, z) dx dz \\
& = \eta \left(\int_{\Omega} \left[\frac{\omega_1^{m-1}}{|x|^{sm}} \left(\frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}} - \omega_{1,\epsilon} \right) + \frac{\omega_2^{m-1}}{|x|^{sm}} \left(\frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}} - \omega_{2,\epsilon} \right) \right] dx \right) \\
& + \lambda \left(\int_{\Omega} \left[h(x, \omega_1) \left(\frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}} - \omega_{1,\epsilon} \right) + h(x, \omega_2) \left(\frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}} - \omega_{2,\epsilon} \right) \right] dx \right). \quad (32)
\end{aligned}$$

Now, in virtue of Lemma 9 and the fact that $\xi \rightarrow \min\{|\xi|, \epsilon^{-1}\}$ is 1-Lipschitz, we show that

$$\psi_r(v_{1,\epsilon}(x) - v_{1,\epsilon}(z)) \left(\frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(x) - \frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(z) \right) \leq |\omega_2(x) - \omega_2(z)|^r \quad (33)$$

and

$$\psi_r(v_{2,\epsilon}(x) - v_{2,\epsilon}(z)) \left(\frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(x) - \frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(z) \right) \leq |\omega_1(x) - \omega_1(z)|^r. \quad (34)$$

In light of Lemma 10 with $f(\xi) = \xi^{r-1}$, we know that

$$\begin{aligned}
& \psi_q(v_{1,\epsilon}(x) - v_{1,\epsilon}(z)) \left(\frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(x) - \frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(z) \right) \\
& \leq |\omega_{2,\epsilon}(x) - \omega_{2,\epsilon}(z)|^r |\omega_1(x) - \omega_1(z)|^{q-r} \quad (35)
\end{aligned}$$

and

$$\begin{aligned}
& \psi_q(v_{2,\epsilon}(x) - v_{2,\epsilon}(z)) \left(\frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(x) - \frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(z) \right) \\
& \leq |\omega_{1,\epsilon}(x) - \omega_{1,\epsilon}(z)|^r |\omega_2(x) - \omega_2(z)|^{q-r}. \quad (36)
\end{aligned}$$

From Young's inequality, inequalities (35) and (36) imply

$$\begin{aligned}
& \psi_q(v_{1,\epsilon}(x) - v_{1,\epsilon}(z)) \left(\frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(x) - \frac{\omega_{2,\epsilon}^r}{v_{1,\epsilon}^{r-1}}(z) \right) \\
& \leq \frac{r}{q} |\omega_{2,\epsilon}(x) - \omega_{2,\epsilon}(z)|^q + \frac{q-r}{q} |\omega_1(x) - \omega_1(z)|^q \quad (37)
\end{aligned}$$

and

$$\begin{aligned}
& \psi_q(v_{2,\epsilon}(x) - v_{2,\epsilon}(z)) \left(\frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(x) - \frac{\omega_{1,\epsilon}^r}{v_{2,\epsilon}^{r-1}}(z) \right) \\
& \leq \frac{r}{q} |\omega_{1,\epsilon}(x) - \omega_{1,\epsilon}(z)|^q + \frac{q-r}{q} |\omega_2(x) - \omega_2(z)|^q. \quad (38)
\end{aligned}$$

As $\omega_{i,\epsilon} \rightarrow \omega_i$ and $v_{i,\epsilon} \rightarrow \omega_i$ as $\epsilon \rightarrow 0$ for $i = 1, 2$, by taking the limit $\epsilon \rightarrow 0$ in (33)–(34) and (37)–(38), we infer that

$$\begin{aligned}
& \psi_r(\omega_1(x) - \omega_1(z)) \left(\frac{\omega_2^r}{\omega_1^{r-1}}(x) - \frac{\omega_2^r}{\omega_1^{r-1}}(z) \right) \leq |\omega_2(x) - \omega_2(z)|^r, \\
& \psi_r(\omega_2(x) - \omega_2(z)) \left(\frac{\omega_1^r}{\omega_2^{r-1}}(x) - \frac{\omega_1^r}{\omega_2^{r-1}}(z) \right) \leq |\omega_1(x) - \omega_1(z)|^r \quad (39)
\end{aligned}$$

and

$$\begin{aligned}
 & \psi_q(\omega_1(x) - \omega_1(z)) \left(\frac{\omega_2^r}{\omega_1^{r-1}}(x) - \frac{\omega_2^r}{\omega_1^{r-1}}(z) \right) \\
 & \leq \frac{r}{q} |\omega_2(x) - \omega_2(z)|^q + \frac{q-r}{q} |\omega_1(x) - \omega_1(z)|^q, \\
 & \psi_q(\omega_2(x) - \omega_2(z)) \left(\frac{\omega_1^r}{\omega_2^{r-1}}(x) - \frac{\omega_1^r}{\omega_2^{r-1}}(z) \right) \\
 & \leq \frac{r}{q} |\omega_1(x) - \omega_1(z)|^q + \frac{q-r}{q} |\omega_2(x) - \omega_2(z)|^q. \tag{40}
 \end{aligned}$$

