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Some Fixed Point Results for Fractals of Interpolative Ćirić–Reich–Rus Mappings in b -Metric Spaces

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Abstract

In this paper, we introduce the concept of a $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -interpolative Ćirić–Reich–Rus contraction iterated function system and an iterated multivalued system using interpolative Ćirić–Reich–Rus operators. The objective of this paper is to construct fractals using the Hutchinson operator and Hutchinson-like operator involving interpolative Ćirić–Reich–Rus contraction mappings, which extend the class of classical contractions, in b -metric space. The use of interpolative Ćirić–Reich–Rus contraction guarantees unique fractal attractors, thereby playing a vital role in the analysis of geometric structures and their wide-range applications in scientific and engineering fields. Using the above new definitions, we present a version of the Collage theorem adapted to iterated function systems satisfying interpolative Ćirić–Reich–Rus contractions. Further, we study the well-posedness of the new interpolative Ćirić–Reich–Rus contraction-iterated function system-Hutchinson problem. Our findings unify, generalize, and extend various earlier results reported in the literature.

Keywords: fractals; attractor; fixed point; interpolative Ćirić–Reich–Rus mapping; b -metric space; Hutchinson operator

MSC: 47H10; 54H25



Academic Editors: Austine Efut
Ofem, S. P. Moshokoa and
M. C. Kekana

Received: 12 May 2026

Revised: 3 June 2026

Accepted: 5 June 2026

Published: 25 June 2026

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1. Introduction and Preliminaries

Fractals are complex geometric structures that exhibit self-similarity across multiple scales; i.e., regardless of the level of magnification, each part resembles the whole. In 1975, the term fractal was introduced by Mandelbrot [1], and fractals are well known for their self-repeating structures and unbounded complexity. The significance of this concept lies in its extensive applicability across various branches of mathematics, such as geometry, dynamical systems, and topology, and in its applications to data compression, signal processing, image compression, and the modeling of human body organs in addition to inspiring fractal art. In 2023, Chandra et al. [2] developed fractal constructions based on Kannan mappings and investigated their smoothness properties. Thangaraj et al. [3]

developed a controlled Kannan iterated function system (KIFS) in the setting of controlled metric spaces (MSs), utilizing Kannan contractive mappings to construct a new type of invariant set, referred to as a controlled Kannan attractor or controlled Kannan fractals. Some other the generalizations of fractals can be found in [4,5] and references there in.

Iterated function systems (IFSs) constitute a powerful method for generating fractals and are grounded in the mathematical theory developed by Hutchinson [6] in 1981. The concept of IFSs was popularized by Barnsley [7] in 1988 as a natural extension of the Banach contraction principle (BCP) [8]. Hutchinson established that a finite collection of contraction mappings defined on $\mathbb{R}^n$ generates a Hutchinson operator (HO) whose fixed point (FP) is called the attractor of the IFS. An IFS is a versatile framework widely used in fractal geometry, image processing, data compression, computer graphics, wavelet analysis, quantum physics, and various other disciplines. Later, this topic has attracted substantial interest among mathematicians and computer experts as seen in various research studies.

FP theory continues to be one of the key areas of ongoing research in nonlinear sciences. A significant amount of research has been carried out to obtain FP results for single and multivalued mappings in MS and its generalizations. BCP is a fundamental result in FP theory. It is significant due to its broad range of application in various fields of mathematics and other scientific disciplines. Indeed, Banach [8] introduced a simple yet ingenious method for locating a desired FP. This technique significantly enhanced the potential for solving diverse problems across various fields of research. This motivated many researchers to broaden and enrich the framework of metric FP theory. Accordingly, BCP has been generalized in two main directions: expanding the domain of the mapping and modifying the contractive conditions. In various settings, replacing the range $\mathcal{X}$ of a mapping with a family of sets possessing a topological structure naturally results in the extension of single-valued mapping to multivalued mappings. In 1969, Nadler [9] was the first to combine the concepts of multivalued mappings and contractions, thereby initiating the study of metric FP theory of multivalued operators. This development enhanced the traditional concept of mappings, in which each input usually determines exactly one output. Nadler's work on multivalued mappings offered a broader and more adaptable way to capture relationships among sets. The motivation for Nadler's research arose from the realization that numerous real-world systems are naturally multivalued. Multivalued mappings are widely used to model complex systems involving diverse types of relationships. Multivalued mapping play an important role in several areas of mathematics, such as topology, optimization and functional analysis.

On the other hand, the mapping in the BCP is continuous. This naturally leads to the following question:

Can a discontinuous mapping on a complete MS admit a FP, while satisfying a contractive condition analogous to BCP?

This question prompted extensive research in the area and was answered in the affirmative by Kannan [10]. According to Kannan, $\mathbb{J}: \mathcal{X} \rightarrow \mathcal{X}$ is Kannan contraction (KC) if $\exists \lambda \in [0, \frac{1}{2})$ s.t. $\forall \hbar, \ell \in \mathcal{X}$

$$\delta(\mathbb{J}\hbar, \mathbb{J}\ell) \leq \lambda[\delta(\hbar, \mathbb{J}\hbar) + \delta(\ell, \mathbb{J}\ell)]$$

where $\mathbb{J}$ is not a continuous mapping. He obtained FP results for KC by proving that if $\mathcal{X}$ is a complete MS and $\mathbb{J}$ is a KC mapping on $\mathcal{X}$, then it has a unique FP. A Kannan mapping may not be continuous on the given space; however, it is continuous at its FP.

In 1972, Reich [11] developed the Banach and KC principles. According to Reich, $\mathbb{J}: \mathcal{X} \rightarrow \mathcal{X}$ is Reich contraction if $\exists \omega_1, \omega_2, \omega_3 \in [0, 1)$, where $\omega_1 + \omega_2 + \omega_3 < 1$, such that

$$\delta(\mathbb{J}\hbar, \mathbb{J}\ell) \leq \omega_1 \delta(\hbar, \mathbb{J}\hbar) + \omega_2 \delta(\ell, \mathbb{J}\ell) + \omega_3 \delta(\hbar, \ell)$$

$\forall \hbar, \ell \in \mathcal{X}$.

