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A Fast L_2-1_σ Finite Element Method for Time Fractional Keller–Segel Equations with Weakly Singular Solutions

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Abstract

We propose a fast L_2-1_σ finite element method for solving the time fractional Keller–Segel equations with a Caputo fractional derivative of $\alpha \in (0, 1)$. Firstly, the fast L_2-1_σ scheme on the graded mesh is used to discretize the time fractional derivative. This approach relies on the sum of exponentials (SOE) skill to speed up the convolution kernel. Thus, we overcome the computational cost caused by the nonlocality of fractional derivatives. Then, by combining finite element discretization in spatial direction, a fully implicit numerical scheme is derived. Subsequently, we establish the stability and an α -robust error analysis of the fully discrete scheme. Finally, we present some numerical examples to demonstrate the correctness of our theoretical results.

Keywords: time fractional Keller–Segel equations; L_2-1_σ scheme; finite element method; stability and error analysis

1. Introduction

In recent years, fractional partial differential equations have been widely applied in different branches, for instance, biology [1], physics [2], control system [3], and finance [4]. And a large number of researchers have shown great interest in theoretical and numerical analysis. Let Ω be a open and bounded domain of $\mathbb{R}^d$ ($d \geq 1$) with a smooth boundary $\partial\Omega$. Given $\alpha \in (0, 1)$, we shall be interested in the numerical approximation for a time-fractional Keller–Segel (TFKS) equation

$$\begin{cases} \partial_t^\alpha u - \Delta u + \nabla \cdot (u \nabla v) = 0, & x \in \Omega, 0 < t < T, \\ \partial_t^\alpha v - \Delta v + v = u, & x \in \Omega, 0 < t < T, \\ u(x, t) = v(x, t) = 0, & x \in \partial\Omega, 0 < t < T, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where u_0, v_0 are a given functions, and $\partial_t^\alpha u$ represents the Caputo fractional derivative, which is defined as

$$\partial_t^\alpha u(t) = \int_0^t w_{1-\alpha}(t-\xi) u'(\xi) d\xi, \quad \text{with } w_\alpha = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \quad (2)$$

When $\alpha = 1$, model (1) is called the classic Keller–Segel (KS) equation, which was first introduced in [5,6] to capture the chemotaxis response of bacteria towards chemical agents, where $u(x, t)$ stands for the density of bacteria and $v(x, t)$ represents the concentration of the



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oxygen. In biology and mathematics, diffusion and chemotaxis are the basis of the motion of bacteria [7,8]. Thus, KS equations provide a powerful framework to understand the intricate interplay between the motion of bacteria, chemical signaling, and emergent pattern formation [9]. However, anomalous diffusion is a ubiquitous phenomenon in the process of biological movement [10–12]. Moreover, we note that anomalous diffusion can be modeled as continuous-time random walks at the macroscopic level [13], so we can modify the diffusion equation by applying a fractional temporal operator. For instance, Langlands and Henry [14] developed fractional chemotaxis diffusion equations with anomalous diffusion for modeling the chemically directed transport of biological organisms. Their work studied the influence of substrate heterogeneity on the dynamics of the KS model by the means of fractional calculus in [15].

Let us recall some work in the literature concerning TFKS equations with the Caputo fractional derivative. By the Aubin–Lions lemma and its variants, the existence of a weak solution was introduced in [16]. Zhou et al. [17] proved the existence of a solution by using the Faedo–Galerkin method with some compactness arguments, and the Mittag–Leffler stability of the solution was given. Their work has established the existence of the nonnegative weak solution in [18]. In [19], some refined results regarding the large time behavior of solutions were derived, and the well-posedness and the asymptotic stability of solutions in Marcinkiewicz spaces were studied. The well-posedness and the blow-up of the solution were established in the setting of Lebesgue and Besov spaces from [20]. El-Sayed et al. [21] obtained the analytical solution by the Adomians decomposition method. The modified homotopy analysis transform method was developed in [22]. Additionally, studies on the fractional KS equations with other fractional derivative operators can be found in [23–26].

Less work has been carried out on the numerical methods of the TFKS equation. To our knowledge, it is only mentioned in [27]. This work proposed a fractional class of explicit Adams–Bashforth and implicit Adams–Moulton methods of first- and second-order accuracy to solve the TFKS equations with the sufficiently smooth solution. Nevertheless, a growing number of studies suggest that the solution of the time fractional partial differential equations is weakly singular near $t = 0$ [28,29], i.e., there exists a constant C , for any $t \in (0, T]$:

$$|\partial_t^l u(t)| \leq C(1 + t^{\alpha-l}), \quad l \in \mathbb{N},$$

To overcome the initial time singularity, the L1, L2-1 σ , L2 and DG methods on the graded meshes or general nonuniform meshes have been developed in [30–35]. The L1, L2 and CQ schemes with proper initial correction using uniform step size can recover the optimal error estimates; see [36–39]. Another important feature is the problem of large computational cost due to the nonlocality of the fractional derivatives. Then, the fast L1, L2 and L2-1 σ schemes based on the sum of exponentials (SOE) approximation were proposed and analyzed in [40–44].

The main difficulties in the study of problem (1) are the weak singularity and nonlocality of the fractional time derivative and the nonlinearity of the equation itself. In order to derive high-precision numerical solutions, we firstly use the nonuniform L2-1 σ scheme to handle the singularity of solutions at the initial time, and this scheme is high-order. Inspired by [45,46], we shall adopt the SOE approximation to speed up the computation of the L2-1 σ scheme, thus reducing the computational storage and computational cost from $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$ to $\mathcal{O}(N_{\text{exp}})$ and $\mathcal{O}(NN_{\text{exp}})$, respectively. Here N denotes the number of time step and N_{exp} stands for the number of the quadrature nodes. Next, we use the finite element to discretize spatial direction and combine it with the fast L2-1 σ method to obtain an implicit scheme. Compared to the explicit scheme, our approach is unconditionally stable, and its accuracy will not decrease. By the improved fractional Grönwall inequality,

we further prove the stability and the α -robust error estimates of the numerical scheme. Finally, some numerical results demonstrate the correctness of the theoretical analysis.

The main contributions of our work are as follows:

1. We consider for the first time the numerical method for solving singular cases in the fractional KS models. Compared to the smooth case in reference [27], our numerical method is more in line with the characteristics of fractional order models.
2. A fast numerical scheme is obtained by constructing a fast nonuniform L2-1 $_{\sigma}$ scheme with the finite element method for the time fractional Keller–Segel equations. This numerical scheme has the advantages of high accuracy and low computational storage and can effectively handle the singularity of the solution at $t \rightarrow 0$.
3. We prove the stability of the numerical scheme for both the $L^2(\Omega)$ and $H^1(\Omega)$ norms under some constraints on the time step ratio and obtain a α -robust error estimate by the fractional Grönwall inequality. We note that this theoretical analysis framework is also applicable to the more complex time fractional-order coupled diffusion systems [47].

The outline of this work is as follows. In Section 2, a fast L2-1 $_{\sigma}$ method and the fully discrete scheme are introduced. In Section 3, we establish the unconditional stability of the fully discrete scheme in both $L^2(\Omega)$ and $H^1(\Omega)$ norms. In Section 4, an α -robust error estimate is provided in both $L^2(\Omega)$ and $H^1(\Omega)$ norms. In Section 5, we propose several numerical examples to verify theoretical analysis. Finally, a summary of this work is made in Section 6.

2. Fully Discrete Scheme for the TFKS Equations

Let $L^p(\Omega)$ be the Lebesgue space with norm $\|\cdot\|_{0,p}$ for $1 \leq p \leq \infty$; we denote as $W^{k,p}(\Omega)$ the Sobolev space with norm $\|\cdot\|_{k,p}$. For $p = 2$, we define $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$. We also denote $\|v\|_{L^2(\Omega)} = \|v\|$ and $\|v\|_{H^k(\Omega)} = \|v\|_k$. For the sake of simplicity, we shall apply the letter C to denote a positive constant that is independent of the mesh size and time size.

