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An Effective Numerical Scheme for Fractional Integro-Differential Equation Systems Using Hermite Wavelets

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Abstract

Numerous mathematical models, such as elasticity, nuclear reactor dynamics, and heat conduction in memory materials, use systems of fractional integro-differential equations, or FIDEs. In this research, the numerical solution of linear and nonlinear systems of FIDEs is obtained by means of an operational matrix approach based on Hermite wavelets. Using the operational matrix of the Hermite wavelets, the FIDE is transformed into an algebraic equation, and the solution of that algebraic equation is obtained. Three numerical examples are provided to show the accuracy and consistency of the suggested technique.

Keywords: Hermite wavelet method; fractional integro-differential equations; operational matrix method

1. Introduction

The study of the complex order or fractional derivative and integral operators falls under the concept of the fractional calculus. Whereas the differentiation of a function at point x only depends on values that are close to that point of interest for an integer derivative, fractional derivatives are non-local. Therefore, the boundary conditions that provide information on the function further out are required in the fractional equations. Engineering research and mathematical models frequently employ fractional differential equations, or FDEs, and fractional integro-differential equations, or FIDEs. To name a few, these models have been applied to HBV infection dynamics and COVID-19 transmission [1,2], numerical analysis of fractional differential equations through spectral methods [3], fractional-order systems and control applications [4], fractal and acoustic wave propagation models [5,6], uncertainty-based financial systems [7], engineering applications [8], biological synchronization phenomena [9], and bioengineering problems [10]. FIDEs are generally impossible to solve analytically, so finding a numerical or approximate solution to these kinds of equations is useful.

Researchers have employed numerical techniques to obtain approximate solutions to the FDEs and FIDEs up to the present. These techniques include hybrid collocation method [11], hybrid Bernstein block-pulse functions [12], Jacobi spectral-collocation [13,14], Bessel collocation method [15], homotopy perturbation technique [16,17], spline collocation [18], Adomian decomposition technique [19,20], homotopy analysis technique [21], Chebyshev pseudo-spectral method [22], Laplace decomposition method [23], differential transform method [24], polynomials methods [25–29], wavelet methods [30–38], least-squares method [39], pseudo-spectral method [40], discrete Galerkin method [41], varia-



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tional iteration method [42,43], and operational Jacobi Tau method [44] among others. Since most of these methods have only one or few fractional differential terms in them, applying those methods to systems of FDEs/FIDEs or multi-term variable order FDEs/FIDEs may quickly become very complex and prone to errors. Spectral base functions are known to be infinitely differentiable, but they have the disadvantage of being nonzero for all the region of interest. The disadvantage of the finite differences and finite elements methods is that their basis functions are not necessarily continuous, although they have compact support. In contrast to other methods, the wavelet methods have the advantage of having smooth basis functions and compact support.

Wavelets are a collection of functions created by stretching and shifting a certain function known as the mother wavelet. Wavelets are an effective technique dating back to the early 1960s, for the approximations to solutions of a variety of equations. Wavelet algorithms have a number of benefits, such as the improved accuracy by increasing the degree of the mother wavelet and the level of resolution, being multi-resolution, being compactly supported orthonormal, and having sparse coefficient matrices of the algebraic equations. Wavelets were originally applied to the solution of differential equations in 1990 and have since been the focus of many numerical methods. In a variety of scientific fields, wavelets with diverse topologies have been used to solve FDEs and FIDEs, including the Legendre wavelet [30,33], Haar wavelet [31,35], CAS wavelet [32], Taylor Wavelet [34], Chebyshev wavelet [36], and Hermite wavelet [37,38].

In this paper, we propose a numerical technique to solve systems of FIDEs using Hermite wavelets. Hermite wavelets have not been utilized much to the best of our knowledge. Furthermore, one of the main benefits of the orthogonal basis functions is that they yield sparser operational matrices. Therefore, in this study, we first build the necessary operational matrices for the numerical approach to solve the FIDEs using block-pulse functions (BPFs) and Hermite wavelets. The original value of the paper stems from the sparser conversion matrices used to map the FIDE to the resulting systems of algebraic equations in vector-matrix form. Then, those systems of algebraic equations are solved to obtain the approximate solutions of FIDEs. Since the approximation is obtained in vector-matrix form, the steps to obtain the solution are straightforward and easier to code, which is another advantage of the proposed method. Compared with the existing wavelet and polynomial approaches reviewed above [30–38], the present contribution is twofold. First, to the best of our knowledge, Hermite wavelets have not previously been applied to coupled systems of FIDEs. Second, the orthogonality of the Hermite basis yields operational matrices with a large proportion of (near-)zero entries, which directly reduces the cost of forming and solving the resulting algebraic system. The specific difficulties addressed here are the coupling between the unknowns u and v , which prevents component-wise solution; the nonlinear product terms, which are handled through the block-pulse product operations; and the simultaneous presence of fractional derivative and integral operators, which are treated within a single unified operational-matrix framework.

In this research, we aim to find the solution of the system of FIDEs defined as:

$$\begin{aligned} D_*^{\alpha_i} u_i(t) &= g_i(t, u_1(t), u_2(t)) + \int_0^t h_i(\eta, u_1(\eta), u_2(\eta)) d\eta, \\ u_i(0) &= c_i, \quad i = 1, 2 \end{aligned} \quad (1)$$

$$\text{where } t, \eta \in [0, 1], 0 < \alpha_i \leq 1, \text{ and } D^\alpha u(t) = \begin{cases} \frac{d^n u(t)}{dt^n}, \alpha = n \in \mathbb{R} \\ \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(\tau)}{(t-\tau)^{1-n+\alpha}} d\tau, 0 \leq n-1 < \alpha < n \end{cases}$$

is the Caputo derivative [33]. The Caputo derivative is adopted here because it allows the initial conditions to be specified directly in terms of integer-order values $u_i(0) = c_i$, which carry a clear physical meaning for the initial-value problem (1). Since the Caputo

derivative of a constant is zero, these initial conditions enter the operational-matrix formulation cleanly. The Grünwald–Letnikov definition, although equivalent under suitable smoothness assumptions, is most natural for direct discretization schemes and would complicate the incorporation of the integer-order initial conditions within the present operational-matrix framework.

The organization of the study is as follows: Hermite wavelets are introduced in Section 2. In Section 3, we define the operational matrices for FIDEs. Section 4 includes the numerical examples for several FIDEs. The paper is concluded with obtained results and feasibility of the method.