Further important generalizations of the BCP were independently investigated by Ćirić, Riech, and Rus [12–16]. Their combined contributions lead to the result presented below, commonly called the Ćirić–Reich–Rus contraction (CRRRC). Then, $\mathbb{J}: \mathcal{X} \rightarrow \mathcal{X}$ is CRRRC if $\exists \lambda \in [0, \frac{1}{3})$ such that

$$\delta(\mathbb{J}\hbar, \mathbb{J}\ell) \leq \lambda [\delta(\hbar, \ell) + \delta(\hbar, \mathbb{J}\hbar) + \delta(\ell, \mathbb{J}\ell)]$$

Using the interpolation technique, Karapinar [17] re-examined Kannan-type contractions in 2018. Before this study, interpolative techniques were mainly used in interpolation theory, which belongs to functional analysis. Karapinar et al. [18] pointed out a gap in Karapinar’s analysis related to the uniqueness assumption of the FP. They achieved this by providing a counter example and formulating the concept of interpolative Ćirić–Reich–Rus contractions (ICRRC). Then, $\mathbb{J}: \mathcal{X} \rightarrow \mathcal{X}$ is ICRRC if $\exists \lambda \in [0, 1)$ and $\omega_1, \omega_2 \in [0, 1)$, such that

$$\delta(\mathbb{J}\hbar, \mathbb{J}\ell) \leq \lambda [\delta(\hbar, \mathbb{J}\hbar)]^{\omega_1} \cdot [\delta(\ell, \mathbb{J}\ell)]^{\omega_2} \cdot [\delta(\hbar, \ell)]^{1-\omega_1-\omega_2}$$

$\forall \hbar, \ell \in \mathcal{X} \setminus \text{Fix}(\mathbb{J})$ where $\text{Fix}(\mathbb{J}) = \{\hbar: \mathbb{J}\hbar = \hbar, \hbar \in \mathcal{X}\}$. Further, Karapinar [19] extended the concept of interpolative contractions by introducing interpolative Kannan–Meir–Keeler contractions in complete MSs. The work of Karapinar is extended to a new direction by Shi et al. [20] through the introduction of fractal interpolative Kannan mappings. In another development, Monika and Rohen [21] extended interpolative Kannan-type F -contraction and Reich–Rus–Ćirić type F -contraction in S -MSs and in depth study about α_s -interpolative Kannan and CRRRC in S -MS by Pradeep et al. [22].

In 1989, Bakhtin [23] introduced the concept of a b -MS as a generalization of the traditional MS. He established the contraction mapping principle in b -MS, extending the classical BCP in MS. Stephen Czerwik [24] made b -MS popular among researchers in FP theory.

Inspired by the works mention above, in this paper, we introduce the concept of $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRRC iterated function system (ICRRC-IFS) and $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRRC iterated multivalued system (ICRRC-IMS) using ICRRC. The objective of this paper is to construct fractals using the HO and Hutchinson-like operator (HLO) involving ICRRC mappings, which extend the class of classical contractions, in b -MS. Using the above new definitions, we present a version of the Collage theorem adapted to IFSs satisfying ICRRC. Further, we study the well-posedness of the new ICRRC-IFS-Hutchinson problem.

Definition 1 ([23,24]). Let $\mathcal{X} \neq \emptyset$ and $\delta: \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ is called a b -metric if for some $s \geq 1$

- (i) $\delta(\hbar, \ell) = 0$ iff $\hbar = \ell$
- (ii) $\delta(\hbar, \ell) = \delta(\ell, \hbar)$
- (iii) $\delta(\hbar, \ell) \leq s[\delta(\hbar, z) + \delta(z, \ell)] \forall \hbar, \ell, z \in \mathcal{X}$.

The pair $(\mathcal{X}, \delta)$ is called a b -MS with $s \geq 1$.

Let $(\mathcal{X}, \delta)$ be a complete MS, and let $C(\mathcal{X})$ represent the set of all non-void compact subsets of $\mathcal{X}$. The Hausdorff metric (HM) $\mathcal{H}_\delta$ on $C(\mathcal{X})$ is defined as

$$\begin{aligned}
\mathcal{H}_\delta(\Theta, \Omega) &= \max\{\mathbb{D}(\Theta, \Omega), \mathbb{D}(\Omega, \Theta)\}, \quad \forall \Theta, \Omega \in C(\mathcal{X}) \\
&= \max\left\{\max_{\hbar \in \Theta} \delta(\hbar, \Omega), \max_{\ell \in \Omega} \delta(\ell, \Theta)\right\} \\
&= \max\left\{\max_{\hbar \in \Theta} \min_{\ell \in \Omega} \delta(\hbar, \ell), \max_{\ell \in \Omega} \min_{\hbar \in \Theta} \delta(\ell, \hbar)\right\}.
\end{aligned}$$

Since the function $\mathcal{H}_\delta$ fulfills all the conditions of a metric, it is referred to as the HM defined on $C(\mathcal{X})$ with respect to metric δ . If $\mathcal{X}$ is complete, then $C(\mathcal{X})$ is also complete.

Lemma 1 ([20]). Suppose $(\mathcal{X}, \delta)$ is an MS and $\Theta, \Omega, C \in C(\mathcal{X})$. Then, the following statements hold

- (i) $\Omega \subseteq C \implies \sup_{\hbar \in \Theta} \delta(\hbar, \Omega) \leq \sup_{\hbar \in \Theta} \delta(\hbar, C)$;
- (ii) $\sup_{\ell \in \Theta \cup \Omega} \delta(\ell, C) = \max\left\{\sup_{\hbar \in \Theta} \delta(\hbar, C), \sup_{z \in \Omega} \delta(z, C)\right\}$;
- (iii) If $\{\Theta_\sigma : \sigma = 1, 2, \dots, \kappa\}$ and $\{\Omega_\sigma : \sigma = 1, 2, \dots, \kappa\}$ are two finite collections of subsets of $\mathcal{X}$, then

$$\mathcal{H}_\delta\left(\bigcup_{\sigma=1}^{\kappa} \Theta_\sigma, \bigcup_{\sigma=1}^{\kappa} \Omega_\sigma\right) \leq \max_{\sigma=1}^{\kappa} \{\mathcal{H}_\delta(\Theta_\sigma, \Omega_\sigma)\}.$$

