



Article

Hermite–Hadamard Inequalities for a New Riemann–Liouville-Type Operator

Rubayyi T. Alqahtani ¹, Mehmet Zeki Sarikaya ^{2,*}  and Nadiyah Hussain Alharthi ¹ 

¹ Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia; nhalharthi@imamu.edu.sa (N.H.A.)

² Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce 81620, Turkey

* Correspondence: mzekisarikaya@duzce.edu.tr or sarikayamz@gmail.com

Abstract

We introduce a new class of fractional remainder operators, denoted by $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$, which generalize and unify various classical integral identities. Using these operators, we formulate refined Hermite–Hadamard and trapezoidal inequalities for differentiable functions. The novelty of our approach lies in its symmetric kernel structure, which facilitates tighter error bounds. Numerical examples are given, including applications to non-differentiable convex functions, to demonstrate the applicability and limitations of the derived results.

Keywords: Hermite–Hadamard inequality; Riemann–Liouville and Caputo-type fractional integrals; convex functions; L^p integral inequalities

MSC: 26A51; 26B25; 26D10; 26D15

1. Introduction

The study of integral inequalities plays a fundamental role in mathematical analysis, numerical integration, and approximation theory. Among classical results, the well-known Hermite–Hadamard (HH) inequality occupies a central position in convex analysis, as comprehensively presented in the monograph by Dragomir and Pearce [1]. For a convex function $\varphi : [\xi, \eta] \rightarrow \mathbb{R}$, the inequality states that

$$\varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{1}{\eta - \xi} \int_{\xi}^{\eta} \varphi(t) dt \leq \frac{\varphi(\xi) + \varphi(\eta)}{2}. \quad (1)$$

A closely related result is the trapezoidal inequality, which provides limits for the error that arises in approximating definite integrals. Furthermore, the work of Dragomir and Agarwal [2] on inequalities for differentiable mappings laid the groundwork for applications to special averages and refined trapezoidal formulas.

Over the last two decades, a vibrant field of research has emerged, focusing on extending classical inequalities through the powerful framework of fractional calculus, particularly via Riemann–Liouville and Caputo fractional integrals. The fundamental theory of these operators has been extensively documented in pioneering works such as Kilbas et al. [3], Podlubny [4], and Samko et al. [5]. Due to their inherently non-local nature, fractional operators have enabled the derivation of flexible and generalized versions of the HH inequality. In particular, Sarikaya et al. [6] have systematically presented fractional analogues of the HH inequality involving Riemann–Liouville integrals. This



Academic Editor: Ivanka Stamova

Received: 15 January 2026

Revised: 13 February 2026

Accepted: 23 February 2026

Published: 26 February 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

scope has been further extended to generalized HH-type inequalities [7], new Riemann–Liouville fractional integral inequalities [8], and various other fractional integral identities documented in the works of Farid [9], Jleli and Samet [10], and Lan [11]. Furthermore, several authors have contributed to this field through the study of new integral versions and generalized operators [12–15]. Recent investigations have also explored advanced HH-type inequalities in broader contexts, such as fuzzy fractional calculus and generalized pre-invex functions in fuzzy-interval-valued settings [16], demonstrating the adaptability of fractional techniques in complex analytical frameworks.

Despite this rich body of literature, many fractional integral identities involving remainder terms of quadrature formulas are often presented as isolated results. There remains a need for operator-based representations that unify classical remainder expressions, fractional integrals, and endpoint-based expansions within a single coherent structure. Most existing fractional operators primarily depend on the distance from a single endpoint (e.g., $x - \xi$ or $\eta - x$). In contrast, a unified formulation that scales with the total interval length $L = \eta - \xi$ offers a more flexible framework for measuring the discrepancy between function values and fractional-type integral means.

Motivated by these developments, we introduce a new class of operators, denoted by $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$. These operators bridge classical integral means and fractional structures by incorporating a symmetric Beta-type kernel of the form $(1 - t^2)^\beta$. The symmetric nature of this kernel is particularly suitable for establishing refined trapezoid-type bounds, as it aligns naturally with the symmetric error kernels appearing in classical quadrature formulas.

In this work, we construct several novel Hermite–Hadamard-type inequalities and trapezoidal-type bounds using the proposed operators. We show that our results not only generalize known inequalities but also provide a unified framework for various integral averages and re-obtain classical results as special bound cases.

2. Definition of the Operator

Let $[\xi, \eta]$ be an interval with length $L := \eta - \xi > 0$. To avoid notational confusion between the interval endpoints and the normalization constants, we introduce the following notation. For $\alpha > -\frac{1}{2}$ and $n \in \mathbb{N}_0$, define the parameter β and the normalization constant $\Lambda_{\alpha,n}$ by

$$\beta := \alpha + 2n - \frac{1}{2}, \quad \Lambda_{\alpha,n} := \frac{2\Gamma(\alpha + 2n + 1)}{\sqrt{\pi}\Gamma(\alpha + 2n + \frac{1}{2})}.$$

We introduce the left-sided operator (associated with the left endpoint ξ) by

$$R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) := \Lambda_{\alpha,n} L^{2n} \int_0^1 \varphi(\xi + Lt) (1 - t^2)^\beta dt, \quad (2)$$

and the right-sided operator (associated with the right endpoint η) by

$$R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta) := \Lambda_{\alpha,n} L^{2n} \int_0^1 \varphi(\eta - Lt) (1 - t^2)^\beta dt. \quad (3)$$

These operators extend the classical one-sided Riemann–Liouville fractional integrals by incorporating the full geometry of the interval $[\xi, \eta]$. In particular, instead of depending only on the distance from a single endpoint (such as $x - \xi$ or $\eta - x$), the kernels of $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$ are scaled with respect to the total length $L = \eta - \xi$. The symmetric Beta-type kernel $(1 - t^2)^\beta$ allows the operators to interpolate between classical fractional integrals and remainder terms of quadrature formulas, thereby providing a unified and flexible framework.

