



Article

Existence and Uniqueness of Mild Solutions for Fractional Impulsive Evolution Equations of Mixed Type with Nonlocal and Delay Conditions in Banach Spaces

Limin Guo ¹ , Lishan Liu ^{2,3,*} and Haibo Gu ³ ¹ School of Science, Changzhou Institute of Technology, Changzhou 213002, China; guolm@czu.cn² School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China³ School of Mathematical Sciences, Xinjiang Normal University, Urumqi 830017, China; hbgu_math@163.com

* Correspondence: mathlls@163.com

Abstract

In this paper, based on the Schauder fixed point theorem, the (generalized) Darbo fixed point theorem, and the (generalized) Banach contraction mapping principle, we study the mixed-type fractional impulse evolution equation with non-local and delay terms, and obtain the existence and uniqueness theorems under whether the operator is compact or not. The order of the derivative in this paper is $0 < \alpha < 1$, this fractional order introduces a series of problems concerning compactness, continuity, and convergence. We overcome these problems using methods such as Hölder inequality and Minkowski inequality. Moreover, under the condition of the non-compact measure, the non-negative constant is extended to an unbounded Lebesgue-integrable function. In addition, when obtaining the uniqueness of the solution through the (generalized) Banach contraction mapping principle, the non-negative constant L in the Lipschitz condition is extended to an unbounded Lebesgue integrable function. Finally, a case study is conducted to demonstrate the validity of the theoretical results.

Keywords: fractional impulsive evolution equations; nonlocal and delay conditions; existence and uniqueness; compact and non-compact semigroups; non-compact measure

MSC: 34A08; 47J35; 65J08; 34A10; 34B37

1. Introduction

In recent years, the study of various solution problems for fractional evolution equations has attracted considerable attention. These include problems with delay terms [1], convergence analysis [2,3], nonlocal conditions [4], multiple solutions [5,6], and others. The main tools employed in these theoretical studies include the monotone iteration method [7], the contraction mapping principle [8], the Schauder fixed-point theorem [9,10], the k -set contractive mapping principle [4], the Krasnoselskii fixed-point theorem [11], and iterative methods [12], among others. Compared with traditional integer-order differential models, fractional differential models exhibit distinct properties, including non-locality (i.e., memory effects) [13], as well as complexity and diversity [14]. Consequently, systems described by fractional differential equations offer advantages such as high accuracy, high efficiency, and robustness. Owing to these superior properties, fractional differential equations have found extensive applications across various engineering disciplines, including viscoelastic



Academic Editor: Carlo Cattani

Received: 11 April 2026

Revised: 1 June 2026

Accepted: 18 June 2026

Published: 23 June 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

materials [15], signal processing [16], control systems [17], circuit analysis [18], market equilibrium [19] and others.

Theoretical research on evolution equations is essential, and many mathematicians are currently engaged in foundational studies of fractional evolution equations. Gou et al. [8] explored an integro-differential equation within the framework of Caputo fractional derivatives. By employing the contraction mapping fixed-point theorem, they established the attainability of solutions under Lipschitz continuity conditions. In reference [4], Kumar and colleagues examined the attainability of fractional differential equations involving the Atangana–Baleanu operator, specifically addressing systems with non-instantaneous impulsive effects and nonlocal boundary constraints. Their analysis rigorously proved well-posedness under Lipschitzian continuity assumptions via fixed-point methods. Similarly, Ravichandran et al. [11] demonstrated the well-posedness of solutions for their system through fixed-point analysis under Lipschitz continuity constraints. Thus, rigorous investigation of fractional differential models carries substantial academic and applied weight, as it simultaneously advances the mathematical foundations of calculus and provides innovative analytical tools for addressing complex phenomena across scientific and engineering disciplines.

The fractional derivative with the Mittag–Leffler kernel has attracted the interest of the mathematical community in various research directions. For example, Liang [20] investigated the new Hermite–Hadamard type inequalities under Mittag–Leffler kernel fractional integral operators. Abdeljawad [21] established the well-posedness theorems for the non-local Cauchy problems governed by Mittag–Leffler integral operators and give out the illustrative examples. Hammad [22] estimated the controllability of nonlocal Mittag–Leffler kernel fractional order stochastic systems. Priya [23] investigated soil pollution due to microplastics using the Mittag–Leffler kernel fractional-order model. Dhayal and Zhu [24] explore a class of Mittag–Leffler kernel fractional order stochastic system and analyze a constructive result for the solution of the system. In [25], Chen pioneered a fractional stochastic discrete-time chaos framework utilizing Mittag–Leffler memory operators to capture financial market anomalies. Akram [26] develops a rigorous framework for deriving closed-form solutions to Mittag–Leffler kernel fractional differential equations under Pythagorean fuzzy uncertainty. Kalamani [27] rigorously validates the well-posedness (existence, uniqueness) and controllability of fractional-order systems within Mittag–Leffler kernel differentiation frameworks. In [28], the deterministic mathematical model of monkeypox virus is constructed using classical and Mittag–Leffler kernel non-integer order differential equations. El-hady et al. [29] discussed stability results for Atangana–Baleanu–Caputo fractional equations with impulse influences. Hussain et al. [30] investigated the controllability of fractional equations with delay terms of the Atangana–Baleanu–Caputo type. Although Mittag–Leffler kernel fractional-order derivatives have been used in various industries, the theoretical landscape surrounding fractional differential systems with such kernels remains underdeveloped. Current investigations into the solution of initial-boundary value problems are still in their early stages, with fundamental aspects of solution existence, regularity, and structural stability requiring systematic consolidation through rigorous mathematical frameworks.

Currently, most of these results have been obtained under the Lipschitz condition and the non-compact measure condition, whereas only a few have been established under the extended Lipschitz condition and the generalized non-compact measure condition.

In [8], Gou studied the following integro-differential equation under Caputo fractional derivative frame

$$\begin{cases} {}^c D_t^\alpha u(t) + Au(t) = f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, t \geq 0, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), k = 1, 2, \dots, m, \\ u(0) = u_0 \in E, \end{cases}$$

where ${}^c D_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1]$, $f, g : [0, \infty) \times E \rightarrow E$, and $q : I \rightarrow E$ are continuous, $t \in I = [0, T]$, $0 < T \leq +\infty$, $u_0 \in E$, $I_k : E \rightarrow E$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $\Delta u_k|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+) = \lim_{m \rightarrow 0} u(t_k + m)$ and $u(t_k^-) = \lim_{m \rightarrow 0} u(t_k - m)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. $A : D(A) \subseteq E \rightarrow E$ is a closed linear operator and $-A$ generates a uniformly bounded C_0 -semigroup $T(t)$ ($t \geq 0$) in E , the authors obtained the existence results by the fixed point theorem of condensation mapping under the Lipschitz condition. In [31], El-hady et al. studied the following fractional differential equation

$$\begin{cases} {}^{ABC} D_t^\alpha u(t) = f(t, u(t), \int_0^t Z(t, s, u(s))ds), t \in (s_i, t_{i+1}), 0 < \alpha \leq 1, \\ u(t) = H_i(t, u(t)), t \in (t_i, s_i], i = 1, 2, \dots, m, \\ u(b) - u'(b) = \int_b^T h(s, u(s))ds, \end{cases}$$

where ${}^{ABC} D_t^\alpha$ is the Atangana–Baleanu fractional derivative of order $\alpha \in (0, 1]$, and $0 = s_0 < t_1 \leq t_2 < \dots < t_m \leq s_m \leq s_{m+1} = T$, $f : [b, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$; $h : [b, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $H_i : [t_i, s_i] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. The authors obtained the existence results by the fixed point theorem of condensation mapping under the Lipschitz condition. In [32], Hamza et al. studied following fractional differential equation

$$\begin{cases} {}^{ABC} D_t^\alpha u(t) = f(t, u(t), u(\tau(t))), 1 < \alpha \leq 2, t \in (0, T], \\ u(t) = \zeta(t), u'(t) = \zeta'(t), t \in [-\delta, 0], \end{cases}$$

where ${}^{ABC} D_t^\alpha$ is the Atangana–Baleanu fractional derivative of order $\alpha \in (1, 2]$, $\zeta \in C^1([-\delta, 0])$ is a prescribed initial function, $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, and the delay function $\tau(t) \leq t$. The authors obtained the existence results by the fixed point theorem of condensation mapping under the Lipschitz condition. In [4], the authors researched the following Atangana–Baleanu fractional differential equations with non-transient impulse and nonlocal conditions

$$\begin{cases} {}^{ABC} D_t^\alpha u(t) = Au(t) + f(t, u(t)), t \in \cup_{k=0}^m (t_k, t_{k+1}], \\ u(t) = \gamma_k(t, u(t)), t \in \cup_{k=0}^m (t_k, t_{k+1}], \\ u(0) = x_0 - g(x), \end{cases}$$

where ${}^{ABC} D_t^\alpha$ is the Atangana–Baleanu fractional derivative of order $\alpha \in (0, 1]$, $A : D(A) \subseteq E \rightarrow E$ is a closed linear operator and $-A$ generates a uniformly bounded C_0 -semigroup $T(t)$ ($t \geq 0$) in E , $T(t)$ ($t \geq 0$) is the solution operator on a Banach space $(E, \|\cdot\|)$, $J = [0, T]$, $T > 0$ is a constant, $0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, $s_0 = 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \dots, m$, $f : \cup_{k=0}^m (t_k, t_{k+1}] \times E \rightarrow E$ satisfies Carathéodory condition, i.e., $f(\cdot, x) : J \rightarrow$

X is Lebesgue measurable and $f(t, \cdot) : E \rightarrow E$ is continuous, $g : PC(J, E) \rightarrow E$ satisfies Lipschitz condition. In [11], the authors investigated the following initial-value problem

$$\begin{cases} {}^{ABc}D_t^\alpha u(t) = f\left(t, u(t), \int_0^t h(t, s, u(s))ds, \int_0^T \ell(t, s, u(s))ds\right), 0 < \alpha < 1, \\ u(0) = x_0, \end{cases}$$

where $t \in [0, 1] = J$, ${}^{ABc}D_t^\alpha$ is the Atangana–Baleanu fractional derivative of order $\alpha \in (0, 1]$, $f \in C[0, 1]$, $f(0, x(0), 0, \int_0^T \ell(0, s, u(s))) = 0$, $T > 0$, $h, \ell : J \times E \rightarrow E$ satisfy Lipschitz condition, $f : J \times E \times E \times E \rightarrow E$ is a continuous function and satisfy the Lipschitz condition, the authors got the existence and uniqueness of this system under the Lipschitz condition.

To date, most existing results have been obtained under standard Lipschitz conditions, with only a few under generalized Lipschitz conditions. Moreover, a defining feature of impulsive dynamical systems is their ability to rigorously incorporate discontinuous state transitions, thereby offering a refined framework for modeling evolutionary processes subject to abrupt perturbations. In this study, we investigate the well-posedness of hybrid Atangana–Baleanu–Caputo fractional differential equations in Banach spaces, incorporating delays, nonlocal boundary constraints, and impulsive discontinuities. The specific equation is as follows:

$$\begin{cases} {}^{ABc}D_\kappa^\alpha u(\kappa) = Au(\kappa) + \Xi(\kappa, u(\kappa), u(\tau_1(\kappa)), \dots, u(\tau_{n-2}(\kappa)), (\overline{\mathcal{H}}u)(\kappa), (\overline{\Lambda}u)(\kappa)), \kappa \in J', \\ \Delta u(\kappa)|_{\kappa=\kappa_i} = I_i(u(\kappa_i)), i = 1, 2, \dots, m, \\ u(0) = u_0 + g(u). \end{cases} \quad (1)$$

where $J = [0, T]$, $J' = J \setminus \{\kappa_i\}$, $\alpha \in (0, 1)$, $u_0 \in E$, $g : PC(J, E) \rightarrow E$ is continuous, $I_i : E \rightarrow E$ ($i = 1, 2, \dots, m$) is a impulse term and continuous, and $0 = \kappa_0 < \kappa_1 < \dots < \kappa_m < \kappa_{m+1} = T$, $I_k(u(\kappa_i)) = u(\kappa_i^+) - u(\kappa_i^-)$, $i = 1, 2, \dots, m$, $u(\kappa_i^-) = \lim_{\lambda \rightarrow 0} u(\kappa_i - \lambda)$ and $u(\kappa_i^+) = \lim_{\lambda \rightarrow 0} u(\kappa_i + \lambda)$ denote the left and right limits of $u(\kappa)$ at $\kappa = \kappa_i$, respectively, nonlinear integral operator $\overline{\mathcal{H}}$ and $\overline{\Lambda}$ is defined as follows:

$$(\overline{\mathcal{H}}u)(\kappa) = \int_0^\kappa \hbar(\kappa, s, u(s))ds, (\overline{\Lambda}u)(\kappa) = \int_0^T \rho(\kappa, s, u(s))ds,$$

where $\hbar : \Theta_1 \times E \rightarrow E$ and $\rho : \Theta_2 \times E \rightarrow E$ are continuous, here $\Theta_1 = \{(\kappa, s) | 0 \leq s \leq \kappa \leq T\}$, $\Theta_2 = \{(\kappa, s) | 0 \leq s, \kappa \leq T\}$, $F : J \times E^{n+1} \rightarrow E$ is continuous, delay term $\tau_i : J \rightarrow J$ is continuous and $0 \leq \tau_i(\kappa) \leq \kappa$ ($i = 1, 2, \dots, n-2$), $\kappa \in J$, and ${}^{ABc}D_\kappa^\alpha$ is the Atangana–Baleanu Caputo type fractional derivative of order α .

In this study, we investigate the existence and uniqueness of mild solutions for fractional impulsive evolution equations with nonlocal and delay conditions in Banach spaces. Theorem 1 establishes the existence of a mild solution for Equation (1) under the assumption that the semigroup is compact. Theorems 2 and 3 provide the existence of a unique solution for Equation (1) under generalized Lipschitz conditions. Theorems 4 and 5 demonstrate the existence of at least one solution for Equation (1) under the generalized measure of noncompactness under the assumption that the semigroup is not compact. Compared with [4,8,11,31], the non-negative constants L of Lipschitz conditions in Section 3.2 and non-compactness measure conditions in Section 3.3 are generalized to the unbounded Lebesgue integrable functions. Compared with [11,31], the equations in this article involve the closed linear operator A . Therefore, this manuscript can be regarded as a generalization of these several reference papers. Compared with [11,31], the equation we are studying contains impulse terms or involves Fredholm operators.

2. Preliminaries and Lemmas

In this article, E represents a Banach space, $C(J, E) = \{u : J \rightarrow E \text{ be continuous}\}$ is also a Banach space, whose norm is defined as $\|u\|_C = \max\{\|u(\kappa)\| : \kappa \in J\}$, $PC(J, E) = \{u : u : J \rightarrow E \text{ is continuous at } \kappa \neq \kappa_k, \text{ is left continuous at } \kappa = \kappa_i, \text{ and } u(\kappa_i^+) \text{ exist, } i = 1, 2, \dots, m\}$, and the norm is $\|u\|_{PC} = \sup_{0 \leq \kappa \leq T} \|u(\kappa)\|$. $J_0 = [\kappa_0, \kappa_1]$, $J_1 = (\kappa_1, \kappa_2]$, $\dots$, $J_m = (\kappa_m, \kappa_{m+1}]$, for any $p \geq 1$, $L^p(J, E)$ is Bochner integrable function space of E value. Let $T_R = \{u \in E : \|u\| \leq R\}$ and $B_R = \{u \in PC(J, E) : \|u\|_{PC} \leq R\}$ for $R > 0$.

Definition 1 ([33]). Given E is a real complete space under the given norm, $Y \subset E$ is any bounded set, the Kuratowski measure of noncompactness of Y is characterized by

$$\omega(Y) = \inf\{\epsilon > 0 : Y = \cup_{i=1}^m E_i, \text{diam}(E_i) \leq \epsilon, i = 1, 2, \dots, m\}.$$

If $U, V \subset X$ are two bounded set, then the assertions are valid:

- (1) If $V \subset U$, then $\omega(U) \leq \omega(V)$;
- (2) $\omega(U) = \omega(\overline{U}) = \omega(\text{Conv}U)$, where $\overline{U}$ and $\text{Conv}U$ represent the closure and convex hull;
- (3) U is relatively compact is equivalent to $\omega(U) = 0$;
- (4) $\omega(\lambda U) = |\lambda|\omega(U)$, λ is real number;
- (5) $\omega(U + V) \leq \omega(U) + \omega(V)$, where $U + V = \{x + y : x \in U, y \in V\}$;
- (6) $\omega(U \cup V) = \max\{\omega(U), \omega(V)\}$.

Lemma 1 ([34]). Let $B \subset PC(J, E)$ form a totally bounded and uniformly equicontinuous family of functions over J_k ($k = 0, 1, \dots, m$). Then $\omega(B(\kappa))$ is continuous on J_k and

$$\omega_{PC}(B) = \sup_{\kappa \in J} \omega(B(t)),$$

where $B(\kappa) = \{u(\kappa) : u \in B\}$, $\kappa \in J$, $\omega_{PC}(B)$ denotes the Kuratowski measure of noncompactness of the set B in $PC(J, E)$ and $\omega(B(\kappa))$ represents the Kuratowski measure of noncompactness of $B(t)$ in E .

Lemma 2 ([35]). Assume $B \subset PC(J, X)$ is bounded and is equicontinuous on J_k ($k = 0, 1, \dots, m$), then $m(\kappa) = \omega(B(\kappa))$ is continuous on J_k ($k = 0, 1, \dots, m$), and

$$\omega\left(\int_J B(s)ds\right) \leq \int_J m(s)ds,$$

where ω represents the Kuratowski noncompactness measure defined over a closed bounded set.

Lemma 3 ([35]). Let $B = \{u_n\} \subset PC(J, E)$ be countable. Assume that there exists $\psi \in L^1(J, \mathbb{R}^+)$ making that $\|u_n(\kappa)\| \leq \psi(\kappa)$ a.e. $t \in J$, $n = 1, 2, \dots$, then $\omega(B(t))$ represents Lebesgue integrable on J and

$$\omega\left(\left\{\int_J u_n(\kappa)d\kappa : n \in \mathbb{N}\right\}\right) \leq 2 \int_J \omega(B(\kappa))d\kappa.$$

Lemma 4 ([36]). The set $B \subset C(J, E)$ is assumed to be bounded. Hence, one can find a countable subset B_0 of B such that $\omega(B_0) = \omega(B)$.

Lemma 5 ([37]). Let G be a closed and convex subset of a real Banach space X , and let the operator $\aleph : G \rightarrow G$ be bounded and continuous. For any bounded subset $\Omega \subset G$, suppose

$$\tilde{\aleph}^1(\Omega) = \aleph(\Omega), \tilde{\aleph}^n(\Omega) = \aleph(\overline{\text{Conv}}\tilde{\aleph}^{n-1}(\Omega)), n = 2, 3, \dots$$

Assume there exist a constant $0 \leq \eta < 1$ and a positive integer n_0 , making that for any bounded subset $\Omega \subset E$,

$$\omega(\tilde{\aleph}^{n_0}(\Omega)) \leq \eta\omega(\Omega),$$

then $\aleph$ has a fixed point in G .