2. Hermite Wavelets

A family of functions called wavelets is generally employed to localize a particular function in both space and time. A zero-mean function $\psi(t)$, which is also constrained in a finite interval, frequently referred to as a “mother wavelet” is used to build the family of wavelets. Scaling (a) and translation (b) parameters are used to obtain different versions of the same function. Wavelets have features that make them more advantageous than traditional Fourier transforms, rendering them particularly helpful for compressing image data. The following can be used to obtain the family of continuous wavelets [37]:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right), a, b \in \mathbb{R}, a \neq 0, \quad (2)$$

Assuming a and b are allowed to be discrete only, the group of discrete wavelets are obtained as:

$$\psi_{kn}(t) = |a_0|^{k/2} \psi(a_0^k t - nb_0), \quad (3)$$

where k and n are positive integers and $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 0$. The definition of Hermite wavelets (HW) on $[0, 1]$ can be given as [37]:

$$\psi_{nm}(t) = \begin{cases} \frac{2^{(\frac{k+1}{2})}}{\sqrt{\pi}} H_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise} \end{cases}, \quad (4)$$

where $H_m(t)$ are Hermite polynomials of degree m with $m = 0, 1, \dots, M-1$ and $n = 1, 2, \dots, 2^{k-1}$. Hermite polynomials are defined by the recurrence formula:

$$H_{i+2}(t) = 2tH_{i+1}(t) - 2(i+1)H_i(t), \quad i = 0, 1, 2, \dots \quad (H_0(t) = 1, \quad H_1(t) = 2t). \quad (5)$$

HW convergence analysis is given in [37] and error analysis can be found in [38]. For completeness, we summarize the relevant bound. If a function $z(t)$ belonging to the weighted space $L_\omega^2[0, 1]$ is approximated by its truncated Hermite wavelet expansion at resolution level k with Hermite polynomials up to degree $M-1$, then the truncation error in the L^2 norm tends to zero as k and M increase, the error being controlled by the magnitude of the neglected wavelet coefficients s_{nm} (which decay with increasing k and m for sufficiently smooth z). The detailed estimate is provided in [38]. For the coupled FIDE system considered here, this bound applies component-wise to the approximations of $u(t)$ and $v(t)$, so that increasing the resolution $\tilde{M} = 2^{k-1}M$ reduces the error of both components simultaneously, consistent with the numerical results reported in Section 4.

3. Function Approximation and Operational Matrices of Hermite Wavelets

Assuming a $z(t)$ function being square integrable in $[0, 1]$, HW can be used to approximate it as:

$$z(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} s_{nm} \psi_{nm}(t) = S^T \Psi(t), \quad (6)$$

where $s_{nm} = \langle z(t), \psi_{nm}(t) \rangle_{L^2_{\omega}[0,1]} = \int_0^1 z(t) \psi_{nm}(t) \omega_n(t) dt$, in which $\langle \cdot, \cdot \rangle_{L^2_{\omega}[0,1]}$ is the inner product in $L^2_{\omega}[0, 1]$ and $\omega_n(t) = e^{-t^2}$ is the weight function. We can approximate $z(t)$ as:

$$z(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} s_{nm} \psi_{nm}(t) = S^T \Psi(t), \quad (7)$$

where $S = [s_{10}, s_{11}, \dots, s_{1(M-1)}, s_{20}, s_{21}, \dots, s_{2(M-1)} \dots s_{2^{k-1}0}, s_{2^{k-1}1}, \dots, s_{2^{k-1}(M-1)}]^T$, $\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \dots, \psi_{1(M-1)}(t), \dots, \psi_{2^{k-1}0}(t), \psi_{2^{k-1}1}(t), \dots, \psi_{2^{k-1}(M-1)}(t)]^T$ are column vectors of size $\tilde{m} = 2^{k-1}M$.

We can then define HW matrix as:

$$\phi_{\tilde{m} \times \tilde{m}} = [\Psi(t_1), \Psi(t_2), \Psi(t_3), \dots, \Psi(t_{\tilde{m}})], \quad (8)$$

where $t_i = \frac{i-0.5}{\tilde{m}}$ $i = 1, 2, \dots, \tilde{m}$ are collocation points.

Block-pulse functions (BPFs) reduce the amount of computation needed for fractional operational matrices of HWs. As a result, the FIDE can be more easily transformed into an algebraic equation.

We can expand a squarely integrable function $z(t)$ into a set of BPFs as follows:

$$z(t) = \sum_{i=1}^{\tilde{m}} z_i b_i(t) = z^T B_{\tilde{m}}(t), \quad (9)$$

where $z = [z_1, z_2, \dots, z_{\tilde{m}}]^T$, $B_{\tilde{m}}(t) = [b_1(t), b_2(t), \dots, b_{\tilde{m}}(t)]^T$ and z_i are given as $z_i = \frac{1}{\tilde{m}} \int_{(i-1)/\tilde{m}}^{i/\tilde{m}} z(t) b_i(t) dt$.

Let $W = [w_1, w_2, \dots, w_{\tilde{m}}]^T$ and $Y = [y_1, y_2, \dots, y_{\tilde{m}}]^T$. Using BPF properties, we can write:

$$W * Y = [w_1 y_1, w_2 y_2, \dots, w_{\tilde{m}} y_{\tilde{m}}]^T, \quad (10)$$

where $*$ operator denotes dot product.

$$W^n = [w_1^n, w_2^n, \dots, w_{\tilde{m}}^n]^T, \quad (11)$$

Provided that the functions $w(t)$ and $y(t)$ belong to $L^2[0, 1]$, we obtain:

$$w(t) = W^T B_{\tilde{m}}(t), \quad y(t) = Y^T B_{\tilde{m}}(t), \quad (12)$$

$$w(t)y(t) = W^T B_{\tilde{m}}(t) Y^T B_{\tilde{m}}(t) = (W^T * Y^T) B_{\tilde{m}}(t), \quad (13)$$

$$[w(t)]^n = [W^T B_{\tilde{m}}(t)]^n = [W^n]^T B_{\tilde{m}}(t). \quad (14)$$

Then we can approximate $I^\alpha \Psi(t)$ [37] as:

$$I^\alpha \Psi(t) \approx P_{\tilde{m} \times \tilde{m}}^\alpha \Psi(t), \quad (15)$$

where I^α denotes the fractional integration. $P_{\tilde{m} \times \tilde{m}}^\alpha$ is called HW operational matrix. Let us now expand the HW operational matrix as:

$$\Psi(t) \approx \phi_{\tilde{m} \times \tilde{m}} B_{\tilde{m}}(t). \quad (16)$$

BP operational matrix associated with fractional integration matrix F^α is used as given in [31]:

$$I^\alpha B_{\check{m}}(t) \approx F^\alpha B_{\check{m}}(t). \quad (17)$$

Using Equations (15)–(17), we obtain:

$$I^\alpha \Psi(t) \approx I^\alpha \phi_{\check{m} \times \check{m}} B_{\check{m}}(t) \approx \phi_{\check{m} \times \check{m}} F^\alpha B_{\check{m}}(t), \quad (18)$$

$$P_{\check{m} \times \check{m}}^\alpha \Psi(t) \approx I^\alpha \Psi(t) \approx \phi_{\check{m} \times \check{m}} F^\alpha B_{\check{m}}(t) = \phi_{\check{m} \times \check{m}} F_{\check{m} \times \check{m}}^{-1} \Psi(t). \quad (19)$$

As a result, the HW operational matrix $P_{\check{m} \times \check{m}}^\alpha$ is obtained as:

$$P_{\check{m} \times \check{m}}^\alpha \approx \phi_{\check{m} \times \check{m}} F_{\check{m} \times \check{m}}^{-1}. \quad (20)$$