Lemma 2 ([9]). Suppose $(\mathcal{X}, \delta)$ is an MS, $\Theta, \Omega \subseteq \mathcal{X}$ and $K > 1$. Then $\forall \hbar \in \Theta, \exists \ell \in \Omega$, such that

$$\delta(\hbar, \ell) \leq K \mathcal{H}_\delta(\Theta, \Omega)$$

Lemma 3 ([20]). For any non-negative real numbers $\{\hbar_\sigma\}$, $\{\ell_\sigma\}$, where $\sigma = 1, 2, \dots, \kappa$, the following inequality holds:

$$\max\left\{\prod_{\sigma=1}^{\kappa} \hbar_\sigma, \prod_{\sigma=1}^{\kappa} \ell_\sigma\right\} \leq \prod_{\sigma=1}^{\kappa} \max\{\hbar_\sigma, \ell_\sigma\}.$$

Definition 2 ([20]). Suppose $(\mathcal{X}, \delta)$ is a complete MS and $\{\mathbb{J}^{(\sigma)} : \mathcal{X} \rightarrow \mathcal{X}; \forall \sigma = 1, 2, \dots, \kappa\}$ is a set of continuous contraction mappings with contraction factors (CFs) λ_σ , $\forall \sigma = 1, 2, \dots, \kappa$. Then $(\mathcal{X}; \mathbb{J}^{(1)}, \mathbb{J}^{(2)}, \dots, \mathbb{J}^{(\kappa)})$ is called IFS.

Theorem 1 ([7]). Let $(\mathcal{X}; \mathbb{J}^{(1)}, \mathbb{J}^{(2)}, \dots, \mathbb{J}^{(\kappa)})$ be an IFS with CFs λ_σ , $\forall \sigma = 1, 2, \dots, \kappa$. Then $\mathbb{H} : C(\mathcal{X}) \rightarrow C(\mathcal{X})$ is defined as

$$\mathbb{H}(\Theta) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta), \quad \forall \Theta \in C(\mathcal{X})$$

which is also a contraction on $(C(\mathcal{X}), \mathcal{H}_\delta)$ with CF $\lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma$. Furthermore, $\mathbb{H}$ has a unique attractor, i.e., $\exists \Theta^* \in C(\mathcal{X})$ such that

$$\Theta^* = (\Theta^*) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta^*), \quad \forall \Theta \in C(\mathcal{X})$$

and is achieved by $\Theta^* = \lim_{\sigma \rightarrow \infty} \mathbb{J}^\sigma(\Omega)$, for an arbitrary initial choice $\Omega \in C(\mathcal{X})$. Here, $\mathbb{H}^\sigma(\Omega)$ is given by $\mathbb{H}^\sigma(\Omega) = \mathbb{H}(\mathbb{H}^{\sigma-1}(\Omega))$.

Definition 3 ([25]). Suppose that $(\mathcal{X}, \delta)$ is an MS. A mapping $\mathbb{J}: \mathcal{X} \rightarrow C(\mathcal{X})$ is called a multivalued interpolative Kannan-type contraction (MIKC) if $\exists \lambda \in [0, 1)$ and $\omega_1 \in (0, 1)$ such that

$$\mathcal{H}_\delta(\mathbb{J}\hbar, \mathbb{J}\ell) \leq \lambda[\delta(\hbar, \mathbb{J}\hbar)]^{\omega_1}[\delta(\ell, \mathbb{J}\ell)]^{1-\omega_1}, \quad \forall \hbar, \ell \in \mathcal{X} \setminus \text{Fix}(\mathbb{J}).$$

Theorem 2 ([26]). In a complete MS $(\mathcal{X}, \delta)$, take Ω as a fixed set in $C(\mathcal{X})$, with $\epsilon \geq 0$ and KIFS $\{\mathcal{X}; \mathbb{J}^{(1)}, \mathbb{J}^{(2)}, \dots, \mathbb{J}^{(\kappa)}\}$, with a CF λ . If Θ is the attractor of the KIFS and

$$\mathcal{H}_\delta\left(\Omega, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Omega)\right) \leq \epsilon$$

then

$$\mathcal{H}_\delta(\Omega, \Theta) \leq \frac{\epsilon(1-\lambda)}{(1-2\lambda)}.$$

This inequality characterizes how the distance from Ω to Θ depends on the constants λ and ϵ , offering a measure of convergence to the attractor Θ .

Using interpolative KC operators, Shi et al. [20] established the interpolative KIFS and IMS as follows:

Definition 4 ([20]). Let $(\mathcal{X}, \delta)$ be an MS and $\{\mathbb{J}^{(\sigma)}: \mathcal{X} \rightarrow \mathcal{X}, \sigma = 1, 2, \dots, \kappa\}$ be a finite collection of interpolative KC operators (IKCs) with contraction $(\lambda_\sigma, \omega_{1\sigma})$, $\forall \sigma = 1, 2, \dots, \kappa$. Then $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$, is called a $(\lambda_\sigma, \omega_{1\sigma})$ -IKC-IFS or IKC-IFS.

Definition 5 ([20]). Suppose that $(\mathcal{X}, \delta)$ is an MS, and $\{\mathbb{J}^{(\sigma)}: \mathcal{X} \rightarrow C(\mathcal{X}), \sigma = 1, 2, \dots, \kappa\}$ is a finite collection of multivalued interpolative KC operators (MIKCs) with the contraction $(\lambda_\sigma, \omega_{1\sigma}) \forall \sigma = 1, 2, \dots, \kappa$. Then, $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$, is called a $(\lambda_\sigma, \omega_{1\sigma})$ -interpolative KC-iterated multivalued system (IKC-IMS).

Definition 6 ([20]). Consider $(\mathcal{X}, \delta)$ be an MS and $\{\mathbb{J}^{(\sigma)}\}$, where $\sigma = 1, 2, \dots, \kappa$, be a finite collection of $(\lambda_\sigma, \omega_{1\sigma})$ -IKCs. Then $\mathbb{H}: C(\mathcal{X}) \rightarrow C(\mathcal{X})$ is defined as

$$\mathbb{H}(\Theta) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta), \quad \forall \Theta \in C(\mathcal{X}), \quad (1)$$

and is called the HO involving IKCs.

