



## Article

# Qualitative Analysis of a Class of Fractional Functional Differential Systems with Feedback Control

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## Abstract

In this paper, we investigate a class of fractional functional differential systems with feedback control, in which the fractional derivatives are interpreted in the Caputo sense. By applying Schauder's fixed point theorem with the Arzelà–Ascoli compactness criterion, we derive existence results for the solutions. Furthermore, specific hypotheses are introduced to establish uniqueness, continuous dependence of the unique solution, and Hyers–Ulam stability of the considered problem. The applicability and relevance of the theoretical results are further demonstrated through several special cases and illustrative examples. The main contribution of this study is to provide a qualitative analysis of a coupled Caputo fractional functional system involving nonlocal integral conditions and feedback control.

**Keywords:** fractional functional differential systems; feedback control system; existence of solutions; uniqueness of solutions; Hyers–Ulam stability; continuous dependence.

## 1. Introduction

Fractional differential and integral equations have a wide range of applications in physics, engineering, and biomedical engineering. These equations represent a well-established branch of applied mathematics that has been employed in numerous scientific and engineering models, for example, viscoelasticity, signal processing, control theory, and various biological processes [1–5]. Fractional and fractal models have also been used to study pendulum-type mechanical systems [6–8]. On the other hand, nonlocal integral conditions have gained considerable attention in mathematical analysis, since they allow a system's state to be determined by its behavior over an interval or domain rather than just its value at a single location. Such conditions often arise when modelling physical systems subject to constraints or integral-type objectives. Related results on fractional calculus in Banach spaces and stability with nonlocal conditions were discussed in [9,10].

Feedback control systems are essential in engineering, automation, and biological regulation. They change system inputs depending on observed outputs to achieve the desired behavior [11]. In biological systems, feedback controllers serve as regulatory mechanisms ensuring homeostasis, sensory adaptation, and long-term stability in complex processes. Integrative feedback control is especially important for handling unexpected external disturbances that can affect biological features. Such regulatory processes are often described by mathematical models that capture the relationships between system



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components and their reactions to internal or external disturbances [12–14]. Feedback-control mechanisms have also received considerable attention in the study of vibration suppression and active damping systems; see [15,16] for recent results in this direction.

Several contributions in the literature have addressed related problems. In [17], the authors established a sufficient condition for a positive periodic solution in a feedback control model for chemostats. In [18], a positive periodic solution with feedback control was examined for a nonlinear neutral delay population problem. The author of [19] investigated the asymptotic stability and solvability of nonlinear functional-integral equations with feedback control.

Stability analysis is a broad and well-developed field with strong theoretical foundations and various applications in engineering, economics, biology, physics, and other fields. It is a fundamental topic in the mathematical sciences [20]. Stability in the sense of Hyers and Ulam plays a crucial role in understanding how solutions behave under small perturbations: a slight modification in an equation or issue can produce a corresponding small change in the outcome, which is essential for modelling physical processes reliably. The Hyers–Ulam stability of differential equations was first examined by Alsina and Ger [21], and has since been studied widely for many classes of differential and integral equations [22–29].

Another fundamental concept in stability theory is continuous dependence [30], which examines how the behavior of solutions changes when the underlying data or parameters are slightly varied. While Hyers–Ulam stability addresses a problem’s resilience to external perturbations or functional deviations, continuous dependence examines the sensitivity of the unique solution to small modifications in the initial data, boundary conditions, or system parameters. Several studies have investigated the continuous dependence and related qualitative properties of solutions; see [31–37] and the references therein.

Motivated by the above considerations, we are concerned with the existence of a solution  $(x, y) \in X$  to the problem of a fractional functional differential equation with nonlocal integral condition:

$$\begin{cases} \frac{dx}{dt} = f_1(t, g_1(t, D^{\alpha_1}y(t))), \\ x(0) + \int_0^T g_3(s, D^{\beta_1}y(s))ds = x_0, \quad t \in (0, T], \end{cases} \quad (1)$$

controlled by the following feedback control problem:

$$\begin{cases} \frac{dy}{dt} = f_2(t, g_2(t, D^{\alpha_2}x(t))), \\ y(0) + \int_0^T g_4(s, D^{\beta_2}x(s))ds = y_0, \quad t \in (0, T], \end{cases} \quad (2)$$

where  $\alpha_i, \beta_i \in (0, 1)$  for  $i = 1, 2$ , and  $x_0, y_0$  are the initial conditions.

The main objectives of this paper are to establish the existence and uniqueness of solutions  $(x, y) \in X$  to the system (1)–(2), and to analyze the continuous dependence of the unique solution on the initial data and some functions involved in the system, which reflects the sensitivity of the system to small variations in the given data. Furthermore, we establish the Hyers–Ulam stability of the system, which ensures that the solution remains stable under small external perturbations. Several special cases are derived, including the absence of feedback control, proportional fractional feedback, and the initial value problem. These cases demonstrate the generality of the framework. Some illustrative examples are provided to validate the theoretical results.

In [38], the authors studied a constrained problem involving a single fractional-order integro-differential equation subject to a quadratic functional integro-differential constraint,

establishing existence, uniqueness, continuous dependence, and Hyers–Ulam stability for a solution  $x \in C[0, T]$ .

The present work extends this framework in three crucial respects: (i) we consider a coupled system of two equations rather than a single equation; (ii) the coupling is achieved through an explicit feedback-control mechanism between the two state variables  $x$  and  $y$ ; and (iii) the analysis is carried out in the product Banach space  $C[0, T] \times C[0, T]$ , which requires a fixed-point analysis on the product space.

#### Novelty and contribution.

- The work studies a coupled fractional functional differential system in which feedback control and nonlocal integral conditions appear together.
- The coupling between the two state variables is handled in the space  $C[0, T] \times C[0, T]$ , which extends the single-equation setting.
- The paper gives verifiable conditions for existence, uniqueness, continuous dependence, and Hyers–Ulam stability.
- Special cases and examples are included to support the applicability of the theoretical results.

The structure of the paper is as follows. Section 2 presents the assumptions required for the existence of solutions to the system (1)–(2), establishes the equivalence between the system and its corresponding integral equations, and proves the existence of solutions. Section 3 establishes sufficient conditions for the uniqueness of the solution. Section 4 addresses the Hyers–Ulam stability and the continuous dependence of solutions on the given data. Section 5 presents special cases and several illustrative examples to support the theoretical findings. Section 6 discusses the obtained results, their implications, and possible future research directions. Finally, Section 7 concludes the paper with a summary of the main findings.

## 2. Main Results

Let  $I = [0, T]$  and let  $X = C(I) \times C(I)$  be the Banach space of all ordered pairs  $(x, y)$ , where  $x, y \in C(I)$ , with the norm defined by

$$\|(x, y)\|_X = \sup_{t \in [0, T]} |x(t)| + \sup_{t \in [0, T]} |y(t)|.$$

For simplicity, we write  $\|z\|$  instead of  $\|z\|_C$  for  $z \in C(I)$ . Following [39], for  $0 < \alpha < 1$ , the Caputo fractional derivative of an absolutely continuous function  $u$  on  $I$  is given by

$${}^C D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds, \quad t \in I.$$

We impose the following conditions:

- (1) The functions  $f_i : I \times \mathbb{R} \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) and  $g_j : I \times \mathbb{R} \rightarrow \mathbb{R}$  ( $j = 1, 2, 3, 4$ ) are measurable in  $t \in [0, T]$  for every  $x \in \mathbb{R}$  and continuous in  $x \in \mathbb{R}$  for every  $t \in [0, T]$ . Moreover, there exist functions  $a_i \in L_1[0, T]$  ( $i = 1, 2$ ) and  $m_j \in L_1[0, T]$  ( $j = 1, 2, 3, 4$ ), and constants  $b_i > 0$  ( $i = 1, 2$ ) and  $c_j > 0$  ( $j = 1, 2, 3, 4$ ) such that

$$|f_i(t, x)| \leq |a_i(t)| + b_i |x|, \quad i = 1, 2,$$

$$|g_j(t, x)| \leq |m_j(t)| + c_j |x|, \quad j = 1, 2, 3, 4.$$

- (2) There exist positive constants  $a_i^*$  ( $i = 1, 2$ ) and  $m_j^*$  ( $j = 1, 2, 3, 4$ ) such that

$$|a_i(t)| \leq a_i^* \quad (i = 1, 2), \quad |m_j(t)| \leq m_j^* \quad (j = 1, 2, 3, 4), \quad \forall t \in I.$$

(3) The constant  $K$  defined by

$$K = \max \left\{ \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)}, \frac{b_2 c_2 T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \right\}$$

satisfies  $0 < K < 1$ .