A weak form of the problem (1) is as follows. Find $u, v \in H_0^1(\Omega)$ such that

$$\begin{cases} (\partial_t^\alpha u, w) + (\nabla u, \nabla w) - (u \nabla v, \nabla w) = 0, & w \in H_0^1(\Omega), \\ (\partial_t^\alpha v, w) + (\nabla v, \nabla w) + (v, w) = (u, w), & w \in H_0^1(\Omega), \\ u(x, 0) = u_0, \quad v(x, 0) = v_0. \end{cases} \quad (3)$$

Let $N \in \mathbb{N}$ be the time step; we define a nonuniform time partition $0 = t_0 \leq t_1 \leq \dots \leq t_N = T$ with the time point $t_k = \left(\frac{k}{N}\right)^r T$, $k = 0, 1, \dots, N$. Here $r \geq 1$ is the grading constant. For $k = 1, 2, \dots, N$, we let $\tau_k = t_k - t_{k-1}$ and $t_{k-\sigma} = (1 - \sigma)\tau_k + t_{k-1}$, $\sigma \in [0, 1]$. We also define $\varphi^k = \varphi(t_k)$ and $\varphi^{k,\sigma} = (1 - \sigma)\varphi_k + \varphi_{k-1}$ for $1 \leq k \leq N$. Setting $\sigma = \frac{\alpha}{2}$ here and after, the L2-1 $_{\sigma}$ approximation of the Caputo derivative (2) is given below [48]. For $n = 1, 2, \dots, N$, we have

$$\begin{aligned} \partial_t^\alpha \varphi^{n-\sigma} &= \int_0^{t_{n-\sigma}} w_{1-\alpha}(t_{n-\sigma} - \xi) \varphi'(\xi) d\xi \\ &\approx \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} w_{1-\alpha}(t_{n-\sigma} - \xi) (\Pi_{2,k} \varphi)'(\xi) d\xi \\ &\quad + \int_{t_{n-1}}^{t_{n-\sigma}} w_{1-\alpha}(t_{n-\sigma} - \xi) (\Pi_{1,n} \varphi)'(\xi) d\xi =: \bar{\partial}_\tau^\alpha \varphi^{n-\sigma}, \end{aligned} \quad (4)$$

where $\Pi_{1,n} \varphi$ stands for the linear interpolate with the nodes t_n, t_{n-1} , $\Pi_{2,k} \varphi$ denotes the quadratic interpolate at t_{k-1}, t_k and t_{k+1} . Setting the step size ratio $\rho_k = \tau_k / \tau_{k+1}$ for

$1 \leq k \leq N-1$ and $\nabla_\tau \varphi^k = \varphi^k - \varphi^{k-1}$ for $1 \leq k \leq N$, the L2-1 $_\sigma$ scheme in (4) can be reformulated as

$$\begin{aligned}\bar{\partial}_\tau^\alpha \varphi^{n-\sigma} &= \sum_{k=1}^{n-1} \left(a_{n-k}^{(n)} \nabla_\tau \varphi^k + \rho_k b_{n-k}^{(n)} \nabla_\tau \varphi^{k+1} - b_{n-k}^{(n)} \nabla_\tau \varphi^k \right) + a_0^{(n)} \nabla_\tau \varphi^n \\ &= \sum_{k=1}^{n-1} A_{n-k}^{(n)} \nabla_\tau \varphi^k + A_0^{(n)} \nabla_\tau \varphi^n\end{aligned}$$

with

$$\begin{aligned}a_0^{(n)} &= \frac{1}{\tau_n} \int_{t_{n-1}}^{t_{n-\sigma}} w_{1-\alpha}(t_{n-\sigma} - \xi) d\xi, \quad a_{n-k}^{(n)} = \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} w_{1-\alpha}(t_{n-\sigma} - \xi) d\xi, \\ b_{n-k}^{(n)} &= \frac{2}{\tau_k(\tau_k + \tau_{k+1})} \int_{t_{k-1}}^{t_k} (\xi - t_{k-1/2}) w_{1-\alpha}(t_{n-\sigma} - \xi) d\xi,\end{aligned}$$

and

$$A_{n-k}^{(n)} = \begin{cases} a_0^{(n)} + \rho_{n-1} b_1^{(n)}, & k = n, \\ a_{n-k}^{(n)} + \rho_{k-1} b_{n-k+1}^{(n)} - b_{n-k}^{(n)}, & 2 \leq k \leq n-1, \\ a_{n-1}^{(n)} - b_{n-1}^{(n)}, & k = 1. \end{cases}$$

As can be seen from the above, the computational storage and cost of the L2-1 $_\sigma$ scheme are $\mathcal{O}(N)$ or $\mathcal{O}(N^2)$, which are too expensive. Thus, we consider the fast L2-1 $_\sigma$ scheme based on the SOE technique to approximate the convolution kernel $t^{-\alpha}$. The following lemmas are mainly adopted.

Lemma 1 ([40]). *For the given parameters $\alpha \in (0, 1)$, ϵ , $\hat{\tau}$ and T , there exists a family of points s_i and weight ω_i ($i = 1, 2, \dots, N_{\text{exp}}$) such that*

$$\left| t^{-\alpha} - \sum_{i=1}^{N_{\text{exp}}} \omega_i e^{-s_i t} \right| \leq \epsilon, \quad \forall t \in [\hat{\tau}, T],$$

where

$$N_{\text{exp}} = \mathcal{O} \left(\log \frac{1}{\epsilon} \left(\log \log \frac{1}{\epsilon} + \log \frac{T}{\hat{\tau}} \right) + \log \frac{1}{\hat{\tau}} \left(\log \log \frac{1}{\epsilon} + \log \frac{1}{\hat{\tau}} \right) \right).$$

Due to N_{exp} is of the order $\mathcal{O}(\log N)$ for $T \gg 1$ or $\mathcal{O}(\log^2 N)$ for $T \approx 1$ when we fix ϵ , the complexity of our algorithm is nearly optimal. Based on the above lemma, the history part in (4) can be written as

$$\begin{aligned}& \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} w_{1-\alpha}(t_{n-\sigma} - \xi) (\Pi_{2,k} u)'(\xi) d\xi \\ & \approx \frac{1}{\Gamma(1-\alpha)} \sum_{i=1}^{N_{\text{exp}}} \int_0^{t_{n-1}} (\Pi_{2,k} u)'(\xi) \omega_i e^{-s_i(t_{n-\sigma}-\xi)} d\xi := \sum_{i=1}^{N_{\text{exp}}} H_i(t_{n-1}),\end{aligned}\tag{5}$$

where $H_i(t_0) = 0$ and

$$H_i(t_{n-1}) = e^{-s_i \tau_{n-\sigma}} H_i(t_{n-2}) + \frac{1}{\Gamma(1-\alpha)} \int_{t_{n-2}}^{t_{n-1}} (\Pi_{2,k} u)'(\xi) \omega_i e^{-s_i(t_{n-\sigma}-\xi)} d\xi.\tag{6}$$

Combining (4) and (5), the fast L2-1 $_\sigma$ scheme can be represented as

$$\bar{\partial}_F^\alpha \varphi^{n-\sigma} = a_0^{(n)} \nabla_\tau \varphi^n + \sum_{i=1}^{N_{\text{exp}}} H_i(t_{n-1}),\tag{7}$$

where $H_i(t_{n-1})$ can be calculated by the recurrence formula (6). Obviously, the fast algorithm reduces computational storage and cost from $\mathcal{O}(N)$ and $\mathcal{O}(N^2)$ to $\mathcal{O}(N_{\text{exp}})$ and $\mathcal{O}(NN_{\text{exp}})$, respectively. Then, we equivalently reformulate (7) into the following convolution form

$$\bar{\partial}_F^\alpha \varphi^{n-\sigma} = B_0^{(n)} \varphi^n + \sum_{i=1}^{n-1} (B_{n-i}^{(n)} - B_{n-i-1}^{(n)}) \varphi^i - B_{n-1}^{(n)} \varphi^0, \quad (8)$$

where

$$B_{n-k}^{(n)} = \begin{cases} a_0^{(n)} + \sum_{i=1}^{N_{\text{exp}}} \rho_{n-1} \tilde{b}_1^{(n)}, & k = n, \\ \sum_{i=1}^{N_{\text{exp}}} e^{-s_i(t_{n-\sigma}-t_{k+1-\sigma})} (\tilde{a}_{n-k}^{(k+1)} + e^{-s_i \tau_{k+1-\sigma}} \rho_{k-1} \tilde{b}_{n-k+1}^{(k)} - \tilde{b}_{n-k}^{(k+1)}), & 2 \leq k \leq n-1, \\ \sum_{i=1}^{N_{\text{exp}}} e^{-s_i(t_{n-\sigma}-t_{2-\sigma})} (\tilde{a}_{n-1}^{(2)} - \tilde{b}_{n-1}^{(2)}), & k = 1, \end{cases}$$

with

$$\tilde{a}_{n-k}^{(k+1)} = \frac{\omega_i}{\tau_k \Gamma(1-\alpha)} \int_{t_{k-1}}^{t_k} e^{-s_i(t_{k+1-\sigma}-\xi)} d\xi,$$

$$\tilde{b}_{n-k}^{(k+1)} = \frac{2\omega_i}{\tau_k(\tau_k + \tau_{k+1}) \Gamma(1-\alpha)} \int_{t_{k-1}}^{t_k} (\xi - t_{k-1/2}) e^{-s_i(t_{k+1-\sigma}-\xi)} d\xi.$$

Then, the discrete convolution kernel $B_{n-k}^{(n)}$ of (8) has the following properties [49]:

$$B_{n-k-1}^{(n)} - B_{n-k}^{(k)} > 0, \quad \text{for } 1 \leq k \leq n-1, \quad (9)$$

$$B_0^{(n)} \leq \frac{26}{11} \int_{t_{n-1}}^{t_n} \frac{w_{1-\alpha}(t_n-s)}{\tau_n} ds \leq \frac{2\tau_n^{-\alpha}}{\Gamma(2-\alpha)} \quad (10)$$

$$\text{with } \epsilon \leq \epsilon^* = \min \left\{ \frac{\alpha}{2(1-\alpha)} w_{1-\alpha}(T), \frac{1}{26} w_{1-\alpha}(T) \right\}.$$

Lemma 2 ([50]). Assume that $|\partial_t^l \varphi(t)| \leq C(1+t^{\alpha-l})$ for $l = 0, 1, 2, 3$. Then, the fast L2-1 $_{\sigma}$ scheme (7) satisfies the following truncation errors:

$$|\partial_t^\alpha \varphi(t^{n-\sigma}) - \bar{\partial}_F^\alpha \varphi^{n-\sigma}| \leq C \left(t_{n-\sigma}^{-\alpha} N^{-\min\{3-\alpha, r\alpha\}} + \epsilon \right), \quad (11)$$

and

$$|\varphi(t^{n-\sigma}) - \varphi^{n,\sigma}| \leq C t_{n-\sigma}^{-\alpha} N^{-\min\{2, r\alpha\}} \quad (12)$$

for $n = 1, 2, \dots, N$.