4. Numerical Examples

4.1. Example 1

As our first example, we have the following system of FIDE, whose initial conditions are $u(0) = 0, v(0) = 0$ and the exact solutions are $u_{ex}(t) = t^{1/2}, v_{ex}(t) = -t^{1/2}, t \in [0, 1]$:

$$\begin{aligned} D^{1/2}u(t) &= \Gamma\left(\frac{3}{2}\right) + \int_0^t [u(\eta) + v(\eta)]d\eta \\ D^{1/2}v(t) &= v(t) + u(t) - \Gamma\left(\frac{3}{2}\right). \end{aligned} \quad (21)$$

Applying the proposed method on the fractional derivatives, we now get,

$$D^{1/2}u(t) \approx E_{\check{m}}^T \Psi(t) D^{1/2}v(t) \approx F_{\check{m}}^T \Psi(t), \quad (22)$$

where $E_{\check{m}}^T = [e_1, e_2, \dots, e_{\check{m}}]$ and $F_{\check{m}}^T = [f_1, f_2, \dots, f_{\check{m}}]$ are the coefficients to be determined for the HWM approximation. Applying initial conditions and Equations (15), (16) and (22) yields:

$$\begin{aligned} u(t) &= I^{1/2} D^{1/2}u(t) + u(0) \approx E_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} \Psi(t) \approx \underbrace{E_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} \phi_{\check{m} \times \check{m}} B_{\check{m}}(t)}_{G_{\check{m}}^T} \\ v(t) &= I^{1/2} D^{1/2}v(t) + v(0) \approx F_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} \Psi(t) \approx \underbrace{F_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} \phi_{\check{m} \times \check{m}} B_{\check{m}}(t)}_{H_{\check{m}}^T}, \end{aligned} \quad (23)$$

where $G_{\check{m}}^T = [g_1, g_2, \dots, g_{\check{m}}]$ and $H_{\check{m}}^T = [h_1, h_2, \dots, h_{\check{m}}]$ are size $1 \times \check{m}$ coefficient vectors. Combining (15) and (23) yields,

$$\int_0^t u(\eta) d\eta \approx \int_0^t E_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} \Psi(\eta) d\eta = E_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} \int_0^t \Psi(\eta) d\eta \approx E_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} P_{\check{m} \times \check{m}}^1 \Psi(t) = \underbrace{E_{\check{m}}^T P_{\check{m} \times \check{m}}^{3/2} \phi_{\check{m} \times \check{m}} B_{\check{m}}(t)}_{Y_{\check{m}}^T}, \quad (24)$$

$$\int_0^t v(\eta) d\eta \approx F_{\check{m}}^T P_{\check{m} \times \check{m}}^{1/2} P_{\check{m} \times \check{m}}^1 \Psi(t) = \underbrace{F_{\check{m}}^T P_{\check{m} \times \check{m}}^{3/2} \phi_{\check{m} \times \check{m}} B_{\check{m}}(t)}_{Z_{\check{m}}^T}. \quad (25)$$

Applying (16) and (22)–(25) in (21) produces a system of algebraic equations of length $2\check{m}$, which is solved to obtain the coefficients $E_{\check{m}}^T$ and $F_{\check{m}}^T$ as:

$$\begin{aligned} E_{\check{m}}^T \phi_{\check{m} \times \check{m}} &= \Gamma\left(\frac{3}{2}\right) [1, 1, \dots, 1]_{1 \times \check{m}} + Y_{\check{m}}^T + Z_{\check{m}}^T, \\ F_{\check{m}}^T \phi_{\check{m} \times \check{m}} &= G_{\check{m}}^T + H_{\check{m}}^T - \Gamma\left(\frac{3}{2}\right) [1, 1, \dots, 1]_{1 \times \check{m}}. \end{aligned} \quad (26)$$

The resulting system of algebraic Equation (26) is solved using Newton–Raphson iterative method to calculate $E_{\check{m}}^T$ and $F_{\check{m}}^T$, which in turn provides the numerical solution

for $u(t)$ and $v(t)$ as given in (23). All computations were carried out in MATLAB R2025b on an Intel Core i7, 2.6 GHz laptop with 32 GB of RAM. The Newton–Raphson iteration was initialized from the unit vector for every example. The solutions converge to machine precision within five iterations for all solved examples. We observed no sensitivity of the converged solution to the initial guess across the tested fractional orders, and the iteration remained stable for all examples.

The exact solution and Hermite Wavelet Method (HWM) results are given in Figure 1, in which the HWM solutions are in agreement with the exact solution. Table 1 displays the absolute errors of $u(t)$ denoted by Er_u and absolute errors of $v(t)$ denoted by Er_v for a number of $\tilde{m}$ parameters. Table 1 shows that as resolution increases, the HWM error decreases as expected. The approximate absolute errors fall into one of three categories: $E - 4, E - 5$ for $\tilde{m} = 24$; $E - 5, E - 6$ for $\tilde{m} = 48$ and $\tilde{m} = 96$; and $E - 6, E - 7$ for $\tilde{m} = 192$. Table 1 and Figure 1 both validate the accuracy of the proposed HWM approximation.

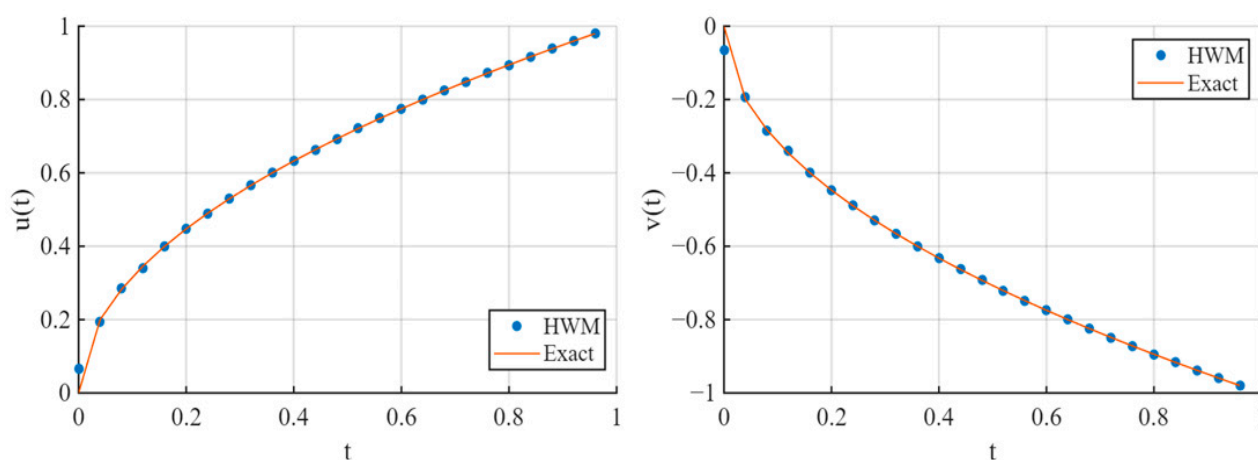


Figure 1. The HWM and exact solution of example 1 for $\tilde{m} = 24$ ($k = 4$, $M = 3$).