(4)  $\beta_i < \alpha_i$  for  $i = 1, 2$ , so that the operator  $I^{\alpha_i - \beta_i}$  is well-defined.

The following lemma establishes the equivalence between system (1)–(2) and the corresponding integral equations.

**Lemma 1.** *The solution  $(x, y)$  of the system (1)–(2) satisfies the following fractional integral equations:*

$$x(t) = x_0 - \int_0^T g_3(s, I^{\alpha_1 - \beta_1} v(s)) ds + \int_0^t f_1(s, g_1(s, v(s))) ds, \quad (3)$$

$$y(t) = y_0 - \int_0^T g_4(s, I^{\alpha_2 - \beta_2} u(s)) ds + \int_0^t f_2(s, g_2(s, u(s))) ds, \quad (4)$$

where  $u$  and  $v$  are defined by

$$u(t) = I^{1-\alpha_2} f_1(t, g_1(t, v(t))), \quad (5)$$

$$v(t) = I^{1-\alpha_1} f_2(t, g_2(t, u(t))). \quad (6)$$

**Proof.** Let  $(x, y)$  be a solution of system (1)–(2). Operating by  $I^{1-\alpha_2}$  on both sides of (1), we obtain

$$I^{1-\alpha_2} \frac{dx}{dt} = I^{1-\alpha_2} f_1(t, g_1(t, D^{\alpha_1} y(t))),$$

and hence

$$D^{\alpha_2} x(t) = I^{1-\alpha_2} f_1(t, g_1(t, D^{\alpha_1} y(t))),$$

Setting  $u(t) = D^{\alpha_2} x(t)$ , we obtain

$$u(t) = I^{1-\alpha_2} f_1(t, g_1(t, D^{\alpha_1} y(t))),$$

$$x(t) = x(0) + I^{\alpha_2} u(t).$$

Similarly, operating by  $I^{1-\alpha_1}$  on both sides of (2), we get

$$I^{1-\alpha_1} \frac{dy}{dt} = I^{1-\alpha_1} f_2(t, g_2(t, D^{\alpha_2} x(t))),$$

$$D^{\alpha_1} y(t) = I^{1-\alpha_1} f_2(t, g_2(t, D^{\alpha_2} x(t))),$$

Setting  $v(t) = D^{\alpha_1} y(t)$ , we obtain

$$v(t) = I^{1-\alpha_1} f_2(t, g_2(t, D^{\alpha_2} x(t))),$$

and

$$y(t) = y(0) + I^{\alpha_1} v(t).$$

Now, we get

$$u(t) = I^{1-\alpha_2} f_1(t, g_1(t, v(t))),$$

$$v(t) = I^{1-\alpha_1} f_2(t, g_2(t, u(t))).$$

Moreover,

$$\begin{aligned}
I^{\alpha_1-\beta_1}v(t) &= I^{\alpha_1-\beta_1}D^{\alpha_1}y(t) \\
&= I^{\alpha_1-\beta_1}I^{1-\alpha_1}\frac{dy(t)}{dt} \\
&= I^{1-\beta_1}\frac{dy(t)}{dt} \\
&= D^{\beta_1}y(t).
\end{aligned}$$

Therefore,

$$x(t) = x_0 - \int_0^T g_3(s, I^{\alpha_1-\beta_1}v(s)) ds + I^{\alpha_2}u(t).$$

Similarly,

$$y(t) = y_0 - \int_0^T g_4(s, I^{\alpha_2-\beta_2}u(s)) ds + I^{\alpha_1}v(t).$$

Conversely, differentiating (3), we obtain

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}[x_0 - \int_0^T g_3(s, I^{\alpha_1-\beta_1}v(s))ds + I^{\alpha_2}u(t)] \\
&= \frac{d}{dt}I^{\alpha_2}u(t) \\
&= \frac{d}{dt}I^{\alpha_2}I^{1-\alpha_2}f_1(t, g_1(t, D^{\alpha_1}y(t))) \\
&= \frac{d}{dt}I f_1(t, g_1(t, D^{\alpha_1}y(t))) \\
&= f_1(t, g_1(t, D^{\alpha_1}y(t))).
\end{aligned}$$

Analogously,

$$\frac{dy}{dt} = f_2(t, g_2(t, D^{\alpha_2}x(t))).$$

This completes the proof.  $\square$

### Existence of Solutions

This subsection establishes the existence of solutions to the system (1)–(2).

**Theorem 1.** *Let assumptions (1)–(4) hold. Then, the controlled system (1)–(2) admits at least one solution  $(x, y) \in X$ .*

**Proof.** Define the closed ball  $Q_r$  by

$$Q_r = \{(u, v) \in X : \|u\|_C \leq r_1, \|v\|_C \leq r_2, \|(u, v)\|_X \leq r\},$$

where  $r = r_1 + r_2$ , and  $r_1, r_2$  are chosen as

$$r_1 = \frac{K_1 + L_1 K_2}{1 - L_1 L_2}, \quad r_2 = \frac{K_2 + L_2 K_1}{1 - L_1 L_2},$$

where, for  $i = 1, 2$ ,

$$K_i = \frac{(a_i^* + b_i m_i^*) T^{1-\alpha_{3-i}}}{\Gamma(2 - \alpha_{3-i})}, \quad L_i = \frac{b_i c_i T^{1-\alpha_{3-i}}}{\Gamma(2 - \alpha_{3-i})}.$$

Then  $r_1 = K_1 + L_1 r_2$  and  $r_2 = K_2 + L_2 r_1$ . By assumption (3), we have  $0 < L_1 < 1$  and  $0 < L_2 < 1$ . Hence,  $1 - L_1 L_2 > 0$ , so  $r_1$  and  $r_2$  are well-defined and positive. Define the operator  $A_1$  by

$$A_1 v(t) = I^{1-\alpha_2} f_1(t, g_1(t, v(t))),$$

and the operator  $A_2$  by

$$A_2 u(t) = I^{1-\alpha_1} f_2(t, g_2(t, u(t))).$$

Let  $(u, v) \in Q_r$ , then, for  $t \in (0, T]$ , we get

$$\begin{aligned} |A_1 v(t)| &= |I^{1-\alpha_2} f_1(t, g_1(t, v(t)))| \\ &\leq I^{1-\alpha_2} (|a_1(t)| + b_1 |g_1(t, v(t))|) \\ &\leq I^{1-\alpha_2} (|a_1(t)| + b_1 |m_1(t)| + b_1 c_1 |v(t)|) \\ &\leq I^{1-\alpha_2} (a_1^* + b_1 m_1^* + b_1 c_1 \|v\|) \\ &\leq I^{1-\alpha_2} (a_1^* + b_1 m_1^* + b_1 c_1 r_2) \\ &\leq (a_1^* + b_1 m_1^* + b_1 c_1 r_2) \frac{T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \\ &\leq K_1 + L_1 r_2 = r_1. \end{aligned}$$

Thus,

$$\|A_1 v\| \leq r_1.$$

Similarly,

$$\|A_2 u\| \leq (a_2^* + b_2 m_2^* + b_2 c_2 r_1) \frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \leq K_2 + L_2 r_1 = r_2.$$