Let U^n (or V^n) be the approximate value of u (or v) at t_n and $U^{n,\sigma} = (1-\sigma)U^n + U^{n-1}$ (or $V^{n,\sigma} = (1-\sigma)V^n + V^{n-1}$). We apply the fast L2-1 $_{\sigma}$ scheme to (3), and an implicit semi-discrete scheme reads as follows. For $n = 1, 2, \dots, N$, find $U^n, V^n \in H_0^1(\Omega)$ such that

$$\begin{cases} (\bar{\partial}_F^\alpha U^{n-\sigma}, w) + (\nabla U^{n,\sigma}, \nabla w) - (U^{n,\sigma} \nabla V^{n,\sigma}, \nabla w) = 0, & w \in H_0^1(\Omega), \\ (\bar{\partial}_F^\alpha V^{n-\sigma}, w) + (\nabla V^{n,\sigma}, \nabla w) + (V^{n,\sigma}, w) = (U^{n,\sigma}, w), & w \in H_0^1(\Omega), \\ U^0 = u_0, & V^0 = v_0. \end{cases} \quad (13)$$

Define the Mittag-Leffler functions

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)}, \quad E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k\alpha)}.$$

By utilizing Duhamel's principle, we have the solution's representation of the problem (1):

$$u(t) = E_\alpha(t^\alpha \Delta)u_0 - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha \Delta) \nabla \cdot (u \nabla v)(s) ds, \quad (14)$$

$$v(t) = E_\alpha(t^\alpha (\Delta - 1))v_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha (\Delta - 1))u(s) ds, \quad (15)$$

Furthermore, we calculate the gradient of (15) to obtain

$$\nabla v(t) = E_\alpha(t^\alpha (\Delta - 1)) \nabla v_0 + \int_0^t (t-s)^{\alpha-1} \nabla E_{\alpha,\alpha}((t-s)^\alpha (\Delta - 1))u(s) ds. \quad (16)$$

Generally, we call the above solutions "mild solution". Based on the mild solution (14)–(16), we can derive a prior bounds of the solution in the problem (1).

Lemma 3 ([19]). Suppose that $u_0 \in L^1(\Omega)$, $v_0, \nabla v_0 \in L^1(\Omega) \cap L^\infty(\Omega)$ are sufficiently small. Then, there exists a unique global solution $(u, v) \in \mathbb{R}^2 \times (0, +\infty)$ to the problem (1) such that

$$\sup_{t>0} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) < \infty, \quad \sup_{t>0} \|\nabla v(t)\|_{L^\infty(\Omega)} < \infty.$$

and has the following time decay behavior

$$\begin{aligned} \sup_{t>0} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)}) &< C(1+t)^{-\alpha/\mu}, \\ \sup_{t>0} \|\nabla v(t)\|_{L^\infty(\Omega)} &\leq C(1+t)^{-\alpha/\mu}. \end{aligned}$$

where $1 < \mu < 2$.

According to the Lemma 3, we assume that ∇v is bounded, i.e., for any $t \in [0, T]$, there exists a constant $M > 0$, such that

$$\|\nabla v\|_{L^\infty(\Omega)} \leq M. \quad (17)$$

Then, we shall derive the following boundedness of $\nabla V^{n,\sigma}$:

$$\|\nabla V^{n,\sigma}\|_{L^\infty(\Omega)} \leq M, \quad (18)$$

where $M = \max_{0 \leq n \leq N} \|\nabla v^{n,\sigma}\|_{L^\infty(\Omega)} + 1$.

Let $\mathcal{T}_h = \bigcup K$ be a uniform mesh partition of Ω with the mesh size h . We define that $V_h \subset H_0^1(\Omega)$ is the finite element space satisfying the homogeneous Dirichlet condition. Define the Ritz projection $R_h : H_0^1(\Omega) \rightarrow V_h$ satisfying

$$(\nabla(u - R_h u), w_h) = 0, \quad w \in V_h.$$

Then, the following interpolation estimates hold [51]

$$\|u - R_h u\| + h\|u - R_h u\|_1 \leq Ch^{k+1}\|u\|_{k+1}. \quad (19)$$

for any $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$.

Together, the finite element method with the semi-discrete scheme (13) and the fully discrete scheme of the target problem (1) are as follows: For $n = 1, 2, \dots, N$, find $U_h^n, V_h^n \in V_h$ such that

$$\begin{cases} (\bar{\partial}_F^\alpha U_h^{n-\sigma}, w_h) + (\nabla U_h^{n,\sigma}, \nabla w_h) - (U_h^{n,\sigma} \nabla V_h^{n,\sigma}, \nabla w_h) = 0, & w_h \in V_h, \\ (\bar{\partial}_F^\alpha V_h^{n-\sigma}, w_h) + (\nabla V_h^{n,\sigma}, \nabla w_h) + (V_h^{n,\sigma}, w_h) = (U_h^{n,\sigma}, w_h), & w_h \in V_h, \\ U_h^n = R_h u_0, & V_h^n = R_h v_0. \end{cases} \quad (20)$$

Similarly, we give the boundedness of $\nabla V_h^{n,\sigma}$ by using the the inverse inequality, adn we have

$$\begin{aligned} \|\nabla V_h^{n,\sigma}\|_{L^\infty(\Omega)} &\leq \|\nabla R_h V^{n,\sigma}\|_{L^\infty(\Omega)} + \|\nabla(R_h V^{n,\sigma} - V_h^{n,\sigma})\|_{L^\infty(\Omega)} \\ &\leq \|\nabla R_h V^{n,\sigma}\|_{L^\infty(\Omega)} + Ch^{-1} \|\nabla(R_h V^{n,\sigma} - V_h^{n,\sigma})\| \\ &\leq \|\nabla R_h V^{n,\sigma}\|_{L^\infty(\Omega)} + Ch^{-1} h^k \|v\|_k. \end{aligned} \quad (21)$$

Thus, when v is smooth enough, there exists a constant $M > 0$, such that

$$\|\nabla V_h^{n,\sigma}\|_{L^\infty(\Omega)} \leq M. \quad (22)$$

3. Stability Analysis of Fully Discrete Scheme

In this section, we shall establish the stability of the fully discrete scheme (20) in both $L^2(\Omega)$ and $H^1(\Omega)$ norms. Inspired by [30], we define a sequence of the discrete complementary convolution kernels $\{P_j^{(n)}\}_{j=1}^n$ by

$$P_0^{(n)} = \frac{1}{B_0^{(n)}}, \quad P_j^{(n)} = \frac{1}{B_0^{(n-j)}} \sum_{k=0}^{j-1} (B_{j-k-1}^{(n-k)} - B_{j-k}^{(n-k)}) P_{n-k}^{(n)}, \quad 1 \leq j \leq n-1.$$

By the theoretical basis provided in [52], we shall derive that the kernel $P_j^{(n)}$ satisfies the following three properties [50]:

$$\begin{aligned} \sum_{j=k}^n B_{j-k}^{(j)} P_{n-j}^{(n)} &= 1 \quad \text{for } 1 \leq n \leq N. \\ \sum_{j=1}^n P_{n-j}^{(n)} t_{j-\sigma}^{-\alpha} &\leq \frac{2^{1+r\alpha} T^{\alpha-l_N} e^r t_N^{l_N} \Gamma(1+l_N-\alpha)}{\Gamma(1+l_N)}, \quad l_N = 1/\ln N. \end{aligned} \quad (23)$$

$$\sum_{j=1}^n P_{n-j}^{(n)} \leq \frac{2t_n^\alpha}{\Gamma(1+\alpha)}. \quad (24)$$

We shall introduce two useful lemmas, which play a significant role in the subsequent theory.

Lemma 4 ([50]). *For any sequence $\{\varphi^n\}_{n=1}^N$, the following holds:*

$$(\bar{\partial}_F^\alpha \varphi^{n-\sigma}, \varphi^{n,\sigma}) \geq \frac{1}{2} \bar{\partial}_F^\alpha \|\varphi^{n-\sigma}\|^2. \quad (25)$$

Lemma 5 ([50]). *Let λ_i be the nonnegative constants with $0 \leq \sum_{i=1}^n \lambda_i \leq \Lambda$, where Λ is a positive constant. Assume that the nonnegative sequences $\{\varphi^k\}_{k=0}^N$, $\{\zeta^n\}_{n=1}^N$, and $\{\eta^n\}_{n=1}^N$ satisfy*

$$\bar{\partial}_F^\alpha (\varphi^{n-\sigma})^2 \leq \sum_{i=1}^n \lambda_i (\varphi^{i,\sigma})^2 + \zeta^n \varphi^{n,\sigma} + (\eta^n)^2 \quad \text{for } n \geq 1.$$

If the maximum time step satisfies $\tau \leq [2\Gamma(2 - \alpha)\Lambda]^{-1/\alpha}$, we can get

$$\varphi^n \leq E_\alpha(2\Lambda t_n^\alpha) \left[\varphi^0 + \max_{1 \leq k \leq n} \sum_{j=1}^k P_{k-j}^{(k)} (\xi^j + \eta^j) + \max_{1 \leq j \leq n} \{\eta^j\} \right] \quad \text{for } 1 \leq n \leq N.$$

In the following, we shall present the stability results of the fully discrete scheme in both $L^2(\Omega)$ and $H^1(\Omega)$ norms.