Table 1. The HWM absolute errors Er_u of $u(t)$ and Er_v of $v(t)$ for several resolutions for example 1.

t	$\tilde{m} = 24$ ($k = 4$, $M = 3$)		$\tilde{m} = 48$ ($k = 5$, $M = 3$)		$\tilde{m} = 96$ ($k = 6$, $M = 3$)		$\tilde{m} = 192$ ($k = 7$, $M = 3$)	
	Er_u	Er_v	Er_u	Er_v	Er_u	Er_v	Er_u	Er_v
0.1	4.10×10^{-4}	4.10×10^{-4}	8.60×10^{-5}	8.60×10^{-5}	3.90×10^{-5}	3.90×10^{-5}	9.68×10^{-6}	9.68×10^{-6}
0.2	1.22×10^{-4}	1.22×10^{-4}	5.52×10^{-5}	5.52×10^{-5}	1.37×10^{-5}	1.37×10^{-5}	3.07×10^{-6}	3.07×10^{-6}
0.3	1.34×10^{-4}	1.34×10^{-4}	2.53×10^{-5}	2.53×10^{-5}	6.48×10^{-6}	6.48×10^{-6}	1.75×10^{-6}	1.75×10^{-6}
0.4	7.80×10^{-5}	7.80×10^{-5}	1.94×10^{-5}	1.94×10^{-5}	4.34×10^{-6}	4.34×10^{-6}	1.09×10^{-6}	1.09×10^{-6}
0.5	1.26×10^{-5}	1.26×10^{-5}	7.43×10^{-6}	7.43×10^{-6}	2.49×10^{-6}	2.49×10^{-6}	7.08×10^{-7}	7.08×10^{-7}
0.6	3.58×10^{-5}	3.58×10^{-5}	9.16×10^{-6}	9.16×10^{-6}	2.47×10^{-6}	2.47×10^{-6}	6.17×10^{-7}	6.17×10^{-7}
0.7	2.78×10^{-5}	2.78×10^{-5}	7.94×10^{-6}	7.94×10^{-6}	1.98×10^{-6}	1.98×10^{-6}	4.79×10^{-7}	4.79×10^{-7}
0.8	2.74×10^{-5}	2.74×10^{-5}	6.14×10^{-6}	6.14×10^{-6}	1.55×10^{-6}	1.55×10^{-6}	3.98×10^{-7}	3.98×10^{-7}
0.9	2.21×10^{-5}	2.21×10^{-5}	5.49×10^{-6}	5.49×10^{-6}	1.31×10^{-6}	1.31×10^{-6}	3.28×10^{-7}	3.28×10^{-7}

Table 2 reports the number of nonzero elements and the corresponding sparsity percentage of the Hermite-wavelet operational matrices $P_{\tilde{m} \times \tilde{m}}^\alpha$ at each resolution for example 1. As can be seen from the table, the operational matrices are sparse at all tested resolutions (≈ 70 – 89%), with sub-quadratic growth in the number of nonzeros. Table 3 represents the figures obtained for Euler- and Legendre-based operational matrices of the same size and the measured CPU times in seconds for assembling and solving the algebraic system. These results confirm that, at matched resolution and accuracy, the Hermite-wavelet operational matrices require less computational time than the compared alternatives.

Table 2. The HWM number of nonzero elements and sparsity percentage for several resolutions for example 1.

Resolution	# Nonzero Elements	# Total Elements	Sparsity %
$k = 4, M = 3$	169	576	70.6597
$k = 5, M = 3$	428	2304	81.4236
$k = 6, M = 3$	1372	9216	85.1128
$k = 7, M = 3$	4100	36,864	88.8780

Table 3. The HWM, Euler and Legendre wavelet CPU elapse times in seconds for several resolutions for example 1.

Resolution	Hermite Wavelet Method Elapse Time [s]	Euler Wavelet Method Elapse Time [s]	Legendre Wavelet Method Elapse Time [s]
$k = 4, M = 3$	8.215537	9.47189	11.708111
$k = 5, M = 3$	39.869815	42.949887	58.055942
$k = 6, M = 3$	282.649858	309.71218	399.044548
$k = 7, M = 3$	2014.197954	2090.579494	2980.915017

4.2. Example 2

Here we have the following system of FIDEs whose initial conditions are $u(0) = 0$, $v(0) = 0$ and the exact solutions for integer-order derivatives are $u_{ex}(t) = t^2$, $v_{ex}(t) = t$, $t \in [0, 1]$:

$$\begin{aligned} D^\alpha u(t) &= \frac{1}{3}u(t)v(t) + \frac{1}{2}v^2(t) + 2v(t) - \int_0^t [u(\eta) + v(\eta)]d\eta, \\ D^\beta v(t) &= \frac{1}{3}u(t)v(t) - u(t) + 1 - \int_0^t [u(\eta) - 2v(\eta)]d\eta, 0 < \alpha \leq 1, 0 < \beta \leq 1. \end{aligned} \quad (27)$$

Using proposed HWM on the fractional derivatives, we get,

$$\begin{aligned} D^\alpha u(t) &\approx E_m^T \Psi(t) \\ D^\beta v(t) &\approx F_m^T \Psi(t). \end{aligned} \quad (28)$$

Using (15), (16) and (28) and initial values yields:

$$\begin{aligned} u(t) &\approx E_m^T P_{m \times m}^\alpha \Psi(t) \approx \underbrace{E_m^T P_{m \times m}^\alpha \phi_{m \times m}}_{G_m^T} B_m(t) \\ v(t) &\approx F_m^T P_{m \times m}^\beta \Psi(t) \approx \underbrace{F_m^T P_{m \times m}^\beta \phi_{m \times m}}_{H_m^T} B_m(t), \end{aligned} \quad (29)$$

$$\int_0^t u(\eta)d\eta \approx \int_0^t E_m^T P_{m \times m}^\alpha \Psi(\eta)d\eta = E_m^T P_{m \times m}^\alpha \int_0^t \Psi(\eta)d\eta \approx E_m^T P_{m \times m}^\alpha P_{m \times m}^1 \Psi(t) = \underbrace{E_m^T P_{m \times m}^{\alpha+1} \phi_{m \times m}}_{Y_m^T} B_m(t), \quad (30)$$

$$\int_0^t v(\eta)d\eta \approx \underbrace{F_m^T P_{m \times m}^{\beta+1} \phi_{m \times m}}_{Z_m^T} B_m(t), \quad (31)$$

and from (13) and (14), we have

$$v^2(t) \approx \left(H_m^T \right)^2 B_m(t), \quad (32)$$

$$u(t)v(t) \approx \left(G_m^T * H_m^T \right) B_m(t). \quad (33)$$

Applying (28)–(33) in (27) produces a system of algebraic equations of length $2\tilde{m}$ as before, which is then solved to obtain the coefficients $E_{\tilde{m}}^T$ and $F_{\tilde{m}}^T$ as:

$$\begin{aligned} E_{\tilde{m}}^T \phi_{\tilde{m} \times \tilde{m}} &= \frac{1}{3} (G_{\tilde{m}}^T * H_{\tilde{m}}^T) + \frac{1}{2} (H_{\tilde{m}}^T)^2 + 2H_{\tilde{m}}^T - Y_{\tilde{m}}^T - Z_{\tilde{m}}^T, \\ F_{\tilde{m}}^T \phi_{\tilde{m} \times \tilde{m}} &= \frac{1}{3} (G_{\tilde{m}}^T * H_{\tilde{m}}^T) - G_{\tilde{m}}^T + [1, 1, \dots, 1]_{1 \times \tilde{m}} - Y_{\tilde{m}}^T + 2Z_{\tilde{m}}^T. \end{aligned} \quad (34)$$