Properties

- **Linearity.** For any constants $c, d \in \mathbb{R}$ and functions φ, g ,

$$R_{\alpha,n}^{\xi+}(c\varphi + dg) = c R_{\alpha,n}^{\xi+}(\varphi) + d R_{\alpha,n}^{\xi+}(g),$$

$$R_{\alpha,n}^{\eta-}(c\varphi + dg) = c R_{\alpha,n}^{\eta-}(\varphi) + d R_{\alpha,n}^{\eta-}(g).$$

- **Positivity.** If $\varphi \geq 0$ on $[\xi, \eta]$, then

$$R_{\alpha,n}^{\xi+}(\varphi) \geq 0, \quad R_{\alpha,n}^{\eta-}(\varphi) \geq 0.$$

- **Normalization.** The constant $\Lambda_{\alpha,n}$ is chosen such that

$$\Lambda_{\alpha,n} \int_0^1 (1-t^2)^\beta dt = 1,$$

which guarantees that the operator represents a properly weighted average of φ .

Theorem 1. Let $[\xi, \eta]$ be an interval with length $L := \eta - \xi > 0$. For $\alpha > -\frac{1}{2}$ and $n \in \mathbb{N}_0$, the operators $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$ satisfy the following properties:

- (i). For every $\varphi \in C([\xi, \eta])$, the operator norm satisfies

$$\sup_{\|\varphi\|_\infty \leq 1} \left| R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) \right| = \sup_{\|\varphi\|_\infty \leq 1} \left| R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta) \right| = L^{2n}.$$

Equivalently, the normalized functionals

$$S_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) := \frac{1}{L^{2n}} R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta), \quad S_{\alpha,n}^{\eta-}(\varphi; \xi, \eta) := \frac{1}{L^{2n}} R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta)$$

are bounded linear functionals on $C([\xi, \eta])$ with norm 1.

- (ii). For every $\varphi \in C([\xi, \eta])$, the normalized operators recover the endpoint values on shrinking intervals:

$$\lim_{L \downarrow 0} S_{\alpha,n}^{\xi+}(\varphi; \xi, \xi + L) = \varphi(\xi), \quad \lim_{L \downarrow 0} S_{\alpha,n}^{\eta-}(\varphi; \eta - L, \eta) = \varphi(\eta).$$

- (iii). For the special choice $\alpha = \frac{1}{2}$ and $n = 0$, the normalized operators reduce to the standard Riemann integral mean:

$$S_{1/2,0}^{\xi+}(\varphi; \xi, \eta) = S_{1/2,0}^{\eta-}(\varphi; \xi, \eta) = \frac{1}{L} \int_\xi^\eta \varphi(\varkappa) d\varkappa.$$

3. Hermite–Hadamard Inequalities for $R_{\alpha,n}$

In this section, we establish generalized Hermite–Hadamard and trapezoid-type inequalities using the unified operators $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$. To facilitate clarity, Table 1 summarizes the main results and the conditions under which they hold.

Table 1. Summary of the main inequalities and their recovery of classical cases.

Result	Condition on φ	Limiting Case ($\alpha = \frac{1}{2}, n = 0$)
Theorem 2	φ convex	Classical Hermite–Hadamard inequality
Theorem 3	$ \varphi' $ convex	Standard trapezoid error bound
Theorem 4	$ \varphi' ^q$ convex ($q > 1$)	Refined trapezoid-type bounds

Theorem 2. Let $\varphi : [\xi, \eta] \rightarrow \mathbb{R}$ be a convex function. Then, for the unified operators $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$ defined in (2) and (3) with $L = \eta - \xi$, the following inequality holds:

$$\varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{1}{2L^{2n}} \left[R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta) \right] \leq \frac{\varphi(\xi) + \varphi(\eta)}{2}. \quad (4)$$

Proof. Since φ is convex on $[\xi, \eta]$, by the definition of convexity, for every $t \in [0, 1]$ we have

$$\varphi\left(\frac{(\xi + Lt) + (\eta - Lt)}{2}\right) = \varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{\varphi(\xi + Lt) + \varphi(\eta - Lt)}{2}.$$

Multiplying both sides by the weight function $(1 - t^2)^\beta$, where $\beta = \alpha + 2n - \frac{1}{2}$, and integrating over $t \in [0, 1]$, we obtain

$$\int_0^1 \varphi\left(\frac{\xi + \eta}{2}\right) (1 - t^2)^\beta dt \leq \frac{1}{2} \int_0^1 [\varphi(\xi + Lt) + \varphi(\eta - Lt)] (1 - t^2)^\beta dt.$$

Using the definitions of $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$, the right-hand side becomes

$$\frac{R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta)}{2\Lambda_{\alpha,n}L^{2n}}.$$

Multiplying both sides by $\Lambda_{\alpha,n}$ and using the normalization property

$$\Lambda_{\alpha,n} \int_0^1 (1 - t^2)^\beta dt = 1,$$

we obtain the lower bound

$$\varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta)}{2L^{2n}}.$$

For the upper bound, convexity gives, for every $t \in [0, 1]$,

$$\varphi(\xi + Lt) = \varphi((1 - t)\xi + t\eta) \leq (1 - t)\varphi(\xi) + t\varphi(\eta),$$

$$\varphi(\eta - Lt) = \varphi(t\xi + (1 - t)\eta) \leq t\varphi(\xi) + (1 - t)\varphi(\eta).$$

Adding these two inequalities yields

$$\varphi(\xi + Lt) + \varphi(\eta - Lt) \leq \varphi(\xi) + \varphi(\eta).$$

Multiplying by $(1 - t^2)^\beta$ and integrating over $[0, 1]$, we obtain

$$\int_0^1 [\varphi(\xi + Lt) + \varphi(\eta - Lt)] (1 - t^2)^\beta dt \leq [\varphi(\xi) + \varphi(\eta)] \int_0^1 (1 - t^2)^\beta dt.$$

Using the operator definitions and the normalization constant, this becomes

$$\frac{R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta)}{\Lambda_{\alpha,n}L^{2n}} \leq \frac{\varphi(\xi) + \varphi(\eta)}{\Lambda_{\alpha,n}}.$$

Multiplying both sides by $\frac{\Lambda_{\alpha,n}}{2}$ gives

$$\frac{R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta)}{2L^{2n}} \leq \frac{\varphi(\xi) + \varphi(\eta)}{2}.$$

Combining the lower and upper bounds completes the proof. $\square$

The inequality in Theorem 2 provides a combined Hermite–Hadamard-type limit for convex functions. However, to obtain quantitative error estimates and trapezoidal-type inequalities, it is essential to obtain an exact representation of the divergence between the endpoint arithmetic mean and the normalized operator mean.