Theorem 1. Let U_h^n, V_h^n be the solution to the fully discrete scheme (20). For $n = 0, 1, 2, \dots, N$, there exists a positive constant τ^* ($\tau \leq \tau^*$), and the following holds

$$\|U_h^n\|_1 + \|V_h^n\| \leq C \left(\|U_h^0\| + \|V_h^0\| \right), \quad (26)$$

$$\|U_h^n\|_1 + \|V_h^n\|_1 \leq C \left(\|U_h^0\|_1 + \|V_h^0\|_1 \right), \quad (27)$$

where C is α -robust constant.

Proof. Setting $w_h = U_h^{n,\sigma}$ in the first equation of the fully discrete scheme (20), and utilizing the Cauchy–Schwartz inequality and the Young’s inequality, we have

$$\begin{aligned} (\bar{\partial}_F^\alpha U_h^{n-\sigma}, U_h^{n,\sigma}) + (\nabla U_h^{n,\sigma}, \nabla U_h^{n,\sigma}) &= (U_h^{n,\sigma} \nabla V_h^{n,\sigma}, \nabla U_h^{n,\sigma}) \\ &\leq \|\nabla V_h^{n,\sigma}\|_{L^\infty(\Omega)} \|U_h^{n,\sigma}\| \|\nabla U_h^{n,\sigma}\| \\ &\leq \varepsilon \|\nabla U_h^{n,\sigma}\|^2 + \frac{M}{\varepsilon} \|U_h^{n,\sigma}\|^2, \quad \varepsilon \in (0, 1) \end{aligned}$$

It follows from the Lemma 5 and (25) that

$$\|U_h^n\| \leq E_\alpha \left(\frac{2M}{\varepsilon} t_n^\alpha \right) \|U_h^0\|. \quad (28)$$

Let $w_h = V_h^{n,\sigma}$ in the second equation of the fully discrete scheme (20). We have

$$(\bar{\partial}_F^\alpha V_h^{n-\sigma}, V_h^{n,\sigma}) + (\nabla V_h^{n,\sigma}, \nabla V_h^{n,\sigma}) + (V_h^{n,\sigma}, V_h^{n,\sigma}) = (U_h^{n,\sigma}, V_h^{n,\sigma}),$$

Then, we use Young’s inequality and (25) to get

$$\bar{\partial}_F^\alpha \|V_h^{n-\sigma}\|^2 \leq \|U_h^{n,\sigma}\|^2 + \|V_h^{n,\sigma}\|^2.$$

By Lemma 5, we have

$$\|V_h^n\| \leq E_\alpha(2t_n^\alpha) \left(\|V_h^0\| + \max_{1 \leq j \leq n} \|U_h^{j,\sigma}\| \right). \quad (29)$$

Together with (28), (26) has been completed.

Let $w_h = \bar{\partial}_F^\alpha U_h^{n-\sigma}$ in the first equation of the fully discrete scheme (20). The Cauchy–Schwartz inequality and the Young’s inequality are used to yield

$$\begin{aligned} (\bar{\partial}_F^\alpha U_h^{n-\sigma}, \bar{\partial}_F^\alpha U_h^{n-\sigma}) + (\nabla U_h^{n,\sigma}, \bar{\partial}_F^\alpha \nabla U_h^{n-\sigma}) &= (U_h^{n,\sigma} \nabla V_h^{n,\sigma}, \bar{\partial}_F^\alpha \nabla U_h^{n-\sigma}) \\ &\leq \|\nabla V_h^{n,\sigma}\|_{L^\infty(\Omega)} \|U_h^{n,\sigma}\| \|\bar{\partial}_F^\alpha \nabla U_h^{n-\sigma}\| \\ &\leq \frac{M}{\varepsilon} \|U_h^{n,\sigma}\|^2 + \varepsilon \|\bar{\partial}_F^\alpha \nabla U_h^{n-\sigma}\|^2 \end{aligned} \quad (30)$$

Here $\varepsilon > 0$ sufficient small. By the Poincaré inequality in (30) and (25), we obtain

$$\bar{\partial}_F^\alpha \|\nabla U_h^{n-\sigma}\|^2 \leq \frac{2M}{\varepsilon} \|\nabla U_h^{n,\sigma}\|^2. \quad (31)$$

Then, based on Lemma 5, we have

$$\|U_h^n\|_1 \leq E_\alpha \left(\frac{4M}{\varepsilon} t_n^\alpha \right) \|U_h^0\|_1.$$

Setting $w_h = -\Delta V_h^{n,\sigma}$ in the second equation of the fully discrete scheme (20), we have

$$(\bar{\partial}_F^\alpha \nabla V_h^{n-\sigma}, \nabla V_h^{n,\sigma}) + (\Delta V_h^{n,\sigma}, \Delta V_h^{n,\sigma}) + (\nabla V_h^{n,\sigma}, \nabla V_h^{n,\sigma}) = (\nabla U_h^{n,\sigma}, \nabla V_h^{n,\sigma}).$$

By using the Cauchy–Schwartz inequality and (25), we thus obtain

$$\bar{\partial}_F^\alpha \|V_h^{n-\sigma}\|_1^2 \leq \|U_h^{n,\sigma}\|_1^2 + \|V_h^{n,\sigma}\|_1^2.$$

By Lemma 5, we get

$$\|V_h^n\|_1 \leq E_\alpha(2t_n^\alpha) \left(\|V_h^0\|_1 + \max_{1 \leq j \leq n} \|U_h^{j,\sigma}\|_1 \right). \quad (32)$$

Then, (27) has been completed. $\square$

4. Error Analysis of the Fully Discrete Scheme

In this section, we focus on the error analysis of the fully discrete scheme. Define

$$\begin{aligned} U^{n-\sigma} - U_h^{n-\sigma} &= U^{n-\sigma} - R_h U^{n-\sigma} + R_h U^{n-\sigma} - U_h^{n-\sigma} = \theta_h^{n-\sigma} + \eta_h^{n-\sigma}. \\ V^{n-\sigma} - V_h^{n-\sigma} &= V^{n-\sigma} - R_h V^{n-\sigma} + R_h V^{n-\sigma} - V_h^{n-\sigma} = \rho_h^{n-\sigma} + \beta_h^{n-\sigma}. \\ U^{n,\sigma} - U_h^{n,\sigma} &= U^{n,\sigma} - R_h U^{n,\sigma} + R_h U^{n,\sigma} - U_h^{n,\sigma} = \theta_h^{n,\sigma} + \eta_h^{n,\sigma}. \\ V^{n,\sigma} - V_h^{n,\sigma} &= V^{n,\sigma} - R_h V^{n,\sigma} + R_h V^{n,\sigma} - V_h^{n,\sigma} = \rho_h^{n,\sigma} + \beta_h^{n,\sigma}. \end{aligned}$$

Firstly, we provide the spatial error estimates for the fully discrete scheme (20).

Theorem 2. Assume that $U^n, V^n \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ are the solution of the semi-discrete scheme (13). Let $U_h^n, V_h^n \in V_h$ be the solution of the fully discrete scheme (20). Then, for $n = 1, 2, \dots, N$, the following error estimates hold:

$$\|U^n - U_h^n\| + \|V^n - V_h^n\| \leq Ch^{k+1}, \quad (33)$$

$$\|U^n - U_h^n\|_1 + \|V^n - V_h^n\|_1 \leq Ch^k, \quad (34)$$

where the constant C is α -robust.

Proof. Subtract the first equation of the semi-discrete scheme (13) from the first equation of the fully discrete scheme (20) to obtain

$$\begin{aligned} (\bar{\partial}_F^\alpha \eta_h^{n-\sigma}, w_h) + (\nabla \eta_h^{n,\sigma}, \nabla w_h) &= (U^{n,\sigma} \nabla V^{n,\sigma} - U_h^{n,\sigma} \nabla V_h^{n,\sigma}, \nabla w_h) - (\bar{\partial}_F^\alpha \theta_h^{n-\sigma}, w_h) \\ &\leq \frac{M}{\varepsilon} \|U^{n,\sigma} - U_h^{n,\sigma}\|^2 + \varepsilon \|\nabla w_h\|^2 + \|\bar{\partial}_F^\alpha \theta_h^{n-\sigma}\| \|w_h\| \\ &\leq \frac{M}{\varepsilon} (\|\eta_h^{n,\sigma}\|^2 + \|\theta_h^{n,\sigma}\|^2) + \varepsilon \|\nabla w_h\|^2 + \|\bar{\partial}_F^\alpha \theta_h^{n-\sigma}\| \|w_h\|. \end{aligned} \quad (35)$$