Definition 7 ([20]). Let $(\mathcal{X}, \delta)$ be an MS and $\{\mathbb{J}^{(\sigma)}\}$, where $\sigma = 1, 2, \dots, \kappa$ is a finite collection of $(\lambda_\sigma, \omega_{1\sigma})$ -MIKCs. Then $\mathbb{H}: \mathcal{X} \rightarrow C(\mathcal{X})$ is defined as

$$\mathbb{H}(z) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(z), \quad \forall z \in \mathcal{X}, \quad (2)$$

and is called the HLO related to MIKCs.

Inspired by the works of Shi et al. [20] who introduce the IKC-IFS and IKC-IMS, we redefine the concept of ICRRC-IFS and IMS using ICRRC operators.

2. Main Results

In this section, firstly, we define the ICRRC-IFS and IMS Using traditional ICRRC as follows:

Definition 8. Let $(\mathcal{X}, \delta)$ be a b -MS with $s \geq 1$ and $\{\mathbb{J}^{(\sigma)}: \mathcal{X} \rightarrow \mathcal{X}, \sigma = 1, 2, \dots, \kappa\}$ be a finite collection of ICRRCs operators with contraction triple $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma}), \forall \sigma = 1, 2, \dots, \kappa$. Then, we call $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$, a $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRRC-IFS or ICRRC-IFS.

Definition 9. Suppose $(\mathcal{X}, \delta)$ is a b -MS with $s \geq 1$ and $\{\mathbb{J}^{(\sigma)}: \mathcal{X} \rightarrow C(\mathcal{X}), \sigma = 1, 2, \dots, \kappa\}$ is a finite collection of multivalued interpolative Ćirić–Reich–Rus contraction operators (MICRRCs) with the contraction triple $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma}) \forall \sigma = 1, 2, \dots, \kappa$. Then, we call $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$, a $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -MICRRC-IFS.

Using this outcome, we present the following result.

Theorem 3. Let $\mathbb{J}: \mathcal{X} \rightarrow \mathcal{X}$ be a continuous $(\lambda, \omega_1, \omega_2)$ -ICRRC. Then, $\mathbb{H}$ is defined by $\mathbb{H}(\Omega) = \{\mathbb{J}(\hbar); \hbar \in \Omega\}, \forall \Omega \in C(\mathcal{X})$ and maps $C(\mathcal{X})$ into itself, and $\mathbb{H}$ is also a $(\lambda, \omega_1, \omega_2)$ -ICRRC on $(C(\mathcal{X}), \mathcal{H}_\delta)$.

Proof. By continuity of $\mathbb{J}$. Then, $\mathbb{H}$ maps $C(\mathcal{X})$ into itself. Next, we have

$$\begin{aligned} \mathbb{D}(\mathbb{J}\hbar, \mathbb{J}\Theta) &= \inf_{\ell \in \Theta} \{\delta(\mathbb{J}\hbar, \mathbb{J}\ell)\} \\ &\leq \inf_{\ell \in \Theta} \left\{ \lambda [\delta(\hbar, \ell)]^{\omega_2} [\delta(\hbar, \mathbb{J}\hbar)]^{\omega_1} [\delta(\ell, \mathbb{J}\ell)]^{1-\omega_1-\omega_2} \right\} \\ &\leq \lambda [\mathcal{H}_\delta(\hbar, \Theta)]^{\omega_2} [\delta(\hbar, \mathbb{J}\hbar)]^{\omega_1} [\mathcal{H}_\delta(\Theta, \mathbb{J}\Theta)]^{1-\omega_1-\omega_2}. \end{aligned} \quad (3)$$

Similarly,

$$\begin{aligned} \mathbb{D}(\mathbb{J}\ell, \mathbb{J}\Omega) &= \inf_{\hbar \in \Omega} \{\delta(\mathbb{J}\ell, \mathbb{J}\hbar)\} \\ &\leq \inf_{\hbar \in \Omega} \left\{ \lambda [\delta(\ell, \hbar)]^{\omega_2} [\delta(\ell, \mathbb{J}\ell)]^{\omega_1} [\delta(\hbar, \mathbb{J}\hbar)]^{1-\omega_1-\omega_2} \right\} \\ &\leq \lambda [\mathcal{H}_\delta(\ell, \Omega)]^{\omega_2} [\delta(\ell, \mathbb{J}\ell)]^{\omega_1} [\mathcal{H}_\delta(\Omega, \mathbb{J}\Omega)]^{1-\omega_1-\omega_2}. \end{aligned} \quad (4)$$

Thus, using (3) and (4), we obtain

$$\begin{aligned} \mathcal{H}_\delta(\mathbb{J}\Omega, \mathbb{J}\Theta) &= \max \left\{ \sup_{\hbar \in \Omega} \mathbb{D}(\mathbb{J}\hbar, \mathbb{J}\Theta), \sup_{\ell \in \Theta} \mathbb{D}(\mathbb{J}\ell, \mathbb{J}\Omega) \right\} \\ &\leq \max \left\{ \sup_{\hbar \in \Omega} \lambda [\mathcal{H}_\delta(\hbar, \Theta)]^{\omega_2} [\delta(\hbar, \mathbb{J}\hbar)]^{\omega_1} [\mathcal{H}_\delta(\Theta, \mathbb{J}\Theta)]^{1-\omega_1-\omega_2}, \sup_{\ell \in \Theta} \lambda [\mathcal{H}_\delta(\ell, \Omega)]^{\omega_2} [\delta(\ell, \mathbb{J}\ell)]^{\omega_1} [\mathcal{H}_\delta(\Omega, \mathbb{J}\Omega)]^{1-\omega_1-\omega_2} \right\} \\ &= \lambda [\mathcal{H}_\delta(\Omega, \Theta)]^{\omega_2} [\mathcal{H}_\delta(\Omega, \mathbb{J}\Omega)]^{\omega_1} [\mathcal{H}_\delta(\Theta, \mathbb{J}\Theta)]^{1-\omega_1-\omega_2}. \end{aligned}$$

This shows that $\mathbb{H}$ is also a $(\lambda, \omega_1, \omega_2)$ -ICRRC.