Lemma 6 ([37]). The set $\Omega \subset PC(J, X)$ is assumed to be bounded and equicontinuous on J_k ($k = 0, 1, \dots, m$), then $\overline{\text{Co}}\Omega \subset PC(J, X)$ is bounded and equicontinuous on J_k ($k = 0, 1, \dots, m$).

Lemma 7 ([38]). For any $R > 0$, the function $f : J \times T_R^{n+1} \rightarrow X$ is bounded and uniformly continuous, $\Omega \subset PC(J, X)$ is bounded and equicontinuous in J_i ($i = 0, 1, \dots, m$), then

$$\{f(\kappa, u(\kappa), u(\tau_1(\kappa)), \dots, u(\tau_{n-1}(\kappa)), (\overline{\mathcal{H}u})(t), (\overline{\Lambda u})(\kappa)) : u \in \Omega\}$$

is bounded and equicontinuous in J_i ($i = 0, 1, \dots, m$).

Lemma 8 ([37]). Suppose that $0 < \ell < 1$ and $h > 0$ are constants. Let

$$S_m = \ell^m + C_m^1 \ell^{m-1} h + \frac{C_m^2 \ell^{m-2} h^2}{2!} + \dots + \frac{h^m}{m!}, m \in \mathbb{N},$$

where $C_m^k = \frac{m!}{k!(m-k)!}$, then $S_m = o\left(\frac{1}{m^s}\right)$ ($m \rightarrow +\infty$), where $s > 1$ is any real number.

Definition 2 ([39]). Assume $x : (0, \infty) \rightarrow \mathbb{R}^1$, and $\alpha > 0$, then the Riemann–Liouville fractional integral of order α is given by

$$I_\kappa^\alpha x(\kappa) = \frac{1}{\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} x(s) ds, \quad (2)$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 3 ([40]). Let $\alpha \in [0, 1]$ and $x' \in H'(a, b)$, where $a \leq b$, then the Atangana–Baleanu Caputo derivative is

$$({}^{ABC}D_\kappa^\alpha x)(\kappa) = \frac{B(\alpha)}{1-\alpha} \int_0^\kappa x'(s) E_\alpha\left(-\alpha \frac{(\kappa-s)^\alpha}{1-\alpha}\right) ds,$$

where B is defined as a normalization function satisfying $B(0) = B(1) = 1$, and E_α denotes the well-known Mittag–Leffler function about a variable α .

Lemma 9 ([41]). For $0 \leq \alpha \leq 1$, then

$$\begin{aligned} (I_\kappa^\alpha ({}^{ABC}D_\kappa^\alpha x))(\kappa) &= x(\kappa) - x(0)E_\alpha(\lambda\kappa^\alpha) - \frac{\alpha}{1-\alpha}x(0)E_{\alpha,\alpha+1}(\lambda\kappa^\alpha) \\ &= x(t) - x(0). \end{aligned}$$

Remark 1. Unlike classical fractional derivatives, the AB formulation's Mittag–Leffler kernel enables precise modeling of power-law memory decay in viscoelastic materials and anomalous diffusion processes, and the commonly Riemann–Liouville and Caputo derivatives have a singular kernel that they does not have the function of fully describing memory.

For the convenience of the following statement, we give some introduction to the operator A .

The symbol A represents a closed linear operator, $-A$ is a finite generator of Banach space $(E, \|\cdot\|)$ for the semigroup $\mathfrak{S}_\alpha(\kappa)$ ($\kappa > 0$), $\|\mathfrak{S}_\alpha(\kappa)\| \leq Me^{\omega\kappa}$ and let $M^* = \sup_{\kappa \in [0, +\infty)} \|\mathfrak{S}_\alpha(\kappa)\|$.

Lemma 10 ([42]). Let $\alpha \in (0, 1)$, $q \in L^p(J, E)$ with $p > \frac{1}{\alpha}$, for linear Cauchy problems with specified initial data in the framework of integral-form solutions

$$\begin{cases} {}^{ABc}D_\kappa^\alpha u(\kappa) = Au(\kappa) + q(\kappa), \kappa \geq a \geq 0, \\ u(a) = u_0 + g(u), \end{cases} \quad (3)$$

then the solution to Equation (3) can be formulated as

$$\begin{aligned} u(\kappa) = & \mathcal{G}\mathfrak{S}_\alpha(\kappa - a)(u_0 + g(u)) + \frac{\mathcal{K}\mathcal{G}(1 - \alpha)}{B(\alpha)\Gamma(\alpha)} \int_a^\kappa (\kappa - s)^{\alpha-1} q(s) ds \\ & + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_a^\kappa \mathfrak{R}_\alpha(\kappa - s) q(s) ds, \kappa \in [a, \infty), \end{aligned}$$

where $\mathcal{G}$ and $\mathcal{K}$ are linear operators:

$$\mathcal{G} = \mu(\mu I - A)^{-1}, \mathcal{K} = -\gamma A(\mu I - A)^{-1} \text{ and } \mu = \frac{B(\alpha)}{1 - \alpha}, \gamma = \frac{\alpha}{1 - \alpha},$$

$$\mathfrak{S}_\alpha(t) = E_\alpha(-\mathcal{K}t^\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{st} s^{\alpha-1} (s^\alpha I - \mathcal{K})^{-1} ds,$$

$$\overline{\mathfrak{S}}_\alpha(t) = E_\alpha(-\mathcal{K}t^\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{st} s^{\alpha-2} (s^\alpha I - \mathcal{K})^{-1} ds,$$

$$\mathfrak{R}_\alpha(t) = t^{\alpha-1} E_{\alpha, \alpha}(-\mathcal{K}t^\alpha) = \frac{1}{2\pi i} \int_\Gamma e^{st} (s^\alpha I - \mathcal{K})^{-1} ds,$$

and Γ is a certain path lying on $\Sigma_{(\alpha, \omega)}$, for more detail see [4].

Remark 2. Combing the formula (2.5) of [42], the derivative definition of Mittag–Leffler kernel, the derivation process of Theorem 3.1 of [8], the definition of this type should be appropriate.

Lemma 11 ([8]). In the context of semigroup theory, operators $\mathfrak{S}_\alpha(\kappa)$ ($\kappa \geq 0$) and $\mathfrak{R}_\alpha(\kappa)$ ($\kappa \geq 0$) satisfy the following conditions:

(1) Given an arbitrary fixed $\kappa \geq 0$, $\mathfrak{S}_\alpha(\kappa)$ and $\mathfrak{R}_\alpha(\kappa)$ exhibit both linearity and boundedness, i.e., one can find a constant $M^* > 0$, for every admissible $\kappa \geq 0$, $u \in E$, such that

$$\|\mathfrak{S}_\alpha(\kappa)u\| \leq M^* \|u\|, \|\mathfrak{R}_\alpha(\kappa)u\| \leq \frac{\alpha M^*}{\Gamma(\alpha + 1)} \|u\|;$$

(2) For every $u \in E$, $\kappa \rightarrow \mathfrak{S}_\alpha(\kappa)u$ and $t \rightarrow \mathfrak{R}_\alpha(\kappa)u$ are continuous mapping from $[0, \infty)$ into E ;

(3) The operators $\mathfrak{S}_\alpha(\kappa)$ ($\kappa \geq 0$) and $\mathfrak{R}_\alpha(\kappa)$ ($\kappa \geq 0$) exhibit strong continuity over the interval $[0, \infty)$.

Lemma 12. Assume a function $u \in PC(J, E)$ is a mild solution for the problem (1), then u satisfies the following formula,

$$u(\kappa) = \begin{cases} \mathcal{G}\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds, \kappa \in [0, \kappa_1], \\ \mathcal{G}^2\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}^2(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \mathfrak{S}_\alpha(\kappa - \kappa_1) \int_0^{\kappa_1} (\kappa_1-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ + \frac{\alpha\mathcal{G}^3}{B(\alpha)} \int_0^{\kappa_1} \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds + \mathcal{G}\mathfrak{S}_\alpha(\kappa - \kappa_1) I_1(u(\kappa_1)) \\ + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_1}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_1}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds, \kappa \in (\kappa_1, \kappa_2], \\ \dots \\ \mathcal{G}^{k+1}\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds \\ + \sum_{i=1}^v \mathcal{G}^{v+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(u(\kappa_i)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_i}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_k}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds, t \in (\kappa_i, \kappa_{i+1}], i = 2, \dots, m, \end{cases} \quad (4)$$

for

$$\bar{q}(s, u(s)) = \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\mathcal{H}u)(s), (\Lambda u)(s)).$$

Proof. By Lemma 10, for $\kappa \in [0, \kappa_1]$, one has

$$u(\kappa) = \mathcal{G}\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds. \quad (5)$$

By (5), then

$$u(\kappa_1) = \mathcal{G}\mathfrak{S}_\alpha(\kappa_1)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^{\kappa_1} (\kappa_1-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^{\kappa_1} \mathfrak{R}_\alpha(\kappa_1-s) \bar{q}(s, u(s)) ds. \quad (6)$$

By impulse term $\Delta u(\kappa)|_{\kappa=\kappa_i} = I_k(u(\kappa_i))$ ($i = 1, 2, \dots, m$) in Equations (1) and (6), one has

$$u(\kappa_1^+) = u(\kappa_1^-) + I_i(u(\kappa_i^-)) = u(\kappa_1) + I_1(u(\kappa_1)).$$

Take $a = \kappa_1$ in Lemma 10, for $\kappa \in (\kappa_1, \kappa_2]$, one has

$$\begin{aligned}
u(\kappa) &= \mathcal{G}\mathfrak{S}_\alpha(\kappa - \kappa_1)(u(\kappa_1^+)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_1}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
&\quad + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_1}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds \\
&= \mathcal{G}\mathfrak{S}_\alpha(\kappa - \kappa_1)(u(\kappa_1) + I_1(u(\kappa_1))) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_1}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
&\quad + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_1}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds \\
&= \mathcal{G}\mathfrak{S}_\alpha(\kappa - \kappa_1) \left(\mathcal{G}\mathfrak{S}_\alpha(\kappa_1)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^{\kappa_1} (\kappa_1-s)^{\alpha-1} \bar{q}(s, u(s)) ds \right. \\
&\quad \left. + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^{\kappa_1} \mathfrak{R}_\alpha(\kappa_1-s) \bar{q}(s, u(s)) ds + I_1(u(\kappa_1)) \right) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_1}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
&\quad + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_1}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds \\
&= \mathcal{G}^2\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}^2(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \mathfrak{S}_\alpha(\kappa - \kappa_1) \int_0^{\kappa_1} (\kappa_1-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
&\quad + \frac{\alpha\mathcal{G}^3}{B(\alpha)} \int_0^{\kappa_1} \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds + \mathcal{G}\mathfrak{S}_\alpha(\kappa - \kappa_1) I_1(u(\kappa_1)) \\
&\quad + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_1}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_1}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds.
\end{aligned} \tag{7}$$

Follow the steps above in turn, for $\kappa \in (\kappa_{i-1}, \kappa_i]$, one arrives

$$\begin{aligned}
u(\kappa) &= \mathcal{G}^k\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}(1-\alpha)}{\Gamma(\alpha)B(\alpha)} \left(\mathcal{G}^k\mathfrak{S}_\alpha(\kappa - \kappa_1) \int_0^{\kappa_1} (\kappa_1-s)^{\alpha-1} \bar{q}(s, u(s)) ds \right. \\
&\quad \left. + \mathcal{G}^{k-1}\mathfrak{S}_\alpha(\kappa - \kappa_2) \int_{\kappa_1}^{\kappa_2} (\kappa_2-s)^{\alpha-1} \bar{q}(s, u(s)) ds + \dots \right. \\
&\quad \left. + \mathcal{G}^2\mathfrak{S}_\alpha(\kappa - \kappa_{k-1}) \int_{\kappa_{k-2}}^{\kappa_{k-1}} (\kappa_{k-1}-s)^{\alpha-1} \bar{q}(s, u(s)) ds \right) \\
&\quad + \frac{\alpha}{B(\alpha)} \left(\mathcal{G}^{k+1} \int_0^{\kappa_1} \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds + \mathcal{G}^k \int_{\kappa_1}^{\kappa_2} \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds + \dots \right. \\
&\quad \left. + \mathcal{G}^3 \int_{\kappa_{k-2}}^{\kappa_{k-1}} \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds \right) \\
&\quad + \mathcal{G}^{k-1}\mathfrak{S}_\alpha(\kappa - \kappa_1) I_1(u(\kappa_1)) + \mathcal{G}^{k-2}\mathfrak{S}_\alpha(\kappa - \kappa_2) I_2(u(\kappa_2)) + \dots + \mathcal{G}\mathfrak{S}_\alpha(\kappa - \kappa_{k-1}) I_2(u(\kappa_{k-1})) \\
&\quad + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_k}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{t_k}^\kappa \mathfrak{R}_\alpha(t-s) \bar{q}(s, u(s)) ds \\
&= \mathcal{G}^k\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+1-i}\mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
&\quad + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds + \sum_{i=1}^k \mathcal{G}^{k-i}\mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(u(\kappa_i)) \\
&\quad + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_{k-1}}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_{k-1}}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds.
\end{aligned} \tag{8}$$

As a result, by (8), one has

$$\begin{aligned}
u(\kappa_k) &= \mathcal{G}^k\mathfrak{S}_\alpha(\kappa_k)(u_0 + g(u)) + \frac{\mathcal{H}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+1-i}\mathfrak{S}_\alpha(\kappa_k - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
&\quad + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(t_k-s) \bar{q}(s, u(s)) ds + \sum_{i=1}^k \mathcal{G}^{k-i}\mathfrak{S}_\alpha(\kappa_k - \kappa_i) I_i(u(\kappa_i)) \\
&\quad + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_{i-1}}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_{i-1}}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u(s)) ds,
\end{aligned} \tag{9}$$

hence, for $\kappa \in (\kappa_i, \kappa_{i+1}]$, by (9), one arrives

$$\begin{aligned} u(\kappa) = & \mathcal{G}^{k+1} \mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{K}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ & + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(\kappa - s) \bar{q}(s, u(s)) ds \\ & + \sum_{i=1}^k g^{k+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(u(t_i)) + \frac{\mathcal{K}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_i}^{\kappa} (\kappa - s)^{\alpha-1} \bar{q}(s, u(s)) ds \\ & + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_i}^{\kappa} \mathfrak{R}_\alpha(\kappa - s) \bar{q}(s, u(s)) ds, t \in (\kappa_i, \kappa_{i+1}], i = 2, 3, \dots \end{aligned} \quad (10)$$

Hence, Equation (1) is mathematically interchangeable with the integral formulation (4), establishing their equivalence under the prescribed analytical framework. $\square$

Now we define an operator $\aleph : PC(J, E) \rightarrow PC(J, E)$, for any $u \in PC(J, E)$, define $\aleph u$ by the right of the Equation (4). Obviously, u is the solutions of Equation (4) in $PC(J, E)$ iff u is a fixed point of the operator $\aleph$ in $PC(J, E)$.

Definition 4. A function $u \in PC(J, E)$ is said to be a mild solution of the problem (1) if u satisfies (4), that is, u is a fixed point of the operator $\aleph$ in $PC(J, E)$.

3. Main Results

To streamline the presentation, we formally define assumptions that will govern our subsequent analysis.

(H₁) There exist integrable functions $\hbar_1(\kappa, \cdot), \rho_1(\kappa, \cdot), \zeta_h(\kappa, s), \zeta_\rho(\kappa, s) \in L(J, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, subject to

$$\begin{aligned} \hbar_0 &= \sup_{\kappa \in [0, T]} \int_0^\kappa \hbar_1(\kappa, s) ds < +\infty, \rho_0 = \sup_{\kappa \in [0, T]} \int_0^\kappa \rho_1(\kappa, s) ds < +\infty, \\ \bar{\hbar}_0 &= \max_{\kappa \in [0, T]} \int_0^\kappa \zeta_h(\kappa, s) ds < +\infty, \bar{\rho}_0 = \max_{\kappa \in [0, T]} \int_0^\kappa \zeta_\rho(\kappa, s) ds < +\infty, \end{aligned} \quad (11)$$

$$\|\hbar(\kappa, s, u)\| \leq \hbar_1(\kappa, s)\|u\| + \zeta_h(\kappa, s), (\kappa, s) \in \Theta_1, u \in E, \quad (12)$$

$$\|\rho(\kappa, s, u)\| \leq \rho_1(\kappa, s)\|u\| + \zeta_\rho(\kappa, s), (\kappa, s) \in \Theta_2, u \in E. \quad (13)$$

(H₂) Functions $I_k : E \rightarrow E$ are continuous and there exist constants $d_{I_k} > 0, \zeta_{I_k} > 0$, satisfies

$$\|I_k(u)\| \leq d_{I_k}\|u\| + \zeta_{I_k}, u \in E, k = 1, 2, \dots, m. \quad (14)$$

(H₃) Continuity of $g : PC(J, E) \rightarrow E$ with there existing constants $d_1 > 0, \zeta_1 > 0$, for all $u \in PC(J, E)$, satisfies

$$\|g(u)\| \leq d_1\|u\|_{PC} + \zeta_1. \quad (15)$$

(H₄) $\mathcal{G}$ and $\mathcal{K}$ are bounded operators, that is, the operator admits bounds $l, \bar{l}$ fulfilling

$$\|\mathcal{G}\| \leq l, \|\mathcal{K}\| \leq \bar{l}.$$

(H₅) Nonlinear term $\Xi : J \times T_R^{n+1} \rightarrow E$ is bounded and uniformly continuous,

$$\gamma = \limsup_{\|\chi_1\| + \|\chi_2\| + \|\chi_3\| + \dots + \|\chi_{n+1}\| \rightarrow +\infty} \left(\sup_{s \in J} \frac{\|\Xi(s, \chi_1, \chi_2, \chi_3, \dots, \chi_{n+1})\|}{\|\chi_1\| + \|\chi_2\| + \|\chi_3\| + \dots + \|\chi_{n+1}\|} \right) < +\infty, \quad (16)$$

and

$$K = \tilde{l}M^*d_1 + \hat{d}\gamma(n-1 + \hbar_0 + \rho_0) + l^m M^* \sum_{k=1}^m d_{I_k} < 1, \quad (17)$$

where

$$\bar{d} = \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(mM^*l^m + (1-\alpha)), M^*\alpha l^2(ml^m + 1)\},$$

$$\tilde{l} = \max\{l, l^{m+1}\}, \hat{d} = \bar{d}\left(\frac{1}{\alpha}T^\alpha + T\right).$$

Remark 3. Condition (H_1) – (H_5) is mainly used to prove that the operator mapping from a bounded region to another bounded region is bounded.

3.1. The Existence Theorem of Mild Solution Under the Compact Resolvent Operator Framework

Theorem 1. Suppose $\{\mathfrak{S}_\alpha(\kappa)\}$ is compact, and the conditions of (H_1) – (H_5) hold, then the model (1) has at least one global mild solution in $B_{R^*} \subset PC(J, E)$.