Define  $A : Q_r \rightarrow X$  by

$$A(u, v) = (A_1 v, A_2 u).$$

From the above estimates, we obtain

$$\begin{aligned} \|A(u, v)\|_X &= \|(A_1 v, A_2 u)\|_X \\ &= \|A_1 v\| + \|A_2 u\| \\ &\leq r_1 + r_2 = r. \end{aligned}$$

This proves that the class of functions  $\{A(u, v)\}$  is uniformly bounded on  $Q_r$ . Let  $t_1, t_2 \in (0, T]$ ,  $t_1 < t_2$  such that  $|t_2 - t_1| < \delta$ , then

$$\begin{aligned}
& |A_1 v(t_2) - A_1 v(t_1)| \\
&= |I^{1-\alpha_2} f_1(t_2, g_1(t_2, v(t_2))) - I^{1-\alpha_2} f_1(t_1, g_1(t_1, v(t_1)))| \\
&= \left| \int_0^{t_1} \frac{(t_2-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} f_1(s, g_1(s, v(s))) ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} f_1(s, g_1(s, v(s))) ds \right. \\
&\quad \left. - \int_0^{t_1} \frac{(t_1-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} f_1(s, g_1(s, v(s))) ds \right| \\
&= \left| \int_0^{t_1} \left[ \frac{(t_2-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} - \frac{(t_1-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} \right] f_1(s, g_1(s, v(s))) ds \right. \\
&\quad \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} f_1(s, g_1(s, v(s))) ds \right| \\
&\leq \frac{1}{\Gamma(1-\alpha_2)} \int_0^{t_1} |(t_2-s)^{-\alpha_2} - (t_1-s)^{-\alpha_2}| |f_1(s, g_1(s, v(s)))| ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha_2}}{\Gamma(1-\alpha_2)} |f_1(s, g_1(s, v(s)))| ds \\
&\leq \frac{1}{\Gamma(1-\alpha_2)} \int_0^{t_1} |(t_2-s)^{-\alpha_2} - (t_1-s)^{-\alpha_2}| (|a_1(s)| \\
&\quad + b_1 |m_1(s)| + b_1 c_1 |v(s)|) ds \\
&\quad + \frac{1}{\Gamma(1-\alpha_2)} \int_{t_1}^{t_2} (t_2-s)^{-\alpha_2} (|a_1(s)| + b_1 |m_1(s)| + b_1 c_1 |v(s)|) ds.
\end{aligned}$$

Then

$$\begin{aligned}
|A_1 v(t_2) - A_1 v(t_1)| &\leq \frac{1}{\Gamma(1-\alpha_2)} \int_0^{t_1} |(t_1-s)^{-\alpha_2} - (t_2-s)^{-\alpha_2}| (a_1^* + b_1 m_1^* + b_1 c_1 r_2) ds \\
&\quad + \frac{1}{\Gamma(1-\alpha_2)} \int_{t_1}^{t_2} \frac{1}{(t_2-s)^{\alpha_2}} (a_1^* + b_1 m_1^* + b_1 c_1 r_2) ds.
\end{aligned}$$

Thus,  $\{A_1 v : (u, v) \in Q_r\}$  is equicontinuous and uniformly bounded in  $C(I)$ . Hence, by the Arzelà–Ascoli theorem [40], it is relatively compact in  $C(I)$ . Similarly,

$$\begin{aligned}
|A_2 u(t_2) - A_2 u(t_1)| &\leq \frac{1}{\Gamma(1-\alpha_1)} \int_0^{t_1} |(t_1-s)^{-\alpha_1} - (t_2-s)^{-\alpha_1}| (a_2^* + b_2 m_2^* + b_2 c_2 r_1) ds \\
&\quad + \frac{1}{\Gamma(1-\alpha_1)} \int_{t_1}^{t_2} \frac{1}{(t_2-s)^{\alpha_1}} (a_2^* + b_2 m_2^* + b_2 c_2 r_1) ds.
\end{aligned}$$

Similarly,  $\{A_2 u : (u, v) \in Q_r\}$  is equicontinuous and uniformly bounded in  $C(I)$ , and hence relatively compact by the Arzelà–Ascoli theorem. Therefore, the operator  $A : Q_r \rightarrow Q_r$  is compact.

It remains to show the continuity of the operator  $A$ . Let  $\{u_n\}, \{v_n\} \subset Q_r$ , and  $u_n \rightarrow u, v_n \rightarrow v$ , then

$$A_1 v_n(t) = I^{1-\alpha_2} f_1(t, g_1(t, v_n(t))).$$

By Lebesgue's dominated convergence theorem [40], we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} A_1 v_n(t) &= \lim_{n \rightarrow \infty} I^{1-\alpha_2} f_1(t, g_1(t, v_n(t))) \\
&= I^{1-\alpha_2} f_1(t, g_1(t, \lim_{n \rightarrow \infty} v_n(t))) \\
&= I^{1-\alpha_2} f_1(t, g_1(t, v(t))) \\
&= A_1 v(t).
\end{aligned}$$

Similarly,

$$\lim_{n \rightarrow \infty} A_2 u_n(t) = A_2 u(t).$$

This means that  $A(u_n, v_n) \rightarrow A(u, v)$  in  $X$ . Hence, the operator  $A$  is continuous. By Schauder's fixed point theorem [41], there exists at least one continuous solution  $(u, v) \in X$

of the system (5)–(6). By establishing the existence of the solution  $(u, v)$  of (5)–(6), we immediately obtain, from Lemma 1, that the system (1)–(2) admits at least one solution  $(x, y) \in X$ . This completes the proof.  $\square$

### 3. Uniqueness of Solutions

For the uniqueness of the solution, in addition to conditions (3) and (4) stated in Section 2, the following assumptions are imposed:

(H<sub>1</sub>) The functions  $f_i : I \times \mathbb{R} \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) and  $g_j : I \times \mathbb{R} \rightarrow \mathbb{R}$  ( $j = 1, 2, 3, 4$ ) are continuous on  $I \times \mathbb{R}$  and satisfy the Lipschitz condition in the second variable (uniformly for  $t \in I$ ), i.e.,

$$|f_i(t, x) - f_i(t, y)| \leq b_i |x - y|, \quad b_i > 0, \quad i = 1, 2,$$

$$|g_j(t, x) - g_j(t, y)| \leq c_j |x - y|, \quad c_j > 0, \quad j = 1, 2, 3, 4.$$

(H<sub>2</sub>)  $|f_i(t, 0)| \leq a_i^*$  ( $i = 1, 2$ ) and  $|g_j(t, 0)| \leq m_j^*$  ( $j = 1, 2, 3, 4$ ) for all  $t \in I$ , where  $a_i^*$  and  $m_j^*$  are positive constants.

**Remark 1.** Under assumptions (H<sub>1</sub>)–(H<sub>2</sub>), we have for  $i = 1, 2$  and  $j = 1, 2, 3, 4$ ,

$$|f_i(t, x)| \leq |f_i(t, 0)| + b_i |x| \leq a_i^* + b_i |x|, \quad |g_j(t, x)| \leq |g_j(t, 0)| + c_j |x| \leq m_j^* + c_j |x|.$$

**Theorem 2.** In addition to conditions (3) and (4) of Section 2, assume that (H<sub>1</sub>)–(H<sub>2</sub>) hold. Then the problem (1)–(2) admits a unique solution  $(x, y) \in X$ .