By taking the limit $\epsilon \rightarrow 0$ in (32) and utilizing Fatou's Lemma in the first, third, fifth and seventh terms, as well as applying the Lebesgue dominated convergence theorem for all the other terms, (39) and (40) yield that

$$\begin{aligned}
 0 & \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(\omega_1(x) - \omega_1(z)) \left(\frac{\omega_2^r}{\omega_1^{r-1}}(x) - \frac{\omega_2^r}{\omega_1^{r-1}}(z) \right) K_r(x, z) \, dz \, dx \\
 & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_1(x) - \omega_1(z)|^r K_r(x, z) \, dz \, dx \\
 & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(\omega_1(x) - \omega_1(z)) \left(\frac{\omega_2^r}{\omega_1^{r-1}}(x) - \frac{\omega_2^r}{\omega_1^{r-1}}(z) \right) K_q(x, z) \, dz \, dx \\
 & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_1(x) - \omega_1(z)|^q K_q(x, z) \, dz \, dx \\
 & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_r(\omega_2(x) - \omega_2(z)) \left(\frac{\omega_1^r}{\omega_2^{r-1}}(x) - \frac{\omega_1^r}{\omega_2^{r-1}}(z) \right) K_r(x, z) \, dz \, dx \\
 & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_2(x) - \omega_2(z)|^r K_r(x, z) \, dz \, dx \\
 & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_q(\omega_2(x) - \omega_2(z)) \left(\frac{\omega_1^r}{\omega_2^{r-1}}(x) - \frac{\omega_1^r}{\omega_2^{r-1}}(z) \right) K_q(x, z) \, dz \, dx \\
 & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\omega_2(x) - \omega_2(z)|^q K_q(x, z) \, dz \, dx \\
 & = \eta \left(\int_{\Omega} \left[\frac{\omega_1^{m-1}}{|x|^{sm}} \left(\frac{\omega_2^r}{\omega_1^{r-1}} - \omega_1 \right) + \frac{\omega_2^{m-2}}{|x|^{sm}} \left(\frac{\omega_1^r}{\omega_2^{r-1}} - \omega_2 \right) \right] dx \right) \\
 & + \lambda \left(\int_{\Omega} h(x, \omega_1) \left(\frac{\omega_2^r}{\omega_1^{r-1}} - \omega_1 \right) + h(x, \omega_2) \left(\frac{\omega_1^r}{\omega_2^{r-1}} - \omega_2 \right) dx \right) \\
 & = \eta \left(\int_{\Omega} \left[\frac{\omega_1^{m-1}}{|x|^{sm}} \left(\frac{\omega_2^r}{\omega_1^{r-1}} - \omega_1 \right) + \frac{\omega_2^{m-2}}{|x|^{sm}} \left(\frac{\omega_1^r}{\omega_2^{r-1}} - \omega_2 \right) \right] dx \right) \\
 & - \lambda \int_{\Omega} \left(\frac{h(x, \omega_1)}{\omega_1^{r-1}} - \frac{h(x, \omega_2)}{\omega_2^{r-1}} \right) (\omega_1^r - \omega_2^r) \, dx.
 \end{aligned}$$

Using this, one has

$$\begin{aligned}
 & \eta \left(\int_{\Omega} \left[\frac{\omega_1^{m-1}}{|x|^{sm}} \left(\frac{\omega_2^r}{\omega_1^{r-1}} - \omega_1 \right) + \frac{\omega_2^{m-2}}{|x|^{sm}} \left(\frac{\omega_1^r}{\omega_2^{r-1}} - \omega_2 \right) \right] dx \right) \\
 & \leq \lambda \int_{\Omega} \left(\frac{h(x, \omega_1)}{\omega_1^{r-1}} - \frac{h(x, \omega_2)}{\omega_2^{r-1}} \right) (\omega_1^r - \omega_2^r) \, dx. \tag{41}
 \end{aligned}$$

When $m = r$, one has

$$\frac{\omega_1^{r-1}}{|x|^{sr}} \left(\frac{\omega_2^r}{\omega_1^{r-1}} - \omega_1 \right) + \frac{\omega_2^{r-1}}{|x|^{sr}} \left(\frac{\omega_1^r}{\omega_2^{r-1}} - \omega_2 \right) = 0.$$

Owing to (41), we derive that for every $\eta \in (0, \tilde{\eta}^*)$ and for any $\lambda > 0$,

$$\int_{\Omega} \left(\frac{h(x, \omega_1)}{\omega_1^{r-1}} - \frac{h(x, \omega_2)}{\omega_2^{r-1}} \right) (\omega_1^r - \omega_2^r) dx \geq 0. \quad (42)$$

Thus, since the function $\xi \mapsto \frac{h(x, \xi)}{\xi^{r-1}}$ is decreasing in $(0, +\infty)$, we deduce that $\omega_1 = \omega_2$. As a result, we conclude that Problem (1) has a unique positive solution for every $\eta \in (0, \tilde{\eta}^*)$ and for any $\lambda > 0$.