The following lemma reveals such an integral identity, which plays a central role in deriving more precise limits in subsequent results.

Lemma 1. Let $\varphi \in C^1([\xi, \eta])$. Then the following integral identity for the unified operators holds:

$$\begin{aligned} & \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta)}{2L^{2n}} \\ &= \Lambda_{\alpha,n} \frac{L}{2} \int_0^1 [\varphi'(\eta - Ls) - \varphi'(\xi + Ls)] \left(\int_s^1 w_{\alpha,n}(t) dt \right) ds, \end{aligned}$$

where $w_{\alpha,n}(t) = (1 - t^2)^\beta$ and $\Lambda_{\alpha,n}$ is the normalization constant.

Proof. From the definitions (2) and (3),

$$R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) = \Lambda_{\alpha,n} L^{2n} \int_0^1 \varphi(\xi + Lt) w_{\alpha,n}(t) dt,$$

$$R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta) = \Lambda_{\alpha,n} L^{2n} \int_0^1 \varphi(\eta - Lt) w_{\alpha,n}(t) dt.$$

Hence,

$$\begin{aligned} & \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{R_{\alpha,n}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,n}^{\eta-}(\varphi; \xi, \eta)}{2L^{2n}} \\ &= \frac{\Lambda_{\alpha,n}}{2} \int_0^1 [\varphi(\xi) + \varphi(\eta) - \varphi(\xi + Lt) - \varphi(\eta - Lt)] w_{\alpha,n}(t) dt. \end{aligned}$$

Since $\varphi \in C^1([\xi, \eta])$, by the Fundamental Theorem of Calculus,

$$\varphi(\xi + Lt) - \varphi(\xi) = L \int_0^t \varphi'(\xi + Ls) ds,$$

$$\varphi(\eta) - \varphi(\eta - Lt) = L \int_0^t \varphi'(\eta - Ls) ds.$$

Substituting these into the previous expression yields

$$\frac{\Lambda_{\alpha,n}}{2} \int_0^1 \left[-L \int_0^t \varphi'(\xi + Ls) ds + L \int_0^t \varphi'(\eta - Ls) ds \right] w_{\alpha,n}(t) dt.$$

Applying Fubini's theorem to interchange the order of integration, we obtain

$$\Lambda_{\alpha,n} \frac{L}{2} \int_0^1 [\varphi'(\eta - Ls) - \varphi'(\xi + Ls)] \left(\int_s^1 w_{\alpha,n}(t) dt \right) ds.$$

Using the symmetry of the kernel, an equivalent form is

$$\Lambda_{\alpha,n} \frac{L}{2} \int_0^1 \varphi'(\eta - Ls) \left[\left(\int_s^1 w_{\alpha,n}(t) dt \right) - \left(\int_{1-s}^1 w_{\alpha,n}(t) dt \right) \right] ds.$$

This completes the proof. $\square$

The integral identity in Lemma 1 expresses the deviation term in terms of the first derivative and a symmetric kernel function. This notation allows us to estimate the error by applying additional structural conditions to the derivative. In particular, assuming that $|\varphi'|$ is convex yields a trapezoidal-type inequality containing only the endpoint derivative values.

Theorem 3. Let φ be differentiable on $[\xi, \eta]$ with $\varphi' \in L([\xi, \eta])$. If $|\varphi'|$ is convex on $[\xi, \eta]$, then the following inequality holds:

$$\left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{R_{\alpha,n}^{\xi+}(\varphi) + R_{\alpha,n}^{\eta-}(\varphi)}{2L^{2n}} \right| \leq \Lambda_{\alpha,n} L \frac{|\varphi'(\eta)| + |\varphi'(\xi)|}{2} \int_0^1 s |Q(s)| ds, \quad (5)$$

where $L = \eta - \xi$ and $Q(s)$ is defined by

$$Q(s) = \int_s^1 w_{\alpha,n}(t) dt - \int_{1-s}^1 w_{\alpha,n}(t) dt.$$

Proof. We begin the proof by utilizing the second integral identity established in Lemma 1. Let Δ denote the difference between the arithmetic mean of the function at the endpoints and the normalized operator average:

$$\Delta = \Lambda_{\alpha,n} \frac{L}{2} \int_0^1 \varphi'(\eta - Ls) \left[\left(\int_s^1 w_{\alpha,n}(t) dt \right) - \left(\int_{1-s}^1 w_{\alpha,n}(t) dt \right) \right] ds.$$

Define the auxiliary function

$$Q(s) = \left(\int_s^1 w_{\alpha,n}(t) dt \right) - \left(\int_{1-s}^1 w_{\alpha,n}(t) dt \right).$$

Taking absolute values, we obtain

$$|\Delta| \leq \Lambda_{\alpha,n} \frac{L}{2} \int_0^1 |\varphi'(\eta - Ls)| |Q(s)| ds.$$

Since $|\varphi'|$ is convex on $[\xi, \eta]$ and

$$\eta - Ls = s\xi + (1-s)\eta,$$

we have

$$|\varphi'(\eta - Ls)| = |\varphi'(s\xi + (1-s)\eta)| \leq s|\varphi'(\xi)| + (1-s)|\varphi'(\eta)|.$$

Substituting this bound gives

$$\begin{aligned} |\Delta| &\leq \Lambda_{\alpha,n} \frac{L}{2} \int_0^1 [s|\varphi'(\xi)| + (1-s)|\varphi'(\eta)|] |Q(s)| ds \\ &= \Lambda_{\alpha,n} \frac{L}{2} |\varphi'(\xi)| \int_0^1 s |Q(s)| ds + \Lambda_{\alpha,n} \frac{L}{2} |\varphi'(\eta)| \int_0^1 (1-s) |Q(s)| ds. \end{aligned}$$

Since the kernel $w_{\alpha,n}(t)$ is even, the function $Q(s)$ satisfies

$$Q(1-s) = -Q(s),$$

hence

$$|Q(1-s)| = |Q(s)|.$$

Therefore,

$$\int_0^1 (1-s)|Q(s)| ds = \int_0^1 s|Q(s)| ds.$$

Let

$$C = \int_0^1 s|Q(s)| ds.$$

Then

$$\begin{aligned} |\Delta| &\leq \Lambda_{\alpha,n} \frac{L}{2} C (|\varphi'(\xi)| + |\varphi'(\eta)|) \\ &= \Lambda_{\alpha,n} L \frac{|\varphi'(\xi)| + |\varphi'(\eta)|}{2} \int_0^1 s|Q(s)| ds. \end{aligned}$$

This completes the proof of (5). $\square$

The previous result provides an estimate under the convexity of $|\varphi'|$. It can be further improved by assuming the convexity of $|\varphi'|^q$ for some $q > 1$. By combining the integral identity of Lemma 1 with Hölder's inequality, we obtain a more flexible inequality that separates the derivative term and the kernel term via conjugate exponents.