By the definition (8), it follows from (9), (10) and (21) that

$$\begin{aligned}
\|\bar{\partial}_F^\alpha \theta_h^{n-\sigma}\| &\leq \left\| B_0^{(n)} \theta_h^n + \sum_{i=1}^{n-1} (B_{n-i-1}^{(n)} - B_{n-i}^{(n)}) \theta_h^i - B_{n-1}^{(n)} \theta_h^0 \right\| \\
&\leq B_0^{(n)} \|\theta_h^n\| + \sum_{i=1}^{n-1} (B_{n-i-1}^{(n)} - B_{n-i}^{(n)}) \|\theta_h^i\| + B_{n-1}^{(n)} \|\theta_h^0\| \\
&\leq \left(B_0^{(n)} + \sum_{i=1}^{n-1} (B_{n-i-1}^{(n)} - B_{n-i}^{(n)}) + B_{n-1}^{(n)} \right) Ch^{k+1} \\
&= 2B_0^{(n)} Ch^{k+1} \leq \frac{4\tau_n^{-\alpha}}{\Gamma(2-\alpha)} Ch^{k+1}.
\end{aligned} \tag{36}$$

Taking $w_h = \eta_h^{n,\sigma}$ in (35) and by (25), we have

$$\begin{aligned}
\bar{\partial}_F^\alpha \|\eta_h^{n-\sigma}\|^2 &\leq \frac{2M}{\varepsilon} \|\eta_h^{n,\sigma}\|^2 + \frac{2M}{\varepsilon} Ch^{2(k+1)} + 4B_0^{(n)} Ch^{k+1} \|\eta_h^{n,\sigma}\| \\
&\leq \left(1 + \frac{2M}{\varepsilon} \right) \|\eta_h^{n,\sigma}\|^2 + \left(\frac{2M}{\varepsilon} + 2B_0^{(n)} \right) Ch^{2(k+1)}.
\end{aligned} \tag{37}$$

Due to $\eta_h^0 = 0$, by Lemma 5 and (24), we have

$$\|\eta_h^n\| \leq E_\alpha \left(2 \left(1 + \frac{2M}{\varepsilon} \right) t_n^\alpha \right) \left(\frac{2M}{\varepsilon} + 2B_0^{(n)} \right) Ch^{k+1}.$$

Subtract the second equation of the semi-discrete scheme (13) from the second equation of the fully discrete scheme (20) to yield

$$(\bar{\partial}_F^\alpha \beta_h^{n-\sigma}, w_h) + (\nabla \beta_h^{n,\sigma}, \nabla w_h) = -(\beta_h^{n,\sigma} + \rho_h^{n,\sigma}, w_h) - (\bar{\partial}_F^\alpha \rho_h^{n-\sigma}, w_h) + (\theta_h^{n,\sigma} + \eta_h^{n,\sigma}, w_h). \tag{38}$$

Let $w_h = \beta_h^{n,\sigma}$ and apply Young's inequality and (25) to get

$$\begin{aligned}
\bar{\partial}_F^\alpha \|\beta_h^{n-\sigma}\|^2 &\leq 2 \left(\frac{3}{2} \|\beta_h^{n,\sigma}\|^2 + \|\rho_h^{n,\sigma}\| \|\beta_h^{n,\sigma}\| + \|\bar{\partial}_F^\alpha \rho_h^{n-\sigma}\| \|\beta_h^{n,\sigma}\| + \|\theta_h^{n,\sigma}\| \|\beta_h^{n,\sigma}\| \right. \\
&\quad \left. + \frac{1}{2} \|\eta_h^{n,\sigma}\|^2 \right).
\end{aligned}$$

Similar to (36), we have

$$\|\bar{\partial}_F^\alpha \rho_h^{n-\sigma}\| \leq 2B_0^{(n)} Ch^{k+1}.$$

Therefore, we obtain that

$$\bar{\partial}_F^\alpha \|\beta_h^{n-\sigma}\|^2 \leq 3\|\beta_h^{n,\sigma}\|^2 + 2(2 + 2B_0^{(n)}) Ch^{k+1} \|\beta_h^{n,\sigma}\| + \|\eta_h^{n,\sigma}\|^2.$$

Due to $\beta_h^0 = 0$, by Lemma 5 and (24), we have

$$\begin{aligned}
\|\beta_h^n\| &\leq E_\alpha(6t_n^\alpha) \left(\|\beta_h^0\| + \max_{1 \leq i \leq n} \sum_{j=1}^i P_{i-j}^{(i)} \left(2(2 + 2B_0^{(n)}) Ch^{k+1} + \|\eta_h^{j,\sigma}\| \right) + \max_{1 \leq j \leq n} \|\eta_h^{j,\sigma}\| \right) \\
&\leq E_\alpha(6t_n^\alpha) \left(\max_{1 \leq i \leq n} \left\{ \frac{4(2 + 2B_0^{(n)}) t_i^\alpha}{\Gamma(1+\alpha)} Ch^{k+1} + \frac{2t_i^\alpha}{\Gamma(1+\alpha)} \max_{1 \leq j \leq i} \|\eta_h^{j,\sigma}\| \right\} + \max_{1 \leq j \leq n} \|\eta_h^{j,\sigma}\| \right) \\
&\leq E_\alpha(6t_n^\alpha) \left(\frac{4(2 + 2B_0^{(n)}) t_n^\alpha}{\Gamma(1+\alpha)} Ch^{k+1} + \left(1 + \frac{2t_n^\alpha}{\Gamma(1+\alpha)} \right) \max_{1 \leq j \leq n} \|\eta_h^{j,\sigma}\| \right) \\
&\leq C_1 \left(h^{k+1} + \max_{1 \leq j \leq n} \|\eta_h^{j,\sigma}\| \right),
\end{aligned} \tag{39}$$

where

$$C_1 = \max \left\{ E_\alpha(6t_n^\alpha) \frac{4(2 + 2B_0^{(n)})t_n^\alpha}{\Gamma(1 + \alpha)} C, E_\alpha(6t_n^\alpha) \left(1 + \frac{2t_n^\alpha}{\Gamma(1 + \alpha)} \right) \right\}.$$

Thus, by using the triangle inequality, (4) and the interpolation error (19), we can obtain (33).

Taking $w_h = \bar{\partial}_F^\alpha \eta_h^{n-\sigma}$ in the first equation of the semi-discrete scheme (13) and the fully discrete scheme (20), and subtracting the two equations to obtain

$$\begin{aligned} & (\bar{\partial}_F^\alpha \eta_h^{n-\sigma}, \bar{\partial}_F^\alpha \eta_h^{n-\sigma}) + (\nabla \eta_h^{n,\sigma}, \bar{\partial}_F^\alpha \nabla \eta_h^{n-\sigma}) \\ &= (U_h^{n,\sigma} \nabla V_h^{n,\sigma} - U_h^{n,\sigma} \nabla V_h^{n,\sigma}, \bar{\partial}_F^\alpha \nabla \eta_h^{n-\sigma}) - (\bar{\partial}_F^\alpha \theta_h^{n-\sigma}, \bar{\partial}_F^\alpha \eta_h^{n-\sigma}) \\ &\leq M(\|\eta_h^{n,\sigma}\| + \|\theta_h^{n,\sigma}\|) \|\bar{\partial}_F^\alpha \nabla \eta_h^{n-\sigma}\| + \|\bar{\partial}_F^\alpha \theta_h^{n-\sigma}\| \|\bar{\partial}_F^\alpha \eta_h^{n-\sigma}\| \\ &\leq \frac{M}{\varepsilon} (\|\eta_h^{n,\sigma}\|^2 + \|\theta_h^{n,\sigma}\|^2) + \varepsilon \|\bar{\partial}_F^\alpha \nabla \eta_h^{n-\sigma}\|^2 + \frac{1}{\varepsilon} \|\bar{\partial}_F^\alpha \theta_h^{n-\sigma}\|^2 + \varepsilon \|\bar{\partial}_F^\alpha \eta_h^{n-\sigma}\|^2. \end{aligned} \quad (40)$$

Here, we set the parameter ε to be sufficient small. Also, we have

$$\|\bar{\partial}_F^\alpha \theta_h^{n-\sigma}\| \leq 2B_0^{(n)} Ch^{k+1}.$$

Thus, by the Poincaré inequality and (25), we obtain

$$\bar{\partial}_F^\alpha \|\nabla \eta_h^{n-\sigma}\|^2 \leq \frac{2M}{\varepsilon} \|\nabla \eta_h^{n,\sigma}\|^2 + \left(\frac{2M}{\varepsilon} + \frac{4B_0^{(n)}}{\varepsilon} \right) Ch^{2(k+1)}.$$

By using the fractional Grönwall inequality, we have

$$\|\nabla \eta_h^n\| \leq E_\alpha \left(\frac{4M}{\varepsilon} t_n^\alpha \right) \left(\frac{2M}{\varepsilon} + \frac{4B_0^{(n)}}{\varepsilon} \right) Ch^{k+1}. \quad (41)$$

Taking $w_h = -\Delta \beta_h^{n,\sigma}$ in Equation (38) and using Young's inequality and interpolation estimate (19), we get

$$\begin{aligned} & (\bar{\partial}_F^\alpha \nabla \beta_h^{n-\sigma}, \nabla \beta_h^{n,\sigma}) \leq \|\nabla \beta_h^{n,\sigma}\|^2 + \|\nabla \rho_h^{n,\sigma}\| \|\nabla \beta_h^{n,\sigma}\| + \|\bar{\partial}_F^\alpha \nabla \rho_h^{n,\sigma}\| \|\nabla \beta_h^{n,\sigma}\| \\ & \quad + \|\nabla \theta_h^{n,\sigma}\| \|\nabla \beta_h^{n,\sigma}\| + \|\nabla \eta_h^{n,\sigma}\| \|\nabla \beta_h^{n,\sigma}\| \\ & \leq \frac{3}{2} \|\nabla \beta_h^{n,\sigma}\|^2 + (2Ch^k + \|\bar{\partial}_F^\alpha \nabla \rho_h^{n,\sigma}\|) \|\nabla \beta_h^{n,\sigma}\| + \frac{1}{2} \|\nabla \eta_h^{n,\sigma}\|^2. \end{aligned}$$

Also, $\|\bar{\partial}_F^\alpha \nabla \rho_h^{n,\sigma}\| \leq 2B_0^{(n)} Ch^k$. Thus, similar to the deduction of (39), it can be concluded that

$$\|\nabla \beta_h^n\| \leq C_1 \left(h^k + \max_{1 \leq j \leq n} \|\nabla \eta_h^{j,\sigma}\| \right).$$

Finally, the triangle inequality and (41) can be used to prove (34). $\square$

Assume that

$$|\partial_t^l u(t)| \leq C_T(1 + t^{\alpha-l}) \quad \text{for } l = 0, 1, 2, 3.$$

$$|\partial_t^l v(t)| \leq C_T(1 + t^{\alpha-l}) \quad \text{for } l = 0, 1, 2, 3.$$

Next, we shall introduce the error estimates of the fully discrete scheme (20) in both $L^2(\Omega)$ and $H^1(\Omega)$ norms.