On the other hand, when $m = q$, we know that

$$\begin{aligned} & \frac{\omega_1^{q-1}}{|x|^{sq}} \left(\frac{\omega_2^r}{\omega_1^{r-1}} - \omega_1 \right) + \frac{\omega_2^{q-1}}{|x|^{sq}} \left(\frac{\omega_1^r}{\omega_2^{r-1}} - \omega_2 \right) \\ &= \frac{1}{|x|^{sq}} (\omega_1^{q-r} - \omega_2^{q-r}) (\omega_2^r - \omega_1^r) \\ &\leq 0. \end{aligned}$$

For any $\eta \in (-\infty, 0]$, we derive the following:

$$\eta \left(\int_{\Omega} \left[\frac{\omega_1^{q-1}}{|x|^{sq}} \left(\frac{\omega_2^r}{\omega_1^{r-1}} - \omega_1 \right) + \frac{\omega_2^{q-1}}{|x|^{sq}} \left(\frac{\omega_1^r}{\omega_2^{r-1}} - \omega_2 \right) \right] dx \right) \geq 0.$$

From this and (41), we therefore infer that (42) is satisfied, and thus, from the previous argument, we directly show that $\omega_1 = \omega_2$ for every $\eta \in (-\infty, 0]$ and for any $\lambda > 0$. $\square$

5. Conclusions

This study aims to establish the existence of at most one positive weak solution to non-local Brézis–Oswald-type fractional problems with fractional (r, q) -Laplace operators and Hardy potentials. To show this, we first determined the semicontinuity property by exploiting the fractional Hardy inequality in [16] and the concentration–compactness principle for fractional, critical and Hardy terms in [30,31]. By using this property and considering eigenvalue problems related to the fractional (r, q) -Laplacian, we proved the existence of at least one positive weak solution for our problem as our first result. Finally, we provide the uniqueness result by applying some versions of the discrete Picone inequalities in [8,13] as the crucial tool. The obtained result in this paper is an improvement of the notable works of [13,17,29] and extends the consequences for the non-local fractional equations in [14,18] to the case for unbalanced double-phase fractional problems with Hardy potentials.

As an additional study, we would like to obtain analogous results as in Theorems 1 and 2 to fractional $(r(\cdot), q(\cdot))$ -Laplacian problems with Hardy potentials where

$$\begin{cases} \mathcal{L}_{r(\cdot, \cdot)}^s \omega + \mathcal{L}_{q(\cdot, \cdot)}^s \omega = \frac{\eta |\omega|^{m(x)-2} \omega}{|x|^{sm(x)}} + \lambda h(x, \omega) & \text{in } \Omega, \\ \omega > 0 & \text{in } \Omega, \\ \omega = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (43)$$

Here, the operator $\mathcal{L}_{m(\cdot,\cdot)}^s$ is defined as

$$\mathcal{L}_{m(\cdot,\cdot)}^s \omega(x) = 2 \lim_{\epsilon \searrow 0} \int_{\{x \in \mathbb{R}^N : |x-z| \geq \epsilon\}} \frac{|\omega(x) - \omega(z)|^{m(x,z)-2} (\omega(x) - \omega(z))}{|z - x|^{N+sm(x,z)}} dz, \quad x \in \mathbb{R}^N,$$

where $s \in (0, 1)$, $m \in \{r, q\}$ with $r(x, z) < q(x, z)$ for all $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$; $m \in C(\mathbb{R}^N \times \mathbb{R}^N)$ is symmetric, i.e., $m(x, z) = m(z, x)$ for all $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$ such that $1 < m^- := \inf_{(x,z) \in \mathbb{R}^N \times \mathbb{R}^N} m(x, z) \leq m^+ := \sup_{(x,z) \in \mathbb{R}^N \times \mathbb{R}^N} m(x, z) < \frac{N}{s}$; $m(x, x) \leq r_s^*(x) := \frac{Nr(x,x)}{N-sr(x,x)}$ for all $x \in \Omega$; and λ is a positive real parameter. To show this, we must analyze the fractional Hardy inequality and the concentration–compactness principle for a fractional Sobolev space with a variable exponent and eigenvalue problems related to the fractional $(r(\cdot), q(\cdot))$ -Laplacian. To our knowledge, there exist no reports about the existence of at most one positive solution to Problem (43).

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