Proof. Our first objective is to construct a parameter $R^* > 0$ verifying $\mathfrak{N} : B_{R^*} \rightarrow B_{R^*}$. By (17) of condition (H_5) , there exists $\gamma' > \gamma$, satisfies

$$K' = lM^*d_1 + d\gamma'(n-1 + h_0 + \rho_0) + lM^* \sum_{k=1}^m d_{I_k} < 1. \quad (18)$$

By (16), there exists $r_0 > 0$, for $\|\chi_1\| + \|\chi_2\| + \dots + \|\chi_{n+1}\| > r_0$ filling

$$\|\Xi(s, \chi_1, \chi_2, \dots, \chi_{n+1})\| < \gamma'(\|\chi_1\| + \|\chi_2\| + \dots + \|\chi_{n+1}\|), s \in J, \quad (19)$$

then for $s \in J, \chi_1, \chi_2, \dots, \chi_{n+1} \in E$, we have

$$\|\Xi(s, \chi_1, \chi_2, \dots, \chi_{n+1})\| \leq \gamma'(\|\chi_1\| + \|\chi_2\| + \dots + \|\chi_{n+1}\|) + M, \quad (20)$$

where

$$M = \sup_{s \in J, \|\chi_1\| + \|\chi_2\| + \dots + \|\chi_{n+1}\| \leq r_0} \|\Xi(s, \chi_1, \chi_2, \dots, \chi_{n+1})\| < +\infty. \quad (21)$$

For fixed $R > 0, \forall u \in B_R = \{u \in PC(J, E) : \|u\|_{PC} \leq R\}, \kappa \in J$, we have

$$\|u(\tau_i(\kappa))\| \leq \|u\|_{PC} \leq R, i = 1, 2, \dots, n-2. \quad (22)$$

By (H_1) , one has

$$\begin{aligned} \|(\overline{\mathcal{H}u})(\kappa)\| &= \left\| \int_0^\kappa \hbar(\kappa, \tau, u(\tau)) d\tau \right\| \leq \int_0^\kappa \|\hbar(\kappa, \tau, u(\tau))\| d\tau \\ &\leq \int_0^\kappa (\hbar_1(\kappa, \tau) \|u\|_{PC} + \zeta_h(\kappa, \tau)) d\tau \\ &\leq \hbar_0 R + \bar{\hbar}_0, t \in J, \end{aligned} \quad (23)$$

where $\hbar_0, \bar{\hbar}_0$ are as (H_1) . Similarly,

$$\|(\overline{\Lambda u})(\kappa)\| = \left\| \int_0^T \rho(\kappa, s, u(s)) ds \right\| \leq \int_0^T \|\rho(\kappa, s, u(s))\| ds \leq \rho_0 R + \bar{\rho}_0, \kappa \in J, \quad (24)$$

where $\rho_0, \bar{\rho}_0$ are as (H_1) . Hence, for $\forall \kappa \in J, u \in B_R$, one has

$$\begin{aligned} & \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| \\ & \leq \gamma' \left(\|u(s)\| + \|u(\tau_1(s))\| + \dots + \|u(\tau_{n-2}(s))\| + \|(\overline{\mathcal{H}}u)(s)\| \right. \\ & \quad \left. + \|(\overline{\Lambda}u)(s)\| \right) + M \\ & \leq \gamma' \left((n-1)R + \hbar_0 R + \bar{\hbar}_0 + \rho_0 R + \bar{\rho}_0 \right) + M \\ & = \gamma' (n-1 + \hbar_0 + \rho_0) R + \gamma' (\bar{\hbar}_0 + \bar{\rho}_0) + M, \end{aligned} \quad (25)$$

where M is a constant, γ' is a constant and $\gamma' > \gamma$. For $u \in B_R$, by the definition of operator $\aleph$ and Formula (25), for every $\kappa \in [\kappa_0, \kappa_1]$, one has

$$\begin{aligned} & \|(\aleph u)(\kappa)\| \\ & \leq lM^*(\|u_0\| + \|g(u)\|) \\ & \quad + \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\ & \quad + \frac{\alpha l^2}{B(\alpha)} \int_0^\kappa \|\aleph_\alpha(\kappa-s)\| \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\ & \leq lM^*(\|u_0\| + \|g(u)\|) \\ & \quad + \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\ & \quad + \frac{\alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\ & \leq lM^*(\|u_0\| + \|g(u)\|) + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(1-\alpha), M^* \alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \\ & \quad \times \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds. \end{aligned}$$

Continuing to simplify above formula, we have

$$\begin{aligned} & \|(\aleph u)(\kappa)\| \\ & \leq lM^*(\|u_0\| + \|g(u)\|) + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(1-\alpha), M^* \alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \\ & \quad \times \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds + l^m M^* \sum_{i=1}^m (d_{I_i} \|u\| + \zeta_{I_i}) \\ & \leq lM^*(\|u_0\| + \|g(u)\|) + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(mM^*l^m + (1-\alpha)), M^* \alpha l^2 (ml^m + 1)\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \\ & \quad \times \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds + l^m M^* \sum_{i=1}^m (d_{I_i} \|u\| + \zeta_{I_i}) \\ & \leq lM^* \left(\|u_0\| + d_1 \|u\|_{PC} + \zeta_1 \right) + l^m M^* \sum_{i=1}^m (d_{I_i} R + \zeta_{I_i}) \\ & \quad + \bar{d} \left(\frac{1}{\alpha} T^\alpha + T \right) \left(\gamma' (n-1 + \hbar_0 + \rho_0) R + \gamma' (\bar{\hbar}_0 + \bar{\rho}_0) + M \right). \end{aligned}$$

After further simplifying, we have

$$\begin{aligned}
 & \|(\aleph u)(\kappa)\| \\
 & \leq \tilde{l}M^* \|u_0\| + \bar{l}M^* \zeta_1 + \hat{d} \left(\gamma'(\bar{h}_0 + \bar{\rho}_0) + M \right) + l^m M^* \sum_{i=1}^m \zeta_{I_i} \\
 & \quad + \left\{ \tilde{l}M^* d_1 + \hat{d} \gamma' \left(n - 1 + \bar{h}_0 + \rho_0 \right) + l^m M^* \sum_{i=1}^m d_{I_i} \right\} R \\
 & \leq C + K'R \leq R,
 \end{aligned} \tag{26}$$

where $C > 0$ is a constant, K' is in (18), $\bar{d}$ is from (H_5) , $\tilde{l} = \max\{l, l^{m+1}\}$, $\hat{d} = \bar{d}(\frac{1}{\alpha}T^\alpha + T)$. By the definition of operator $\aleph$, formula (25) and the conditions (H_1) – (H_5) , for $u \in B_R$, $\kappa \in (\kappa_i, \kappa_{i+1}]$, one has

$$\begin{aligned}
 & \|(\aleph u)(\kappa)\| \\
 & \leq l^{m+1}M^* (\|u_0\| + \|g(u)\|) \\
 & \quad + \frac{mM^* l^{m+1}\bar{l} + \bar{l}\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
 & \quad + \frac{m\alpha l^{m+2} + \alpha l^2}{B(\alpha)} \int_0^\kappa \|\Re_\alpha(\kappa-s)\| \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
 & \quad + lM^* \sum_{0 < \kappa_i < \kappa} \|I_i(u(\kappa_i))\| \\
 & \leq lM^* (\|u_0\| + \|g(u)\|) \\
 & \quad + \frac{mM^* l^{m+1}\bar{l} + \bar{l}\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
 & \quad + \frac{m\alpha l^{m+2}M^* + \alpha l^2M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
 & \quad + l^k M^* \sum_{i=1}^m (d_{I_i} \|u\| + \zeta_{I_i}) \\
 & \leq l^{m+1}M^* (\|u_0\| + \|g(u)\|) + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}\bar{l}(mM^* l^m + (1-\alpha)), M^* \alpha l^2 (ml^m + 1)\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \\
 & \quad \times \left\| \Xi(\kappa, u(\kappa), u(\tau_1(\kappa)), \dots, u(\tau_{n-2}(\kappa)), (\overline{\mathcal{H}}u)(\kappa), (\overline{\Lambda}u)(\kappa)) \right\| ds + l^m M^* \sum_{i=1}^m (d_{I_i} \|u\| + \zeta_{I_i}) \\
 & \leq l^{m+1}M^* \left(\|u_0\| + d_1 \|u\|_{PC} + \zeta_1 \right) + l^m M^* \sum_{i=1}^m (d_{I_i} R + \zeta_{I_i}) \\
 & \quad + \bar{d} \left(\frac{1}{\alpha} T^\alpha + T \right) \left(\gamma' (n-1 + \bar{h}_0 + \rho_0) R + \gamma' (\bar{h}_0 + \bar{\rho}_0) + M \right) \\
 & \leq \tilde{l}M^* \|u_0\| + \bar{l}M^* \zeta_1 + \hat{d} \left(\gamma' (\bar{h}_0 + \bar{\rho}_0) + M \right) + l^m M^* \sum_{i=1}^m \zeta_{I_i} \\
 & \quad + \left\{ \tilde{l}M^* d_1 + \hat{d} \gamma' \left(n - 1 + \bar{h}_0 + \rho_0 \right) + l^m M^* \sum_{i=1}^m d_{I_i} \right\} R \\
 & \leq C + K'R \leq R,
 \end{aligned} \tag{27}$$

where $C > 0$ is as (26), K' is in (18), $\bar{d}$ is from (H_5) , $\bar{l}$ is from (26), $\hat{d} = \bar{d}(\frac{1}{\alpha}T^\alpha + T)$. By (18), (26) and (27), we only need take

$$R^* > (1 - K')^{-1}C, \tag{28}$$

then for any $u \in B_{R^*}$, we have $\|\aleph u\| \leq R^*$, that is, $\aleph : B_{R^*} \rightarrow B_{R^*}$.

As evidenced by the above arguments, $\aleph : B_{R^*} \rightarrow B_{R^*}$ is norm-bounded in the operator topology. We proceed to demonstrate the continuity of $\aleph : B_{R^*} \rightarrow B_{R^*}$ within the operator topology. Let $u_n \rightarrow u$ in B_{R^*} , then $\|u_n\|_{PC} \leq R^*$ ($n = 1, 2, \dots$), $\|u\|_{PC} \leq R^*$, combine the definition of (17), for any $\kappa \in (\kappa_0, \kappa_1]$, one arrives

$$\begin{aligned}
& \|\aleph u_n(\kappa) - \aleph u(\kappa)\| \leq \|\mathcal{G}\mathfrak{S}_\alpha(\kappa)\| \|g(u_n) - g(u)\| \\
& + \left\| \frac{\mathcal{K}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \right. \\
& \left. \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \right. \\
& + \left\| \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^\kappa \|\aleph_\alpha(\kappa-s)\| \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \right. \\
& \left. \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \right. \\
& \leq IM^* \|g(u_n) - g(u)\| \\
& + \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
& \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& + \frac{\alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
& \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \tag{29} \\
& \leq IM^* \|g(u_n) - g(u)\| \\
& + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(1-\alpha), M^* \alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \right. \\
& \left. \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& \leq IM^* \|g(u_n) - g(u)\| \\
& + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(mM^*l^m + (1-\alpha)), M^* \alpha l^2(ml^m + 1)\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \right. \\
& \left. \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& \leq \tilde{I}M^* \|g(u_n) - g(u)\| \\
& + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
& \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& \leq I + II,
\end{aligned}$$

where $I = \tilde{I}M^* \|g(u_n) - g(u)\| + l^m M^* \sum_{i=1}^m \|I_i(u_n(\kappa_i)) - I_i(u(\kappa_i))\|$,

$$\begin{aligned}
II = & \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \times \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
& \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds,
\end{aligned}$$

$\bar{d}$ is as Formula (16).

For any $\kappa \in (\kappa_i, \kappa_{i+1}]$, one has

$$\begin{aligned}
& \|\mathfrak{N}u_n(\kappa) - \mathfrak{N}u(\kappa)\| \leq \|\mathcal{G}^{k+1}\mathfrak{S}_\alpha(\kappa)\| \|g(u_n) - g(u)\| \\
& + \left\| \frac{\mathcal{K}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \right\| \left\| \sum_{i=1}^k \mathcal{G}^{k+2-i}\mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s))) \right. \\
& \left. (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& + \left\| \frac{\mathcal{K}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \right\| \left\| \int_{\kappa_k}^\kappa (\kappa - s)^{\alpha-1} \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right. \right. \\
& \left. \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \right. \\
& + \left\| \frac{\alpha}{B(\alpha)} \right\| \left\| \sum_{i=1}^k \|\mathcal{G}\|^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \|\mathfrak{R}_\alpha(\kappa - s)\| \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right. \right. \\
& \left. \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \right. \\
& + \left\| \frac{\alpha\mathcal{G}^2}{B(\alpha)} \right\| \left\| \int_{\kappa_k}^\kappa \|\mathfrak{R}_\alpha(\kappa - s)\| \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right. \right. \\
& \left. \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \right. \\
& + \sum_{i=1}^k \|\mathcal{G}^{k+1-i}\mathfrak{S}_\alpha(\kappa - \kappa_i)\| \|I_i(u_n(\kappa_i)) - I_i(u(\kappa_i))\| \\
& \leq l^{m+1}M^* \|g(u_n) - g(u)\| + l^m M^* \sum_{i=1}^m \|I_i(u_n(\kappa_i)) - I_k(u(\kappa_i))\| \\
& + \frac{mM^*l^{m+1}\bar{l} + \bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
& \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& + \frac{m\alpha l^{m+2}M^* + \alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
& \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& \leq l^{m+1}M^* \|g(u_n) - g(u)\| + l^m M^* \sum_{i=1}^m \|I_i(u_n(\kappa_i)) - I_i(u(\kappa_i))\| \\
& + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{l\bar{l}(mM^*l^m + (1-\alpha)), M^*\alpha l^2(ml^m + 1)\} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \right. \\
& \left. \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& = l^{m+1}M^* \|g(u_n) - g(u)\| + l^m M^* \sum_{i=1}^m \|I_i(u_n(\kappa_i)) - I_k(u(\kappa_i))\| \\
& + \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
& \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
& \leq I + II,
\end{aligned} \tag{30}$$

where I, II are as (29), $\bar{d}$ is as Formula (16).

From (H_2) and (H_3) , we know that g and I_k are continuous, then we easily get

$$I = \tilde{l}M^* \|g(u_n) - g(u)\| + l^m M^* \sum_{i=1}^m \|I_i(u_n(\kappa_i)) - I_i(u(\kappa_i))\| \rightarrow 0, n \rightarrow \infty, \tag{31}$$

By (22)–(25), we have

$$\begin{aligned}
 II &= \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
 &\quad \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
 &\leq 2\bar{d} \left(\frac{1}{\alpha} T^\alpha + T \right) T(\gamma'(n-1 + \hbar_0 + \rho_0)\bar{U} + \gamma'(\bar{\hbar}_0 + \bar{\rho}_0) + M) \\
 &= 2\hat{d}T(\gamma'(n-1 + \hbar_0 + \rho_0)\bar{U} + \gamma'(\bar{\hbar}_0 + \bar{\rho}_0) + M) \\
 &< +\infty,
 \end{aligned} \tag{32}$$

where $\hat{d}$ is from formula (16). From $\Xi : J \times T_R^{n+1} \rightarrow E$ is bounded and uniformly continuous, it is easy to see that for any fixed $s \in J$,

$$\begin{aligned}
 &\left\| ((\kappa - s)^{\alpha-1} + 1) \left[\Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \right. \\
 &\quad \left. \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right] \right\| \rightarrow 0, \quad n \rightarrow \infty,
 \end{aligned} \tag{33}$$

then by (32), (33) and the Lebesgue control convergence theorem, we have

$$\begin{aligned}
 II &= \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left\| \Xi(s, u_n(s), u_n(\tau_1(s)), \dots, u_n(\tau_{n-2}(s)), (\overline{\mathcal{H}}u_n)(s), (\overline{\Lambda}u_n)(s)) \right. \\
 &\quad \left. - \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) \right\| ds \\
 &\rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned} \tag{34}$$

Formulas (28)–(34) imply that $\aleph : B_{R^*} \rightarrow B_{R^*}$ is continuous. By the above proof, we have $\aleph(B_{R^*})$ is bounded. Now we prove the set $\aleph(B_{R^*}) \subset PC(J, E)$ is the precompactness set. Given an arbitrary but fixed κ ($0 < \kappa \leq T$) and any ϵ : $0 < \epsilon < \kappa$, let $u \in B_{R^*}$, one defines

$$\aleph_\epsilon u(\kappa) = \begin{cases} \mathcal{G}\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^{\kappa-\epsilon} (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
 + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^{\kappa-\epsilon} \mathfrak{R}_\alpha(t-s) \bar{q}(s, u(s)) ds, \kappa \in (0, \kappa_1], \\
 \mathcal{G}^{k+1}\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i}\mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
 + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(\kappa - s) \bar{q}(s, u(s)) ds \\
 + \sum_{i=1}^k \mathcal{G}^{k+1-i}\mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(u(\kappa_i^-)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_i}^{\kappa-\epsilon} (\kappa-s)^{\alpha-1} \bar{q}(s, u(s)) ds \\
 + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_i}^{\kappa-\epsilon} \mathfrak{R}_\alpha(\kappa - s) \bar{q}(s, u(s)) ds, \kappa \in (\kappa_i, \kappa_{i+1}], i = 1, 2, \dots, m, \end{cases} \tag{35}$$

for

$$\bar{q}(s, u(s)) = \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)).$$

According to the compactness of $\mathfrak{S}_\alpha(t)$, for any $\epsilon(0 < \epsilon < \kappa)$, the set $\{(\mathfrak{N}_\epsilon u)(t) : u \in B_{R^*}\}$ is precompact on E , for any $u \in B_{R^*}, \kappa \in J$, one has

$$\begin{aligned}
 & \|(\mathfrak{N}u)(\kappa) - (\mathfrak{N}_\epsilon u)(\kappa)\| \\
 & \leq \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa-\epsilon}^{\kappa} (\kappa-s)^{\alpha-1} \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\
 & \quad + \frac{\alpha l^2}{B(\alpha)} \int_{\kappa-\epsilon}^{\kappa} \|\mathfrak{R}_\alpha(\kappa-s) \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\
 & \leq \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa-\epsilon}^{\kappa} (\kappa-s)^{\alpha-1} \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\
 & \quad + \frac{\alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_{\kappa-\epsilon}^{\kappa} \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\
 & \leq \frac{l}{B(\alpha)\Gamma(\alpha)} \max\{\bar{l}(1-\alpha), \alpha l M^*\} \\
 & \quad \times \int_{\kappa-\epsilon}^{\kappa} ((\kappa-s)^{\alpha-1} + 1) \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\
 & = \tilde{d} \int_{\kappa-\epsilon}^{\kappa} ((\kappa-s)^{\alpha-1} + 1) \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\
 & \leq \tilde{d} [\gamma'(n-1 + \hbar_0 + \rho_0) R^* + \gamma'(\bar{\hbar}_0 + \bar{\rho}_0) + M] \int_{\kappa-\epsilon}^{\kappa} ((\kappa-s)^{\alpha-1} + 1) ds \\
 & = \frac{\tilde{d}}{\alpha} [\gamma'(n-1 + \hbar_0 + \rho_0) R^* + \gamma'(\bar{\hbar}_0 + \bar{\rho}_0) + M] (\epsilon^\alpha + \epsilon),
 \end{aligned} \tag{36}$$

in which $\tilde{d} = \frac{l}{B(\alpha)\Gamma(\alpha)} \max\{\bar{l}(1-\alpha), \alpha l M^*\}$, hence, for any $\kappa \in J$, $\{(\mathfrak{N}u)(\kappa) : u \in B_{R^*}\}$ is totally bounded, that is, for any $\kappa \in J$, the set $\{(\mathfrak{N}u)(\kappa) : u \in B_{R^*}\}$ is precompact in E . By Arzela–Ascoli theorem, we have $\{\mathfrak{N}u : u \in B_{R^*}\}$ is to be relatively compact on $PC(J, E)$. Hence, $\mathfrak{N} : B_{R^*} \rightarrow B_{R^*}$ is completely continuous operator. As a result, Equation (1) gives rise to at least one global mild solution on $B_{R^*} \subset PC(J, E)$ by the Schauder fixed point theorem. The proof is complete. $\square$

Remark 4. Theorem 1 is completed under the condition of $\mathfrak{S}_\alpha(t)$ being compact; hence, we do not need the non-compactness measure conditions and the Lipschitz conditions.