The solution of (34) is obtained as in the previous example. The solution for $\alpha = 1$, $\beta = 1$ is plotted in Figure 2 and for several fractional α, β values in Figure 3. As seen in Figure 2, the HWM solution is accurate for $\alpha = 1$, $\beta = 1$. Figure 3 demonstrates that the HWM is accurate for the full range of α and β values. In the first two graphs, $u(t)$ and $v(t)$ are plotted for different α, β and in the third and fourth graphs, $u(t)$ and $v(t)$ are plotted for the same α, β , which shows that when α, β approach 1, the fractional solutions also approach the graphs plotted for integer orders in Figure 2. The absolute errors for several resolutions are given for $\alpha = 1$, $\beta = 1$ in Table 4, whereas the HWM results for a few fractional α, β values are listed in Table 5. As the tables demonstrate, the method is accurate and produces better error values as the resolution is increased.

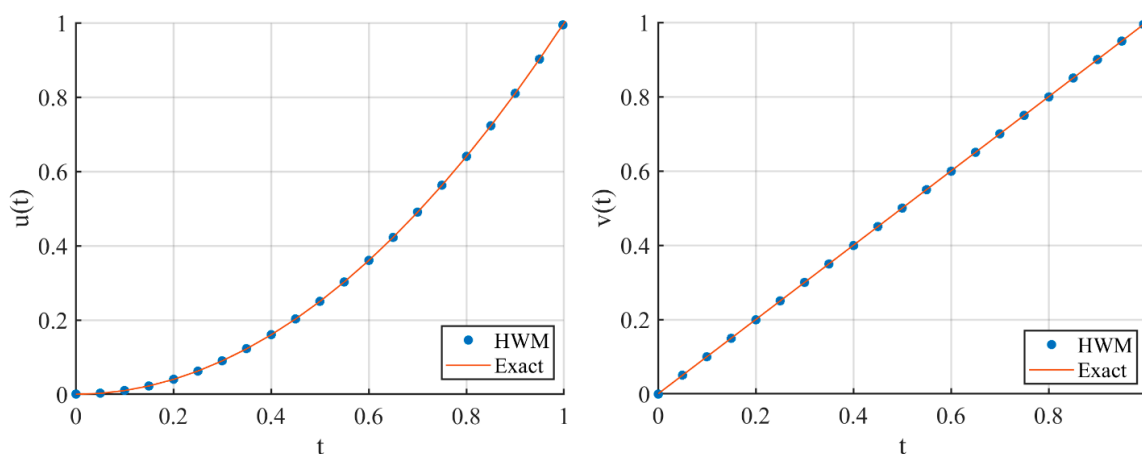


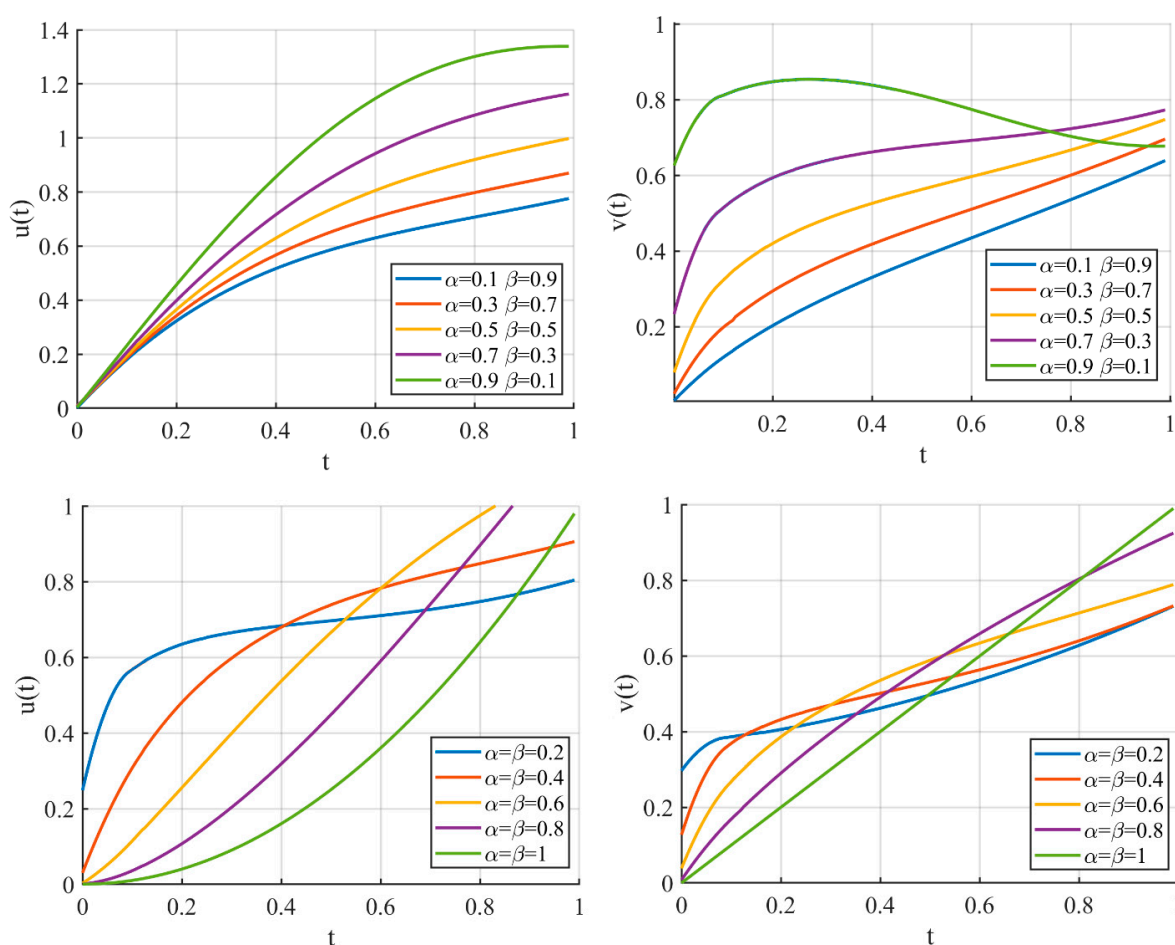
Figure 2. The HWM and exact solution of example 2 for $\alpha = 1$, $\beta = 1$, $\tilde{m} = 24$ ($k = 4, M = 3$).