Theorem 3. Assume that u, v are the solution of the weak form (3). Let U_h^n, V_h^n be the solution of the fully discrete scheme (20). Then, for $n = 0, 1, 2, \dots, N$, the following error estimates hold:

$$\|u(t_n) - U_h^n\| + \|v(t_n) - V_h^n\| \leq C \left(N^{-\min\{2, r\alpha\}} + h^{k+1} + \epsilon \right). \quad (42)$$

$$\|u(t_n) - U_h^n\|_1 + \|v(t_n) - V_h^n\|_1 \leq C \left(N^{-\min\{2, r\alpha\}} + h^k + \epsilon \right). \quad (43)$$

where C is an α -robust constant.

Proof. Define

$$\begin{aligned} E_u^{n,\sigma} &= u^{n,\sigma} - U^{n,\sigma}, & E_v^{n,\sigma} &= v^{n,\sigma} - V^{n,\sigma}. \\ E_u^{n-\sigma} &= u^{n-\sigma} - U^{n-\sigma}, & E_v^{n-\sigma} &= v^{n-\sigma} - V^{n-\sigma}. \end{aligned}$$

Subtract the second equation of the semi-discrete scheme (13) from the second equation of the weak form (3) to yield

$$(\partial_t^\alpha v^{n-\sigma} - \bar{\partial}_F^\alpha V^{n-\sigma}, w) + (\nabla v^{n-\sigma} - \nabla V^{n,\sigma}, \nabla w) + (v^{n-\sigma} - V^{n,\sigma}, w) = (u^{n-\sigma} - U^{n,\sigma}, w). \quad (44)$$

Due to

$$\begin{aligned} \partial_t^\alpha v^{n-\sigma} - \bar{\partial}_F^\alpha V^{n-\sigma} &= \partial_t^\alpha v^{n-\sigma} - \bar{\partial}_F^\alpha v^{n-\sigma} + \bar{\partial}_F^\alpha v^{n-\sigma} - \bar{\partial}_F^\alpha V^{n-\sigma}. \\ \nabla v^{n-\sigma} - \nabla V^{n,\sigma} &= \nabla v^{n-\sigma} - \nabla v^{n,\sigma} + \nabla v^{n,\sigma} - \nabla V^{n,\sigma}. \\ v^{n-\sigma} - V^{n,\sigma} &= v^{n-\sigma} - v^{n,\sigma} + v^{n,\sigma} - V^{n,\sigma}. \\ u^{n-\sigma} - U^{n,\sigma} &= u^{n-\sigma} - u^{n,\sigma} + u^{n,\sigma} - U^{n,\sigma}. \end{aligned}$$

Taking $w_h = E_v^{n,\sigma}$, we can obtain

$$\begin{aligned} &(\bar{\partial}_F^\alpha E_v^{n-\sigma}, E_v^{n,\sigma}) + (\nabla E_v^{n,\sigma}, \nabla E_v^{n,\sigma}) + (E_v^{n,\sigma}, E_v^{n,\sigma}) \\ &= -(\partial_t^\alpha v^{n-\sigma} - \bar{\partial}_F^\alpha v^{n-\sigma}, E_v^{n,\sigma}) + (\Delta(v^{n-\sigma} - v^{n,\sigma}), E_v^{n,\sigma}) \\ &\quad - (v^{n-\sigma} - v^{n,\sigma}, E_v^{n,\sigma}) + (u^{n-\sigma} - u^{n,\sigma}, E_v^{n,\sigma}) + (E_u^{n,\sigma}, E_v^{n,\sigma}). \end{aligned} \quad (45)$$

According to the truncation errors (11) and (12), it follows from (25) that

$$\begin{aligned} \bar{\partial}_F^\alpha \|E_v^{n-\sigma}\|^2 &\leq 2 \left(3C t_{n-\sigma}^{-\alpha} N^{-\min\{2, r\alpha\}} + C t_{n-\sigma}^{-\alpha} N^{-\min\{3-\alpha, r\alpha\}} + \epsilon \right) \|E_v^{n,\sigma}\| \\ &\quad + \|E_v^{n,\sigma}\|^2 + \|E_u^{n,\sigma}\|^2 \\ &\leq \|E_v^{n,\sigma}\|^2 + \left(8C t_{n-\sigma}^{-\alpha} N^{-\min\{2, r\alpha\}} + \epsilon \right) \|E_v^{n,\sigma}\| + \|E_u^{n,\sigma}\|^2. \end{aligned}$$

Thus, by the Lemma 5, (23) and $\|E_v^0\| = 0$, we can obtain

$$\begin{aligned} \|E_v^n\| &\leq E_\alpha(2t_n^\alpha) \left[\max_{1 \leq k \leq n} \sum_{j=1}^k P_{k-j}^{(k)} \left(8C_T t_{j-\sigma}^{-\alpha} N^{-\min\{2, r\alpha\}} + \epsilon + \|E_u^{j,\sigma}\| \right) \right. \\ &\quad \left. + \max_{1 \leq j \leq n} \|E_u^{j,\sigma}\| \right] \\ &\leq 8C_T E_\alpha(2t_n^\alpha) \frac{2^{1+r\alpha} T^{\alpha-l_N} e^r t_N^{l_N} \Gamma(1+l_N-\alpha)}{\Gamma(1+l_N)} N^{-\min\{2, r\alpha\}} + \epsilon \\ &\quad + E_\alpha(2t_n^\alpha) \left(\frac{2t_n^\alpha}{\Gamma(1+\alpha)} + 1 \right) \max_{1 \leq j \leq n} \|E_u^{j,\sigma}\| \\ &\leq C_2 \left(N^{-\min\{2, r\alpha\}} + \epsilon + \max_{1 \leq j \leq n} \|E_u^{j,\sigma}\| \right). \end{aligned} \quad (46)$$

Here

$$C_2 = \max \left\{ 8C_T E_\alpha (2t_n^\alpha) \frac{2^{1+r\alpha} T^{\alpha-l_N} e^r t_N^{l_N} \Gamma(1+l_N-\alpha)}{\Gamma(1+l_N)}, E_\alpha (2t_n^\alpha) \left(\frac{2t_n^\alpha}{\Gamma(1+\alpha)} + 1 \right) \right\}.$$

Subtracting the first equation of the semi-discrete scheme (13) from the second equation of the weak form (3) and shifting the equation terms to obtain

$$\begin{aligned} (\bar{\partial}_F^\alpha E_u^{n-\sigma}, w) + (\nabla E_u^{n,\sigma}, w) &= (u^{n-\sigma} \nabla v^{n-\sigma} - U^{n,\sigma} \nabla V^{n,\sigma}, w) + (\bar{\partial}_F^\alpha u^{n-\sigma} - \partial_t^\alpha u^{n-\sigma}, w) \\ &\quad + (\nabla(u^{n,\sigma} - u^{n-\sigma}), w) \\ &\leq M \left(Ct_{n-\sigma}^{-\alpha} N^{-\min\{2,r\alpha\}} + \|E_u^{n,\sigma}\| \right) \|w\| + (Ct_{n-\sigma}^{-\alpha} N^{-\min\{3-\alpha,r\alpha\}} + \epsilon) \|w\| \\ &\quad + Ct_{n-\sigma}^{-\alpha} N^{-\min\{2,r\alpha\}} \|w\| \\ &\leq M \|E_u^{n,\sigma}\| \|w\| + (M+2)(Ct_{n-\sigma}^{-\alpha} N^{-\min\{2,r\alpha\}} + \epsilon) \|w\|. \end{aligned} \quad (47)$$

Taking $w = E_u^{n,\sigma}$ and by (25), we obtain

$$\bar{\partial}_F^\alpha \|E_u^{n-\sigma}\|^2 \leq 2M \|E_u^{n,\sigma}\|^2 + (M+2)(Ct_{n-\sigma}^{-\alpha} N^{-\min\{2,r\alpha\}} + \epsilon) \|E_u^{n,\sigma}\|.$$

Then, by the fractional Grönwall inequality, we have

$$\|E_u^n\| \leq E_\alpha(4Mt_n^\alpha) \left(\frac{2t_n^\alpha}{\Gamma(1+\alpha)} + 1 \right) (M+2) (Ct_{n-\sigma}^{-\alpha} N^{-\min\{2,r\alpha\}} + \epsilon).$$

Finally, the conclusion (42) can be obtained by utilizing the triangular inequality, (46) and (33).