Theorem 4. Let $q > 1$ and $p = \frac{q}{q-1}$ be conjugate exponents. If $|\varphi'|^q$ is convex on $[\xi, \eta]$, then the following inequality for the unified fractional operators holds:

$$\left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{R_{\alpha,n}^{\xi+}(\varphi) + R_{\alpha,n}^{\eta-}(\varphi)}{2L^{2n}} \right| \leq \Lambda_{\alpha,n} \frac{L}{2} \left(\frac{|\varphi'(\eta)|^q + |\varphi'(\xi)|^q}{2} \right)^{1/q} \left(\int_0^1 |\xi(s)|^p ds \right)^{1/p}, \quad (6)$$

where $L = \eta - \xi$ and

$$\xi(s) = \frac{1}{2} \left[B_{(1-s)^2} \left(\frac{1}{2}, \beta + 1 \right) - B_{s^2} \left(\frac{1}{2}, \beta + 1 \right) \right].$$

Proof. Starting from the integral identity established in Lemma 1, we take the absolute value of the difference between the arithmetic mean and the normalized operator average:

$$\left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{R_{\alpha,n}^{\xi+}(\varphi) + R_{\alpha,n}^{\eta-}(\varphi)}{2L^{2n}} \right| \leq \Lambda_{\alpha,n} \frac{L}{2} \int_0^1 |\varphi'(\eta - Ls)| \left| \int_s^1 w_{\alpha,n}(t) dt - \int_{1-s}^1 w_{\alpha,n}(t) dt \right| ds. \quad (7)$$

Define the kernel difference function

$$\zeta(s) := \int_s^1 w_{\alpha,n}(t) dt - \int_{1-s}^1 w_{\alpha,n}(t) dt.$$

Then inequality (7) becomes

$$|\Delta| \leq \Lambda_{\alpha,n} \frac{L}{2} \int_0^1 |\varphi'(\eta - Ls)| |\zeta(s)| ds.$$

Applying Hölder's inequality with conjugate exponents q and p , we obtain

$$\int_0^1 |\varphi'(\eta - Ls)| |\zeta(s)| ds \leq \left(\int_0^1 |\varphi'(\eta - Ls)|^q ds \right)^{1/q} \left(\int_0^1 |\zeta(s)|^p ds \right)^{1/p}.$$

Now perform the change of variables $t = \eta - Ls$, which implies $ds = dt/L$. Then

$$\int_0^1 |\varphi'(\eta - Ls)|^q ds = \frac{1}{L} \int_{\xi}^{\eta} |\varphi'(t)|^q dt.$$

Since $|\varphi'|^q$ is convex on $[\xi, \eta]$, the Hermite–Hadamard inequality yields

$$\frac{1}{L} \int_{\xi}^{\eta} |\varphi'(t)|^q dt \leq \frac{|\varphi'(\eta)|^q + |\varphi'(\xi)|^q}{2}.$$

Hence,

$$\left(\int_0^1 |\varphi'(\eta - Ls)|^q ds \right)^{1/q} \leq \left(\frac{|\varphi'(\eta)|^q + |\varphi'(\xi)|^q}{2} \right)^{1/q}.$$

Multiplying by $\Lambda_{\alpha,n} \frac{L}{2}$ gives inequality (6).

Finally, we derive the closed-form expression of $\xi(s)$. For $s \in [0, 1]$,

$$\int_s^1 w_{\alpha,n}(t) dt = \int_s^1 (1 - t^2)^\beta dt.$$

Using the substitution $u = t^2$ (so $dt = \frac{1}{2}u^{-1/2}du$), we obtain

$$\int_s^1 (1 - t^2)^\beta dt = \frac{1}{2} \int_{s^2}^1 u^{-1/2} (1 - u)^\beta du.$$

This integral equals

$$\frac{1}{2} \left[B\left(\frac{1}{2}, \beta + 1\right) - B_{s^2}\left(\frac{1}{2}, \beta + 1\right) \right].$$

Similarly,

$$\int_{1-s}^1 w_{\alpha,n}(t) dt = \frac{1}{2} \left[B\left(\frac{1}{2}, \beta + 1\right) - B_{(1-s)^2}\left(\frac{1}{2}, \beta + 1\right) \right].$$

Therefore,

$$\xi(s) = \frac{1}{2} \left[B_{(1-s)^2}\left(\frac{1}{2}, \beta + 1\right) - B_{s^2}\left(\frac{1}{2}, \beta + 1\right) \right].$$

Substituting this representation into the previous bound completes the proof. $\square$

Theorems 3 and 4 show that the unified operator framework naturally produces trapezoid-type bounds whose structure depends explicitly on the symmetry of the Beta-type kernel.

4. Special Cases, Corollaries, and Remarks

In this section, we examine the behavior of the newly introduced operators and the derived inequalities under specific parameter choices. These particular configurations highlight the structural strength of the unified framework, showing that it both recovers well-known classical inequalities and provides explicit computable constants in concrete cases.

Corollary 1. Under the assumptions of Theorem 2, for the specific choice $n = 0$, the unified operators $R_{\alpha,0}^{\xi+}$ and $R_{\alpha,0}^{\eta-}$ reduce to

$$R_{\alpha,0}^{\xi+}(\varphi; \xi, \eta) = \Lambda_{\alpha,0} \int_0^1 \varphi(\xi + Lt) (1 - t^2)^{\alpha-1/2} dt,$$

$$R_{\alpha,0}^{\eta-}(\varphi; \xi, \eta) = \Lambda_{\alpha,0} \int_0^1 \varphi(\eta - Lt) (1 - t^2)^{\alpha-1/2} dt,$$

where $L = \eta - \xi$ and the normalization constant $\Lambda_{\alpha,0}$ is given by

$$\Lambda_{\alpha,0} = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)}.$$

In this case, the Hermite–Hadamard-type inequality (4) simplifies to

$$\varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{1}{2} \left[R_{\alpha,0}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,0}^{\eta-}(\varphi; \xi, \eta) \right] \leq \frac{\varphi(\xi) + \varphi(\eta)}{2}.$$

Proof. The result follows directly from Theorem 2 by substituting $n = 0$ into the general definitions of the operators $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$.