Table 4. The HWM absolute errors Er_u of $u(t)$ and Er_v of $v(t)$ for several resolutions for example 2.

t	$\tilde{m} = 24$ ($k = 4, M = 3$)		$\tilde{m} = 48$ ($k = 5, M = 3$)		$\tilde{m} = 96$ ($k = 6, M = 3$)		$\tilde{m} = 192$ ($k = 7, M = 3$)	
	Er_u	Er_v	Er_u	Er_v	Er_u	Er_v	Er_u	Er_v
0	4.34×10^{-4}	4.03×10^{-8}	1.08×10^{-4}	2.73×10^{-9}	2.71×10^{-5}	1.77×10^{-10}	6.78×10^{-6}	1.13×10^{-11}
0.1	4.22×10^{-4}	3.00×10^{-5}	1.06×10^{-4}	7.49×10^{-6}	2.64×10^{-5}	1.87×10^{-6}	6.6×10^{-6}	4.68×10^{-7}
0.2	4.01×10^{-4}	6.18×10^{-5}	1.00×10^{-4}	1.55×10^{-5}	2.51×10^{-5}	3.86×10^{-6}	6.27×10^{-6}	9.66×10^{-7}
0.3	3.70×10^{-4}	9.52×10^{-5}	9.25×10^{-5}	2.38×10^{-5}	2.31×10^{-5}	5.95×10^{-6}	5.78×10^{-6}	1.49×10^{-6}
0.4	3.27×10^{-4}	1.30×10^{-4}	8.19×10^{-5}	3.25×10^{-5}	2.05×10^{-5}	8.13×10^{-6}	5.12×10^{-6}	2.03×10^{-6}
0.5	2.73×10^{-4}	1.66×10^{-4}	6.84×10^{-5}	4.15×10^{-5}	1.71×10^{-5}	1.04×10^{-5}	4.27×10^{-6}	2.59×10^{-6}
0.6	2.06×10^{-4}	2.03×10^{-4}	5.16×10^{-5}	5.08×10^{-5}	1.29×10^{-5}	1.27×10^{-5}	3.23×10^{-6}	3.17×10^{-6}
0.7	1.26×10^{-4}	2.41×10^{-4}	3.15×10^{-5}	6.02×10^{-5}	7.86×10^{-6}	1.5×10^{-5}	1.97×10^{-6}	3.76×10^{-6}
0.8	3.00×10^{-5}	2.79×10^{-4}	7.52×10^{-6}	6.98×10^{-5}	1.88×10^{-6}	1.74×10^{-5}	4.71×10^{-7}	4.36×10^{-6}
0.9	8.18×10^{-5}	3.18×10^{-4}	2.04×10^{-5}	7.95×10^{-5}	5.11×10^{-6}	1.99×10^{-5}	1.28×10^{-6}	4.97×10^{-6}

Table 5. The HWM results for several α , β for example 2.

t	$\alpha = 0.65, \beta = 0.75$		$\alpha = 0.75, \beta = 0.85$		$\alpha = 0.85, \beta = 0.95$		$\alpha = 1, \beta = 1$	
	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$
0	0.00117	0.012627	0.000746	0.004868	0.000569	0.00102	0.000434	-4×10^{-8}
0.1	0.064706	0.188992	0.035837	0.147894	0.019455	0.114085	0.010422	0.09997
0.2	0.167084	0.308943	0.106559	0.262935	0.066132	0.219201	0.040401	0.199938
0.3	0.287629	0.406364	0.201208	0.36523	0.136059	0.319962	0.09037	0.299905
0.4	0.417677	0.489179	0.313975	0.458411	0.226646	0.417176	0.160327	0.39987
0.5	0.551508	0.561322	0.440626	0.544271	0.335878	0.511216	0.250273	0.499834
0.6	0.685058	0.625219	0.577859	0.623911	0.462097	0.602311	0.360206	0.599797
0.7	0.814772	0.682927	0.722331	0.698255	0.603509	0.690655	0.490126	0.699759
0.8	0.938177	0.736016	0.871263	0.768005	0.758509	0.776396	0.64003	0.799721
0.9	1.05341	0.785988	1.021965	0.833836	0.925409	0.859681	0.809918	0.899682

**Figure 3.** The HWM results of example 2 for several α , β with $m = 24$ ($k = 4$, $M = 3$).

As in the previous example, here we represent the number of nonzero elements and the corresponding sparsity ratio of the Hermite-wavelet operational matrices at each resolution to justify our sparsity claim in Table 6. As can be seen from the table, sparsity increases with increasing resolution and again the nonzero count grows sub-quadratically for this example as well. Table 7 represents the figures obtained for Euler- and Legendre-based operational matrices of the same size and the measured CPU times in seconds for assembling and solving the algebraic system. These results confirm that, at matched resolution and accuracy, the Hermite-wavelet operational matrices consistently require less computational time than the compared alternatives for the nonlinear example too.

Table 6. The HWM number of nonzero elements and sparsity percentages for several resolutions for example 2.

Resolution	# Nonzero Elements	# Total Elements	Sparsity %
$k = 4, M = 3$	104	576	81.9444
$k = 5, M = 3$	336	2304	85.4167
$k = 6, M = 3$	1184	9216	87.1528
$k = 7, M = 3$	4416	36,864	88.0208

Table 7. The HWM, Euler and Legendre wavelet CPU elapse times in seconds for several resolutions for example 2.

Resolution	Hermite Wavelet Method Elapse Time [s]	Euler Wavelet Method Elapse Time [s]	Legendre Wavelet Method Elapse Time [s]
$k = 4, M = 3$	9.481028	18.779217	25.843842
$k = 5, M = 3$	51.091404	71.617228	85.845127
$k = 6, M = 3$	334.753675	517.551922	519.259359
$k = 7, M = 3$	2669.946753	3019.070408	3721.14891

4.3. Example 3

$$\begin{aligned} D^\alpha u(t) &= u^2(t) + v^2(t) - \int_0^t u(\eta) d\eta, \\ D^\beta v(t) &= -\frac{1}{2}v^2(t) - u(t) + \frac{1}{2} - \int_0^t u(\eta)v(\eta) d\eta, \quad 0 < \alpha \leq 1, 0 < \beta \leq 1. \end{aligned} \quad (35)$$

The system of FIDE example employed here has the initial conditions $u(0) = 0$, $v(0) = 1$ and exact solution for integer-order derivatives $u_{ex}(t) = \sin t$, $v_{ex}(t) = \cos t$, $t \in [0, 1]$. Employing the suggested technique on the fractional derivatives for this example yields

$$D^\alpha u(t) \approx E_m^T \Psi(t) D^\beta v(t) \approx F_m^T \Psi(t). \quad (36)$$