We define the HO and HLO involving ICRRC mappings as follows. $\square$

Definition 10. Consider $(\mathcal{X}, \delta)$ to be a b -MS with $s \geq 1$ and $\{\mathbb{J}^{(\sigma)}\}$, where $\sigma = 1, 2, \dots, \kappa$, be a finite collection of $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRRCs. Then $\mathbb{H}: C(\mathcal{X}) \rightarrow C(\mathcal{X})$ is defined as

$$\mathbb{H}(\Theta) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta), \forall \Theta \in C(\mathcal{X}), \quad (5)$$

and is called the HO involving ICRRCs.

Definition 11. Let $(\mathcal{X}, \delta)$ be a b -MS with $s \geq 1$ and $\{\mathbb{J}^{(\sigma)}\}$, where $\sigma = 1, 2, \dots, \kappa$ is a finite collection of $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -MICRRCs. Then $\mathbb{H}: \mathcal{X} \rightarrow C(\mathcal{X})$ is defined as

$$\mathbb{H}(z) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(z), \quad \forall z \in \mathcal{X}, \quad (6)$$

and is called the HLO related to MICRRCs.

Further, we present the following results by using the HO involving ICRRRCs as well as MICRRCs formed by the ICRRRC operators.

Theorem 4. Suppose $(\mathcal{X}, \delta)$ is a b -MS with $s \geq 1$ and $\{\mathbb{J}^{(\sigma)}\}$, where $\sigma = 1, 2, \dots, \kappa$ is a finite collection of $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRRRCs. Then $\mathbb{H}: C(\mathcal{X}) \rightarrow C(\mathcal{X})$ is defined as

$$\mathbb{H}(\Omega) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Omega), \quad \forall \Omega \in C(\mathcal{X}), \quad (7)$$

is also a $(\lambda, \omega_1, \omega_2)$ -ICRRRC on $(C(\mathcal{X}), \mathcal{H}_\delta)$, with $\lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$.

Proof. From Lemmas 1 and 3.

$$\begin{aligned} \mathcal{H}_\delta(\mathbb{H}(\Omega), \mathbb{H}(\Theta)) &= \mathcal{H}_\delta\left(\bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Omega), \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta)\right) \\ &\leq \max_{\sigma=1}^{\kappa} \mathcal{H}_\delta(\mathbb{J}^{(\sigma)}(\Omega), \mathbb{J}^{(\sigma)}(\Theta)) \\ &\leq \max_{\sigma=1}^{\kappa} \lambda_\sigma [\mathcal{H}_\delta(\Omega, \Theta)]^{\omega_{2\sigma}} [\mathcal{H}_\delta(\Omega, \mathbb{J}^{(\sigma)}\Omega)]^{\omega_{1\sigma}} [\mathcal{H}_\delta(\Theta, \mathbb{J}^{(\sigma)}\Theta)]^{1-\omega_{1\sigma}-\omega_{2\sigma}} \\ &\leq \lambda \max_{\sigma=1}^{\kappa} [\mathcal{H}_\delta(\Omega, \Theta)]^{\omega_{2\sigma}} [\mathcal{H}_\delta(\Omega, \mathbb{J}^{(\sigma)}\Omega)]^{\omega_{1\sigma}} [\mathcal{H}_\delta(\Theta, \mathbb{J}^{(\sigma)}\Theta)]^{1-\omega_{1\sigma}-\omega_{2\sigma}} \\ &\quad (\text{where } \lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma, \omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}, \omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma} \text{ and } \omega_1, \omega_2 \in [0, 1) \text{ s.t. } \omega_1 + \omega_2 < 1) \\ &\leq \lambda \prod_{\sigma=1}^{\kappa} \left\{ [\mathcal{H}_\delta(\Omega, \Theta)]^{\omega_2} [\mathcal{H}_\delta(\Omega, \mathbb{J}^{(\sigma)}\Omega)]^{\omega_1} [\mathcal{H}_\delta(\Theta, \mathbb{J}^{(\sigma)}\Theta)]^{1-\omega_1-\omega_2} \right\} \\ &= \lambda \prod_{\sigma=1}^{\kappa} \left\{ [\mathcal{H}_\delta(\Omega, \Theta)]^{\omega_2} \left[\mathcal{H}_\delta(\Omega, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}\Omega) \right]^{\omega_1} \left[\mathcal{H}_\delta(\Theta, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}\Theta) \right]^{1-\omega_1-\omega_2} \right\} \\ &= \lambda [\mathcal{H}_\delta(\Omega, \Theta)]^{\omega_2} \left[\mathcal{H}_\delta(\Omega, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}\Omega) \right]^{\omega_1} \left[\mathcal{H}_\delta(\Theta, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}\Theta) \right]^{1-\omega_1-\omega_2} \\ &= \lambda [\mathcal{H}_\delta(\Omega, \Theta)]^{\omega_2} [\mathcal{H}_\delta(\Omega, \mathbb{H}(\Omega))]^{\omega_1} [\mathcal{H}_\delta(\Theta, \mathbb{H}(\Theta))]^{1-\omega_1-\omega_2}. \end{aligned}$$

Thus, $\mathbb{H}$ is also a $(\lambda, \omega_1, \omega_2)$ -ICRRRC, where $\lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$.

The HO involving ICRRRC is also a CRRC. So, we simply call it the $(\lambda, \omega_1, \omega_2)$ -ICRRRC-Hutchinson operator (ICRRRC-HO), where $\lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$. $\square$

Theorem 5. Consider the $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRRRC-IFS $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$ on a complete b -MS $(\mathcal{X}, \delta)$ with $s \geq 1$ and the $(\lambda, \omega_1, \omega_2)$ -ICRRRC $\mathbb{H}: C(\mathcal{X}) \rightarrow C(\mathcal{X})$ defined by $\mathbb{H}(\Omega) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Omega)$, $\forall \Omega \in C(\mathcal{X})$, where $\lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$. Then, $\mathbb{H}$ has a unique attractor of the IFS, say $\Omega^* \in C(\mathcal{X})$,

satisfying $\Omega^* = \mathbb{H}(\Omega^*) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Omega^*)$ and is given by $\Omega^* = \lim_{\sigma \rightarrow \infty} \mathbb{H}^{\sigma}(\Theta)$ for an arbitrary initial choice of $\Theta \in C(\mathcal{X})$.