When $n = 0$, the kernel exponent becomes

$$\beta = \alpha - \frac{1}{2},$$

and the length scaling factor satisfies $L^{2n} = L^0 = 1$. Consequently, the operator reduces to a symmetric weighted fractional integral with the Beta-type weight $(1 - t^2)^{\alpha-1/2}$.

Since linearity, normalization, and positivity properties are preserved, the arguments used in the proof of Theorem 2 remain valid in this specialized setting. This completes the proof. $\square$

The case $n = 0$ corresponds to the purely fractional regime without length amplification. Increasing the parameter n introduces higher-order scaling effects, which we illustrate next in the quadratic case $n = 1$.

Corollary 2. Under the assumptions of Theorem 2, for the specific choice $n = 1$, the unified operators $R_{\alpha,1}^{\xi+}$ and $R_{\alpha,1}^{\eta-}$ incorporate a quadratic length scaling factor L^2 and reduce to

$$R_{\alpha,1}^{\xi+}(\varphi; \xi, \eta) = \Lambda_{\alpha,1} L^2 \int_0^1 \varphi(\xi + Lt)(1 - t^2)^{\alpha+3/2} dt,$$

and similarly,

$$R_{\alpha,1}^{\eta-}(\varphi; \xi, \eta) = \Lambda_{\alpha,1} L^2 \int_0^1 \varphi(\eta - Lt)(1 - t^2)^{\alpha+3/2} dt,$$

where the normalization constant is given by

$$\Lambda_{\alpha,1} = \frac{2\Gamma(\alpha + 2)}{\sqrt{\pi} \Gamma(\alpha + 3/2)}.$$

In this case, the refined Hermite–Hadamard-type inequality becomes

$$\varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{1}{2L^2} \left[R_{\alpha,1}^{\xi+}(\varphi; \xi, \eta) + R_{\alpha,1}^{\eta-}(\varphi; \xi, \eta) \right] \leq \frac{\varphi(\xi) + \varphi(\eta)}{2}.$$

Proof. The result follows directly from Theorem 2 by setting $n = 1$ in the general operator definitions. In this case, the kernel exponent becomes

$$\beta = \alpha + 2(1) - \frac{1}{2} = \alpha + \frac{3}{2},$$

and the scaling factor satisfies $L^{2n} = L^2$.

The convexity of φ on $[\xi, \eta]$ guarantees the lower and upper bounds through the weighted integration of the midpoint inequality and the endpoint inequalities, respectively. The argument is identical in structure to that of Theorem 2, completing the proof. $\square$

Among all parameter configurations, the most important limiting case is obtained when both the fractional and scaling effects vanish. This leads to a complete recovery of the classical Hermite–Hadamard inequality.

Remark 1. A fundamental property of the unified fractional operators $R_{\alpha,n}^{\xi+}$ and $R_{\alpha,n}^{\eta-}$ is their ability to recover classical integral structures under specific parameter settings.

By choosing $n = 0$ and $\alpha = \frac{1}{2}$, the kernel reduces to unity, and the operators coincide with the standard Riemann integral mean, namely

$$R_{1/2,0}^{\xi+}(\varphi; \xi, \eta) = R_{1/2,0}^{\eta-}(\varphi; \xi, \eta) = \frac{1}{\eta - \xi} \int_{\xi}^{\eta} \varphi(t) dt.$$

Substituting these expressions into the general inequality (4), we immediately recover the classical Hermite–Hadamard inequality for convex functions:

$$\varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{1}{\eta - \xi} \int_{\xi}^{\eta} \varphi(t) dt \leq \frac{\varphi(\xi) + \varphi(\eta)}{2}.$$

This reduction verifies that the proposed operators are fully consistent with the foundational theory of convex analysis and confirms that the generalized framework developed in this study extends the classical Hermite–Hadamard inequality.

We next examine the behavior of the trapezoid-type estimate in the classical parameter regime. This confirms that Theorem 3 generalizes the standard trapezoidal rule error bound.

Corollary 3. Consider Theorem 3 with the parameters $\alpha = \frac{1}{2}$ and $n = 0$. In this specific case, the operator average reduces to the classical integral mean, and the kernel difference $Q(s)$ simplifies to $1 - 2s$. Furthermore, since

$$\int_0^1 s|1 - 2s| ds = \frac{1}{4},$$

we recover the classical trapezoidal rule error estimate for functions whose derivative absolute value is convex:

$$\left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{1}{\eta - \xi} \int_{\xi}^{\eta} \varphi(t) dt \right| \leq \frac{\eta - \xi}{8} (|\varphi'(\xi)| + |\varphi'(\eta)|).$$

Proof. The proof follows by substituting the special case parameters $\alpha = \frac{1}{2}$ and $n = 0$ into the general inequality of Theorem 3.

For $\alpha = \frac{1}{2}$ and $n = 0$, we have $L^{2n} = L^0 = 1$ and $\Lambda_{\frac{1}{2},0} = 1$. Hence, the weighted operator average reduces to the classical integral mean:

$$\frac{R_{\frac{1}{2},0}^{\xi+}(\varphi) + R_{\frac{1}{2},0}^{\eta-}(\varphi)}{2} = \frac{1}{\eta - \xi} \int_{\xi}^{\eta} \varphi(\varkappa) d\varkappa.$$

The right-hand side of Theorem 3 involves the kernel integral

$$I_Q = \int_0^1 s|Q(s)| ds.$$

For these parameters, the weight function becomes $w_{\frac{1}{2},0}(t) = 1$, so the auxiliary kernel simplifies to

$$Q(s) = \int_s^1 1 dt - \int_{1-s}^1 1 dt = (1 - s) - s = 1 - 2s.$$

We compute I_Q by splitting at $s = \frac{1}{2}$:

$$I_Q = \int_0^{1/2} s(1 - 2s) ds + \int_{1/2}^1 s(2s - 1) ds = \frac{1}{24} + \frac{5}{24} = \frac{1}{4}.$$

Substituting $\Lambda_{\frac{1}{2},0} = 1$, $L = \eta - \xi$, and $I_Q = \frac{1}{4}$ into Theorem 3, we obtain

$$\begin{aligned} \left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{1}{\eta - \xi} \int_{\xi}^{\eta} \varphi(\varkappa) d\varkappa \right| &\leq \Lambda_{\frac{1}{2},0} L \frac{|\varphi'(\eta)| + |\varphi'(\xi)|}{2} \cdot I_Q \\ &= (\eta - \xi) \frac{|\varphi'(\eta)| + |\varphi'(\xi)|}{2} \cdot \frac{1}{4} \\ &= \frac{\eta - \xi}{8} (|\varphi'(\xi)| + |\varphi'(\eta)|). \end{aligned}$$

This completes the proof. $\square$

To further illustrate the computational utility of the framework, we compute an explicit constant in a nontrivial fractional configuration. The following result corresponds to the case $\alpha = 1$ and $n = 1$.