By combining the initial conditions with (13)–(16) and (36), we obtain:

$$\begin{aligned} u(t) &\approx E_m^T P_{m \times m}^\alpha \Psi(t) \approx \underbrace{E_m^T P_{m \times m}^\alpha \phi_{m \times m}}_{G_m^T} B_m(t) \\ v(t) &\approx F_m^T P_{m \times m}^\beta \Psi(t) + 1 \approx \underbrace{F_m^T P_{m \times m}^\beta \phi_{m \times m}}_{H_m^T} B_m(t) + 1, \end{aligned} \quad (37)$$

$$u(t)v(t) \approx \left[G_m^T * (H_m^T + 1) \right] B_m(t) = (G_m^T * H_m^T) B_m(t) + G_m^T B_m(t), \quad (38)$$

$$u^2(t) \approx (G_m^T)^2 B_m(t), \quad (39)$$

$$v^2(t) \approx (H_m^T + 1)^2 B_m(t) \approx (H_m^T)^2 B_m(t) + 2H_m^T B_m(t) + [1, 1, \dots, 1]_{1 \times m} B_m(t), \quad (40)$$

$$\int_0^t u(\eta) d\eta \approx \underbrace{E_m^T P_{m \times m}^{\alpha+1} \phi_{m \times m}}_{Y_m^T} B_m(t), \quad (41)$$

$$\begin{aligned} \int_0^t u(\eta)v(\eta) d\eta &\approx \int_0^t [(G_m^T * H_m^T) B_m(\eta) + G_m^T B_m(\eta)] d\eta \\ &= (G_m^T * H_m^T) \int_0^t B_m(\eta) d\eta + G_m^T \int_0^t B_m(\eta) d\eta \\ &\approx (G_m^T * H_m^T) \int_0^t \phi_{m \times m}^{-1} \psi(\eta) d\eta + G_m^T \int_0^t \phi_{m \times m}^{-1} \psi(\eta) d\eta \\ &\approx (G_m^T * H_m^T) \phi_{m \times m}^{-1} P_{m \times m}^1 \phi_{m \times m} B_m(t) + G_m^T \phi_{m \times m}^{-1} P_{m \times m}^1 \phi_{m \times m} B_m(t) \\ &\approx (G_m^T * H_m^T) \phi_{m \times m}^{-1} P_{m \times m}^1 \phi_{m \times m} B_m(t) + Y_m^T B_m(t), \end{aligned} \quad (42)$$

As in the previous examples, we derive the system of algebraic equations corresponding the FIDE by applying (36)–(42) in (35):

$$\begin{aligned} E_{\tilde{m}}^T \phi_{\tilde{m} \times \tilde{m}} &= (G_{\tilde{m}}^T)^2 + (H_{\tilde{m}}^T)^2 + 2H_{\tilde{m}}^T + [1, 1, \dots, 1]_{1 \times \tilde{m}} - Y_{\tilde{m}}(t) \\ F_{\tilde{m}}^T \phi_{\tilde{m} \times \tilde{m}} &= -\frac{1}{2}(H_{\tilde{m}}^T)^2 - H_{\tilde{m}}^T - G_{\tilde{m}}^T - (G_{\tilde{m}}^T * H_{\tilde{m}}^T) \phi_{\tilde{m} \times \tilde{m}}^{-1} P_{\tilde{m} \times \tilde{m}}^1 \phi_{\tilde{m} \times \tilde{m}} - Y_{\tilde{m}}^T. \end{aligned} \quad (43)$$

We finally obtain the HWM solution of $u(t)$ and $v(t)$ by solving (43) for the unknown $E_{\tilde{m}}^T$ and $F_{\tilde{m}}^T$ coefficients, as given by (37).

Figure 4 shows the plot of the FIDE solution for $\alpha = 1$, $\beta = 1$, and Figure 5 shows the plot for a few fractional α, β values with HWM for the example. The figures demonstrate how accurate the HWM solution is both for $\alpha = 1$, $\beta = 1$ and various fractional α, β values, where the HWM solution approaches the exact solution achieved as fractional derivatives approach 1. Table 8 provides the absolute errors at multiple resolutions, whereas Table 9 lists the HWM results for a few fractional α, β values. The tables show that the procedure is accurate and that increasing the resolution results in improved error values. Since the operational matrices for Examples 2 and 3 are constructed in the same manner, the sparsity and CPU time values obtained for Example 3 coincide with those reported for Example 2 in Tables 6 and 7. Separate tables are therefore not included for this example.

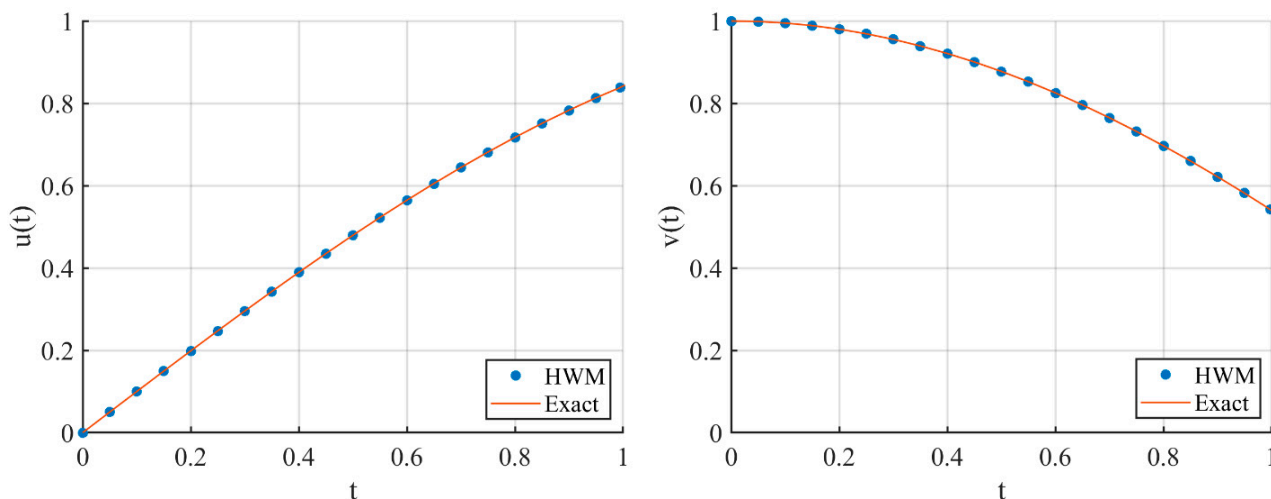


Figure 4. The HWM and exact solution of example 3 for $\alpha = 1$, $\beta = 1$, $\tilde{m} = 24$ ($k = 4$, $M = 3$).