Proof. Let $\Omega_0 \in C(\mathcal{X})$ be any set and construct the sequence $\Omega_{\sigma+1} = \mathbb{H}(\Omega_{\sigma})$, $\forall \sigma \in \mathbb{N}$. By using Theorem 4, clearly $\mathbb{H}$ is a $(\lambda, \omega_1, \omega_2)$ -ICRRC, where $\lambda = \max_{\sigma=1}^{\kappa} \lambda_{\sigma}$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$. Then, we have

$$\begin{aligned} \mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+1}) &= \mathcal{H}_{\delta}(\mathbb{H}(\Omega_{\sigma-1}), \mathbb{H}(\Omega_{\sigma})) \\ &\leq \lambda [\mathcal{H}_{\delta}(\Omega_{\sigma-1}, \Omega_{\sigma})]^{\omega_2} [\mathcal{H}_{\delta}(\Omega_{\sigma-1}, \mathbb{H}(\Omega_{\sigma-1}))]^{\omega_1} [\mathcal{H}_{\delta}(\Omega_{\sigma}, \mathbb{H}(\Omega_{\sigma}))]^{1-\omega_1-\omega_2} \\ &= \lambda [\mathcal{H}_{\delta}(\Omega_{\sigma-1}, \Omega_{\sigma})]^{\omega_2} [\mathcal{H}_{\delta}(\Omega_{\sigma-1}, \Omega_{\sigma})]^{\omega_1} [\mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+1})]^{1-\omega_1-\omega_2} \\ &= \lambda [\mathcal{H}_{\delta}(\Omega_{\sigma-1}, \Omega_{\sigma})]^{\omega_1+\omega_2} [\mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+1})]^{1-\omega_1-\omega_2} \\ \Rightarrow [\mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+1})]^{\omega_1+\omega_2} &\leq \lambda [\mathcal{H}_{\delta}(\Omega_{\sigma-1}, \Omega_{\sigma})]^{\omega_1+\omega_2} \\ \Rightarrow [\mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+1})] &\leq \lambda^{\frac{1}{\omega_1+\omega_2}} [\mathcal{H}_{\delta}(\Omega_{\sigma-1}, \Omega_{\sigma})] \\ &\leq \lambda^{\frac{2}{\omega_1+\omega_2}} [\mathcal{H}_{\delta}(\Omega_{\sigma-2}, \Omega_{\sigma-1})] \\ &\vdots \\ &\leq \lambda^{\frac{\sigma}{\omega_1+\omega_2}} [\mathcal{H}_{\delta}(\Omega_0, \Omega_1)]. \end{aligned}$$

Since $\lambda < 1$, and by taking $\sigma \rightarrow \infty$ in the above inequality, we obtain $\lim_{\sigma \rightarrow \infty} \mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+1}) = 0$.

Next, we show that $\{\Omega_{\sigma}\}_{\sigma=0}^{\infty}$ is a Cauchy in $(C(\mathcal{X}), \mathcal{H}_{\delta})$. For $r > \sigma$, we have

$$\begin{aligned} \mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+r}) &\leq s \mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega_{\sigma+1}) + s^2 \mathcal{H}_{\delta}(\Omega_{\sigma+1}, \Omega_{\sigma+2}) + s^3 \mathcal{H}_{\delta}(\Omega_{\sigma+2}, \Omega_{\sigma+3}) + \dots + s^r \mathcal{H}_{\delta}(\Omega_{\sigma+r-1}, \Omega_{\sigma+r}) \\ &\leq s \lambda^{\frac{\sigma}{\omega_1+\omega_2}} \mathcal{H}_{\delta}(\Omega_0, \Omega_1) + s^2 \lambda^{\frac{\sigma+1}{\omega_1+\omega_2}} \mathcal{H}_{\delta}(\Omega_0, \Omega_1) + s^3 \lambda^{\frac{\sigma+2}{\omega_1+\omega_2}} \mathcal{H}_{\delta}(\Omega_0, \Omega_1) + \dots + s^r \lambda^{\frac{\sigma+r-1}{\omega_1+\omega_2}} \mathcal{H}_{\delta}(\Omega_0, \Omega_1) \\ &= s \lambda^{\frac{\sigma}{\omega_1+\omega_2}} \mathcal{H}_{\delta}(\Omega_0, \Omega_1) \left(1 + s \lambda^{\frac{1}{\omega_1+\omega_2}} + s^2 \lambda^{\frac{2}{\omega_1+\omega_2}} + s^3 \lambda^{\frac{3}{\omega_1+\omega_2}} + \dots + s^{r-1} \lambda^{\frac{r-1}{\omega_1+\omega_2}} \right) \\ &= s \lambda^{\frac{\sigma}{\omega_1+\omega_2}} \mathcal{H}_{\delta}(\Omega_0, \Omega_1) \left(\frac{1 - \left(s \lambda^{\frac{1}{\omega_1+\omega_2}} \right)^r}{1 - s \lambda^{\frac{1}{\omega_1+\omega_2}}} \right) \rightarrow 0 \text{ as } \sigma \rightarrow \infty. \end{aligned}$$

Therefore, $\{\Omega_{\sigma}\}$ is a Cauchy and by completeness of $C(\mathcal{X})$, we can find $\Omega^* \in C(\mathcal{X})$ such that $\Omega_{\sigma} \rightarrow \Omega^*$ as $\sigma \rightarrow \infty$.