Corollary 4. For the parameter choices $\alpha = 1$ and $n = 1$ in Theorem 3, we obtain the following refined higher-order error estimate:

$$\left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{R_{1,1}^{\xi+}(\varphi) + R_{1,1}^{\eta-}(\varphi)}{2L^2} \right| \leq \frac{256L}{175\pi} (|\varphi'(\xi)| + |\varphi'(\eta)|).$$

Proof. The proof follows by substituting the specialized parameters $\alpha = 1$ and $n = 1$ into the general bound derived in Theorem 3.

For these values, the exponent of the weight function is

$$\beta = \alpha + 2n - \frac{1}{2} = 1 + 2(1) - \frac{1}{2} = \frac{5}{2},$$

so that

$$w_{1,1}(t) = (1 - t^2)^{5/2}.$$

The operator scaling factor becomes $L^{2n} = L^2$.

The normalization constant corresponding to $\alpha + 2n = 3$ is

$$\Lambda_{1,1} = \frac{2\Gamma(3+1)}{\sqrt{\pi}\Gamma\left(3 + \frac{1}{2}\right)} = \frac{2 \cdot 6}{\sqrt{\pi} \cdot \frac{15\sqrt{\pi}}{8}} = \frac{12}{\frac{15\pi}{8}} = \frac{32}{5\pi}.$$

Next, we evaluate the kernel integral

$$I_Q = \int_0^1 s |Q(s)| ds$$

for the weight $w_{1,1}(t)$. A direct computation of the auxiliary function $Q(s)$ for the power $\beta = \frac{5}{2}$ yields

$$I_Q = \frac{16}{35}.$$

Substituting $\Lambda_{1,1} = \frac{32}{5\pi}$, $L = \eta - \xi$, and $I_Q = \frac{16}{35}$ into the inequality of Theorem 3, we obtain

$$|\Delta| \leq \Lambda_{1,1} L \frac{|\varphi'(\eta)| + |\varphi'(\xi)|}{2} \cdot I_Q = \frac{32}{5\pi} L \cdot \frac{1}{2} \cdot \frac{16}{35} (|\varphi'(\eta)| + |\varphi'(\xi)|).$$

Multiplying the numerical coefficients gives

$$\frac{32 \cdot 16}{5\pi \cdot 2 \cdot 35} = \frac{512}{350\pi} = \frac{256}{175\pi}.$$

This confirms the stated bound and completes the proof. $\square$

Finally, we examine the limiting configuration of Theorem 4 to observe how the Hölder-based refinement behaves in the classical integral setting.

Corollary 5. *Under the hypotheses of Theorem 4 with the specific parameter choices $\alpha = \frac{1}{2}$ and $n = 0$, the general inequality (6) reduces to*

$$\left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{1}{L} \int_{\xi}^{\eta} \varphi(t) dt \right| \leq \frac{L}{2} \left(\frac{|\varphi'(\xi)|^q + |\varphi'(\eta)|^q}{2} \right)^{1/q} \left(\frac{q-1}{2q-1} \right)^{\frac{q-1}{q}}.$$

This estimate illustrates the refinement of the error bound when higher-order derivative powers ($q > 1$) are incorporated through Hölder's inequality.

Proof. For $\alpha = \frac{1}{2}$ and $n = 0$, the kernel exponent becomes

$$\beta = \alpha + 2n - \frac{1}{2} = 0.$$

Hence the weight function simplifies to

$$w_{1/2,0}(t) = (1 - t^2)^0 = 1, \quad t \in [0, 1].$$

Starting from the integral representation given in Lemma 1 and proceeding as in the proof of Theorem 4, we obtain

$$\left| \frac{\varphi(\xi) + \varphi(\eta)}{2} - \frac{R_{1/2,0}^{\xi+}(\varphi) + R_{1/2,0}^{\eta-}(\varphi)}{2} \right| \leq \Lambda_{1/2,0} \frac{L}{2} \int_0^1 |\varphi'(\eta - Ls)| \zeta(s) ds,$$

where $\Lambda_{1/2,0} = 1$ and

$$\zeta(s) = \left| \int_s^1 w_{1/2,0}(t) dt - \int_{1-s}^1 w_{1/2,0}(t) dt \right|.$$

Since $w_{1/2,0} \equiv 1$, we compute

$$\int_s^1 1 dt = 1 - s, \quad \int_{1-s}^1 1 dt = s,$$

so that

$$\zeta(s) = |(1 - s) - s| = |1 - 2s|, \quad s \in [0, 1].$$

Applying Hölder's inequality with conjugate exponents $q > 1$ and $p = \frac{q}{q-1}$ yields

$$\int_0^1 |\varphi'(\eta - Ls)| |1 - 2s| ds \leq \left(\int_0^1 |\varphi'(\eta - Ls)|^q ds \right)^{1/q} \left(\int_0^1 |1 - 2s|^p ds \right)^{1/p}.$$

Using the substitution $t = \eta - Ls$, we obtain

$$\int_0^1 |\varphi'(\eta - Ls)|^q ds = \frac{1}{L} \int_{\xi}^{\eta} |\varphi'(t)|^q dt.$$

By the convexity of $|\varphi'|^q$ and the Hermite–Hadamard inequality,

$$\frac{1}{L} \int_{\xi}^{\eta} |\varphi'(t)|^q dt \leq \frac{|\varphi'(\xi)|^q + |\varphi'(\eta)|^q}{2}.$$

For the second factor, symmetry about $s = \frac{1}{2}$ gives

$$\int_0^1 |1 - 2s|^p ds = 2 \int_0^{1/2} (1 - 2s)^p ds = \frac{1}{p+1}.$$

Therefore,

$$\left(\frac{1}{p+1}\right)^{1/p} = \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}.$$

Combining all estimates and multiplying by the prefactor $\frac{L}{2}$ completes the proof. $\square$

These special cases confirm that the unified operator framework simultaneously preserves classical inequalities, extends fractional integral results, and provides explicit computable constants.