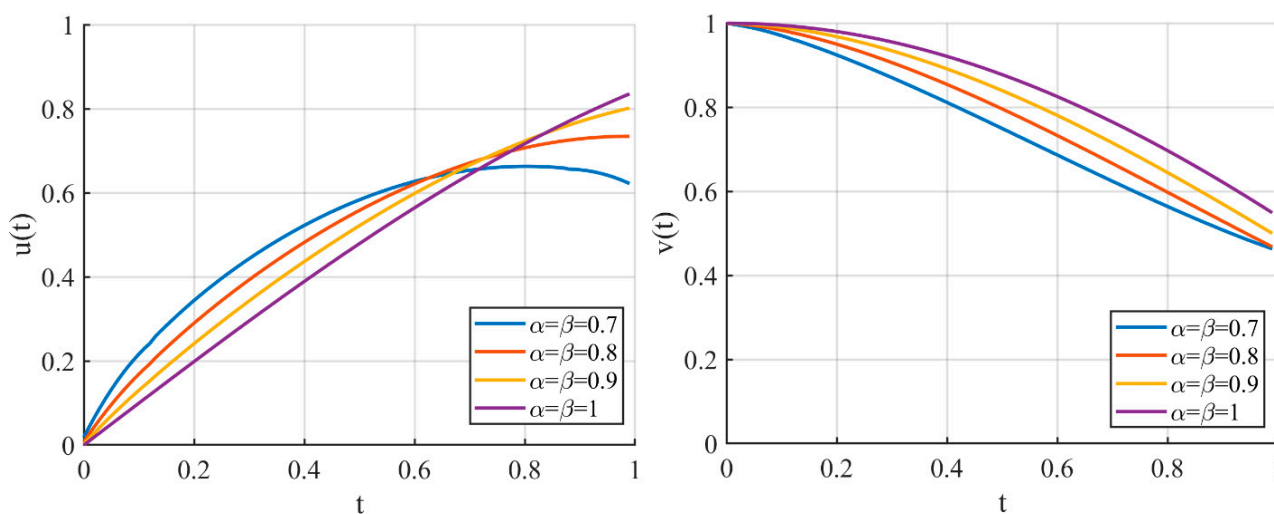


Figure 5. The HWM results of example 3 for several α, β with $\tilde{m} = 24$ ($k = 4$, $M = 3$).

Table 8. The HWM absolute errors Er_u of $u(t)$ and Er_v of $v(t)$ for several resolutions for example 3.

t	$\dot{m} = 24$ ($k = 4, M = 3$)		$\dot{m} = 48$ ($k = 5, M = 3$)		$\dot{m} = 96$ ($k = 6, M = 3$)		$\dot{m} = 192$ ($k = 7, M = 3$)	
	Er_u	Er_v	Er_u	Er_v	Er_u	Er_v	Er_u	Er_v
0	1.80×10^{-4}	8.49×10^{-4}	2.26×10^{-5}	2.16×10^{-4}	2.82×10^{-6}	5.42×10^{-5}	3.53×10^{-7}	1.36×10^{-5}
0.1	2.34×10^{-4}	8.53×10^{-4}	6.71×10^{-5}	2.13×10^{-4}	1.67×10^{-5}	5.32×10^{-5}	4.03×10^{-6}	1.33×10^{-5}
0.2	5.35×10^{-4}	7.98×10^{-4}	1.33×10^{-4}	2.00×10^{-4}	3.22×10^{-5}	5.02×10^{-5}	8.07×10^{-6}	1.25×10^{-5}
0.3	7.59×10^{-4}	7.22×10^{-4}	1.91×10^{-4}	1.80×10^{-4}	4.88×10^{-5}	4.48×10^{-5}	1.22×10^{-5}	1.12×10^{-5}
0.4	1.05×10^{-3}	5.87×10^{-4}	2.56×10^{-4}	1.50×10^{-4}	6.40×10^{-5}	3.76×10^{-5}	1.61×10^{-5}	9.34×10^{-6}
0.5	1.43×10^{-3}	3.41×10^{-4}	3.39×10^{-4}	9.91×10^{-5}	8.25×10^{-5}	2.63×10^{-5}	2.03×10^{-5}	6.76×10^{-6}
0.6	1.50×10^{-3}	2.67×10^{-4}	3.83×10^{-4}	6.21×10^{-5}	9.58×10^{-5}	1.56×10^{-5}	2.38×10^{-5}	3.97×10^{-6}
0.7	1.78×10^{-3}	1.68×10^{-5}	4.45×10^{-4}	4.62×10^{-6}	1.10×10^{-4}	1.85×10^{-6}	2.76×10^{-5}	4.54×10^{-7}
0.8	2.00×10^{-3}	2.21×10^{-4}	5.01×10^{-4}	5.61×10^{-5}	1.26×10^{-4}	1.48×10^{-5}	3.15×10^{-5}	3.7×10^{-6}
0.9	2.27×10^{-3}	5.49×10^{-4}	5.63×10^{-4}	1.31×10^{-4}	1.41×10^{-4}	3.28×10^{-5}	3.53×10^{-5}	8.31×10^{-6}

Table 9. The HWM results for several α , β for example 3.

t	$\alpha = 0.7, \beta = 0.7$		$\alpha = 0.8, \beta = 0.8$		$\alpha = 0.9, \beta = 0.9$		$\alpha = 1, \beta = 1$	
	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$	$u(t)$	$v(t)$
0	0.018856	0.999461	0.008140	0.999658	0.002588	0.999726	−0.000023	0.999784
0.1	0.216503	0.970107	0.168850	0.982859	0.130336	0.990441	0.099766	0.994791
0.2	0.343933	0.924215	0.289940	0.949803	0.241199	0.967781	0.198536	0.979867
0.3	0.443428	0.870259	0.393124	0.906055	0.342939	0.934166	0.295329	0.955156
0.4	0.522404	0.811286	0.482173	0.854143	0.436520	0.891006	0.389163	0.920911
0.5	0.583336	0.749411	0.558142	0.795871	0.521831	0.839441	0.479086	0.877483
0.6	0.627060	0.686335	0.621198	0.732682	0.598461	0.780444	0.564259	0.825274
0.7	0.653668	0.623921	0.670995	0.666141	0.665758	0.715121	0.643773	0.764838
0.8	0.662933	0.563862	0.706960	0.597676	0.723036	0.644509	0.716855	0.696763
0.9	0.654782	0.507923	0.728328	0.528838	0.769519	0.569748	0.782764	0.621741

5. Conclusions

This study is motivated by the intention to use Hermite wavelets in the numerical solution of FIDEs. Because the Hermite wavelets are orthogonal, the operational matrices needed for each of the fractional terms in the FIDE to translate it into an algebraic equation are sparser. The method's quick convergence and reduced computational burden are achieved thanks to the orthogonality property. The numerical examples show that the HW approach is quite precise, even for modest numbers of collocation points. Furthermore, the numerical method also results in the final algebraic equation as a vector-matrix equation, which provides an easy-to-follow compact representation and easier coding process. The quantitative results confirm the central claim of this work. Across all tested resolutions, the Hermite-wavelet operational matrices proved markedly sparse—between roughly 70% and 89% of their entries vanish—and this sparsity translated directly into a measurable computational saving: at each resolution the proposed scheme assembled and solved the algebraic system faster than the Euler- and Legendre-based methods of equal size, while retaining comparable accuracy. The coupled and nonlinear structure of the test problems, which would ordinarily preclude a simple component-wise treatment, was accommodated without difficulty within the single operational-matrix formulation, the nonlinear interactions being resolved through the block-pulse product operations. Taken together, these findings indicate that the Hermite-wavelet approach offers an efficient and readily implementable route to the numerical solution of coupled systems of fractional integro-differential equations, with the sparsity of its operational matrices as the principal source of its advantage.

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Abbreviations

The following abbreviations are used in this manuscript:

BPF	Block-Pulse Function
FDE	Fractional Differential Equation
FIDE	Fractional Integro-Differential Equation
HW	Hermite Wavelet
HWM	Hermite Wavelet Method

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