Next, we show that $\Omega^* \in \text{Fix}(\mathbb{H})$. Using the definition of $(\lambda, \omega_1, \omega_2)$ -ICRRC, we have

$$\begin{aligned} \mathcal{H}_{\delta}(\Omega_{\sigma+1}, \mathbb{H}(\Omega^*)) &= \mathcal{H}_{\delta}(\mathbb{H}(\Omega_{\sigma}), \mathbb{H}(\Omega^*)) \\ &\leq \lambda [\mathcal{H}_{\delta}(\Omega_{\sigma}, \Omega^*)]^{\omega_2} [\mathcal{H}_{\delta}(\Omega_{\sigma}, \mathbb{H}(\Omega_{\sigma}))]^{\omega_1} [\mathcal{H}_{\delta}(\Omega^*, \mathbb{H}(\Omega^*))]^{1-\omega_1-\omega_2}. \end{aligned}$$

Taking $\sigma \rightarrow \infty$, we have obtain $\mathcal{H}_{\delta}(\Omega^*, \mathbb{H}(\Omega^*)) = 0$, so $\Omega^* \in \text{Fix}(\mathbb{H})$, and Ω^* is an attractor of the ICRR-IFS. Taking $\Omega, \Theta \in \text{Fix}(\mathbb{H})$ with $\Omega \neq \Theta$. By definition,

$$\begin{aligned} \mathcal{H}_{\delta}(\Omega, \Theta) &= \mathcal{H}_{\delta}(\mathbb{H}(\Omega), \mathbb{H}(\Theta)) \\ &\leq \lambda [\mathcal{H}_{\delta}(\Omega, \Theta)]^{\omega_2} [\mathcal{H}_{\delta}(\Omega, \mathbb{J}\Omega)]^{\omega_1} [\mathcal{H}_{\delta}(\Theta, \mathbb{J}\Theta)]^{1-\omega_1-\omega_2} \\ &= \lambda [\mathcal{H}_{\delta}(\Omega, \Theta)]^{\omega_2} [\mathcal{H}_{\delta}(\Omega, \Omega)]^{\omega_1} [\mathcal{H}_{\delta}(\Theta, \Theta)]^{1-\omega_1-\omega_2} \\ &= 0 \\ \Rightarrow \mathcal{H}_{\delta}(\Omega, \Theta) &= 0 \end{aligned}$$

a contradiction. Hence, the FP of $\mathbb{H}$ or the attractor of the ICRR-IFS is unique. $\square$

Example 1. Let $\mathcal{X} = [0, \infty)$ and $\delta(\hbar, \ell) = (\hbar - \ell)^2$. Consider a system of three operators defined as

$$\begin{aligned}\mathbb{J}^{(1)}(\hbar) &= \begin{cases} 0, & \text{if } \hbar \in [0, 1) \\ e^{\hbar} - 1, & \text{if } \hbar \geq 1 \end{cases} \\ \mathbb{J}^{(2)}(\hbar) &= \begin{cases} 0, & \text{if } \hbar \in [0, 2) \\ e^{\hbar} - 2, & \text{if } \hbar \geq 2 \end{cases} \\ \text{and, } \mathbb{J}^{(3)}(\hbar) &= \begin{cases} 0, & \text{if } \hbar \in [0, 3) \\ e^{\hbar} - 3, & \text{if } \hbar \geq 3 \end{cases}\end{aligned}$$

on $\mathcal{X}$. Suppose that operators $\mathbb{J}^{(1)}$, $\mathbb{J}^{(2)}$ and $\mathbb{J}^{(3)}$ are CRRC. Firstly, check $\mathbb{J}^{(1)}$ with $\hbar = 0$ and $\ell = 1$.

Then,

$$\begin{aligned}\delta(\mathbb{J}^{(1)}\hbar, \mathbb{J}^{(1)}\ell) &= \delta(\mathbb{J}^{(1)}0, \mathbb{J}^{(1)}1) = \delta(0, e - 1) = (e - 1)^2 = e^2 - 2e + 1 \\ \delta(\hbar, \mathbb{J}^{(1)}\hbar) &= \delta(0, \mathbb{J}^{(1)}0) = \delta(0, 0) = 0 \\ \delta(\ell, \mathbb{J}^{(1)}\ell) &= \delta(1, \mathbb{J}^{(1)}1) = \delta(1, e - 1) = (1 - e + 1)^2 = (e - 2)^2 = e^2 - 4e + 4 \\ \delta\hbar, \ell) &= \delta(0, 1) = (0 - 1)^2 = 1.\end{aligned}$$

Now,

$$\begin{aligned}e^2 - 2e + 1 &\leq \omega_1 \times 0 + \omega_2(e^2 - 4e + 4) + (1 - \omega_1 - \omega_2) \times 1 \\ \Rightarrow e^2 - 2e + 1 &\leq e^2\omega_2 - 4e\omega_2 + 4\omega_2 + 1 - \omega_1 - \omega_2 \\ \Rightarrow e^2 - 2e &\leq e^2\omega_2 - 4e\omega_2 + 3\omega_2 - \omega_1\end{aligned}$$

which is contradiction as $\omega_1, \omega_2 \in [0, 1)$ such that $\omega_1 + \omega_2 < 1$. So, $\mathbb{J}^{(1)}$ is not a CRRC mapping. Secondly, check $\mathbb{J}^{(2)}$ with $\hbar = 0$ and $\ell = 2$.

Then,

$$\begin{aligned}\delta(\mathbb{J}^{(2)}\hbar, \mathbb{J}^{(2)}\ell) &= \delta(\mathbb{J}^{(2)}0, \mathbb{J}^{(2)}2) = \delta(0, e^2 - 2) = (e^2 - 2)^2 = e^4 - 4e^2 + 4 \\ \delta(\hbar, \mathbb{J}^{(2)}\hbar) &= \delta(0, \mathbb{J}^{(2)}0) = \delta(0, 0) = 0 \\ \delta(\ell, \mathbb{J}^{(2)}\ell) &= \delta(2, \mathbb{J}^{(2)}2) = \delta(2, e^2 - 2) = (4 - e^2)^2 = e^4 - 8e^2 + 16 \\ \delta(\hbar, \ell) &= \delta(0, 2) = (0 - 2)^2 = 4.\end{aligned}$$

Now,

$$\begin{aligned}e^4 - 4e^2 + 4 &\leq \omega_1 \times 0 + \omega_2(e^4 - 8e^2 + 16) + (1 - \omega_1 - \omega_2) \times 4 \\ \Rightarrow e^4 - 4e^2 + 4 &\leq e^4\omega_2 - 8e^2\omega_2 + 16\omega_2 + 4 - 4\omega_1 - 4\omega_2 \\ \Rightarrow e^4 - 4e^2 &\leq e^4\omega_2 - 8e^2\omega_2 + 12\omega_2 - 4\omega_1\end{aligned}$$

which is contradiction as $\omega_1, \omega_2 \in [0, 1)$ such that $\omega_1 + \omega_2 < 1$. So, $\mathbb{J}^{(2)}$ is not a CRRC mapping. Lastly, check $\mathbb{J}^{(3)}$ with $\hbar = 0$ and $\ell = 3$.