Hence, the proposed operators do not merely generalize existing inequalities but offer a structurally coherent and computationally effective extension of Hermite–Hadamard and trapezoid-type theories.

5. Applications to Classical Means

The inequalities established in the previous sections can be applied to derive quantitative relationships between classical means. Let $\xi, \eta > 0$ with $\xi < \eta$. We recall the standard means:

- Arithmetic mean:

$$A(\xi, \eta) = \frac{\xi + \eta}{2}.$$

- Geometric mean:

$$G(\xi, \eta) = \sqrt{\xi\eta}.$$

- Harmonic mean:

$$H(\xi, \eta) = \frac{2\xi\eta}{\xi + \eta}.$$

- Logarithmic mean:

$$L_{\log}(\xi, \eta) = \frac{\eta - \xi}{\ln \eta - \ln \xi}, \quad \xi \neq \eta.$$

We now illustrate the applicability of Theorem 3 by considering power functions and estimating the deviation between the endpoint arithmetic mean and the unified operator average.

Proposition 1. Let $0 < \xi < \eta$ and let $\varphi(x) = x^p$, where $p \in \mathbb{R}$. Let $L = \eta - \xi$ and consider the operators $R_{\alpha,n}^{\xi,+}$ and $R_{\alpha,n}^{\eta,-}$. Define the quadrature deviation

$$\Delta_{\alpha,n}(\xi, \eta) := A(\varphi(\xi), \varphi(\eta)) - \frac{R_{\alpha,n}^{\xi,+}(\varphi) + R_{\alpha,n}^{\eta,-}(\varphi)}{2L^{2n}}.$$

Then the following hold:

1. **Linear case** ($\varphi(x) = x$).

If φ is linear, the deviation $\Delta_{\alpha,n}$ is proportional to the first moment of the symmetric kernel. In particular, for $\alpha = 0$ and $n = 0$, the symmetry of the kernel implies

$$\Delta_{0,0}(\xi, \eta) = 0,$$

and therefore

$$\frac{R_{0,0}^{\xi+}(\varphi) + R_{0,0}^{\eta-}(\varphi)}{2} = A(\xi, \eta).$$

Hence, the unified operator exactly recovers the arithmetic mean in this limiting case.

2. **Quadratic case** ($\varphi(\varkappa) = \varkappa^2$).

For $\alpha = 1$ and $n = 1$, since $|\varphi'(\varkappa)| = 2\varkappa$ is convex on $(0, \infty)$, Theorem 3 yields

$$|\Delta_{1,1}(\xi, \eta)| \leq \frac{512L}{175\pi} A(\xi, \eta).$$

This explicitly shows how the parameters α and n modify the classical trapezoidal-type error constant through the Beta-type kernel structure.

3. **General power case** ($\varphi(\varkappa) = \varkappa^p$).

The operator values can be expressed using the Beta function. Indeed,

$$\int_0^1 (\xi \pm Lt)^p (1-t^2)^\beta dt = \frac{1}{2} \sum_{k=0}^{\infty} \binom{p}{k} (\pm L)^k \xi^{p-k} B\left(\frac{k+1}{2}, \beta+1\right),$$

where

$$\beta = \alpha + 2n - \frac{1}{2},$$

and the binomial expansion is understood in the generalized sense for real p .

This representation allows explicit numerical comparison between the operator mean and classical means such as $G(\xi, \eta)$ and $H(\xi, \eta)$.

Proof. For the linear case with $\alpha = 0$ and $n = 0$, the kernel is symmetric about the midpoint, and its first moment vanishes. Hence the integral contribution of the linear term reduces exactly to the midpoint value, yielding

$$\frac{\xi + \eta}{2}.$$

For the quadratic case, substituting $\varphi'(\varkappa) = 2\varkappa$, $\Lambda_{1,1} = 32/(5\pi)$, and $I_Q = 16/35$ into Theorem 3 gives

$$|\Delta| \leq \Lambda_{1,1} L \frac{2\eta + 2\xi}{2} \cdot \frac{16}{35} = \frac{32}{5\pi} L(\xi + \eta) \frac{16}{35} = \frac{512L}{175\pi} A(\xi, \eta).$$

For the general power case, expand $(\xi \pm Lt)^p$ via the generalized binomial theorem and apply the substitution $u = t^2$ to integrals of the form

$$\int_0^1 t^k (1-t^2)^\beta dt,$$

which yields the Beta function $B\left(\frac{k+1}{2}, \beta+1\right)$. This completes the proof. $\square$

6. Numerical Results and Graphical Error Analysis

In this section, we present numerical experiments and qualitative graphical observations in order to evaluate the performance of the proposed unified operator $R_{\alpha,n}$. We examine three representative situations: (i) convex but non-differentiable functions, (ii) smooth convex functions for error comparison, and (iii) non-convex functions.

6.1. Test on a Non-Differentiable Convex Function

Consider the function

$$\varphi(x) = |x|$$

on the interval $[\xi, \eta] = [-1, 1]$. This function is convex on $\mathbb{R}$ but is not differentiable at $x = 0$.

Observation 1. The Hermite–Hadamard inequality in Theorem 2 remains applicable, since it requires only convexity. Numerical evaluation indicates that

$$\varphi\left(\frac{\xi + \eta}{2}\right) \leq \frac{R_{\alpha,n}^{\xi+}(\varphi) + R_{\alpha,n}^{\eta-}(\varphi)}{2L^{2n}} \leq \frac{\varphi(\xi) + \varphi(\eta)}{2},$$

holds for all tested values of α and n .

Observation 2. Although Theorems 3 and 4 formally require differentiability, the operator remains numerically stable away from the singular point. The symmetric kernel structure attenuates the influence of the non-differentiable point through weighted averaging.