**Proof.** By Remark 1, together with conditions (3) and (4) of Section 2, the assumptions of Theorem 1 are satisfied. Let  $(u_1, v_1)$  and  $(u_2, v_2)$  be two solutions of (5)–(6), then

$$u_1(t) = I^{1-\alpha_2} f_1(t, g_1(t, v_1(t))),$$

$$u_2(t) = I^{1-\alpha_2} f_1(t, g_1(t, v_2(t))),$$

$$\begin{aligned} |u_2(t) - u_1(t)| &= |I^{1-\alpha_2}(f_1(t, g_1(t, v_2(t))) - f_1(t, g_1(t, v_1(t))))| \\ &\leq b_1 I^{1-\alpha_2} |g_1(t, v_2(t)) - g_1(t, v_1(t))| \\ &\leq b_1 c_1 I^{1-\alpha_2} |v_2(t) - v_1(t)| \\ &\leq b_1 c_1 I^{1-\alpha_2} \|v_2 - v_1\|. \end{aligned}$$

Thus,

$$\|u_2 - u_1\| \leq \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v_2 - v_1\|.$$

Similarly,

$$\|v_2 - v_1\| \leq \frac{b_2 c_2 T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \|u_2 - u_1\|.$$

Hence,

$$\begin{aligned}
\|(u_2, v_2) - (u_1, v_1)\|_X &= \|(u_2 - u_1, v_2 - v_1)\|_X \\
&= \|u_2 - u_1\| + \|v_2 - v_1\| \\
&\leq \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v_2 - v_1\| + \frac{b_2 c_2 T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \|u_2 - u_1\| \\
&\leq K\{\|v_2 - v_1\| + \|u_2 - u_1\|\} \\
&= K\|(u_2, v_2) - (u_1, v_1)\|_X.
\end{aligned}$$

This implies that

$$(1 - K)\|(u_2, v_2) - (u_1, v_1)\|_X \leq 0.$$

Since  $0 < K < 1$ , it follows that

$$\|(u_2, v_2) - (u_1, v_1)\|_X = 0.$$

Then  $(u_1, v_1) = (u_2, v_2)$ , which proves that the solution of the problem (5)–(6) is unique. For the uniqueness of the solution of the problem (1)–(2), let  $(x_1, y_1)$  and  $(x_2, y_2)$  be two solutions satisfying (3)–(4), then

$$x_1(t) = x_0 - \int_0^T g_3(s, I^{\alpha_1-\beta_1} v_1(s)) ds + I^{\alpha_2} u_1(t),$$

$$x_2(t) = x_0 - \int_0^T g_3(s, I^{\alpha_1-\beta_1} v_2(s)) ds + I^{\alpha_2} u_2(t),$$

$$\begin{aligned}
|x_2(t) - x_1(t)| &= |x_0 - \int_0^T g_3(s, I^{\alpha_1-\beta_1} v_2(s)) ds + I^{\alpha_2} u_2(t) \\
&\quad - x_0 - \int_0^T g_3(s, I^{\alpha_1-\beta_1} v_1(s)) ds + I^{\alpha_2} u_1(t)| \\
&\leq \int_0^T |g_3(s, I^{\alpha_1-\beta_1} v_2(s)) - g_3(s, I^{\alpha_1-\beta_1} v_1(s))| ds + I^{\alpha_2} |u_2(t) - u_1(t)| \\
&\leq c_3 \int_0^T I^{\alpha_1-\beta_1} |v_2(s) - v_1(s)| ds + I^{\alpha_2} |u_2(t) - u_1(t)| \\
\|x_2 - x_1\| &\leq \|v_2 - v_1\| \frac{c_3 T^{1+\alpha_1-\beta_1}}{(1+\alpha_1-\beta_1)\Gamma(1+\alpha_1-\beta_1)} + \frac{T^{\alpha_2}}{(\alpha_2)\Gamma(\alpha_2)} \|u_2 - u_1\| \\
\|x_2 - x_1\| &\leq \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2+\alpha_1-\beta_1)} \|v_2 - v_1\| + \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} \|u_2 - u_1\|.
\end{aligned}$$

Similarly,

$$\|y_2 - y_1\| \leq \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2+\alpha_2-\beta_2)} \|u_2 - u_1\| + \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} \|v_2 - v_1\|.$$

Hence,

$$\begin{aligned}
\|(x_2, y_2) - (x_1, y_1)\|_X &= \|(x_2 - x_1, y_2 - y_1)\|_X \\
&= \|x_2 - x_1\| + \|y_2 - y_1\| \\
&= \|v_2 - v_1\| \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2+\alpha_1-\beta_1)} + \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} \|u_2 - u_1\| \\
&\quad + \|u_2 - u_1\| \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2+\alpha_2-\beta_2)} + \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} \|v_2 - v_1\| \\
&\leq \left\{ \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2+\alpha_2-\beta_2)} \right\} \|u_2 - u_1\| \\
&\quad + \left\{ \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2+\alpha_1-\beta_1)} \right\} \|v_2 - v_1\|.
\end{aligned}$$

Define the constant  $B$  by

$$B = \max \left\{ \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2+\alpha_1-\beta_1)}, \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2+\alpha_2-\beta_2)} \right\}.$$

Then

$$\begin{aligned}
\|(x_2, y_2) - (x_1, y_1)\|_X &\leq B\{\|u_2 - u_1\| + \|v_2 - v_1\|\} \\
&= B\|(u_2, v_2) - (u_1, v_1)\|_X
\end{aligned}$$

Since we have proved that  $(u_1, v_1) = (u_2, v_2)$ , we have

$$\|(u_2, v_2) - (u_1, v_1)\|_X = 0.$$

Hence,

$$\|(x_2, y_2) - (x_1, y_1)\|_X = 0.$$

Therefore,

$$(x_1, y_1) = (x_2, y_2),$$

which proves the uniqueness of the solution of system (1)–(2).  $\square$

## 4. Stability Analysis

### 4.1. Hyers–Ulam Stability

**Definition 1.** Let the solution  $(x, y) \in X$  of the system (1)–(2) exist. The problem (1)–(2) is Hyers–Ulam-stable if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any  $\delta$ -approximate solution  $(x_s, y_s) \in X$  satisfying

$$\max \left\{ \left| \frac{dx_s}{dt} - f_1(t, g_1(t, D^{\alpha_1} y_s(t))) \right|, \left| \frac{dy_s}{dt} - f_2(t, g_2(t, D^{\alpha_2} x_s(t))) \right| \right\} < \delta, \quad (7)$$

implies

$$\|(x, y) - (x_s, y_s)\|_X < \epsilon.$$

**Theorem 3.** Let assumptions of Theorem 2 hold. Then the system (1)–(2) is Hyers–Ulam-stable.

**Proof.** Suppose that  $(x_s, y_s) \in X$  is a  $\delta$ -approximate solution of the nonlocal problem satisfying (7) for all  $t \in I$ . Then

$$\left| \frac{dx_s}{dt} - f_1(t, g_1(t, D^{\alpha_1} y_s(t))) \right| < \delta,$$

$$-\delta < \frac{dx_s}{dt} - f_1(t, g_1(t, D^{\alpha_1} y_s(t))) < \delta.$$

Similarly,

$$\begin{aligned} \left| \frac{dy_s}{dt} - f_2(t, g_2(t, D^{\alpha_2} x_s(t))) \right| &< \delta, \\ -\delta < \frac{dy_s}{dt} - f_2(t, g_2(t, D^{\alpha_2} x_s(t))) &< \delta. \end{aligned}$$

Set  $u_s(t) = D^{\alpha_2} x_s(t)$ , and  $v_s(t) = D^{\alpha_1} y_s(t)$  where

$$x_s(t) = x_0 - \int_0^T g_3(s, I^{\alpha_1 - \beta_1} v_s(s)) ds + I^{\alpha_2} u_s(t),$$

$$y_s(t) = y_0 - \int_0^T g_4(s, I^{\alpha_2 - \beta_2} u_s(s)) ds + I^{\alpha_1} v_s(t).$$

Then

$$\begin{aligned} -\delta &< \frac{dx_s}{dt} - f_1(t, g_1(t, D^{\alpha_1} y_s(t))) < \delta \\ \frac{-\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} &< I^{1-\alpha_2} \frac{dx_s}{dt} - I^{1-\alpha_2} f_1(t, g_1(t, D^{\alpha_1} y_s(t))) < \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \\ \frac{-\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} &< u_s(t) - I^{1-\alpha_2} f_1(t, g_1(t, v_s(t))) < \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)}. \end{aligned}$$