Next, let $w_h = -\Delta E_v^{n,\sigma}$ in Equation (45) and taking $w_h = \bar{\partial}_F^\alpha E_u^{n-\sigma}$ in Equation (47). We adopt the proof in (42), and we can obtain

$$\|\nabla E_u^n\| \leq C(N^{-\min\{2,r\alpha\}} + \epsilon).$$

and

$$\|\nabla E_v^n\| \leq C(N^{-\min\{2,r\alpha\}} + \epsilon).$$

As a result, we can obtain (43) by combining (34) with the triangle inequality. $\square$

5. Numerical Experiment

In this section, we shall provide some numerical examples to illustrate the effectiveness of our numerical scheme and the reliability of our theoretical results. To derive the numerical solution, we use the fast L2-1 σ scheme to approximate the time Caputo fractional derivative and apply the linear finite element to discretize spatial direction. Moreover, we choose the tolerance error $\epsilon = 10^{-12}$ and the cut-off time $\hat{\tau} = 10^{-12}$ to speed up the convolution computation of the L2-1 σ scheme. At each time level, the nonlinear algebraic systems is solved by using a fixed-point algorithms with the termination error 10^{-12} . For simplicity, we define

$$\begin{aligned} \|E_u\| &= \max_{1 \leq n \leq N} \|u(t_n) - U_h^n\|, \quad \|E_u\|_1 = \max_{1 \leq n \leq N} \|u(t_n) - U_h^n\|_1 \\ \|E_v\| &= \max_{1 \leq n \leq N} \|v(t_n) - V_h^n\|, \quad \|E_v\|_1 = \max_{1 \leq n \leq N} \|v(t_n) - V_h^n\|_1 \end{aligned}$$

Let us consider the following time fractional order Keller–Segel problem.

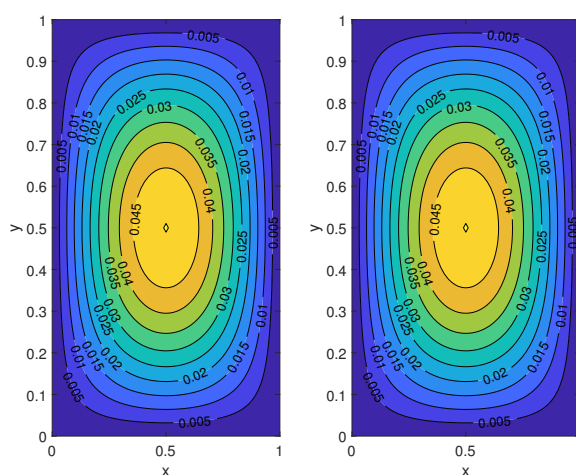
$$\begin{cases} \partial_t^\alpha u - \Delta u + \nabla \cdot (u \nabla v) = f, & x \in \Omega, 0 < t < T, \\ \partial_t^\alpha v - \Delta v + v - u = g, & x \in \Omega, 0 < t < T. \end{cases}$$

Thus, the right-hand sides, f and g are determined from the choice for u and v .

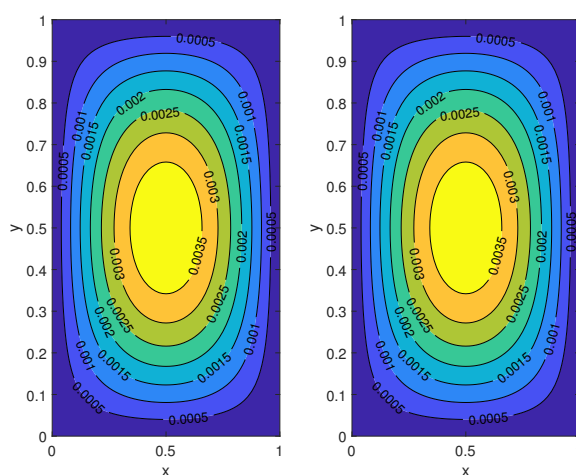
Example 1. Assume that the domain $\Omega = (0,1)^2$, we then consider the smooth solutions are given by

$$u = 4\pi(t^2 + t^\alpha) \sin(\pi x) \sin(\pi y), \quad \text{and} \quad v = (t^2 + t^\alpha) \sin(\pi x) \sin(\pi y).$$

Taking the final time $T = 110^{-3}$ and the time mesh parameter $r = 6/\alpha$, the numerical solution and exact solution are shown in Figure 1, which are consistent. Here, we mainly consider the situation of $\alpha \rightarrow 1^-$. In Table 1, the spatial error convergence orders of u, v are approximately 2 and 1 in the $L^2(\Omega)$ norm and the $H^1(\Omega)$ norm, respectively, which is consistent with Theorem 2. In addition, we provide the time error results of u, v in the $L^2(\Omega)$ norm in Table 2. From the results, it can be seen that the error convergence order is close to 2, which is consistent with Theorem 3.



(a) The left side is the numerical solution U_h^n and the right side is the exact solution u^n for $t = T$.



(b) The left side is the numerical solution V_h^n and the right side is the exact solution v^n for $t = T$.

Figure 1. The comparison figures of the exact solution and the numerical solution with $N = 100$, $h = \frac{1}{64}$, $\alpha = 0.8$ for Example 1.

Table 1. The spatial errors and convergence orders with the time steps $N = 100$ for Example 1.

α	h	$\ E_u\ $	Rate	$\ E_u\ _1$	Rate	$\ E_v\ $	Rate	$\ E_v\ _1$	Rate
0.8	1/4	1.3527×10^{-3}	–	4.5514×10^{-2}	–	1.0776×10^{-4}	–	3.6208×10^{-3}	–
	1/8	2.6193×10^{-4}	2.3686	2.2136×10^{-2}	1.0399	2.0876×10^{-5}	2.3679	1.7614×10^{-3}	1.0396
	1/16	5.9070×10^{-5}	2.1487	1.0954×10^{-2}	1.0149	4.6869×10^{-6}	2.1551	8.7172×10^{-4}	1.0148
	1/32	1.4979×10^{-5}	1.9795	5.4615×10^{-3}	1.0041	1.1450×10^{-6}	2.0333	4.3463×10^{-4}	1.0041
0.9	1/4	6.6022×10^{-4}	–	2.3059×10^{-2}	–	5.2545×10^{-5}	–	1.8348×10^{-3}	–
	1/8	1.2279×10^{-4}	2.4267	1.1145×10^{-2}	1.0490	9.7748×10^{-6}	2.4264	8.8684×10^{-4}	1.0489
	1/16	2.6402×10^{-5}	2.2175	5.4992×10^{-3}	1.0191	2.1014×10^{-6}	2.2177	4.3761×10^{-4}	1.0190
	1/32	6.3036×10^{-6}	2.0664	2.7391×10^{-3}	1.0055	4.9967×10^{-7}	2.0723	2.1797×10^{-4}	1.0055
0.95	1/4	4.6514×10^{-4}	–	1.6385×10^{-2}	–	3.7016×10^{-5}	–	1.3038×10^{-3}	–
	1/8	8.5618×10^{-5}	2.4417	7.9053×10^{-3}	1.0515	6.8140×10^{-6}	2.4416	6.2907×10^{-4}	1.0514
	1/16	1.8064×10^{-5}	2.2448	3.8966×10^{-3}	1.0206	1.4378×10^{-6}	2.2447	3.1008×10^{-4}	1.0206
	1/32	4.2409×10^{-6}	2.0907	1.9399×10^{-3}	1.0062	3.3717×10^{-7}	2.0923	1.5437×10^{-4}	1.0062

Table 2. The time errors and convergence orders with the spatial mesh size $h = \frac{1}{100}$ for Example 1.

α	N	$\ E_u\ $	Rate	$\ E_v\ $	Rate
0.8	4	4.6535×10^{-4}	–	4.0864×10^{-5}	–
	8	1.4175×10^{-4}	1.7149	1.2460×10^{-5}	1.7135
	16	3.7849×10^{-5}	1.9051	3.2314×10^{-6}	1.9471
	32	1.0781×10^{-5}	1.8118	7.9979×10^{-7}	2.0145
0.9	4	1.2799×10^{-4}	–	1.0654×10^{-5}	–
	8	3.8748×10^{-5}	1.7238	3.2209×10^{-6}	1.7258
	16	1.0290×10^{-5}	1.9129	8.4453×10^{-7}	1.9312
	32	2.7964×10^{-6}	1.8796	2.1743×10^{-7}	1.9576
0.95	4	4.6330×10^{-5}	–	3.7983×10^{-6}	–
	8	1.3992×10^{-5}	1.7274	1.1443×10^{-6}	1.7309
	16	3.7566×10^{-6}	1.8971	3.0351×10^{-7}	1.9147
	32	1.0897×10^{-6}	1.7855	8.4279×10^{-8}	1.8485

Finally, we provide a computing time comparison between the $L2-1_\sigma$ scheme and the fast $L2-1_\sigma$ scheme in Table 3, and we can see that the fast algorithm requires significantly less computation time.