3.2. The Well-Posedness of Mild Solutions Under the Weakened Lipschitz Conditions

The conditions for this section are as follows.

(C₁) Assume that $\Xi : J \times E^{n+1} \rightarrow E$ has a continuous property, and we can find non-negative and is integrable functions with respect to Lebesgue measure $L_i \in L^p(J, \mathbb{R}^+)(i = 1, 2, \dots, n+1)$ with $p > \frac{1}{\alpha}$, for any $u_i, v_i \in E, \kappa \in J$, such that

$$\|\Xi(\kappa, u_1, u_2, \dots, u_{n+1}) - \Xi(\kappa, v_1, v_2, \dots, v_{n+1})\| \leq \sum_{i=1}^{n+1} L_i(\kappa) \|u_i - v_i\|. \tag{37}$$

(C₂) There exist integrable functions $L_h(\kappa, \cdot), L_\rho(\kappa, \cdot) \in L(J, \mathbb{R}^+), \mathbb{R}^+ = [0, +\infty)$, for any $u, v \in E$, such that

$$L_{h_0} = \max_{\kappa \in [0, T]} \int_0^\kappa L_h(\kappa, s) ds < +\infty, L_{\rho_0} = \max_{\kappa \in [0, T]} \int_0^T L_\rho(\kappa, s) ds < +\infty,$$

$$\|\hbar(\kappa, s, u) - \hbar(\kappa, s, v)\| \leq L_h(\kappa, s) \|u - v\|, (\kappa, s) \in \Theta_1, \tag{38}$$

$$\|\rho(\kappa, s, u) - \rho(\kappa, s, v)\| \leq L_\rho(\kappa, s) \|u - v\|, (\kappa, s) \in \Theta_2. \tag{39}$$

(C₃) It is possible to select parameters $L_{I_i} > 0$, for any $u, v \in E$, subject to

$$\|I_i(u) - I_i(v)\| \leq L_{I_i} \|u - v\|, i = 1, 2, \dots, m. \tag{40}$$

(C₄) It is possible to select a parameter $L_g > 0$, for any $u, v \in PC(J, E)$, such that

$$\|g(u) - g(v)\| \leq L_g \|u - v\|_{PC}. \quad (41)$$

(C₅) For the Lebesgue integrable function $L_i \in L^p(J, \mathbb{R}^+)$ ($i = 1, 2, \dots, n+1$) in (C₁), the constants in (C₁)–(C₄) and (H₄) satisfy

$$\varrho = M^* \tilde{l} \left(L_g + \sum_{i=1}^m L_{I_k} \right) + \bar{d} \iota < 1, \quad (42)$$

where $\bar{d}, \tilde{l}$ are as (16) and

$$\iota = l_0 \left(\int_0^T L^p(s) ds \right)^{\frac{1}{p}},$$

where L_{h_0}, L_{ρ_0} are from (C₂),

$$L(\kappa) = \sum_{i=1}^{n-1} L_i(\kappa) + L_{h_0} L_n(\kappa) + L_{\rho_0} L_{n+1}(\kappa), \quad l_0 = 2^{\frac{q-1}{q}} \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1} + T \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(C₆) The constants L_{I_k} in (C₃) and l in (H₄), satisfy

$$l^m M^* \sum_{i=1}^m L_{I_i} = j < 1.$$

Lemma 13. Let $0 < \alpha < 1$, suppose that $L \in L^p(J, \mathbb{R}^+)$ with $p > \frac{1}{\alpha}$, then for every $\kappa \in J$

$$\int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) L(s) ds \leq l_0 \left(\int_0^T L^p(s) ds \right)^{\frac{1}{p}}.$$

Proof. By the Hölder inequality and Minkowski inequality, one has

$$\begin{aligned} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) L(s) ds &\leq \left(\int_0^\kappa ((\kappa - s)^{\alpha-1} + 1)^q ds \right)^{\frac{1}{q}} \left(\int_0^\kappa L^p ds \right)^{\frac{1}{p}} \\ &\leq 2^{\frac{q-1}{q}} \left(\int_0^\kappa ((\kappa - s)^{q(\alpha-1)} + 1^q) ds \right)^{\frac{1}{q}} \left(\int_0^T L^p(s) ds \right)^{\frac{1}{p}} \\ &= 2^{\frac{q-1}{q}} \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1} + T \right)^{\frac{1}{q}} \left(\int_0^T L^p(s) ds \right)^{\frac{1}{p}} \\ &= l_0 \left(\int_0^T L^p(s) ds \right)^{\frac{1}{p}}, \end{aligned}$$

where $l_0 = 2^{\frac{q-1}{q}} \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1} + T \right)^{\frac{1}{q}}, \frac{1}{p} + \frac{1}{q} = 1. \quad \square$

Theorem 2. Suppose conditions (C₁)–(C₅) and (H₄) hold, then the model (1) admits exactly one globally mild solution in the phase space $PC(J, E)$, for every $u_0 \in PC(J, E)$, the sequence of iterations $\{u_n\}$ be defined by

$$u_n(\kappa) = \begin{cases} \mathcal{G}\mathfrak{S}_\alpha(\kappa)(u_0 + g(u)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u_{n-1}(s)) ds \\ + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^t \mathfrak{R}_\alpha(t-s) \bar{q}(s, u_{n-1}(s)) ds, \kappa \in (0, \kappa_1], \\ \mathcal{G}^{k+1}\mathfrak{S}_\alpha(\kappa)(u_0 + g(u_{n-1})) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \bar{q}(s, u_{n-1}(s)) ds \\ + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(t-s) \bar{q}(s, u_{n-1}(s)) ds \\ + \sum_{i=1}^k \mathcal{G}^{k+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(u(\kappa_i^-)) + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_i}^\kappa (\kappa-s)^{\alpha-1} \bar{q}(s, u_{n-1}(s)) ds \\ + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_i}^\kappa \mathfrak{R}_\alpha(\kappa-s) \bar{q}(s, u_{n-1}(s)) ds, \kappa \in (\kappa_i, \kappa_{i+1}), i = 1, 2, \dots, m, \end{cases} \quad (43)$$

for

$$\bar{q}(s, u(s)) = \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}u})(s), (\overline{\Lambda u})(s)),$$

uniformly converges to a unique global mild solution $u^*(t)$ of the model (1) about the norm of E on $t \in J$ and for q that defined by (42), one has

$$\|u_n - u^*\| = o(q^n), n \rightarrow +\infty. \quad (44)$$

Proof. For every $\kappa \in [\kappa_0, \kappa_1]$ ($\kappa_0 = 0$), $u, v \in PC(J, E)$, by Lemma 13, one has

$$\begin{aligned} & \|(\mathfrak{N}u)(\kappa) - (\mathfrak{N}v)(\kappa)\| \\ & \leq IM^* \|g(u) - g(v)\| \\ & \quad + \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}u})(s), \\ & \quad (\overline{\Lambda u})(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}v})(s), (\overline{\Lambda v})(s))\| ds \\ & \quad + \frac{\alpha l^2}{B(\alpha)} \int_0^\kappa \|\mathfrak{R}_\alpha(\kappa-s)\| \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}u})(s), \\ & \quad (\overline{\Lambda u})(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}v})(s), (\overline{\Lambda v})(s))\| ds \\ & \leq IM^* L_g \|u - v\|_{PC} + \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} (L_1(s) \|u - v\|_{PC} \\ & \quad + L_2(s) \|u - v\|_{PC} + \dots + L_{n-1}(s) \|u - v\|_{PC} \\ & \quad + L_n(s) \|\overline{\mathcal{H}u} - \overline{\mathcal{H}v}\|_{PC} + L_{n+1}(s) \|\overline{\Lambda u} - \overline{\Lambda v}\|_{PC}) ds \\ & \quad + \frac{\alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (L_1(s) \|u - v\|_{PC} + L_2(s) \|u - v\|_{PC} + \dots \\ & \quad + L_{n-1}(s) \|u - v\|_{PC} + L_n(s) \|\overline{\mathcal{H}u} - \overline{\mathcal{H}v}\|_{PC} + L_{n+1}(s) \|\overline{\Lambda u} - \overline{\Lambda v}\|_{PC}) ds \\ & \leq \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(1-\alpha), M^* \alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) \|u - v\|_{PC} \\ & \quad + L_2(s) \|u - v\|_{PC} + \dots + L_{n-1}(s) \|u - v\|_{PC} + L_n(s) \|\overline{\mathcal{H}u} - \overline{\mathcal{H}v}\|_{PC} \\ & \quad + L_{n+1}(s) \|\overline{\Lambda u} - \overline{\Lambda v}\|_{PC}) ds + M^* l \sum_{k=1}^m L_{I_k} \|u - v\|_{PC} + IM^* L_g \|u - v\|_{PC} \\ & = \tilde{l} M^* L_g \|u - v\|_{PC} + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) \|u - v\|_{PC} + L_2(s) \|u - v\|_{PC} + \dots \\ & \quad + L_{n-1}(s) \|u - v\|_{PC} + L_n(s) \|\overline{\mathcal{H}u} - \overline{\mathcal{H}v}\|_{PC} + L_{n+1}(s) \|\overline{\Lambda u} - \overline{\Lambda v}\|_{PC}) ds \\ & \leq M^* \tilde{l} L_g \|u - v\|_{PC} \\ & \quad + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) + L_2(s) + \dots + L_{h_0} L_n(s) + L_{\rho_0} L_{n+1}(s)) \|u - v\|_{PC} ds \\ & \leq M^* \tilde{l} L_g \|u - v\|_{PC} + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) L(s) ds \cdot \|u - v\|_{PC} \\ & \leq (M^* \tilde{l} L_g + \bar{d} \iota) \|u - v\|_{PC} \leq \varrho \|u - v\|_{PC}, \end{aligned} \quad (45)$$

where $\bar{d}, \tilde{l}, \iota, \varrho, L(s)$ are from the condition (C₅), L_{h_0}, L_{ρ_0} are from (C₂).

For any $\kappa \in (\kappa_i, \kappa_{i+1}]$, $u, v \in PC(J, E)$, as the same way with (45), by Lemma 13, one has

$$\begin{aligned}
& \|(\mathfrak{N}u)(\kappa) - (\mathfrak{N}v)(\kappa)\| \\
& \leq l^{m+1} M^* \|g(u) - g(v)\| \\
& \quad + \frac{m M^* l^{m+1} \bar{l} + \bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), \\
& \quad (\overline{\Lambda}u)(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}}v)(s), (\overline{\Lambda}v)(s))\| ds \\
& \quad + \frac{m\alpha l^{m+2} + \alpha l^2}{B(\alpha)} \int_0^\kappa \|\mathfrak{R}_\alpha(\kappa-s)\| \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), \\
& \quad (\overline{\Lambda}u)(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}}v)(s), (\overline{\Lambda}v)(s))\| ds \\
& \quad + l^m \left\| \sum_{0 < \kappa_i < t} \mathfrak{S}_\alpha(\kappa - \kappa_i) (I_k(u(\kappa_k)) - I_k(v(\kappa_k))) \right\| \\
& \leq l^{m+1} M^* L_g \|u - v\|_{PC} + \frac{m M^* l^{m+1} \bar{l} + \bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} (L_1(s) \|u - v\|_{PC} \\
& \quad + L_2(s) \|u - v\|_{PC} + \dots + L_{n-1}(s) \|u - v\|_{PC} \\
& \quad + L_n(s) \|\overline{\mathcal{H}}u - \overline{\mathcal{H}}v\|_{PC} + L_{n+1}(s) \|\overline{\Lambda}u - \overline{\Lambda}v\|_{PC}) ds \\
& \quad + \frac{m\alpha l^{m+2} M^* + \alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (L_1(s) \|u - v\|_{PC} + L_2(s) \|u - v\|_{PC} + \dots \\
& \quad + L_{n-1}(s) \|u - v\|_{PC} + L_n(s) \|\overline{\mathcal{H}}u - \overline{\mathcal{H}}v\|_{PC} + L_{n+1}(s) \|\overline{\Lambda}u - \overline{\Lambda}v\|_{PC}) ds \\
& \quad + M^* l^m \sum_{i=1}^m L_{I_i} \|u(\kappa_i) - v(\kappa_i)\| \tag{46} \\
& \leq \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(m M^* l^m + (1-\alpha)), M^* \alpha l^2 (m l^m + 1)\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) \|u - v\|_{PC} \\
& \quad + L_2(s) \|u - v\|_{PC} + \dots + L_{n-1}(s) \|u - v\|_{PC} + L_n(s) \|\overline{\mathcal{H}}u - \overline{\mathcal{H}}v\|_{PC} \\
& \quad + L_{n+1}(s) \|\overline{\Lambda}u - \overline{\Lambda}v\|_{PC}) ds + M^* l \sum_{i=1}^m L_{I_i} \|u - v\|_{PC} + l M^* L_g \|u - v\|_{PC} \\
& = l^{m+1} M^* L_g \|u - v\|_{PC} + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) \|u - v\|_{PC} + L_2(s) \|u - v\|_{PC} + \dots \\
& \quad + L_{n-1}(s) \|u - v\|_{PC} + L_n(s) \|\overline{\mathcal{H}}u - \overline{\mathcal{H}}v\|_{PC} + L_{n+1}(s) \|\overline{\Lambda}u - \overline{\Lambda}v\|_{PC}) ds \\
& \quad + M^* l^m \sum_{i=1}^m L_{I_i} \|u - v\|_{PC} \\
& \leq M^* l^{m+1} L_g \|u - v\|_{PC} + M^* l^m \sum_{i=1}^m L_{I_i} \|u - v\|_{PC} + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) L(s) ds \cdot \|u - v\|_{PC} \\
& \leq M^* \bar{l} L_g \|u - v\|_{PC} + M^* l^m \sum_{i=1}^m L_{I_i} \|u - v\|_{PC} + \bar{d} l_0 \left(\int_0^T L^p(s) ds \right)^{\frac{1}{p}} \cdot \|u - v\|_{PC} \\
& \leq \left(M^* \bar{l} \left(L_g + \sum_{i=1}^m L_{I_i} \right) + \bar{d} l_0 \right) \|u - v\|_{PC} \\
& = \varrho \|u - v\|_{PC},
\end{aligned}$$

where $\bar{d}, \bar{l}, l, \varrho, L(s)$ are from the condition (C_5) . Hence, by (45) and (46), we have $\|\mathfrak{N}u - \mathfrak{N}v\|_{PC} \leq \varrho \|u - v\|_{PC}$. According to (C_5) , $\mathfrak{N}$ admits a unique fixed point $u^* \in PC(J, E)$ by using of the Banach compression fixed point theorem, that is, a mild solution is admitted for model (1), and subsequently (44) holds. $\square$

Remark 5. Theorem 2 is proved by the Banach compression principle under the generalized Lipschitz conditions.

Theorem 3. Suppose the condition (C_1) – (C_3) and (C_6) , (H_4) hold, $g(u) = 0$, $L_{n+1}(\kappa) \equiv 0$ in (C_1) , then the model (4) admits a unique global mild solution in $PC(J, E)$, that means model (1) admits a unique global mild solution; moreover, for any $u_0 \in PC(J, E)$, the iterative sequence $\{u_n\}$ defined by (43) uniformly converges to a unique global mild solution $u^*(t)$ of the model (4) about the norm of E on $t \in J$; that means model (1) admits a unique mild solution, and for any real number $s > 1$, one has

$$\|u_n - u^*\| = o\left(\frac{1}{n^s}\right), n \rightarrow +\infty.$$

Proof. For every $\kappa \in [\kappa_0, \kappa_1]$ ($\kappa_0 = 0$), $u, v \in PC(J, E)$, one has

$$\begin{aligned} & \|(\mathfrak{N}u)(\kappa) - (\mathfrak{N}v)(\kappa)\| \\ & \leq \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), \right. \\ & \quad \left. (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}}v)(s), (\overline{\Lambda}v)(s)) \right\| ds \\ & \quad + \frac{\alpha l^2}{B(\alpha)} \int_0^\kappa \|\mathfrak{R}_\alpha(\kappa-s)\| \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), \right. \\ & \quad \left. (\overline{\Lambda}u)(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}}v)(s), (\overline{\Lambda}v)(s)) \right\| ds \\ & \leq \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left(L_1(s)\|u-v\|_{PC} + L_2(s)\|u-v\|_{PC} + \dots + L_{n-1}(s)\|u-v\|_{PC} \right. \\ & \quad \left. + L_n(s)\|\overline{\mathcal{H}}u - \overline{\mathcal{H}}v\|_{PC} \right) ds \\ & \quad + \frac{\alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \left(L_1(s)\|u-v\|_{PC} + L_2(s)\|u-v\|_{PC} + \dots + L_{n-1}(s)\|u-v\|_{PC} \right. \\ & \quad \left. + L_n(s)\|(\overline{\mathcal{H}}u)(s) - (\overline{\mathcal{H}}v)(s)\| \right) ds \\ & \leq \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(1-\alpha), M^*\alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) + L_2(s) + \dots \\ & \quad + L_{n_0}L_n(s)) \|u-v\|_{PC} ds \\ & = \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \bar{B}(s) \|u(s) - v(s)\| ds + j \|u-v\|_{PC}, \end{aligned} \tag{47}$$

where $\bar{d}$ is shown in (C_5) , $j = l^m M^* \sum_{i=1}^m L_{I_i} < 1$ in (C_6) , $B(\kappa) = L_1(\kappa) + L_2(\kappa) + L_3(\kappa) + \dots + L_{n-1}(\kappa) + L_n(\kappa)L_{n_0}$.