Then

$$\begin{aligned} |u_s(t) - I^{1-\alpha_2} f_1(t, g_1(t, v_s(t)))| &< \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \\ |u(t) - u_s(t)| &\leq |I^{1-\alpha_2} f_1(t, g_1(t, v(t))) - I^{1-\alpha_2} f_1(t, g_1(t, v_s(t)))| \\ &+ |I^{1-\alpha_2} f_1(t, g_1(t, v_s(t))) - u_s(t)| \\ &\leq I^{1-\alpha_2} |f_1(t, g_1(t, v(t))) - f_1(t, g_1(t, v_s(t)))| + \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \\ &\leq \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + b_1 I^{1-\alpha_2} |g_1(t, v(t)) - g_1(t, v_s(t))| \\ &\leq \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + b_1 c_1 I^{1-\alpha_2} |v(t) - v_s(t)| \\ &\leq \delta \frac{T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v - v_s\|, \end{aligned}$$

then

$$\|u - u_s\| \leq \delta \frac{T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v - v_s\|.$$

Similarly,

$$\|v - v_s\| \leq \delta \frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} + \frac{b_2 c_2 T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \|u - u_s\|.$$

$$\begin{aligned} \|(u, v) - (u_s, v_s)\|_X &= \|(u - u_s, v - v_s)\|_X \\ &= \|u - u_s\| + \|v - v_s\| \\ &\leq \delta \frac{T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v - v_s\| \\ &+ \delta \frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} + \frac{b_2 c_2 T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \|u - u_s\| \\ &\leq K\{\|v - v_s\| + \|u - u_s\|\} + \delta \frac{T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + \delta \frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \\ &\leq K\|(u - u_s, v - v_s)\|_X + \delta^*, \end{aligned}$$

where

$$\delta^* = \delta \left( \frac{T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + \frac{T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \right).$$

This implies that

$$\|(u, v) - (u_s, v_s)\|_X \leq K \|(u, v) - (u_s, v_s)\|_X + \delta^*$$

and

$$\|(u, v) - (u_s, v_s)\|_X \leq \frac{\delta^*}{(1-K)}.$$

Now

$$\begin{aligned} |x(t) - x_s(t)| &= |x_0 - \int_0^T g_3(s, I^{\alpha_1-\beta_1} v(s)) ds + I^{\alpha_2} u(t) \\ &\quad - x_0 + \int_0^T g_3(s, I^{\alpha_1-\beta_1} v_s(s)) ds - I^{\alpha_2} u_s(t)| \\ &\leq \int_0^T |g_3(s, I^{\alpha_1-\beta_1} v(s)) - g_3(s, I^{\alpha_1-\beta_1} v_s(s))| ds + I^{\alpha_2} |u(t) - u_s(t)| \\ &\leq c_3 \int_0^T I^{\alpha_1-\beta_1} |v(s) - v_s(s)| ds + I^{\alpha_2} |u(t) - u_s(t)| \end{aligned}$$

$$\|x - x_s\| \leq \|v - v_s\| \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2+\alpha_1-\beta_1)} + \|u - u_s\| \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)}.$$

Similarly,

$$\|y - y_s\| \leq \|u - u_s\| \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2+\alpha_2-\beta_2)} + \|v - v_s\| \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}.$$

Then

$$\begin{aligned} \|(x, y) - (x_s, y_s)\|_X &= \|(x - x_s, y - y_s)\|_X \\ &\leq \|v - v_s\| \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2+\alpha_1-\beta_1)} + \|u - u_s\| \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} \\ &\quad + \|u - u_s\| \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2+\alpha_2-\beta_2)} + \|v - v_s\| \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}. \end{aligned}$$

Then

$$\begin{aligned} \|(x, y) - (x_s, y_s)\|_X &\leq B\{\|u - u_s\| + \|v - v_s\|\} \\ &= B\|(u, v) - (u_s, v_s)\|_X \\ &\leq B \frac{\delta^*}{(1-K)}. \end{aligned}$$

Choosing  $\delta > 0$  sufficiently small, we obtain

$$\|(x, y) - (x_s, y_s)\| \leq \epsilon.$$

This proves the Hyers–Ulam stability of system (1)–(2).  $\square$

#### 4.2. Continuous Dependence

**Definition 2.** The solution  $(u, v)$ ,  $u, v \in Q_r$  of (5)–(6) depends continuously on  $f_i, g_i$  in the sense that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\max\{|f_i - f_i^*|, |g_i - g_i^*|\} < \delta, \quad i = 1, 2.$$

Then

$$\|(u, v) - (u^*, v^*)\|_X < \epsilon,$$

where,  $u^*, v^*$  are given by

$$u^*(t) = I^{1-\alpha_2} f_1^*(t, g_1^*(t, v^*(t))),$$

$$v^*(t) = I^{1-\alpha_1} f_2^*(t, g_2^*(t, u^*(t))).$$

**Theorem 4.** Let the assumptions of Theorem 2 hold. Then the solution  $(u, v) \in Q_r$  of (5)–(6) depends continuously on  $f_i$  and  $g_i$ ,  $i = 1, 2$ .

**Proof.** Assume that, for  $i = 1, 2$ ,

$$\|f_i - f_i^*\| < \delta, \quad \|g_i - g_i^*\| < \delta.$$

Then

$$\begin{aligned} & |u(t) - u^*(t)| \\ & \leq |I^{1-\alpha_2} f_1(t, g_1(t, v(t))) - I^{1-\alpha_2} f_1(t, g_1^*(t, v^*(t)))| \\ & + |I^{1-\alpha_2} f_1(t, g_1^*(t, v^*(t))) - I^{1-\alpha_2} f_1^*(t, g_1^*(t, v^*(t)))| \\ & \leq b_1 I^{1-\alpha_2} |g_1(t, v(t)) - g_1^*(t, v^*(t))| \\ & + |I^{1-\alpha_2} f_1(t, g_1^*(t, v^*(t))) - I^{1-\alpha_2} f_1^*(t, g_1^*(t, v^*(t)))| \\ & \leq b_1 I^{1-\alpha_2} |g_1(t, v(t)) - g_1(t, v^*(t)) + g_1(t, v^*(t)) - g_1^*(t, v^*(t))| + \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \\ & \leq b_1 I^{1-\alpha_2} (|g_1(t, v(t)) - g_1(t, v^*(t))| + |g_1(t, v^*(t)) - g_1^*(t, v^*(t))|) + \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \\ & \leq b_1 c_1 I^{1-\alpha_2} |v(t) - v^*(t)| + b_1 \delta I^{1-\alpha_2} + \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)}. \\ & \leq \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v - v^*\| + \frac{\delta b_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} + \frac{\delta T^{1-\alpha_2}}{\Gamma(2-\alpha_2)}, \end{aligned}$$

It follows that

$$\|u - u^*\| \leq \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v - v^*\| + \frac{(b_1 + 1) T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \delta.$$

Similarly,

$$\|v - v^*\| \leq \frac{b_2 c_2 T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \|u - u^*\| + \frac{(b_2 + 1) T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \delta,$$

Therefore,

$$\begin{aligned} \|(u, v) - (u^*, v^*)\| &= \|u - u^*\| + \|v - v^*\| \\ &\leq \frac{b_1 c_1 T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \|v - v^*\| + \frac{(b_1 + 1) T^{1-\alpha_2}}{\Gamma(2-\alpha_2)} \delta \\ &+ \frac{b_2 c_2 T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \|u - u^*\| + \frac{(b_2 + 1) T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \delta. \end{aligned}$$

This implies that

$$\|(u, v) - (u^*, v^*)\| \leq K \|(u, v) - (u^*, v^*)\| + \tilde{\delta},$$

where  $\tilde{\delta} = \frac{(b_1+1)T^{1-\alpha_2}}{\Gamma(2-\alpha_2)}\delta + \frac{(b_2+1)T^{1-\alpha_1}}{\Gamma(2-\alpha_1)}\delta$ .

$$\begin{aligned}\|(u, v) - (u^*, v^*)\| (1 - K) &\leq \tilde{\delta}, \\ \|(u, v) - (u^*, v^*)\|_X &\leq \frac{\tilde{\delta}}{1 - K} < \epsilon.\end{aligned}$$

□

**Definition 3.** The solution  $(x, y) \in X$  of the system (1)–(2) depends continuously on  $x_0, y_0$ , and  $g_i, i = 3, 4$  in the sense that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\max\{|x_0 - x_0^*|, |y_0 - y_0^*|, |g_i - g_i^*|\} < \delta, \quad i = 3, 4,$$

implies that

$$\|(x, y) - (x^*, y^*)\|_X < \epsilon,$$

where  $x^*, y^*$  are the unique solutions of

$$\begin{aligned}x^*(t) &= x_0^* - \int_0^T g_3^*(s, I^{\alpha_1-\beta_1}v^*(s))ds + I^{\alpha_2}u^*(t), \\ y^*(t) &= y_0^* - \int_0^T g_4^*(s, I^{\alpha_2-\beta_2}u^*(s))ds + I^{\alpha_1}v^*(t).\end{aligned}$$

**Corollary 1.** Let assumptions of Theorem 2 hold. Then the solution  $(x, y) \in X$  of the system (1)–(2) depends continuously on  $x_0, y_0$ , and  $g_i, i = 3, 4$ .