Example 2. Assume that the domain $\Omega = (0, 1)^2$; we then consider the non-smooth solutions are given by

$$u = \begin{cases} 5(t^2 + t^\alpha)xy(x - 0.5)(y - 1), & \text{in } \Omega_1, \\ (t^2 + t^\alpha)(x - 0.5)(x - 1)y(y - 1), & \text{in } \Omega_2, \end{cases}$$

and

$$v = \begin{cases} (t^2 + t^\alpha)xy(x - 0.5)(y - 1), & \text{in } \Omega_1, \\ 0.2(t^2 + t^\alpha)(x - 0.5)(x - 1)y(y - 1), & \text{in } \Omega_2, \end{cases}$$

where $\Omega_1 = [0, 0.5] \times [0, 1]$ and $\Omega_2 = [0.5, 1] \times [0, 1]$.

Table 3. The computation time of the L2-1 $_{\sigma}$ /FEM and the fast L2-1 $_{\sigma}$ /FEM scheme with $h = 1/8$, $\alpha = 0.8$ for Example 1.

N	800	1000	1200	1400	1600
L2-1 $_{\sigma}$ /FEM scheme	134.128 s	208.3286 s	338.0771 s	673.9065 s	900.8299 s
Fast L2-1 $_{\sigma}$ /FEM scheme	94.9092 s	167.8129 s	222.4437 s	310.5037 s	328.4481 s

In the same way, we take the final time $T = 1 \times 10^{-3}$ and the time mesh parameter $r = 6/\alpha$. From Figure 2, it can be seen that the numerical solution is consistent with the exact solution. Based on the results in Table 4, we can also achieve the convergence orders of the second order and the first order in the $L^2(\Omega)$ norm and $H^1(\Omega)$ norm for the case of the non-smooth solutions. By comparing the data in Tables 2, 5, and 6, it can be concluded that for the case of the non-smooth solutions, a finer spatial mesh discretization is required to achieve a temporal convergence order close to 2.

Table 4. The spatial errors and convergence orders with the time steps $N = 100$ for Example 2.

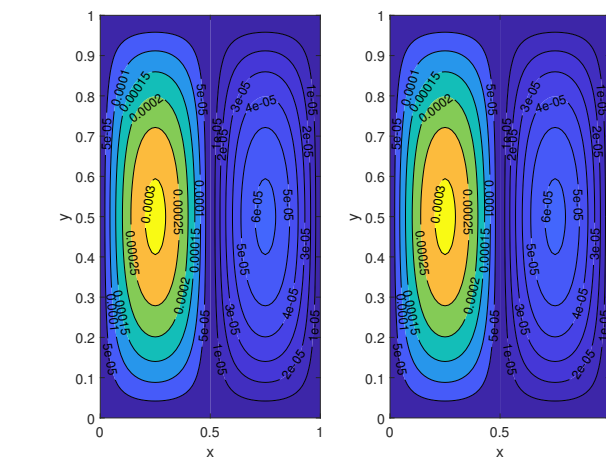
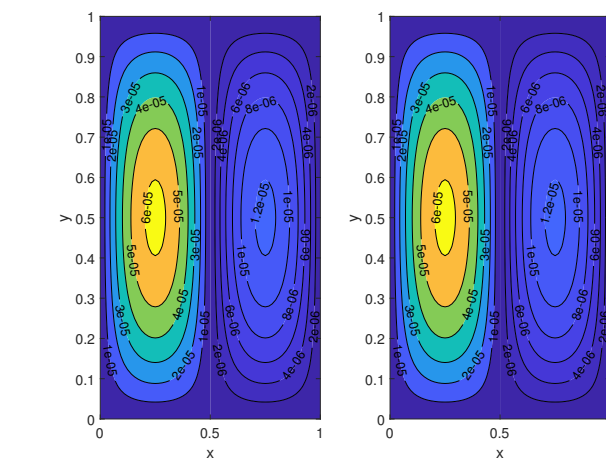
α	h	$\ E_u\ $	Rate	$\ E_u\ _1$	Rate	$\ E_v\ $	Rate	$\ E_v\ _1$	Rate
0.8	1/4	2.6299×10^{-5}	–	6.3752×10^{-4}	–	5.2618×10^{-6}	–	1.2748×10^{-4}	–
	1/8	5.2585×10^{-6}	2.3223	3.1209×10^{-4}	1.0305	1.0528×10^{-6}	2.3213	6.2412×10^{-5}	1.0303
	1/16	1.1810×10^{-6}	2.1547	1.5406×10^{-4}	1.0185	2.3662×10^{-7}	2.1536	3.0811×10^{-5}	1.0184
	1/32	2.8548×10^{-7}	2.0485	7.6730×10^{-5}	1.0056	5.7217×10^{-8}	2.0481	1.5346×10^{-5}	1.0056
0.9	1/4	1.2896×10^{-5}	–	3.2526×10^{-4}	–	2.5793×10^{-6}	–	6.5049×10^{-5}	–
	1/8	2.4591×10^{-6}	2.3907	1.5789×10^{-4}	1.0427	4.9192×10^{-7}	2.3905	3.1577×10^{-5}	1.0427
	1/16	5.2117×10^{-7}	2.2383	7.7482×10^{-5}	1.0270	1.0429×10^{-7}	2.2379	1.5496×10^{-5}	1.0269
	1/32	1.2231×10^{-7}	2.0912	3.8500×10^{-5}	1.0090	2.4481×10^{-8}	2.0908	7.7001×10^{-6}	1.0090
0.95	1/4	9.0968×10^{-6}	–	2.3155×10^{-4}	–	1.8194×10^{-6}	–	4.6309×10^{-5}	–
	1/8	1.7121×10^{-6}	2.4096	1.1223×10^{-4}	1.0449	3.4244×10^{-7}	2.4095	2.2446×10^{-5}	1.0448
	1/16	3.5306×10^{-7}	2.2778	5.4960×10^{-5}	1.0301	7.0629×10^{-8}	2.2775	1.0992×10^{-5}	1.0300
	1/32	8.1251×10^{-8}	2.1195	2.7276×10^{-5}	1.0107	1.6257×10^{-8}	2.1192	5.4552×10^{-6}	1.0107

Table 5. The time errors and convergence orders with the spatial mesh size $h = \frac{1}{100}$ for Example 2.

α	N	$\ E_u\ $	Rate	$\ E_v\ $	Rate
0.8	4	1.9447×10^{-6}	–	4.0582×10^{-7}	–
	8	5.7268×10^{-7}	1.7637	1.2009×10^{-7}	1.7567
	16	1.5138×10^{-7}	1.9195	3.1711×10^{-8}	1.9211
	32	4.9146×10^{-8}	1.6231	1.0095×10^{-8}	1.6513
0.9	4	6.3564×10^{-7}	–	1.2934×10^{-7}	–
	8	1.9213×10^{-7}	1.7261	3.9116×10^{-8}	1.7253
	16	5.2613×10^{-8}	1.8686	1.0696×10^{-8}	1.8707
	32	1.8532×10^{-8}	1.5054	3.7373×10^{-9}	1.5170
0.95	4	2.4057×10^{-7}	–	4.8654×10^{-8}	–
	8	7.3227×10^{-8}	1.7160	1.4809×10^{-8}	1.7161
	16	2.1337×10^{-8}	1.7790	4.3061×10^{-9}	1.7820
	32	9.8102×10^{-9}	1.1210	1.9678×10^{-9}	1.1298

Table 6. The time errors and convergence orders with the spatial mesh size $h = \frac{1}{200}$ for Example 2.

α	N	$\ E_u\ $	Rate	$\ E_v\ $	Rate
0.8	4	1.9387×10^{-6}	–	4.046×10^{-7}	–
	8	5.6607×10^{-7}	1.7760	1.1876×10^{-7}	1.7685
	16	1.4301×10^{-7}	1.9849	3.0036×10^{-8}	1.9832
	32	3.539×10^{-8}	2.0147	7.3613×10^{-9}	2.0287
0.9	4	6.3406×10^{-7}	–	1.2902×10^{-7}	–
	8	1.9028×10^{-7}	1.7365	3.8746×10^{-8}	1.7355
	16	4.9802×10^{-8}	1.9338	1.0136×10^{-8}	1.9345
	32	1.2933×10^{-8}	1.9452	2.6221×10^{-9}	1.9508
0.95	4	2.3972×10^{-7}	–	4.8484×10^{-8}	–
	8	7.2087×10^{-8}	1.7335	1.4581×10^{-8}	1.7334
	16	1.9153×10^{-8}	1.9122	3.8711×10^{-9}	1.9133
	32	5.3333×10^{-9}	1.8444	1.0739×10^{-9}	1.8499

(a) The left side is the numerical solution U_h^n and the right side is the exact solution u^n .(b) The left side is the numerical solution V_h^n and the right side is the exact solution v^n .**Figure 2.** The comparison figures of the exact solution and the numerical solution for $N = 100$, $h = \frac{1}{64}$, $\alpha = 0.8$ for Example 2.

6. Conclusions

This work presents a fast L2-1 $_{\sigma}$ /FEM scheme for solving the time fractional Keller–Segel equations with weakly singular solutions. Based on an improved fractional Grönwall inequality, we establish stability and the α robust error estimates for the numerical formats. Finally, some numerical examples are presented to illustrate the reliability of our algorithms and the correctness of our theoretical results. However, there are still some minor flaws here, such as the small final time of our numerical simulation. For this problem, we will further study the blow-up of the numerical solution to derive a long-term numerical simulation.

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