For any $\kappa \in (\kappa_i, \kappa_{i+1}]$ ($i = 1, 2, \dots, m$), $u, v \in PC(J, E)$, one has

$$\begin{aligned}
 & \|(\aleph u)(\kappa) - (\aleph v)(\kappa)\| \\
 & \leq \frac{mM^*l^{m+1}\bar{l} + \bar{l}l(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), \right. \\
 & \quad \left. (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}}v)(s), (\overline{\Lambda}v)(s)) \right\| ds \\
 & \quad + \frac{m\alpha l^{m+2} + \alpha l^2}{B(\alpha)} \int_0^\kappa \|\Re_\alpha(\kappa-s)\| \left\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), \right. \\
 & \quad \left. (\overline{\Lambda}u)(s)) - \Xi(s, v(s), v(\tau_1(s)), \dots, v(\tau_{n-2}(s)), (\overline{\mathcal{H}}v)(s), (\overline{\Lambda}v)(s)) \right\| ds \\
 & \quad + \left\| \sum_{i=1}^k g^{k+1-i} \Im_\alpha(\kappa - \kappa_k) (I_k(u(\kappa_k)) - I_k(v(\kappa_k))) \right\| \\
 & \leq \frac{mM^*l^{m+1}\bar{l} + \bar{l}l(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} (L_1(s)\|u-v\|_{PC} + L_2(s)\|u-v\|_{PC} + \dots + L_{n-1}(s)\|u-v\|_{PC} \\
 & \quad + L_n(s)\|u-v\|_{PC}) ds \\
 & \quad + \frac{m\alpha l^{m+2}M^* + \alpha l^2M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (L_1(s)\|u-v\|_{PC} + L_2(s)\|u-v\|_{PC} + \dots + L_{n-1}(s)\|u-v\|_{PC} \\
 & \quad + L_n(s)\|(\overline{\mathcal{H}}u)(s) - (\overline{\mathcal{H}}v)(s)\|) ds + M^*l^m \sum_{i=1}^k L_{I_i}\|u-v\|_{PC} \\
 & \leq \frac{1}{\Gamma(\alpha)\bar{B}(\alpha)} \max\{l\bar{l}(mM^*l^m + (1-\alpha)), M^*\alpha l^2(ml^m + 1)\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) + L_2(s) + \dots \\
 & \quad + L_{n_0}(s))\|u-v\|_{PC} ds + M^*l^m \sum_{i=1}^m L_{I_i}\|u-v\|_{PC} \\
 & = \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \bar{B}(s)\|u(s) - v(s)\| ds + J\|u-v\|_{PC},
 \end{aligned} \tag{48}$$

where $\bar{d}, \bar{B}(t), J$ are from (47).

Then, by (47) and (48), for any $\kappa \in J, u, v \in PC(J, E)$, one has

$$\|(\aleph u)(\kappa) - (\aleph v)(\kappa)\| \leq \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \bar{B}(s)\|u(s) - v(s)\| ds + J\|u-v\|_{PC}. \tag{49}$$

By $0 < j < 1$, there exists $0 < \epsilon_1 < 1$, such that $0 < \ell = \epsilon_1 + j < 1$. From $\bar{B}(\kappa) = L_1(\kappa) + L_2(\kappa) + L_3(\kappa) + \dots + L_{n-1}(\kappa) + L_n(\kappa)L_{h_0} \in L^p(J, \mathbb{R})$, there has a continuous functions $\phi : J \rightarrow \mathbb{R}^+$, such that

$$\bar{d}l_0 \left(\int_0^T |\bar{B}(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} < \epsilon_1. \tag{50}$$

By the definition of the operator $\aleph$ and (49), (50) and Lemma 13, for any $\kappa \in J, u, v \in PC(J, E)$, one arrives

$$\begin{aligned}
 & \|(\aleph u)(\kappa) - (\aleph v)(\kappa)\| \\
 & \leq \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) |\bar{B}(s) - \phi(s)| \|u(s) - v(s)\| ds + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \phi(s) \|u(s) - v(s)\| ds \\
 & \quad + J\|u-v\|_{PC} \\
 & \leq \bar{d}l_0 \left(\int_0^T |\bar{B}(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} \|u-v\|_{PC} + \bar{d}l_0 \left(\int_0^\kappa ds \right)^{\frac{1}{q}} \|u-v\|_{PC} + J\|u-v\|_{PC} \\
 & \leq \bar{d}l_0 \left(\int_0^T |\bar{B}(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} \|u-v\|_{PC} + \bar{d}l_0 \left(\int_0^\kappa ds \right) \|u-v\|_{PC} + J\|u-v\|_{PC} \\
 & \leq \epsilon_1 \|u-v\|_{PC} + \bar{b}t \|u-v\|_{PC} + J\|u-v\|_{PC} \\
 & \leq (\ell + \bar{b}\kappa) \|u-v\|_{PC} = \left(C_1^0 \ell + C_1^1 \frac{\bar{b}\kappa}{1!} \right) \|u-v\|_{PC},
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $b = \max\{|\phi(\kappa)| : \kappa \in [0, T]\}$, $\bar{b} = \bar{d}bl_0$. Let us prove it by mathematical induction. For any natural number $n \in \mathbb{N}$, $\kappa \in J$, one has

$$\|(\aleph^n u)(\kappa) - (\aleph^n v)(\kappa)\| \leq \left(\ell^n + C_n^1 \ell^{n-1} (\bar{b}\kappa) + \frac{C_n^2 \ell^{n-2} (\bar{b}\kappa)^2}{2!} + \dots + \frac{(\bar{b}\kappa)^n}{n!} \right) \|u - v\|_{PC}. \quad (51)$$

Assume Formula (51) holds when $n = i$, that is, the following Formula (52) holds as follows:

$$\|(\aleph^i u)(\kappa) - (\aleph^i v)(\kappa)\| \leq \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}\kappa) + \frac{C_i^2 \ell^{i-2} (\bar{b}\kappa)^2}{2!} + \dots + \frac{(\bar{b}\kappa)^i}{i!} \right) \|u - v\|_{PC}. \quad (52)$$

Now we prove the Formula (51) hold when $n = i + 1$. By the definition of the operator $\aleph$, and the conditions (C_1) – (C_4) and (C_6) , (H_5) , for any $\kappa \in J$, one arrives

$$\begin{aligned} & \|(\aleph^{i+1} u)(\kappa) - (\aleph^{i+1} v)(\kappa)\| = \|\aleph(\aleph^i u)(\kappa) - \aleph(\aleph^i v)(\kappa)\| \\ & \leq \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left(L_1(s) \|(\aleph^i u)(s) - (\aleph^i v)(s)\| \right. \\ & \quad + L_2(s) \|(\aleph^i u)(\tau_1(s)) - (\aleph^i v)(\tau_1(s))\| + \dots + L_{n-1}(s) \|(\aleph^i u)(\tau_{n-2}(s)) - (\aleph^i v)(\tau_{n-2}(s))\| \\ & \quad \left. + L_n(s) \|(\aleph^i u)(s) - (\aleph^i v)(s)\| \right) ds \\ & \quad + M^* l \sum_{i=1}^m L_{I_i} \|\aleph^i u(\kappa_i) - \aleph^i v(\kappa_i)\| \\ & \leq \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) |\bar{B}(s) - \phi(s)| ds \|(\aleph^i u) - (\aleph^i v)\| + b \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \|(\aleph^i u)(s) - (\aleph^i v)(s)\| ds \\ & \quad + J \|(\aleph^i u) - (\aleph^i v)\| \\ & \leq \bar{d} l_0 \left(\int_0^T |\bar{B}(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} ds \|(\aleph^i u) - (\aleph^i v)\| + b \bar{d} l_0 \left(\int_0^\kappa ds \right)^{\frac{1}{q}} \|(\aleph^i u) - (\aleph^i v)\| \\ & \quad + J \|(\aleph^i u) - (\aleph^i v)\| \\ & \leq \bar{d} l_0 \left(\int_0^T |\bar{B}(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} ds \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) \|u - v\|_{PC} \\ & \quad + \bar{b} \int_0^t \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) \|u - v\|_{PC} ds \\ & \quad + J \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) \|u - v\|_{PC} \\ & \leq \epsilon_1 \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) \|u - v\|_{PC} \\ & \quad + \bar{b} \int_0^t \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) \|u - v\|_{PC} ds \\ & \quad + J \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) \|u - v\|_{PC} \\ & \leq \ell \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) \|u - v\|_{PC} \\ & \quad + \bar{b} \int_0^t \left(\ell^i + C_i^1 \ell^{i-1} (\bar{b}s) + \frac{C_i^2 \ell^{i-2} (\bar{b}s)^2}{2!} + \dots + \frac{(\bar{b}s)^i}{i!} \right) ds \|u - v\|_{PC} \\ & \leq \left(C_{i+1}^0 \ell^{i+1} + C_{i+1}^1 \ell^i (\bar{b}s) + \frac{C_{i+1}^2 \ell^{i-1} (\bar{b}s)^2}{2!} + \dots + C_{i+1}^{i+1} \frac{(\bar{b}s)^{i+1}}{(i+1)!} \right) \|u - v\|_{PC}, \end{aligned} \quad (53)$$

hence, for any natural number $n \in \mathbb{N}$, the Formula (51) holds. Then, let natural number $n \in \mathbb{N}$ be arbitrary, one has

$$\|(\aleph^n u) - (\aleph^n v)\|_{PC} \leq \left(\ell^n + C_n^1 \ell^{n-1} (bT) + \frac{C_n^2 \ell^{n-2} (bT)^2}{2!} + \dots + \frac{(bT)^n}{n!} \right) \|u - v\|_{PC}.$$

By Lemma 8, for $s > 1$, one has

$$\|\aleph^n u - \aleph^n v\|_{PC} = o\left(\frac{1}{n^s}\right) \|u - v\|_{PC}, n \rightarrow \infty. \quad (54)$$

Hence, by the generalized Banach compression fixed point theorem, $\aleph$ possesses a solitary fixed point u^* on $PC(J, E)$, then the model (1) admits a unique global mild solution $u^*(t)$, and (54) holds. $\square$

Remark 6. Theorem 3 is proved by the generalized Banach compression principle under the condition of $g = 0$ and $L_{n+1}(\kappa) \equiv 0$ in (C_5) . Moreover, the condition (C_6) of Theorem 3 is weaker than the condition (C_5) of Theorem 2. Hence, compared with related literature, this part generalizes the constants L of the Lipschitz conditions to the Lebesgue integrable functions $L_i \in L^p(J, \mathbb{R}^+)$ ($i = 1, 2, \dots, n+1$) with $p > \frac{1}{\alpha}$.

3.3. The Attainability of Mild Solutions Is Established via Non-Compactness Criteria

For this part, we need to add the following conditions.

(H_6) There exist integrable functions $L_h(\kappa, \cdot), L_\rho(\kappa, \cdot) \in L(J, \mathbb{R}^+), \mathbb{R}^+ = [0, +\infty)$, for every bounded set $S \subset E$, satisfies

$$L_{h_0} = \max_{t \in [0, T]} \int_0^\kappa L_h(\kappa, s) ds < +\infty, L_{\rho_0} = \max_{\kappa \in [0, T]} \int_0^T L_\rho(\kappa, s) ds < +\infty,$$

$$\omega(h(\kappa, s, S)) \leq L_h(\kappa, s)\omega(S), (\kappa, s) \in \Theta_1 \quad \omega(\rho(\kappa, s, S)) \leq L_\rho(\kappa, s)\omega(S), (\kappa, s) \in \Theta_2. \quad (55)$$

(H_7) For appropriately chosen $L_{I_k} > 0$ ($k = 0, 1, \dots, m$), given any bounded set $S \subset E$, satisfies

$$\omega(I_k(S)) \leq L_{I_k}\omega(S), k = 0, 1, \dots, m. \quad (56)$$

(H_8) There exists a constant $L_g > 0$, for any bounded set $S^* \subset PC(J, E)$, satisfies

$$\omega(g(S^*)) \leq L_g\omega_{PC}(S^*). \quad (57)$$

(H_9) There exist integrable function $L_i \in L^p(J, \mathbb{R}^+)$ ($i = 1, 2, \dots, n+1$) with $p > \frac{1}{\alpha}$, for any bounded set $S_i \subset E, \kappa \in J$, such that

$$\omega(\Xi(\kappa, S_1, S_2, S_3, \dots, S_{n+1})) \leq \sum_{i=1}^{n+1} L_i(t)\omega(S_i). \quad (58)$$

Theorem 4. Suppose the condition (H_1) – (H_9) and (C_5) hold, then the model (1) admits at least one global mild solution in $B_{R^*} \subset PC(J, E)$.

Proof. As demonstrated in the study of Theorem 1, we have $\aleph : \overline{Co}\aleph(B_{R^*}) \rightarrow \overline{Co}\aleph(B_{R^*}) \subset B_{R^*}$ is bounded and continuous operator,

Next we show the set $\aleph B_{R^*}$ is equicontinuous. In fact, for any $\forall u \in B_{R^*}, \kappa', \kappa'' \in J_0, \kappa' \leq \kappa''$, one arrives

$$\begin{aligned}
& \|\aleph u(\kappa'') - \aleph u(\kappa')\| = \left\| \mathcal{G}\mathfrak{Z}_\alpha(\kappa'')(u_0 + g(u)) - \mathcal{G}\mathfrak{Z}_\alpha(\kappa')(u_0 + g(u)) \right. \\
& + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^{\kappa''} (\kappa'' - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& - \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^{\kappa'} (\kappa' - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^{\kappa''} \mathfrak{R}_\alpha(\kappa'' - s) \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& - \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^{\kappa'} \mathfrak{R}_\alpha(\kappa' - s) \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& \leq I \|\mathfrak{Z}_\alpha(\kappa'') - \mathfrak{Z}_\alpha(\kappa')\| \|u_0 + g(u)\| \\
& + \left\| \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^{\kappa'} ((\kappa'' - s)^{\alpha-1} - (\kappa' - s)^{\alpha-1}) \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \right. \\
& + \left\| \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa'}^{\kappa''} (\kappa'' - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \right. \\
& + \left\| \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^{\kappa'} \|\mathfrak{R}_\alpha(\kappa'' - s) - \mathfrak{R}_\alpha(\kappa' - s)\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \right. \\
& + \left\| \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa'}^{\kappa''} \|\mathfrak{R}_\alpha(\kappa'' - s)\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \right. \\
& \leq \tilde{I} \|\mathfrak{Z}_\alpha(\kappa'') - \mathfrak{Z}_\alpha(\kappa')\| (\|u_0\| + d_1\|u\| + \zeta_1) + Y_1 + Y_2 + Y_3 + Y_4.
\end{aligned} \tag{59}$$

For any $\forall u \in B_{R^+}$, $\kappa', \kappa'' \in J_i$, $\kappa' \leq \kappa''$, one reaches

$$\begin{aligned}
& \|\aleph u(\kappa'') - \aleph u(\kappa')\| = \left\| \mathcal{G}^{k+1}\mathfrak{Z}_\alpha(\kappa'')(u_0 + g(u)) - \mathcal{G}^{k+1}\mathfrak{Z}_\alpha(\kappa')(u_0 + g(u)) \right. \\
& + \frac{\mathcal{H}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} (\mathfrak{Z}_\alpha(\kappa'' - \kappa_i) - \mathfrak{Z}_\alpha(\kappa' - \kappa_i)) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), \\
& (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_k}^{\kappa''} (\kappa'' - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& - \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_k}^{\kappa'} (\kappa' - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_k}^{\kappa''} \mathfrak{R}_\alpha(\kappa'' - s) \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& - \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_k}^{\kappa'} \mathfrak{R}_\alpha(\kappa' - s) \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\
& + \sum_{i=1}^k g^{k+1-i} \mathfrak{Z}_\alpha(\kappa'' - \kappa_i) I_i(u(\kappa_i)) - \sum_{i=1}^k g^{k+1-i} \mathfrak{Z}_\alpha(\kappa' - \kappa_i) I_i(u(\kappa_i)) \Big\| \\
& \leq I^{m+1} \|\mathfrak{Z}_\alpha(\kappa'') - \mathfrak{Z}_\alpha(\kappa')\| \|u_0 + g(u)\| \\
& + \left\| \frac{\mathcal{H}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} (\mathfrak{Z}_\alpha(\kappa'' - \kappa_i) - \mathfrak{Z}_\alpha(\kappa' - \kappa_i)) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), \right. \\
& (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \Big\| + \left\| \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_k}^{\kappa'} ((\kappa'' - s)^{\alpha-1} - (\kappa' - s)^{\alpha-1}) \right. \\
& \times \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \Big\| \\
& + \left\| \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa'}^{\kappa''} (\kappa'' - s)^{\alpha-1} \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \right. \\
& + \left\| \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_k}^{\kappa'} \|\mathfrak{R}_\alpha(\kappa'' - s) - \mathfrak{R}_\alpha(\kappa' - s)\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \right. \\
& + \left\| \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa'}^{\kappa''} \|\mathfrak{R}_\alpha(\kappa'' - s)\| \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \right. \\
& + \left\| \sum_{i=1}^k g^{k+1-i} \mathfrak{Z}_\alpha(\kappa'' - \kappa_i) I_i(u(\kappa_i)) - \sum_{i=1}^k g^{k+1-i} \mathfrak{Z}_\alpha(\kappa' - \kappa_i) I_i(u(\kappa_i)) \right\| \\
& \leq \tilde{I} \|\mathfrak{Z}_\alpha(\kappa'') - \mathfrak{Z}_\alpha(\kappa')\| (\|u_0\| + d_1\|u\| + \zeta_1) + Y_0 + Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \\
& \leq \tilde{I} \|\mathfrak{Z}_\alpha(\kappa'') - \mathfrak{Z}_\alpha(\kappa')\| (\|u_0\| + d_1 R^* + \zeta_1) + Y_0 + Y_1 + Y_2 + Y_3 + Y_4 + Y_5.
\end{aligned} \tag{60}$$