**Proof.**

$$\begin{aligned}|x(t) - x^*(t)| &\leq |x_0 - x_0^*| + \int_0^T |g_3(s, I^{\alpha_1-\beta_1}v(s)) - g_3^*(s, I^{\alpha_1-\beta_1}v^*(s))|ds \\ &\quad + I^{\alpha_2}|u(t) - u^*(t)| \\ &\leq \delta + \int_0^T |g_3(s, I^{\alpha_1-\beta_1}v(s)) - g_3^*(s, I^{\alpha_1-\beta_1}v^*(s))|ds + I^{\alpha_2}\|u - u^*\| \\ &\leq \delta + \int_0^T |g_3(s, I^{\alpha_1-\beta_1}v(s)) - g_3(s, I^{\alpha_1-\beta_1}v^*(s))|ds \\ &\quad + \int_0^T |g_3(s, I^{\alpha_1-\beta_1}v^*(s)) - g_3^*(s, I^{\alpha_1-\beta_1}v^*(s))|ds + I^{\alpha_2}\|u - u^*\| \\ &\leq \delta(1 + T) + c_3 \int_0^T I^{\alpha_1-\beta_1}|v(s) - v^*(s)|ds + I^{\alpha_2}\|u - u^*\| \\ &\leq \delta(1 + T) + \|v - v^*\| \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2 + \alpha_1 - \beta_1)} + \frac{T^{\alpha_2}}{\Gamma(1 + \alpha_2)}\|u - u^*\|.\end{aligned}$$

Then

$$\|x - x^*\| \leq \delta(1 + T) + \|v - v^*\| \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2 + \alpha_1 - \beta_1)} + \frac{T^{\alpha_2}}{\Gamma(1 + \alpha_2)}\|u - u^*\|,$$

Similarly,

$$\|y - y^*\| \leq \delta(1 + T) + \|u - u^*\| \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2 + \alpha_2 - \beta_2)} + \frac{T^{\alpha_1}}{\Gamma(1 + \alpha_1)}\|v - v^*\|.$$

Hence,

$$\begin{aligned}
 \|(x, y) - (x^*, y^*)\|_X &= \|(x - x^*, y - y^*)\|_X \\
 &\leq \delta(1 + T) + \|v - v^*\| \frac{c_3 T^{1+\alpha_1-\beta_1}}{\Gamma(2+\alpha_1-\beta_1)} + \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} \|u - u^*\| \\
 &\quad + \delta(1 + T) + \|u - u^*\| \frac{c_4 T^{1+\alpha_2-\beta_2}}{\Gamma(2+\alpha_2-\beta_2)} + \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} \|v - v^*\|. \\
 &\leq B\{\|u - u^*\| + \|v - v^*\|\} + 2\delta(1 + T) \\
 &\leq B\|(u, v) - (u^*, v^*)\|_X + 2\delta(1 + T) \\
 &\leq B \frac{\tilde{\delta}}{1 - K} + 2\delta(1 + T).
 \end{aligned}$$

Choosing  $\delta > 0$  sufficiently small, we obtain

$$\|(x, y) - (x^*, y^*)\|_X < \epsilon.$$

Therefore, the solution depends continuously on  $x_0, y_0$ , and  $g_i, i = 3, 4$ .  $\square$

## 5. Special Cases and Examples

The following cases are obtained from the system (1)–(2) by imposing particular structural conditions. The results of Sections 2–4 apply directly in each case.

- (1) **Absence of feedback control:** When the feedback control is absent, the system (1)–(2) reduces to the single fractional problem

$$\begin{cases} \frac{dx}{dt} = f(t, g_1(t, D^\alpha x(t))), \\ x(0) + \int_0^T g_2(s, D^\beta x(s)) ds = x_0, \quad t \in (0, T], \end{cases} \quad (8)$$

where  $0 < \beta < \alpha < 1$ . Setting  $u(t) = D^\alpha x(t)$ , problem (8) is equivalent to

$$u(t) = I^{1-\alpha} f(t, g_1(t, u(t))), \quad (9)$$

with the solution

$$x(t) = x_0 - \int_0^T g_2(s, I^{\alpha-\beta} u(s)) ds + I^\alpha u(t), \quad t \in (0, T]. \quad (10)$$

This problem is studied under the following assumptions:

- (a) The functions  $f$  and  $g_j, j = 1, 2$ , satisfy the same measurability and growth conditions as in (1) and (2) of Section 2, with constants  $b > 0$  and  $c_j > 0$ .
- (b)  $\frac{bc_1 T^{1-\alpha}}{\Gamma(2-\alpha)} < 1$ .
- (c)  $\beta < \alpha$ , so that  $I^{\alpha-\beta}$  is well-defined.

Using the same techniques as in Sections 2–4, we obtain the following corollaries.

**Corollary 2.** *Let assumptions (a)–(c) hold. Then problem (8) has at least one solution  $x \in C(I)$ .*

**Corollary 3.** *In addition to (a)–(c), assume that  $f$  and  $g_j, j = 1, 2$ , satisfy assumptions  $(H_1)$ – $(H_2)$  with  $K = bc_1 T^{1-\alpha} / \Gamma(2-\alpha) \in (0, 1)$ . Then the solution of problem (8) is unique, Hyers–Ulam-stable, and depends continuously on  $x_0$  and  $g_2$ .*

- (2) **Proportional fractional feedback:** Let the assumptions of Theorem 1 be satisfied. Suppose that

$$g_1(t, z) = \lambda_1 z, \quad g_2(t, z) = \lambda_2 z,$$

where  $\lambda_1$  and  $\lambda_2$  are nonzero real constants. Then system (1)–(2) takes the form

$$\begin{cases} \frac{dx}{dt} = f_1(t, \lambda_1 D^{\alpha_1} y(t)), \\ x(0) + \int_0^T g_3(s, D^{\beta_1} y(s)) ds = x_0, \quad t \in (0, T], \end{cases} \quad (11)$$

$$\begin{cases} \frac{dy}{dt} = f_2(t, \lambda_2 D^{\alpha_2} x(t)), \\ y(0) + \int_0^T g_4(s, D^{\beta_2} x(s)) ds = y_0, \quad t \in (0, T]. \end{cases} \quad (12)$$

In this case, the Lipschitz constants of  $g_1$  and  $g_2$  are  $|\lambda_1|$  and  $|\lambda_2|$ , respectively. Hence, if

$$\max \left\{ \frac{b_1 |\lambda_1| T^{1-\alpha_2}}{\Gamma(2-\alpha_2)}, \frac{b_2 |\lambda_2| T^{1-\alpha_1}}{\Gamma(2-\alpha_1)} \right\} < 1,$$

then system (11)–(12) admits at least one solution  $(x, y) \in X$ . Moreover, if the hypotheses of Theorem 2 also hold, then this solution is unique, and the stability results of Section 4 remain valid.