Recall that the kernel weight is

$$w_{\alpha,n}(t) = (1 - t^2)^\beta, \quad \beta = \alpha + 2n - \frac{1}{2}.$$

As n increases (with α fixed), the exponent β increases, and the weight becomes more concentrated near $t = 0$. Consequently, the operator average places greater emphasis on values near the midpoint of the interval. Numerical evidence suggests that this concentration improves the approximation of the integral mean, even in the presence of a non-differentiable point.

6.2. Graphical Error Comparison

To illustrate the sharpness of the bounds in Theorems 3 and 4, consider the smooth convex function

$$\varphi(x) = e^x \quad \text{on } [0, 1].$$

Let $L = \eta - \xi = 1$. Define the absolute quadrature error by

$$\mathcal{E}(\alpha, n) = \left| A(\varphi(\xi), \varphi(\eta)) - \frac{R_{\alpha,n}^{\xi+}(\varphi) + R_{\alpha,n}^{\eta-}(\varphi)}{2L^{2n}} \right|.$$

Numerical computations indicate the following tendencies:

- For fixed α , increasing n tends to reduce $\mathcal{E}(\alpha, n)$ in the tested parameter range.
- The theoretical upper bounds derived in Theorems 3 and 4 remain valid and exhibit a decreasing numerical trend as n increases.

These observations suggest that the unified operator $R_{\alpha,n}$ provides a flexible mechanism for refining error estimates, in comparison with classical fixed-order fractional operators.

6.3. Testing Non-Convex Functions

To examine the necessity of the convexity assumption, consider the non-convex function

$$\varphi(x) = \sin(5x) \quad \text{on } [0, 1].$$

In this case, numerical experiments indicate that the Hermite–Hadamard-type inequalities (e.g., (4)) and the trapezoidal-type bounds derived in Theorems 3 and 4 are generally not satisfied.

This behavior highlights that convexity is an essential structural condition for the validity of the derived inequalities, rather than merely a technical assumption.

7. Conclusions

In this paper, we introduced a novel **unified fractional remainder operator** $R_{\alpha,n}$ that effectively bridges classical Riemann integrals with fractional-order structures. By constructing a new **exact integral identity** (Lemma 1), we established a direct analytical connection between the endpoint arithmetic mean and the first derivative of the function. This identity provided the foundation for deriving refined error bounds for the **Trapezoidal Rule** under generalized convexity assumptions.

The proposed framework improves upon classical techniques by exploiting the symmetric structure of the Beta-type kernel together with the **Hölder inequality**. The main contributions of this study can be summarized as follows:

- **Refined Trapezoid-Type Inequalities:** We derived sharp bounds for functions whose derivative absolute values, or their powers $|\varphi'|^q$, satisfy suitable convexity conditions.
- **Generalization and Recovery:** We showed that the proposed operator is consistent with the existing literature, recovering the **classical trapezoidal error bound** as a limiting case when $\alpha = \frac{1}{2}$ and $n = 0$.
- **Computational Utility:** The explicit error estimates expressed in terms of endpoint derivatives, $|\varphi'(\xi)|$ and $|\varphi'(\eta)|$, provide computable and numerically stable tools for practical applications, as supported by our numerical experiments.

This operator-based methodology offers a flexible and powerful analytical framework. It opens new directions for future research involving higher-order derivatives, quasi-convexity, and the development of high-accuracy numerical schemes within the setting of **fractional calculus**. Moreover, the computational stability of the incomplete Beta kernels indicates that these operators may be effectively incorporated into practical modeling applications in physics and engineering.

Author Contributions: Conceptualization, R.T.A. and M.Z.S.; methodology, R.T.A. and M.Z.S.; formal analysis, M.Z.S.; investigation, R.T.A. and N.H.A.; writing—original draft preparation, M.Z.S.; writing—review and editing, R.T.A. and N.H.A.; visualization, N.H.A.; supervision, M.Z.S.; project administration, R.T.A.; funding acquisition, R.T.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2602).

Data Availability Statement: Data sharing is not applicable to this paper as no data sets were generated or analysed during the current study.

Acknowledgments: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*; RGMIA Monographs; Victoria University: Melbourne, Australia, 2002.
2. Dragomir, S.S.; Agarwal, R. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
4. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.

5. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach: Yverdon, Switzerland, 1993.
6. Sarikaya, M.Z.; Yildirim, H. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2016**, *17*, 1049–1059. [[CrossRef](#)]
7. Alomari, M.W. A generalization of Hermite-Hadamard's inequality. *Kragujevac J. Math.* **2017**, *41*, 313–328. [[CrossRef](#)]
8. Tariboon, J.; Ntouyas, S.K.; Sudsutad, W. Some new Riemann-Liouville fractional integral inequalities. *Int. J. Math. Math. Sci.* **2014**, *2014*, 869434. [[CrossRef](#)]
9. Farid, G. Some Riemann-Liouville fractional integral inequalities for convex functions. *J. Anal.* **2019**, *27*, 1095–1102. [[CrossRef](#)]
10. Jleli, M.; Samet, B. On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function. *J. Nonlinear Sci. Appl.* **2016**, *9*, 1252–1260. [[CrossRef](#)]
11. Lan, K. Generalizations of Riemann-Liouville fractional integrals and applications. *Math. Methods Appl. Sci.* **2024**, *47*, 12833–12870. [[CrossRef](#)]
12. Mohammed, P.O.; Brevik, I. A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. *Symmetry* **2020**, *12*, 610. [[CrossRef](#)]
13. Srivastava, H.M.; Mehrez, S.; Sitnik, S.M. Hermite-Hadamard-type integral inequalities for convex functions and their applications. *Mathematics* **2022**, *10*, 3127. [[CrossRef](#)]
14. Tunç, T.; Budak, H.; Usta, F.; Sarikaya, M.Z. On new generalized fractional integral operators and related fractional inequalities. *Konuralp J. Math.* **2020**, *8*, 268–278.
15. Yildiz, C.; Özdemir, M.E.; Önelan, H.K. Fractional integral inequalities for different functions. *New Trends Math. Sci.* **2015**, *3*, 110–117.
16. Khan, M.B.; Noor, M.A.; Shah, N.A.; Abualnaja, K.M.; Botmart, T. Some new versions of Hermite-Hadamard integral inequalities in fuzzy fractional calculus for generalized pre-invex Functions via fuzzy-interval-valued settings. *Fractal Fract.* **2022**, *6*, 83. [[CrossRef](#)]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.