First, since $\Im_\alpha(\cdot)$ is bounded and strongly continuous (Lemma 11), one arrives $\|\Im_\alpha(\kappa'') - \Im_\alpha(\kappa')\| \rightarrow 0$ as $\kappa' \rightarrow \kappa''$, this means $\tilde{l}\|\Im_\alpha(\kappa'') - \Im_\alpha(\kappa')\|(\|u_0\| + d_1 R^* + \zeta_1) \rightarrow 0$. From (25), for $u \in B_{R^*}$, $\kappa', \kappa'' \in J_i$, we get

$$\begin{aligned} Y_0 &= \left\| \frac{\mathcal{K}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \right\| \left\| \sum_{i=1}^k \mathcal{G}^{k+2-i} (\Im_\alpha(\kappa'' - \kappa_i) - \Im_\alpha(\kappa' - \kappa_i)) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \right. \\ &\quad \times \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \\ &\leq \frac{\tilde{l}(1-\alpha)l^{m+1}}{B(\alpha)\Gamma(\alpha)} (\gamma'(n-1 + \hbar_0 + \rho_0)R^* + \gamma'(\bar{\hbar}_0 + \bar{\rho}_0) + M) \sum_{i=1}^m \|\Im_\alpha(\kappa'' - \kappa_i) - \Im_\alpha(\kappa' - \kappa_i)\| \int_0^T (\kappa - s)^{\alpha-1} ds \\ &= \frac{l^m}{\alpha} \sum_{i=1}^m \|\Im_\alpha(\kappa'' - \kappa_i) - \Im_\alpha(\kappa' - \kappa_i)\|, \end{aligned} \quad (61)$$

where $\wp = \frac{\tilde{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} (\gamma'(n-1 + \hbar_0 + \rho_0)R^* + \gamma'(\zeta_\hbar T + \zeta_\rho T) + M)$. Similarly, from (25), for $u \in B_{R^*}$, $\kappa', \kappa'' \in J_i$, we have

$$\begin{aligned} Y_1 &\leq \frac{\tilde{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \left\| \int_0^{\kappa'} ((\kappa' - s)^{\alpha-1} - (\kappa'' - s)^{\alpha-1}) \right. \\ &\quad \times \Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s)) ds \Big\| \\ &\leq \wp \int_0^{\kappa'} ((\kappa' - s)^{\alpha-1} - (\kappa'' - s)^{\alpha-1}) ds \\ &\leq \frac{\wp}{\alpha} ((\kappa'')^\alpha - (\kappa')^\alpha) + (\kappa'' - \kappa')^\alpha, \end{aligned} \quad (62)$$

where $\wp$ is from (61). Since κ^α has uniformly continuous property on J , we have $(\kappa'')^\alpha - (\kappa')^\alpha \rightarrow 0$ as $\kappa' \rightarrow \kappa''$, simply, we have $(\kappa'' - \kappa')^\alpha \rightarrow 0$ as $\kappa' \rightarrow \kappa''$, that means $\lim_{\kappa' \rightarrow \kappa''} Y_1 = 0$. Moreover,

$$\begin{aligned} Y_2 &= \frac{\tilde{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa'}^{\kappa''} (\kappa'' - s)^{\alpha-1} \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\ &\leq \frac{\wp}{\alpha} (\kappa'' - \kappa')^\alpha \rightarrow 0 \text{ as } \kappa' \rightarrow \kappa'', \end{aligned} \quad (63)$$

that means $\lim_{\kappa' \rightarrow \kappa''} Y_2 = 0$. In addition,

$$\begin{aligned} Y_3 &\leq \frac{\alpha l^2}{B(\alpha)} \int_0^{\kappa'} \|\Re_\alpha(\kappa'' - s) - \Re_\alpha(\kappa' - s)\| \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\ &\leq \bar{\wp} \int_0^T \|\Re_\alpha(\kappa'' - s) - \Re_\alpha(\kappa' - s)\| ds \\ &\leq \bar{\wp} T \|\Re_\alpha(\kappa'' - s) - \Re_\alpha(\kappa' - s)\|, \end{aligned} \quad (64)$$

where $\bar{\wp} = \frac{\alpha l^2}{B(\alpha)} (\gamma'(n-1 + \hbar_0 + \rho_0)R^* + \gamma'(\bar{\hbar}_0 + \bar{\rho}_0) + M)$, since $\Re_\alpha(\cdot)$ is bounded and strongly continuous (Lemma 11), one arrives $\|\Re_\alpha(\kappa'' - s) - \Re_\alpha(\kappa' - s)\| \rightarrow 0$ uniformly for $s \in [0, T]$ as $t' \rightarrow t''$, this means $\lim_{\kappa' \rightarrow \kappa''} Y_3 = 0$. Subsequently,

$$\begin{aligned} Y_4 &= \frac{\alpha l^2}{B(\alpha)} \int_{\kappa'}^{\kappa''} \|\Re_\alpha(\kappa'' - s)\| \|\Xi(s, u(s), u(\tau_1(s)), \dots, u(\tau_{n-2}(s)), (\overline{\mathcal{H}}u)(s), (\overline{\Lambda}u)(s))\| ds \\ &\leq \frac{M^* \bar{\wp}}{\Gamma(\alpha)} (\kappa'' - \kappa') \rightarrow 0 \text{ as } \kappa' \rightarrow \kappa'', \end{aligned} \quad (65)$$

where $\bar{\wp}$ is from (64). Finally, for any $\forall u \in B_{R^*}$, $\kappa', \kappa'' \in J_k$, $\kappa' \leq \kappa''$,

$$\begin{aligned}
Y_5 &= \left\| \sum_{i=1}^k g^{k+1-i} \mathfrak{S}_\alpha(\kappa'' - \kappa_i) I_i(u(\kappa_i)) - \sum_{i=1}^k g^{k+1-i} \mathfrak{S}_\alpha(\kappa' - \kappa_i) I_i(u(\kappa_i)) \right\| \\
&\leq l^m \left\| \sum_{i=1}^k (\mathfrak{S}_\alpha(\kappa'' - \kappa_i) - \mathfrak{S}_\alpha(\kappa' - \kappa_i)) I_i(u(\kappa_i)) \right\| \\
&\leq l^m \sum_{i=1}^m \left\| \mathfrak{S}_\alpha(\kappa'' - \kappa_i) - \mathfrak{S}_\alpha(\kappa' - \kappa_i) \right\| (d_{I_i} R^* + \zeta_{I_i}).
\end{aligned} \tag{66}$$

Combining boundedness with strong continuity of $\mathfrak{S}_\alpha(\cdot)$, one has $l^m \sum_{i=1}^m \|\mathfrak{S}_\alpha(\kappa'' - \kappa_i) - \mathfrak{S}_\alpha(\kappa' - \kappa_i)\| (d_{I_i} R^* + \zeta_{I_i}) \rightarrow 0$.

Hence, by (59)–(66), $\mathfrak{N}(B_{R^*})$ is equicontinuous in J_i for $i = 1, 2, \dots, m$, then by Lemma 6, we know that $\overline{\text{Co}}\mathfrak{N}(B_{R^*})$ is also equicontinuous in J_i for $i = 1, 2, \dots, m$.

Then, by Lemma 6, $\overline{\text{Co}}\mathfrak{N}(B_{R^*}) \subset B_{R^*}$ forms a uniformly bounded and uniformly equicontinuous family on J_k for $k = 1, 2, \dots, m$. According to the demonstrate the definition of the operator $\mathfrak{N}$, the conditions (H_1) – (H_9) and the properties of non-compact measures, for any bounded set $\Omega \subset \overline{\text{Co}}\mathfrak{N}(B_{R^*}) \subset B_{R^*}$, $\kappa \in [\kappa_0, \kappa_1]$, by (H_4) , (H_5) , (H_8) , Lemmas 2 and 11, one has

$$\begin{aligned}
\omega(\mathfrak{N}\Omega(\kappa)) &= \omega\left\{ \mathcal{G}\mathfrak{S}_\alpha(\kappa)(u_0 + g(\Omega)) \right. \\
&\quad + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
&\quad \left. + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^\kappa \mathfrak{R}_\alpha(\kappa-s) \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \right\} \\
&\leq M^* l \omega(g(\Omega)) \\
&\quad + \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds \\
&\quad + \frac{\alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds \\
&\leq M^* l L_g \omega_{PC}(\Omega) \\
&\quad + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}((1-\alpha)), M^* \alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \left(L_1(s) \omega(\Omega(s)) \right. \\
&\quad \left. + L_2(s) \omega(\Omega(\tau_1(s))) + \dots + L_{n-1}(s) \omega(\Omega(\tau_{n-2}(s))) + L_n(s) \omega(\overline{\mathcal{H}}\Omega)(s) + L_{n+1}(s) \omega(\overline{\Lambda}\Omega)(s) \right) ds \\
&= M^* l L_g \omega_{PC}(\Omega) \\
&\quad + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \left(L_1(s) \omega(\Omega(s)) + L_2(s) \omega(\Omega(\tau_1(s))) + \dots + L_{n-1}(s) \omega(\Omega(\tau_{n-2}(s))) \right. \\
&\quad \left. + L_n(s) \omega(\overline{\mathcal{H}}\Omega)(s) + L_{n+1}(s) \omega(\overline{\Lambda}\Omega)(s) \right) ds \\
&\leq M^* \bar{l} L_g \omega_{PC}(\Omega) + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) (L_1(s) + L_2(s) + \dots + L_{\hbar_0} L_n(s) + L_{\rho_0} L_{n+1}(s)) ds \times \omega(\Omega(s)) \\
&= M^* \bar{l} L_g \omega_{PC}(\Omega) + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) L(s) ds \times \omega(\Omega(s)) \\
&\leq \left\{ M^* \bar{l} L_g + \bar{d} \iota \right\} \omega_{PC}(\Omega) \\
&= \varrho \omega_{PC}(\Omega),
\end{aligned} \tag{67}$$

where $\iota, L(s)$ are from the condition (C_5) , $L_{\hbar_0}, L_{\rho_0}$ are from (C_2) .

For $\kappa \in (\kappa_i, \kappa_{i+1}] (i = 1, 2, \dots, m)$, by $(H_4), (H_5), (H_7), (H_8)$, Lemmas 2 and 11, one has

$$\begin{aligned}
& \omega(\aleph\Omega(\kappa)) = \omega\left\{\mathcal{G}^{k+1}\mathfrak{S}_\alpha(\kappa)(u_0 + g(\Omega))\right. \\
& + \frac{\mathcal{K}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i}\mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
& + \frac{\mathcal{K}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_k}^{\kappa} (\kappa - s)^{\alpha-1} \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
& + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(\kappa - s) \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
& + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_k}^{\kappa} \mathfrak{R}_\alpha(\kappa - s) \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
& \left. + \sum_{i=1}^k \mathcal{G}^{k+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_k) I_k(\Omega(\kappa_k))\right\} \\
& \leq M^* l^{m+1} \omega(g(\Omega)) \\
& + \frac{mM^* l^{m+1} \bar{l} + \bar{l} \bar{l} (1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds \\
& + \frac{m\alpha l^{m+2} M^* + \alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds \\
& + \omega\left(\sum_{i=1}^k \mathcal{G}^{k+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(\Omega(t_i))\right) \\
& \leq M^* l^{m+1} L_g \omega_{PC}(\Omega) \\
& + \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l} \bar{l} (mM^* l^m + (1-\alpha)), M^* \alpha l^2 (ml^m + 1)\} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left(L_1(s) \omega(\Omega(s))\right. \\
& + L_2(s) \omega(\Omega(\tau_1(s))) + \dots + L_{n-1}(s) \omega(\Omega(\tau_{n-2}(s))) + L_n(s) \omega(\overline{\mathcal{H}}\Omega)(s) + L_{n+1}(s) \omega(\overline{\Lambda}\Omega)(s)) ds \\
& \left. + M^* l^m \left(\sum_{0 < \kappa_i < \kappa} L_{I_i} \omega(\Omega(\kappa_i))\right)\right) \\
& = M^* l^{m+1} L_g \omega_{PC}(\Omega) \\
& + \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left(L_1(s) \omega(\Omega(s)) + L_2(s) \omega(\Omega(\tau_1(s))) + \dots + L_{n-1}(s) \omega(\Omega(\tau_{n-2}(s)))\right. \\
& \left. + L_n(s) \omega(H\Omega)(s) + L_{n+1}(s) \omega(\Lambda\Omega)(s)\right) ds + M^* l^m \left(\sum_{0 < \kappa_i < \kappa} L_{I_i} \omega(\Omega(t_k))\right) \\
& \leq M^* \tilde{l} L_g \omega_{PC}(\Omega) + \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) (L_1(s) + L_2(s) + \dots + L_{n_0}(s) + L_{\rho_0} L_{n+1}(s)) ds \times \omega(\Omega(s)) \\
& = M^* \tilde{l} L_g \omega_{PC}(\Omega) + \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) L(s) ds \times \omega(\Omega(s)) + M^* l^m \left(\sum_{i=1}^m L_{I_i} \omega(\Omega(\kappa_i))\right) \\
& = M^* \tilde{l} L_g \omega_{PC}(\Omega) + \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) L(s) ds \times \omega(\Omega(s)) + M^* l^m \left(\sum_{i=1}^m L_{I_i} \omega(\Omega(\kappa_i))\right) \\
& \leq \left\{M^* \tilde{l} \left(L_g + \sum_{i=1}^m L_{I_i}\right) + \bar{d} l\right\} \omega_{PC}(\Omega) \\
& = \varrho \omega_{PC}(\Omega),
\end{aligned} \tag{68}$$

where $\iota, L(s)$ are from the condition (C_5) , L_{h_0}, L_{ρ_0} are from (C_2) . By Lemma 5, for any bounded set $\Omega \subset \overline{Co}\mathfrak{N}(B_{R^*}) \subset B_{R^*}$, by (67), (68) and Lemma 1, one arrives

$$\omega_{PC}(\mathfrak{N}(\Omega)) \leq \varrho \omega_{PC}(\Omega).$$

According to (42), one has $0 < \varrho < 1$, then by the Darbo fixed point theorem, one achieves the operator $\mathfrak{N}$ admits at least one fixed point $u^* \in B_{R^*}$, that is, u^* is admitted a global mild solution of the model (1) on $B_{R^*} \subset PC(J, E)$, and the validation is finished. $\square$

Remark 7. Theorem 4 is finished based on the Darbo fixed point theorem.

Theorem 5. Suppose the conditions $(H_1)–(H_9), (C_6)$ hold, $g : PC(J, E) \rightarrow E$ is compact; moreover, $L_{n+1}(t) \equiv 0$ in (H_9) , then the model (1) has at least one global mild solution in $B_{R^*} \subset PC(J, E)$.

Proof. According to Theorem 1, we have $\mathfrak{N} : B_{R^*} \rightarrow B_{R^*}$ is a bounded continuous operator. From Theorem 4, we have $\overline{Co}\mathfrak{N}(B_{R^*})$ is equicontinuous on J_k , for $\forall \Omega \subset \overline{Co}\mathfrak{N}(B_{R^*})$, then by Lemmas 5 and 6, we have $\tilde{\mathfrak{N}}^n(\Omega)$ ($n = 1, 2, \dots$) is bounded and equicontinuous on J_k , and it follows from Lemma 1 that

$$\omega_{PC}(\tilde{\mathfrak{N}}^n(\Omega)) = \max_{t \in J} \omega((\tilde{\mathfrak{N}}^n(\Omega))(\kappa), n = 1, 2, \dots \quad (69)$$

Now we show there is a constant $0 \leq \eta < 1$ and an integer number $n_0 > 0$, for any $\Omega \subset \overline{Co}\mathfrak{N}(B_{R^*})$, such that

$$\omega_{PC}(\tilde{\mathfrak{N}}^{n_0}(\Omega)) \leq \eta \omega_{PC}(\Omega).$$

By non-compactness measure and the conditions of $(H_1)–(H_9)$, for any $\kappa \in [\kappa_0, \kappa_1]$, one has

$$\begin{aligned} \omega(\mathfrak{N}\Omega(\kappa)) &= \omega\left\{\mathcal{G}\mathfrak{Z}_\alpha(\kappa)(u_0 + g(\Omega))\right. \\ &\quad + \frac{\mathcal{H}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\ &\quad \left. + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_0^\kappa \mathfrak{R}_\alpha(\kappa-s) \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds\right\} \\ &\leq \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\ &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds + \frac{\alpha l^2}{B(\alpha)} \int_0^\kappa \mathfrak{R}_\alpha(\kappa-s) \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\ &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds \\ &\leq \frac{\bar{l}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\ &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds + \frac{\alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\ &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds \\ &\leq \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(1-\alpha), M^* \alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \\ &\quad \times \left(L_1(s) \omega_{PC}(\Omega) + L_2(s) \omega_{PC}(\Omega) + \dots + L_{n-1}(s) \omega_{PC}(\Omega) + L_n(s) \omega_{PC}(\overline{\mathcal{H}}\Omega)\right) ds \\ &= \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \left(L_1(s) \omega_{PC}(\Omega) + L_2(s) \omega_{PC}(\Omega) + \dots \right. \\ &\quad \left. + L_{n-1}(s) \omega_{PC}(\Omega) + L_n(s) \omega_{PC}(\overline{\mathcal{H}}\Omega)\right) ds \\ &\leq \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) B_1(s) \omega(\Omega(s)) ds + J \omega_{PC}(\Omega), \end{aligned} \quad (70)$$

where $\bar{d}$ is from the condition (C_5) , $J = l^m M^* \sum_{i=1}^m L_{l_i} < 1$ in (C_6) , $B_1(s) = L_1(s) + L_2(s) + \dots + L_{n-1}(s) + L_{h_0} L_n(s)$.

For any $\kappa \in (\kappa_i, \kappa_{i+1}]$, we have

$$\begin{aligned}
 \omega(\aleph\Omega(\kappa)) &= \omega\left\{\mathcal{G}^{k+1}\mathfrak{S}_\alpha(\kappa)(u_0 + g(\Omega))\right. \\
 &\quad + \frac{\mathcal{K}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i}\mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\
 &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
 &\quad + \frac{\mathcal{K}\mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_i}^{\kappa} (\kappa - s)^{\alpha-1} \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
 &\quad + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(\kappa - s) \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
 &\quad + \frac{\alpha\mathcal{G}^2}{B(\alpha)} \int_{\kappa_i}^{\kappa} \mathfrak{R}_\alpha(\kappa - s) \Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s)) ds \\
 &\quad \left. + \sum_{i=1}^k g^{k+1-i} I_i(\Omega(\tau_i))\right\} \\
 &\leq \frac{mM^*l^{m+1}\bar{l} + \bar{l}l(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\
 &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds + \frac{m\alpha l^{m+2} + \alpha l^2}{B(\alpha)} \int_0^\kappa \mathfrak{R}_\alpha(\kappa_2 - s) \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\
 &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds + \omega\left(\sum_{i=1}^k g^{k+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(\Omega(\kappa_i))\right) \tag{71} \\
 &\leq \frac{mM^*l^{m+1}\bar{l} + \bar{l}l(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\
 &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds + \frac{m\alpha l^{m+2}M^* + \alpha l^2M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \omega(\Xi(s, \Omega(s), \Omega(\tau_1(s)), \dots, \Omega(\tau_{n-2}(s)), \\
 &\quad (\overline{\mathcal{H}}\Omega)(s), (\overline{\Lambda}\Omega)(s))) ds + \omega\left(\sum_{i=1}^k g^{k+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i(\Omega(\kappa_i))\right) \\
 &\leq \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}l(mM^*l^m + (1-\alpha)), M^*\alpha l^2(ml^m + 1)\} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \\
 &\quad \times \left(L_1(s)\omega_{PC}(\Omega) + L_2(s)\omega_{PC}(\Omega) + \dots + L_{n-1}(s)\omega_{PC}(\Omega) + L_n(s)\omega_{PC}(\overline{\mathcal{H}}\Omega)\right) ds \\
 &\quad + M^*l^m \sum_{i=1}^m L_{I_i}\omega_{PC}(\Omega) \\
 &= \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \left(L_1(s)\omega_{PC}(\Omega) + L_2(s)\omega_{PC}(\Omega) + \dots \right. \\
 &\quad \left. + L_{n-1}(s)\omega_{PC}(\Omega) + L_n(s)\omega_{PC}(\overline{\mathcal{H}}\Omega)\right) ds + M^*l^m \sum_{0 < \kappa_i < \kappa} L_{I_i}\omega_{PC}(\Omega), \\
 &\leq \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) B_1(s) \omega(\Omega(s)) ds + M^*l^m \sum_{i=1}^m L_{I_i}\omega_{PC}(\Omega),
 \end{aligned}$$

where $\bar{d}, L(s), j$ is as (70). Then, by (70) and (71), for any $\kappa \in J, u, v \in PC(J, E)$, one arrives

$$\omega(\aleph\Omega(\kappa)) \leq \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) B_1(s) \omega(\aleph\Omega(\kappa)) ds + j\omega(\aleph\Omega(\kappa)). \tag{72}$$

By $0 < j < 1$, there exists $0 < \epsilon_0 < 1$, such that $0 < \ell = \epsilon_0 + j < 1$. From $B_1(\kappa) = L_1(\kappa) + L_2(\kappa) + L_3(\kappa) + \dots + L_{n-1}(\kappa) + L_n(\kappa)L_{h_0} \in L^p(J, \mathbb{R})$, A continuous functions can be constructed $\phi : J \rightarrow \mathbb{R}^+$, subject to

$$\bar{d}l_0 \left(\int_0^T |B_1(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} < \epsilon_0.$$

By (71) and (72), one arrive

$$\begin{aligned}
 & \omega((\tilde{\mathfrak{N}}^1(\Omega))(\kappa)) \\
 & \leq \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) |B_1(s) - \phi(s)| \omega(\Omega(s)) ds + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) |\phi(s)| \omega(\Omega(s)) ds \\
 & \quad + J\omega(\Omega(s)) \\
 & \leq \bar{d} l_0 \left(\int_0^T |B_1(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} \omega(\Omega(s)) + \bar{d} \mathfrak{h} l_0 \left(\int_0^\kappa \omega(\Omega(s)) ds \right)^{\frac{1}{p}} + J\omega(\Omega(s)) \\
 & \leq \bar{d} l_0 \left(\int_0^T |B_1(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} \omega(\Omega(s)) + \bar{d} \mathfrak{h} l_0 \left(\int_0^\kappa ds \right)^{\frac{1}{p}} + J\omega(\Omega(s)) \\
 & \leq \epsilon_0 \|u - v\|_{PC} + \bar{\mathfrak{h}} \kappa \omega(\Omega(s)) + J\omega(\Omega(s)) \\
 & \leq (\ell + \bar{\mathfrak{h}} \kappa) \omega(\Omega(s)) = \left(C_1^0 \ell + C_1^1 \frac{\bar{\mathfrak{h}} \kappa}{1!} \right) \omega(\Omega(s)), \kappa \in J,
 \end{aligned} \tag{73}$$

where $\mathfrak{h} = \max\{|\phi(\kappa)| : \kappa \in J\}$, $\bar{\mathfrak{h}} = \bar{d} \mathfrak{h} l_0$.