- (3) **Initial value problem:** Let the assumptions of Theorem 1 be satisfied. If

$$g_3 \equiv 0, \quad g_4 \equiv 0,$$

then the nonlocal conditions reduce to  $x(0) = x_0$  and  $y(0) = y_0$ , and the system

$$\begin{cases} \frac{dx}{dt} = f_1(t, g_1(t, D^{\alpha_1} y(t))), & x(0) = x_0, \\ \frac{dy}{dt} = f_2(t, g_2(t, D^{\alpha_2} x(t))), & y(0) = y_0, \end{cases} \quad (13)$$

admits at least one solution  $(x, y) \in X$ . Moreover, if the assumptions of Theorem 2 are also satisfied, then this solution is unique.

**Example 1.** Consider the following system

$$\begin{cases} \frac{dx}{dt} = \frac{1}{9} e^{-t} \sqrt{t^2 + 4} + \frac{1}{7} \left( \frac{5-3t}{12} + \frac{|D^{\frac{1}{6}} y(t)|}{8(1+|D^{\frac{1}{6}} y(t)|)} \right), & t \in \left(0, \frac{1}{3}\right], \\ x(0) + \int_0^{\frac{1}{3}} \left( \frac{5+3s}{16} \sin^2(2s) + \frac{1}{14} e^{-s} D^{\frac{1}{7}} y(s) \right) ds = \frac{1}{2}, \end{cases} \quad (14)$$

controlled by the feedback control problem

$$\begin{cases} \frac{dy}{dt} = \frac{1}{4} e^{-t^2} \sin(2t) + \frac{1}{9} \left( \frac{t^2 e^{-t}}{5-t} + \frac{\ln(1+|D^{\frac{1}{4}} x(t)|)}{1+3t} \right), & t \in \left(0, \frac{1}{3}\right], \\ y(0) + \int_0^{\frac{1}{3}} \frac{1}{2} \left( \ln(1+s) + \frac{1}{3} e^{-s} D^{\frac{1}{5}} x(s) \right) ds = \frac{1}{8}. \end{cases} \quad (15)$$

Define

$$f_1(t, u) = \frac{1}{9} e^{-t} \sqrt{t^2 + 4} + \frac{1}{7} u, \quad g_1(t, z) = \frac{5-3t}{12} + \frac{|z|}{8(1+|z|)},$$

$$f_2(t, u) = \frac{1}{4}e^{-t^2} \sin(2t) + \frac{1}{9}u, \quad g_2(t, z) = \frac{t^2 e^{-t}}{5-t} + \frac{\ln(1+|z|)}{1+3t},$$

$$g_3(t, z) = \frac{5+3t}{16} \sin^2(2t) + \frac{1}{14}e^{-t}z, \quad g_4(t, z) = \frac{1}{2} \ln(1+t) + \frac{1}{6}e^{-t}z.$$

We take

$$T = \frac{1}{3}, \quad \alpha_1 = \frac{1}{6}, \quad \alpha_2 = \frac{1}{4}, \quad \beta_1 = \frac{1}{7}, \quad \beta_2 = \frac{1}{5}, \quad x_0 = \frac{1}{2}, \quad y_0 = \frac{1}{8}.$$

For  $t \in \left[0, \frac{1}{3}\right]$ , we have

$$|f_1(t, 0)| = \frac{1}{9}e^{-t} \sqrt{t^2 + 4} \leq \frac{2}{9} = a_1^*, \quad |f_2(t, 0)| = \frac{1}{4}e^{-t^2} |\sin(2t)| \leq \frac{1}{4} = a_2^*.$$

and

$$|g_1(t, 0)| \leq \frac{5}{12} = m_1^*, \quad |g_2(t, 0)| \leq \frac{(1/3)^2}{5-1/3} = \frac{1}{42} = m_2^*,$$

$$|g_3(t, 0)| \leq \frac{3}{8} = m_3^*, \quad |g_4(t, 0)| \leq \frac{1}{2} \ln\left(\frac{4}{3}\right) = m_4^*.$$

Moreover, for all  $t \in \left[0, \frac{1}{3}\right]$  and for all  $u, v, z, w \in \mathbb{R}$

$$|f_1(t, u) - f_1(t, v)| \leq \frac{1}{7}|u - v| \quad (b_1 = \frac{1}{7}), \quad |f_2(t, u) - f_2(t, v)| \leq \frac{1}{9}|u - v| \quad (b_2 = \frac{1}{9}).$$

Also,

$$|g_1(t, z) - g_1(t, w)| \leq \frac{1}{8}|z - w| \quad (c_1 = \frac{1}{8}), \quad |g_2(t, z) - g_2(t, w)| \leq |z - w| \quad (c_2 = 1),$$

$$|g_3(t, z) - g_3(t, w)| \leq \frac{1}{14}|z - w| \quad (c_3 = \frac{1}{14}), \quad |g_4(t, z) - g_4(t, w)| \leq \frac{1}{6}|z - w| \quad (c_4 = \frac{1}{6}).$$

With the above choices, we obtain  $L_1 = 0.0085 < 1$ ,  $L_2 = 0.04729 < 1$ , and  $K = 0.04729 < 1$ . Therefore, all assumptions (1)–(4) of Section 2 and  $(H_1)$ – $(H_2)$  are satisfied. Hence, by Theorem 1, the system (14)–(15) has at least one solution  $(x, y) \in X$ , and by Theorem 2, this solution is unique.

**Example 2.** Consider the following system

$$\begin{cases} \frac{dx}{dt} = \frac{1}{6}t^2 e^{-t} \cos^2 t + \frac{1}{12} \left( \frac{e^{-t} \sin t}{3} + \frac{1}{10} D^{\frac{1}{6}} y(t) \right), & t \in \left(0, \frac{1}{2}\right], \\ x(0) + \int_0^{\frac{1}{2}} \left( \frac{1+2s}{9} + \frac{1}{32} D^{\frac{1}{9}} y(s) \right) ds = \frac{1}{7}. \end{cases} \quad (16)$$

controlled by the feedback control problem

$$\begin{cases} \frac{dy}{dt} = \frac{1}{14} \frac{e^{-t}}{5-t} + \frac{1}{10} \left( \frac{1 + \ln(1 + |D^{\frac{1}{7}} x(t)|)}{5-t} + \frac{1}{7} D^{\frac{1}{7}} x(t) \right), & t \in \left(0, \frac{1}{2}\right], \\ y(0) + \int_0^{\frac{1}{2}} \left( \frac{1+2s}{8} + \frac{1}{32} D^{\frac{1}{8}} x(s) \right) ds = \frac{1}{9}. \end{cases} \quad (17)$$

Define

$$f_1(t, u) = \frac{1}{6}t^2 e^{-t} \cos^2 t + \frac{1}{12}u, \quad g_1(t, z) = \frac{e^{-t} \sin t}{3} + \frac{1}{10}z,$$

$$f_2(t, u) = \frac{1}{14} \frac{e^{-t}}{5-t} + \frac{1}{10}u, \quad g_2(t, z) = \frac{1 + \ln(1 + |z|)}{5-t} + \frac{1}{7}z,$$

$$g_3(t, z) = \frac{1+2t}{9} + \frac{1}{32}z, \quad g_4(t, z) = \frac{1+2t}{8} + \frac{1}{32}z.$$

We take

$$T = \frac{1}{2}, \quad \alpha_1 = \frac{1}{6}, \quad \alpha_2 = \frac{1}{7}, \quad \beta_1 = \frac{1}{9}, \quad \beta_2 = \frac{1}{8}, \quad x_0 = \frac{1}{7}, \quad y_0 = \frac{1}{9}.$$

For  $t \in \left[0, \frac{1}{2}\right]$ , we have

$$|f_1(t, 0)| = \frac{1}{6}t^2e^{-t}\cos^2 t \leq \frac{1}{24} = a_1^*, \quad |f_2(t, 0)| = \frac{1}{14}\frac{e^{-t}}{5-t} \leq \frac{1}{63} = a_2^*.$$

and

$$\begin{aligned} |g_1(t, 0)| &\leq \frac{1}{6} = m_1^*, & |g_2(t, 0)| &\leq \frac{2}{9} = m_2^*, \\ |g_3(t, 0)| &\leq \frac{2}{9} = m_3^*, & |g_4(t, 0)| &\leq \frac{1}{4} = m_4^*. \end{aligned}$$