Assume for natural number $j \in \mathbb{N}$, one has

$$\omega((\tilde{\mathfrak{N}}^j)(\kappa)) \leq \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\mathfrak{h}} \kappa) + \frac{C_j^2 \ell^{j-2} (\bar{\mathfrak{h}} \kappa)^2}{2!} + \dots + \frac{(\bar{\mathfrak{h}} \kappa)^j}{j!} \right) \omega_{PC}(\Omega), \kappa \in J, \tag{74}$$

according to the definition of operation, the condition $(H_1)-(H_9)$, by the same steps as (70) and (71), for $\kappa \in [\kappa_0, \kappa_1]$, one arrives

$$\begin{aligned}
 & \omega(\tilde{\mathfrak{N}}^{j+1}(\Omega))(\kappa) = \omega \left\{ \mathcal{G} \mathfrak{Z}_\alpha(\kappa) \left(u_0 + g(\tilde{\mathfrak{N}}^j(\Omega)) \right) \right. \\
 & \quad + \frac{\mathcal{H} \mathcal{G} (1-\alpha)}{B(\alpha) \Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \Xi \left(s, \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), \right. \\
 & \quad \left. \left(\mathcal{H} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), (\overline{\Lambda} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) ds \right. \\
 & \quad \left. + \frac{\alpha \mathcal{G}^2}{B(\alpha)} \int_0^\kappa \mathfrak{R}_\alpha(\kappa-s) \Xi \left(s, \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), \right. \right. \\
 & \quad \left. \left. \left(\mathcal{H} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), (\overline{\Lambda} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) ds \right\} \right. \\
 & \leq \frac{\bar{l} \Gamma(1-\alpha)}{B(\alpha) \Gamma(\alpha)} \int_0^\kappa (\kappa-s)^{\alpha-1} \omega \left(\Xi \left(s, \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \right. \right. \\
 & \quad \left. \left. \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), \left(\mathcal{H} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), (\overline{\Lambda} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) \right) ds \right. \\
 & \quad \left. + \frac{\alpha l^2 M^*}{B(\alpha) \Gamma(\alpha)} \int_0^\kappa \omega \left(\Xi \left(s, \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), \right. \right. \right. \\
 & \quad \left. \left. \left(\mathcal{H} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega)(s), (\overline{\Lambda} \overline{c\partial} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) \right) ds \right) \\
 & \leq \frac{1}{\Gamma(\alpha) B(\alpha)} \max\{\bar{l} \Gamma(1-\alpha), M^* \alpha l^2\} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) \\
 & \quad \times \left(L_1(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) + L_2(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s))) + \dots + L_{n-2}(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s))) \right. \\
 & \quad \left. + L_{n-1}(s) \omega(H \tilde{\mathfrak{N}}^j(\Omega)(s)) \right) ds \\
 & \leq \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) B_1(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) ds + J \omega_{PC}(\tilde{\mathfrak{N}}^j(\Omega)) \\
 & \leq \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) |B_1(s) - \phi(s)| \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) ds + \bar{d} \int_0^\kappa ((\kappa-s)^{\alpha-1} + 1) |\phi(s)| \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) ds \\
 & \quad + J \omega_{PC}(\tilde{\mathfrak{N}}^j(\Omega)) \\
 & \leq \bar{d} l_0 \left(\int_0^T |B_1(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} \omega(\tilde{\mathfrak{N}}^j(\Omega)) + \bar{d} \mathfrak{h} l_0 \int_0^\kappa \omega(\tilde{\mathfrak{N}}^j) ds
 \end{aligned}$$

$$\begin{aligned}
& + J \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) \\
& \leq \epsilon_0 \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) + \bar{\eta} \int_0^\kappa \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) ds \\
& \quad + J \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) \\
& \leq \ell \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) \\
& \quad + \left(C_j^0 \ell^j \frac{\bar{\eta} \kappa}{1!} + C_j^1 \ell^{j-1} \frac{(\bar{\eta} \kappa)^2}{2!} + \dots + C_j^j \ell^0 \frac{(\bar{\eta} \kappa)^{j+1}}{(j+1)!} \right) \omega_{PC}(\Omega) \\
& \leq \left(C_{j+1}^0 \ell^{j+1} + C_{j+1}^1 \ell^j (\bar{\eta} \kappa) + \frac{C_{j+1}^2 \ell^{j-1} (\bar{\eta} \kappa)^2}{2!} + \dots + C_{j+1}^{j+1} \ell^0 \frac{(\bar{\eta} \kappa)^{j+1}}{(j+1)!} \right) \omega_{PC}(\Omega).
\end{aligned} \tag{75}$$

For $\kappa \in (\kappa_i, \kappa_{i+1}]$ ($i = 1, 2, \dots, m$), one arrives

$$\begin{aligned}
& \omega(\tilde{\mathfrak{N}}^{j+1}(\Omega))(\kappa) = \omega \left\{ \mathcal{G}^{k+1} \mathfrak{S}_\alpha(t) \left(u_0 + g(\tilde{\mathfrak{N}}^j(\Omega)) \right) \right. \\
& \quad + \frac{\mathcal{H}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+2-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) \int_{\kappa_{i-1}}^{\kappa_i} (\kappa_i - s)^{\alpha-1} \\
& \quad \times \Xi \left(s, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(s), \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), (\mathcal{H} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s), (\bar{\Lambda} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) ds \\
& \quad + \frac{\mathcal{H} \mathcal{G}(1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_{\kappa_k}^\kappa (\kappa - s)^{\alpha-1} \Xi \left(s, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(s), \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), \right. \\
& \quad \left. (\mathcal{H} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s), (\bar{\Lambda} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) ds + \frac{\alpha}{B(\alpha)} \sum_{i=1}^k \mathcal{G}^{k+3-i} \int_{\kappa_{i-1}}^{\kappa_i} \mathfrak{R}_\alpha(\kappa - s) \\
& \quad \times \Xi \left(s, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(s), \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), (\mathcal{H} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s), (\bar{\Lambda} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) ds \\
& \quad + \frac{\alpha \mathcal{G}^2}{B(\alpha)} \int_{\kappa_i}^\kappa \mathfrak{R}_\alpha(\kappa - s) \Xi \left(s, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(s), \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), \right. \\
& \quad \left. (\mathcal{H} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s), (\bar{\Lambda} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) ds + \sum_{i=1}^k g^{k+1-i} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i \left(\bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\kappa_i) \right) \Big\} \\
& \leq \frac{m M^* l^{m+1} \bar{l} + \bar{l} l (1-\alpha)}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa (\kappa - s)^{\alpha-1} \omega \left(\Xi \left(s, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(s), \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \right. \right. \\
& \quad \left. \left. \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), (\mathcal{H} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s), (\bar{\Lambda} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) \right) ds \\
& \quad + \frac{m \alpha l^{m+2} M^* + \alpha l^2 M^*}{B(\alpha)\Gamma(\alpha)} \int_0^\kappa \omega \left(\Xi \left(s, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(s), \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s)), \dots, \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s)), \right. \right. \\
& \quad \left. \left. (\mathcal{H} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s), (\bar{\Lambda} \bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega))(s) \right) \right) ds + \omega \left(\sum_{0 < \kappa_i < t} \mathcal{G} \mathfrak{S}_\alpha(\kappa - \kappa_i) I_i \left(\bar{c} \bar{o} \tilde{\mathfrak{N}}^j(\Omega)(\kappa_i) \right) \right) \\
& \leq \frac{1}{\Gamma(\alpha) B(\alpha)} \max \{ \bar{l} l (m M^* l^m + (1-\alpha)), M^* \alpha l^2 (m l^m + 1) \} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) \\
& \quad \times \left(L_1(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) + L_2(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(\tau_1(s))) + \dots + L_{n-2}(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(\tau_{n-2}(s))) \right. \\
& \quad \left. + L_{n-1}(s) \omega(H \tilde{\mathfrak{N}}^j(\Omega)(s)) \right) ds + M^* l^m \sum_{0 < \kappa_i < \kappa} L_{i_i} \omega(\tilde{\mathfrak{N}}^j(\Omega)(\kappa_i)) \\
& = \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) B_1(s) \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) ds + M^* l^m \left(\sum_{k=1}^m L_{i_k} \omega(\tilde{\mathfrak{N}}^j(\Omega)(s_k)) \right) \\
& \leq \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) |B_1(s) - \phi(s)| \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) ds + \bar{d} \int_0^\kappa ((\kappa - s)^{\alpha-1} + 1) |\phi(s)| \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) ds \\
& \quad + J \|u - v\|_{PC} \\
& \leq \bar{d} l_0 \left(\int_0^T |B(s) - \phi(s)|^p ds \right)^{\frac{1}{p}} \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) + \bar{d} l_0 \left(\int_0^\kappa \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) ds \right) + J \omega(\tilde{\mathfrak{N}}^j(\Omega)(s)) \\
& \quad + J \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega)
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon_0 \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) + \bar{\eta} \int_0^\kappa \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) ds \\
&\quad + J \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) \\
&\leq \ell \left(\ell^j + C_j^1 \ell^{j-1} (\bar{\eta} \kappa) + \dots + \frac{(\bar{\eta} \kappa)^j}{j!} \right) \omega_{PC}(\Omega) \\
&\quad + \left(C_j^0 \ell^j \frac{\bar{\eta} \kappa}{1!} + C_j^1 \ell^{j-1} \frac{(\bar{\eta} \kappa)^2}{2!} + \dots + C_j^j \ell^0 \frac{(\bar{\eta} \kappa)^{j+1}}{(j+1)!} \right) \omega_{PC}(\Omega) \\
&\leq \left(C_{j+1}^0 \ell^{j+1} + C_{j+1}^1 \ell^j (\bar{\eta} \kappa) + \frac{C_{j+1}^2 \ell^{j-1} (\bar{\eta} \kappa)^2}{2!} + \dots + C_{j+1}^{j+1} \ell^0 \frac{(\bar{\eta} \kappa)^{j+1}}{(j+1)!} \right) \omega_{PC}(\Omega), \kappa \in J,
\end{aligned} \tag{76}$$

where $L(s)$ is from Formula (70).

Hence, by principle of mathematical induction, (75) and (76), for any natural number $n \in \mathbb{N}$ and $\kappa \in J$, one has

$$\omega((\tilde{\aleph}^n)(\Omega))(\kappa) \leq \left(\ell^n + C_n^1 \ell^{n-1} (\bar{\eta} \kappa) + \frac{C_n^2 \ell^{n-2} (\bar{\eta} \kappa)^2}{2!} + \dots + \frac{(\bar{\eta} \kappa)^n}{n!} \right) \omega_{PC}(\Omega), \tag{77}$$

for any natural number $n \in \mathbb{N}$. By (69), one has

$$\omega_{PC}((\tilde{\aleph}^n)(\Omega)) \leq \left(\ell^n + C_n^1 \ell^{n-1} (\bar{\eta} T) + \frac{C_n^2 \ell^{n-2} (\bar{\eta} T)^2}{2!} + \dots + \frac{(\bar{\eta} T)^n}{n!} \right) \omega_{PC}(\Omega), \tag{78}$$

by Lemma 8, for any $s > 1$, as $n \rightarrow \infty$,

$$\omega_{PC}((\tilde{\aleph}^n)(\Omega)) = o\left(\frac{1}{n^s}\right) \omega_{PC}(\Omega).$$

Hence, there exist $0 < \eta < 1$ and natural number $n_0 \in \mathbb{N}$ satisfies $\omega_{PC}(\tilde{\aleph}^{n_0}(\Omega)) \leq \eta \omega_{PC}(\Omega)$ from Lemma 5, that is, the model (1) has at least one global mild solution on $B_{R^*} \subset PC(J, E)$. $\square$

Remark 8. Theorem 5 is completed based on the generalized Darbo fixed point theorem. Theorem 4 is without the condition $L_{n+1} = 0$, but need the condition (C_5) , and Theorem 5 need the condition (C_6) . The condition (C_6) of Theorem 5 is weaker than (C_5) of Theorem 4. Compared with previous relevant literature, this part generalizes the L of the Lipschitz constants to unbounded Lebesgue integrable functions $L_i(s)$ ($i = 1, 2, \dots, i+1$).

4. An Illustration

Example 1. Consider the following Mittag–Leffler kernel fractional-derivative model

$$\begin{cases}
{}^{ABc}D_\kappa^\alpha u(x, \kappa) - a(\kappa) \Delta u(x, \kappa) = \frac{u(x, \kappa)}{1 + 8\sqrt{\kappa}} + \frac{1}{6} \kappa^{-\frac{1}{2}} e^{-\kappa} \sin(u(x, \kappa - \tau_1)) \\
\quad + \frac{1}{6} \kappa^{-\frac{1}{2}} e^{-\kappa} \cos(u(x, \kappa - \tau_2)) + \frac{1}{2 + \kappa^2} \int_0^\kappa \kappa u(x, s) \cos s ds + \frac{\kappa^{-\frac{1}{2}}}{1 + e^{2\kappa}} \int_0^1 \frac{s}{3} u(x, s) ds, \\
\kappa \in (0, 1), \kappa \neq \frac{1}{3}, \\
\Delta u(x, \kappa)|_{\kappa=\kappa_i} = I(u(x, s)) = u(x, (1/3)^+) - u(x, (1/3)^-), i = 1, \\
u(x, 0) = \varphi(x) + g(u(x, \kappa)), 0 < x < 1, \\
u(0, \kappa) = u(\pi, \kappa) = 0, (x, \kappa) \in (0, 1) \times J',
\end{cases} \tag{79}$$

where $\alpha = \frac{3}{4}$. Let $E = L^2([0, 1])$, $Au(\kappa) = -a(\kappa)\bar{\Delta}u$, $A : D(A) \subset X \rightarrow X$, $u(\cdot, \kappa) = u(\kappa)$, $D(A) = \{u \in E : u'' \in E, u(0) = u(1) = 0\}$, then A is infinitesimal generator of the bounded linear operator $\mathfrak{S}_\alpha(\kappa)$, and $\tau_1(\kappa) = \kappa - \tau_1 \leq \kappa$, $\tau_2(\kappa) = \kappa - \tau_2 \leq \kappa$, $(\mathcal{H}u)(\kappa) = \int_0^\kappa \kappa u(x, s) \cos s ds$, $(\Lambda u(x, s))(\kappa) = \int_0^1 \frac{s}{3} u(x, s) ds$, $g(u(x, s)) = \frac{1}{8}(u(x, s) + \cos u(x, s))$, $\Xi(\kappa, x_1, x_2, x_3, x_4, x_5) = \frac{x_1}{1+8\sqrt{\kappa}} + \frac{1}{6}\kappa^{-\frac{1}{2}}e^{-\kappa} \sin x_2 + \frac{1}{6}\kappa^{-\frac{1}{2}}e^{-\kappa} \cos x_3 + \frac{1}{2+\kappa^2}x_4 + \frac{\kappa^{-\frac{1}{2}}}{1+e^{2\kappa}}x_5$, $I(u(x, s)) = \frac{1}{8}u(x, s)$, then the model (79) can be written by the following abstract form

$$\begin{cases} {}^{ABC}D_\kappa^\alpha u(\kappa) + Au(t) = \frac{u(t)}{1+8\sqrt{\kappa}} + \frac{1}{6}\kappa^{-\frac{1}{2}}e^{-\kappa} \sin(u(\kappa - \tau_1)) \\ + \frac{1}{6}\kappa^{-\frac{1}{2}}e^{-\kappa} \cos(u(\kappa - \tau_2)) + \frac{1}{2+\kappa^2} \int_0^\kappa \kappa u(s) \cos s ds + \frac{\kappa^{-\frac{1}{2}}}{1+e^{2\kappa}} \int_0^1 \frac{s}{3} u(s) ds, \\ \Delta u(\kappa)|_{\kappa=1/3} = I(u(1/3)), k=1, \\ u(0) = u_0 + g(u), \end{cases} \quad (80)$$

hence, for any $u_i, v_i \in E$ ($i = 1, 2, 3, 4, 5$), $\kappa \in (0, 1]$, we easily get

$$\|\hbar(\kappa, s, u)\| = \|\kappa u(x, s)\| \leq \kappa \|u\| = \hbar_1(\kappa, s) \|u\| + \zeta_h(\kappa, s), (\kappa, s) \in \Theta_1, u \in E,$$

$$\|\rho(\kappa, s, u)\| = \|\frac{s}{3}u(x, s)\| \leq \frac{s}{3} \|u\| = \rho_1(\kappa, s) \|u\| + \zeta_\rho(\kappa, s), (\kappa, s) \in \Theta_2, u \in E.$$

Obviously, $\hbar_1(\kappa, s) = \kappa$, $\zeta_h(\kappa, s) = 0$, $\rho_1(\kappa, s) = \frac{s}{3}$, $\zeta_\rho(\kappa, s) = 0$, then $\hbar_0 = \sup_{\kappa \in [0, 1]} \int_0^\kappa \hbar_1(\kappa, s) ds = 1$, $\rho_0 = \sup_{\kappa \in [0, 1]} \int_0^1 \rho_1(\kappa, s) ds = \frac{1}{6}$, $\bar{\hbar}_0 = \bar{\rho}_0 = 0$, hence, (H_1) holds. Moreover, $\|I(u)\| = \|\frac{1}{8}u\| \leq \frac{1}{8}\|u\|$, then, $d_I = \frac{1}{8}$, $\zeta_1 = 0$. Hence, the condition (H_2) holds. $\|g(u)\| = \frac{1}{8}\|(u + \cos u)\| \leq \frac{1}{8}\|u\| + \frac{1}{8}$, then $d_1 = \frac{1}{8}$, hence, the condition (H_3) holds. Moreover

$$\begin{aligned} \gamma &= \limsup_{\|x_1\| + \|x_2\| + \|x_3\| + \|x_4\| < R; \|x_5\| \rightarrow +\infty} \left(\sup_{s \in J} \frac{\frac{x_1}{1+8\sqrt{\kappa}} + \frac{1}{6}\kappa^{-\frac{1}{2}}e^{-\kappa} \sin x_2 + \frac{1}{6}\kappa^{-\frac{1}{2}}e^{-\kappa} \cos x_3 + \frac{1}{2+\kappa^2}x_4 + \frac{\kappa^{-\frac{1}{2}}}{1+e^{2\kappa}}x_5}{\|x_1\| + \|x_2\| + \|x_3\| + \|x_4\| + \|x_5\|} \right) \\ &= \frac{1}{2} < +\infty, \end{aligned}$$

and take $p = q = 2$, $M^* = \frac{1}{5}$, $l = \frac{1}{4}$, $\bar{l} = \frac{1}{5}$, $m = 1$, then

$$\tilde{l} = \max\{l, l^{m+1}\} = \frac{1}{4},$$

$$\begin{aligned} \bar{d} &= \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}\bar{l}(mM^*l^m + (1 - \alpha)), M^*\alpha l^2(ml^m + 1)\} \\ &= \frac{1}{\Gamma(\frac{3}{4})B(\frac{3}{4})} \max\{\frac{1}{4} \times \frac{1}{5}(\frac{1}{5} \times \frac{1}{4} + \frac{1}{4}), \frac{1}{5} \times \frac{3}{4}(\frac{1}{4})^2(\frac{1}{4} + 1)\} \approx 0.1483, \\ \hat{d} &= \bar{d} \left(\frac{1}{\alpha} T^\alpha + 1 \right) \approx 0.1483 \left(\frac{4}{3} + 1 \right) \approx 0.3460, \end{aligned}$$

hence

$$\begin{aligned} K &= \tilde{l}M^*d_1 + \hat{d}\gamma \left(n - 1 + \hbar_0 + \rho_0 \right) + l^m M^* \sum_{k=1}^m d_{I_k} \\ &< \frac{1}{4} \times \frac{1}{5} \times \frac{1}{8} + 0.3460 \times \frac{1}{2} \left(3 + 1 + \frac{1}{6} \right) + \frac{1}{4} \times \frac{1}{5} \times \frac{1}{8} \approx 0.7271 < 1, \end{aligned}$$

then, all the condition of Theorem 1 hold, the Mittag–Leffler kernel fractional-derivative model (79) at least has one fixed point by the Schauder fixed point theorem.