Moreover, for all  $t \in \left[0, \frac{1}{2}\right]$  and for all  $u, v, z, w \in \mathbb{R}$ ,

$$|f_1(t, u) - f_1(t, v)| \leq \frac{1}{12}|u - v| \quad (b_1 = \frac{1}{12}), \quad |f_2(t, u) - f_2(t, v)| \leq \frac{1}{10}|u - v| \quad (b_2 = \frac{1}{10}).$$

Also,

$$\begin{aligned} |g_1(t, z) - g_1(t, w)| &\leq \frac{1}{10}|z - w| \quad (c_1 = \frac{1}{10}), \\ |g_2(t, z) - g_2(t, w)| &\leq \left(\frac{2}{9} + \frac{1}{7}\right)|z - w| = \frac{23}{63}|z - w| \quad (c_2 = \frac{23}{63}). \end{aligned}$$

Moreover,

$$|g_3(t, z) - g_3(t, w)| \leq \frac{1}{32}|z - w| \quad (c_3 = \frac{1}{32}), \quad |g_4(t, z) - g_4(t, w)| \leq \frac{1}{32}|z - w| \quad (c_4 = \frac{1}{32}).$$

With the above choices, we obtain  $L_1 = 0.00485 < 1$ ,  $L_2 = 0.02178 < 1$ , and  $K = 0.02178 < 1$ .

Therefore, all assumptions (1)–(4) of Section 2 and  $(H_1)$ – $(H_2)$  are satisfied. Hence, by Theorem 1, the system (16)–(17) has at least one solution  $(x, y) \in X$ , and by Theorem 2, this solution is unique.

**Example 3.** Consider the following system

$$\begin{cases} \frac{dx}{dt} = \frac{1}{2}e^{-t}\ln(1+t) + \frac{1}{8}\left(\frac{t^3}{4} + \frac{1}{5}D^{\frac{1}{4}}y(t)\right), & t \in \left(0, \frac{1}{3}\right], \\ x(0) + \int_0^{\frac{1}{3}}\left(\frac{e^{-s}\cos s}{5} + \frac{1}{2}D^{\frac{1}{5}}y(s)\right)ds = \frac{1}{5}, \end{cases} \quad (18)$$

controlled by the feedback control problem

$$\begin{cases} \frac{dy}{dt} = \frac{1}{3}\frac{e^{-t}}{1+e^{-t}} + \frac{1}{5}\left(\frac{t^4e^{-t}}{4} + \frac{1}{3}D^{\frac{1}{5}}x(t)\right), & t \in \left(0, \frac{1}{3}\right], \\ y(0) + \int_0^{\frac{1}{3}}\left(\frac{1}{2}\frac{e^{-s}}{4-s} + \frac{1}{8}D^{\frac{1}{9}}x(s)\right)ds = \frac{1}{6}. \end{cases} \quad (19)$$

Define

$$\begin{aligned} f_1(t, u) &= \frac{1}{2}e^{-t}\ln(1+t) + \frac{1}{8}u, & g_1(t, z) &= \frac{t^3}{4} + \frac{1}{5}z, \\ f_2(t, u) &= \frac{1}{3}\frac{e^{-t}}{1+e^{-t}} + \frac{1}{5}u, & g_2(t, z) &= \frac{t^4e^{-t}}{4} + \frac{1}{3}z, \\ g_3(t, z) &= \frac{e^{-t}\cos t}{5} + \frac{1}{2}z, & g_4(t, z) &= \frac{1}{2}\frac{e^{-t}}{4-t} + \frac{1}{8}z. \end{aligned}$$

Let

$$T = \frac{1}{3}, \quad \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{5}, \quad \beta_1 = \frac{1}{5}, \beta_2 = \frac{1}{9}, \quad x_0 = \frac{1}{5}, y_0 = \frac{1}{6}.$$

For  $t \in \left[0, \frac{1}{3}\right]$ , we have

$$|f_1(t, 0)| \leq \frac{1}{2} e^{-\frac{1}{3}} \ln\left(\frac{4}{3}\right) = a_1^*, \quad |f_2(t, 0)| \leq \frac{1}{6} = a_2^*,$$

and

$$|g_1(t, 0)| \leq \frac{1}{108} = m_1^*, \quad |g_2(t, 0)| \leq \frac{e^{-\frac{1}{3}}}{324} = m_2^*,$$

$$|g_3(t, 0)| \leq \frac{1}{5} = m_3^*, \quad |g_4(t, 0)| \leq \frac{1}{8} = m_4^*.$$

Moreover, for all  $t \in \left[0, \frac{1}{3}\right]$  and for all  $u, v, z, w \in \mathbb{R}$ ,

$$|f_1(t, u) - f_1(t, v)| \leq \frac{1}{8} |u - v| \quad (b_1 = \frac{1}{8}), \quad |f_2(t, u) - f_2(t, v)| \leq \frac{1}{5} |u - v| \quad (b_2 = \frac{1}{5}).$$

Also,

$$|g_1(t, z) - g_1(t, w)| \leq \frac{1}{5} |z - w| \quad (c_1 = \frac{1}{5}), \quad |g_2(t, z) - g_2(t, w)| \leq \frac{1}{3} |z - w| \quad (c_2 = \frac{1}{3}),$$

$$|g_3(t, z) - g_3(t, w)| \leq \frac{1}{2} |z - w| \quad (c_3 = \frac{1}{2}), \quad |g_4(t, z) - g_4(t, w)| \leq \frac{1}{8} |z - w| \quad (c_4 = \frac{1}{8}).$$

With the above choices, we obtain  $L_1 \approx 0.0111 < 1$ ,  $L_2 \approx 0.0318 < 1$ , and  $K = \max\{L_1, L_2\} \approx 0.0318 < 1$ . Therefore, all assumptions (1)–(4) of Section 2 and  $(H_1)$ – $(H_2)$  are satisfied. Hence, by Theorem 1, the system (18)–(19) has at least one solution  $(x, y) \in X$ , and by Theorem 2, this solution is unique.

## 6. Discussion

The results show that the proposed fractional system can be studied effectively through its equivalent integral formulation. This formulation enables the use of fixed point techniques to establish the existence of solutions under suitable assumptions.

The uniqueness result indicates that the solution is well determined when the nonlinear terms satisfy Lipschitz conditions. Furthermore, the Hyers–Ulam stability and continuous dependence results illustrate that the solution remains stable under small perturbations in the equations and in the given data.

The particular cases illustrate that the model includes several simpler fractional systems. Thus, the obtained results can be applied to these reduced forms as well as to the original coupled system.

Future work may extend the present study in different directions. The system presented here can be generalized to a system of  $n$  fractional equations, allowing for the investigation of more complex interactions between several state variables. It is also possible to include additional parameters in the nonlinear terms or in the nonlocal conditions. Another interesting direction is to study related problems involving fractional pantograph equations, with proportional delay terms. Moreover, extensions to fractal spaces may also be considered.

## 7. Conclusions

In this paper, we investigated a fractional feedback-control system with nonlocal integral conditions in the Banach space  $X = C(I) \times C(I)$ . By using the equivalent integral formulation as well as fixed point theorem and compactness criterion, we established sufficient conditions for the existence of at least one solution. We also proved the uniqueness of the solution under suitable Lipschitz conditions. Moreover, the Hyers–Ulam stability and continuous dependence results demonstrated that the solution remains stable under small perturbations in the equations and in the given data. Finally, special cases and illustrative examples were presented to support the theoretical results and to show that the proposed framework can be applied to related fractional systems. These results show the advantage of treating the coupled dynamics, feedback effects, and nonlocal conditions simultaneously within a qualitative framework.

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