Example 2. Continue to consider model (80), for any $u_i, v_i \in E (i = 1, 2, 3, 4, 5), \kappa \in (0, 1]$, we easily get

$$\begin{aligned} & \|\Xi(\kappa, u_1, u_2, u_3, u_4, u_5) - \Xi(\kappa, v_1, v_2, v_3, v_4, v_5)\| \\ & \leq \frac{1}{1+8\sqrt{\kappa}} \|u_1 - v_1\| + \frac{1}{6} \kappa^{-\frac{1}{2}} e^{-\kappa} \|u_2 - v_2\| + \frac{1}{6} \kappa^{-\frac{1}{2}} e^{-\kappa} \|u_3 - v_3\| + \frac{1}{2+\kappa^2} \|\mathcal{H}u_4 - \mathcal{H}v_4\| + \frac{\kappa^{-\frac{1}{2}}}{1+e^{2\kappa}} \|\Lambda u_5 - \Lambda v_5\|, \\ & \|\hbar(\kappa, s, u) - \hbar(\kappa, s, v)\| \leq \|u - v\|, \kappa, s \in J = [0, 1], \kappa \neq \frac{1}{3}, u, v \in E, \\ & \|\rho(\kappa, s, u) - \rho(\kappa, s, v)\| \leq \frac{1}{3} \|u - v\|, \kappa, s \in J = [0, 1], \kappa \neq \frac{1}{3}, u, v \in E, \\ & \|g(u) - g(v)\| \leq \frac{1}{4} \|u - v\|, u, v \in E, \\ & \|I(u) - I(v)\| \leq \frac{1}{8} \|u - v\|, u, v \in E. \end{aligned}$$

By simple calculation, we have

$$\begin{aligned} \bar{d} &= \frac{1}{\Gamma(\alpha)B(\alpha)} \max\{\bar{l}(mM^*l^m + (1-\alpha)), M^*\alpha l^2(ml^m + 1)\} \\ &= \frac{1}{\Gamma(\frac{3}{4})B(\frac{3}{4})} \max\{M^*l + \frac{1}{4}, M^*\frac{1}{4}l^2(l+1)\}, \\ \tilde{l} &= \max\{l, l^2\}, \end{aligned}$$

Taking $p = q = 2, M^* = \frac{1}{5}, l = \frac{1}{4}$, we have

$$l_0 = 2^{\frac{q-1}{q}} \left(\frac{T^{(\alpha-1)q+1}}{(\alpha-1)q+1} + T \right)^{\frac{1}{q}} = 6\sqrt{2}, \tilde{l} = \frac{1}{4}, \bar{d} = \frac{3}{10\Gamma(\frac{3}{4})B(\frac{3}{4})} \approx 0.27,$$

$$L(\kappa) = \sum_{i=1}^3 L_i(\kappa) + L_{\hbar_0} L_4(\kappa) + L_{\rho_0} L_5(\kappa) = \frac{1}{8+8\sqrt{\kappa}} + \frac{4\kappa^{-\frac{1}{2}}e^{-\kappa}}{6} + \frac{1}{2+\kappa^2} + \frac{1}{\kappa^{\frac{1}{2}}(1+e^{2\kappa})},$$

$$\iota = l_0 \left(\int_0^1 L^2(s) ds \right)^{\frac{1}{2}} = 6\sqrt{2} \left(\int_0^1 \left(\frac{1}{8+8\sqrt{\kappa}} + \frac{\kappa^{-\frac{1}{2}}e^{-\kappa}}{6} + \frac{1}{2+\kappa^2} + \frac{1}{4\kappa^{\frac{1}{2}}(1+e^{2\kappa})} \right)^2 d\kappa \right)^{\frac{1}{2}} \approx 2.84,$$

then we have

$$\varrho = M^*\tilde{l}(L_g + L_l) + \bar{d}\iota = M^*\tilde{l}\left(\frac{1}{4} + \frac{1}{8}\right) + 2.84\bar{d} = \frac{3}{8}M^*\tilde{l} + 2.84\bar{d} \approx 0.7855 < 1. \quad (81)$$

Then by Theorem 2, we have model (79) admits a unique global mild solution $u^* \in PC(J, E)$, then the sequence defined by Theorem (12) converges uniformly to the only global mild solution $u^*(t)$ on the norm of E on the interval $[0, 1]$, and for any real number $s > 1$, one has

$$\|u_n - u^*\| = o\left(\frac{1}{n^s}\right), n \rightarrow +\infty.$$

5. Conclusions

In the case of compact and non-compact resolvent operators, the existence and uniqueness are obtained for Mittag–Leffler kernel fractional differential equations of mixed type with delay subject to nonlocal and impulsive boundary value conditions. The equations we

studied are completed in the framework of Mittag-Leffler kernel fractional differentiation; this type of fractional differentiation has the advantage of having non-singular kernel and can fully describe memory, and at the same time, our equation is a generalization of many fractional differential equations compared with the references; the order of the derivative being $0 < \alpha < 1$, this fractional order will bring a series of difficulties in compactness, continuity, convergence property and so on and we overcome these difficulties by the Hölder inequality, Minkowski inequality and the like. The results of Theorem 1 is obtained under the condition of semigroup being compact; moreover, mathematical induction is used in the process of proving the existence and uniqueness of solutions in Theorems 3 and 5. The uniqueness result is obtained by the Banach compression mapping principle, generalized Banach compression mapping principle in Theorems 2 and 3, respectively. The constant L in the Lipschitz conditions are generalized to Lebesgue integrable functions. Subsequently, the non-compactness measure is used to prove the existence of Theorems 4 and 5 under the semigroup being non-compact based on the Darbo fixed point theorem and generalized Darbo fixed point theorem, respectively. Theorems 4 and 5 are completed under conditions (C_5) and (C_6) , respectively, and condition (C_6) is weaker than condition (C_5) , that means the equation is generalized; at the same time, the constants L in the noncompactness measure is also generalized to Lebesgue integrable functions in this process. Hence, the main innovation of this article is to generalize the types of equations and weakens the conditions; furthermore, we extend the constants L in the Lipschitz conditions and the noncompactness measure conditions to Lebesgue integrable functions $L_i(s) (i = 1, 2, \dots, i + 1)$.

6. Future Work

In the future, we plan to conduct further research on the methods for solving the numerical solutions of the equation, and moreover, we will integrate with the actual system and apply the theory to practice.

Author Contributions: Resources, writing-original draft, L.G. and L.L.; theoretical derivation, methodology, validation, funding acquisition, L.L. and H.G.; formal analysis, revise the English expression, L.G. and H.G. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (Nos. 12101086), Natural Science Foundation of Jiangsu Province (No. BK20241773), Major Project of Basic Science (Natural Science) Research in Jiangsu Universities (No. 24KJA120001).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Huang, H.; Fu, X. Nonconvex optimal control problems for semi-linear neutral integro-differential systems with infinite delay. *J. Diff. Equa.* **2025**, *416*, 1137–1177.
2. Guo, L.; Li, C.; Qiao, N.; Zhao, J. Convergence analysis of positive solution for Caputo-Hadamard fractional differential equation. *Nonl. Anal. Mode. Cont.* **2025**, *20*, 212–230.
3. Guo, L.; Wang, Y.; Liu, H.; Li, C.; Zhao, J.; Chu, H. On iterative positive solutions for a class of singular infinite-point p-Laplacian fractional differential equation with singular source terms. *J. Appl. Anal. Comput.* **2023**, *13*, 2827–2842.
4. Kumar, A.; Pandey, D. Existence of mild solution of Atangana-Baleanu fractional differential equations with non-instantaneous impulses and with non-local conditions. *Chaos Solitons Fractals* **2020**, *132*, 109551.
5. Guo, L.; Zhang, B.; Li, C.; Chen, J. Multiple and unique nontrivial solutions for fractional differential equations with singular property and derivatives contained in the nonlinear term. *Demonstr. Math.* **2025**, *58*, 20250186. [[CrossRef](#)]
6. Li, C.; Guo, L. Positive solutions and their existence of a nonlinear Hadamard fractional-order differential equation with a singular source item using spectral analysis. *Fractal Fract.* **2024**, *8*, 377. [[CrossRef](#)]
7. Zhang, X.; Li, Y.; Chen, P. Existence of extremal mild solutions for the initial value problem of evolution equations with non-instantaneous impulses. *J. Fixed Point Theory Appl.* **2017**, *19*, 3013–3027. [[CrossRef](#)]

8. Gou, H.; Li, B. Local and global existence of mild solution to impulsive fractional semilinear integro- differential equation with noncompact semigroup. *Comm. Nonli. Sci. Nume. Simu.* **2017**, *42*, 204–214. [\[CrossRef\]](#)
9. Sharma, A.; Mishra, S.; Shukla, A. Schauders fixed point theorem approach for stability analysis of nonlinear fractional difference equations. *Chaos Solitons Fractals* **2024**, *188*, 115586. [\[CrossRef\]](#)
10. Wang, Y.; Guo, L. Solvability of the caputo fractional differential system with riemann-stieltjes boundary conditions. *J. Appl. Anal. Comp.* **2026**, *16*, 34–44. [\[CrossRef\]](#)
11. Ravichandran, C.; Logeswari, K.; Jarad, F. New results on existence in the framework of Atangana- Baleanu derivative for fractional integro-differential equations. *Chaos Solitons Fractals* **2019**, *125*, 194–200. [\[CrossRef\]](#)
12. Ugbo, J.; Obayi, J.; Udo, M.; Nabwey, H.A.; Ofem, A.; Narain, O. On a faster iterative method for solving fractional delay differential equations in banach spaces. *Fractal Fract.* **2024**, *8*, 166. [\[CrossRef\]](#)
13. Abdul, H.; Bashir, A. Mild solution and finite-approximate controllability of higher-order fractional integrodifferential equations with nonlocal conditions. *Evol. Equa. Cont. Theory* **2024**, *13*, 457324.
14. Liu, Q.; Zuo, Z. Partially regular weak solutions and liouville-type theorem to the stationary fractional navier cstokes equations in dimensions four and five. *J. Diff. Equa.* **2025**, *421*, 291–335.
15. Li, Z.; Yu, J.; Liu, Y.; Zhang, X.; Zhang, B.; Elmaimouni, L. Determination of fractional order for layered viscoelastic materials in cement grouted resin anchor bolt by using guided wave technology. *Mech. Syst. Signal Proc.* **2025**, *224*, 112101. [\[CrossRef\]](#)
16. Zhao, W.; Li, X.; Gao, Y.; Hao, C. Fast gridless direction-of-arrival estimation for wideband linear frequency modulated signals based on fractional fourier transform. *Signal Image Video Proc.* **2025**, *19*, 57.
17. Narayanan, G.; Ali, M.S.; Ahn, S.; Joo, Y.H.; Karthikeyan, R.; Rajchakit, G. Fractional impulsive controller design of fractional-order fuzzy systems with average dwell-time strategy and its application to wind energy systems. *Commu. Nonl. Sci. Nume. Simu.* **2025**, *140*, 108394.
18. Weng, Y.; Li, R.; Li, X. A review of research on fractional-order circuits and fractional-order wireless power transfer system. *Int. Forum Electr. Eng. Autom. (IFEEA)* **2023**, 744–748. [\[CrossRef\]](#)
19. Liu, X.; Ye, G.; Liu, W.; Guo, Y.; Shi, F. On Atangana-Baleanu fractional granular calculus and its applications to fuzzy economic models in market equilibrium. *J. Compu. Appl. Math.* **2024**, *450*, 116017.
20. Liang, C.; Shaokat, S.; Razzaq, A. Hakami, K. Some new integral inequalities for convex functions via abk-fractional operator. *J. Math. Anal. Appl.* **2025**, *542*, 128876.
21. Abdeljawad, T.; Al-Mdallal, Q. Discrete mittag-leffler kernel type fractional difference initial value problems and gronwalls inequality. *Compu. Appl. Math.* **2018**, *339*, 218–230.
22. Hammad, A.; Alshehri, G. Application of the mittag-leffler kernel in stochastic differential systems for approximating the controllability of nonlocal fractional derivatives. *Chaos Solitons Fractals* **2024**, *182*, 114775. [\[CrossRef\]](#)
23. Priya, P.; Sabarmathi, A. Optimal control on abc fractal fractional order model of micro-plastic pollution in soil and its effect on the nutrient cycle. *Compu. Appl. Math.* **2024**, *450*, 115997. [\[CrossRef\]](#)
24. Dhayal, R.; Zhu, Q. The averaging principle of Atangana-Baleanu fractional stochastic integro- differential systems with delay. *J. Anal.* **2024**, *32*, 3565–3576.
25. Chen, L.; Muhammad, A.; Abdon, A.; Sunil, K. A new financial chaotic model in Atangana-Baleanu stochastic fractional differential equations. *Alex. Engi. J.* **2021**, *60*, 5193–5204. [\[CrossRef\]](#)
26. Akram, M.; Muhammad, G.; Ahmad, D. Analytical solution of the Atangana-Baleanu-Caputo fractional differential equations using Pythagorean fuzzy sets. *Granul. Comput.* **2023**, *8*, 667–687.
27. Kalamani, P.; Raj, A. Kumar, P. Results on the existence, uniqueness, and controllability of neutral fractional differential equations in the sense of Atangana-Baleanu derivative. *J. Anal.* **2023**, *33*, 105–120. [\[CrossRef\]](#)
28. Khadija, T.; Samra, G.; Rooh, A. An Atangana-Baleanu derivative-based fractal-fractional order model for the monkey pox virus: A case study of USA. *Part. Diff. Equa. Appl. Math.* **2024**, *9*, 100623.
29. Hilala, K.; Kajounia, A.; Baihia, A.; Lmoua, H. Analytical approach for a coupled system of hybrid fractional integro-differential equations with AtanganaBaleanuCaputo derivative. *Filomat* **2025**, *39*, 2681–2694.
30. Hussain, S.; Sarwar, M.; Shah, S.K.; Abodayeh, K.; Ansari, K.J.; Promsakon, C.; Sitthiwirattam, T. A study on the controllability of AtanganaBaleanu Caputo fractional neutral differential equations with delay. *Chaos Solitons Fractals* **2025**, *201*, 117365. [\[CrossRef\]](#)
31. El-hady, E.; Venkatachalam, K.; Murugapandian, G.S.; Lazar, T.A.; Lazar, V.; Guran, L. β -Ulam stability results for atangana-baleanucaputo fractional equations with non-instantaneous impulsive boundary conditions. *Alex. Engi. J.* **2025**, *125*, 347–353.
32. Hamza, A. E.; Abdo, M. S.; Younis, B.; Aldwoah, K.; Osman, O.; Adam, A.; Saber, H. Qualitative Analysis of Second-Order Atangana–Baleanu Fractional Delay Equations. *Fractal Fract.* **2026**, *10*, 150.
33. Guo, D.; Cho, Y.; Zhu, J. *Partial Ordering Methods in Nonlinear Problems*; Nov a Science Publishers: New York, NY, USA, 2004.
34. Lakshmikantham, V.; Bainov, D.; Simeonov, P. *Theory of Impulsive Differential Equations*; World Scientific: London, UK, 1989.
35. Heinz, H. On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions. *Nonl. Anal. Theo. Meth. Appl.* **1983**, *7*, 1351–1371. [\[CrossRef\]](#)

36. Chen, X.; Cheng, L. On countable determination of the Kuratowski measure of noncompactness. *J. Math. Anal. Appl.* **2021**, *504*, 125370. [[CrossRef](#)]
37. Liu, L.; Guo, F.; Wu, C.; Wu, Y. Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. *J. Math. Anal. Appl.* **2005**, *309*, 638–649. [[CrossRef](#)]
38. Liu, L. Iterative method for solutions and coupled quasi-solutions of nonlinear integro-differential equations of mixed type in banach spaces. *Non. Anal. Theo. Meth. Appl.* **2000**, *42*, 583–598.
39. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science BV: Amsterdam, The Netherlands, 2006.
40. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [[CrossRef](#)]
41. Basc, Y.; Ogrekci, S.; Msr, A. On Hyers-Ulam stability for fractional differential equations including the new Caputo-Fabrizio fractional derivative. *Mediterr. J. Math.* **2019**, *16*, 131.
42. Shu, X.; Lai, Y.; Chen, Y. The existence of mild solutions for impulsive fractional partial differential equations. *Nonl. Anal. Theo. Meth. Appl.* **2011**, *74*, 2003–2011. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.