Then,

$$\begin{aligned}\delta(T^{(3)}\mathfrak{h}, \mathbb{J}^{(3)}\ell) &= \delta(\mathbb{J}^{(3)}0, \mathbb{J}^{(3)}3) = \delta(0, e^3 - 3) = (e^3 - 3)^2 = e^6 - 6e^2 + 9 \\ \delta(\mathfrak{h}, \mathbb{J}^{(3)}\mathfrak{h}) &= \delta(0, \mathbb{J}^{(3)}0) = \delta(0, 0) = 0 \\ \delta(\ell, \mathbb{J}^{(3)}\ell) &= \delta(3, \mathbb{J}^{(3)}3) = \delta(3, e^3 - 3) = (6 - e^3)^2 = e^6 - 12e^3 + 36 \\ \delta(\mathfrak{h}, \ell) &= \delta(0, 3) = (0 - 3)^2 = 9.\end{aligned}$$

Now,

$$\begin{aligned}e^6 - 6e^2 + 9 &\leq \omega_1 \times 0 + \omega_2(e^6 - 12e^3 + 36) + (1 - \omega_1 - \omega_2) \times 9 \\ \Rightarrow e^6 - 6e^2 + 9 &\leq e^6\omega_2 - 12e^3\omega_2 + 36\omega_2 + 9 - 9\omega_1 - 9\omega_2 \\ \Rightarrow e^6 - 6e^2 &\leq e^6\omega_2 - 12e^3\omega_2 + 27\omega_2 - 9\omega_1\end{aligned}$$

which is contradiction as $\omega_1, \omega_2 \in [0, 1)$ such that $\omega_1 + \omega_2 < 1$. So, $\mathbb{J}^{(3)}$ is not a CRRC mapping.

It is evident that $\mathbb{J}^{(1)}, \mathbb{J}^{(2)}$ and $\mathbb{J}^{(3)}$ satisfy ICCRC with $\omega_1 = \omega_2 = \frac{1}{2}$ and $\lambda = \frac{1}{3}$. Hence, this forms an ICRRC-IFS.

On the other hand, the corresponding $(\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$ -ICRRC-Hutchinson-like operator is define as

$$\mathbb{H}(\mathbf{z}) = \mathbb{J}^{(1)}(\mathbf{z}) \cup \mathbb{J}^{(2)}(\mathbf{z}) \cup \mathbb{J}^{(3)}(\mathbf{z}), \quad \forall \mathbf{z} \in \mathcal{X}.$$

The attractor of the ICRRC-HO is $\{0\}$.

Next, we present some results concerning the HO related to the IMS

Theorem 6. Suppose $(\mathcal{X}, \delta)$ is a b -MS with $s \geq 1$ and $\{\mathbb{J}^{(\sigma)}\}$, where $\sigma = 1, 2, \dots, \kappa$ is a finite collection of $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -MICRRCs. Then $\mathbb{H}: \mathcal{X} \rightarrow C(\mathcal{X})$ is defined as

$$\mathbb{H}(\mu) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\mu), \quad \forall \mu \in \mathcal{X}, \quad (8)$$

is also a $(\lambda, \omega_1, \omega_2)$ -MICRRC on $(\mathcal{X}, \delta)$, where $\lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$.

Proof. By Lemmas 1 and 3, we get

$$\begin{aligned}\mathcal{H}_\delta(\mathbb{H}(\mu), \mathbb{H}(\nu)) &= \mathcal{H}_\delta\left(\bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\mu), \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\nu)\right) \\ &\leq \max_{\sigma=1}^{\kappa} \mathcal{H}_\delta\left(\mathbb{J}^{(\sigma)}(\mu), \mathbb{J}^{(\sigma)}(\nu)\right) \\ &\leq \max_{\sigma=1}^{\kappa} \lambda_\sigma [\delta(\mu, \nu)]^{\omega_{2\sigma}} [\delta(\mu, \mathbb{J}^{(\sigma)}(\mu))]^{\omega_{1\sigma}} [\delta(\nu, \mathbb{J}^{(\sigma)}(\nu))]^{1-\omega_{1\sigma}-\omega_{2\sigma}} \\ &\leq \lambda \max_{\sigma=1}^{\kappa} [\delta(\mu, \nu)]^{\omega_2} [\delta(\mu, \mathbb{J}^{(\sigma)}(\mu))]^{\omega_1} [\delta(\nu, \mathbb{J}^{(\sigma)}(\nu))]^{1-\omega_1-\omega_2} \\ &\quad (\text{where } \lambda = \max_{\sigma=1}^{\kappa} \lambda_\sigma, \omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}, \omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma} \text{ and } \omega_1, \omega_2 \in [0, 1) \text{ s.t. } \omega_1 + \omega_2 < 1)\end{aligned}$$

$$\begin{aligned}
&\leq \lambda \prod_{\sigma=1}^{\kappa} \left\{ [\delta(\mu, \nu)]^{\omega_2}, [\delta(\mu, \mathbb{J}^{(\sigma)}(\mu))]^{\omega_1}, [\delta(\nu, \mathbb{J}^{(\sigma)}(\nu))]^{1-\omega_1-\omega_2} \right\} \\
&= \lambda \prod_{\sigma=1}^{\kappa} \left\{ [\delta(\mu, \nu)]^{\omega_2}, \left[\delta\left(\mu, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\mu)\right) \right]^{\omega_1}, \left[\delta\left(\nu, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\nu)\right) \right]^{1-\omega_1-\omega_2} \right\} \\
&= \lambda \cdot [\delta(\mu, \nu)]^{\omega_2} \cdot \left[\delta\left(\mu, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\mu)\right) \right]^{\omega_1} \cdot \left[\delta\left(\nu, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\nu)\right) \right]^{1-\omega_1-\omega_2} \\
&= \lambda \cdot [\delta(\mu, \nu)]^{\omega_2} \cdot [\delta(\mu, \mathbb{H}(\mu))]^{\omega_1} \cdot [\delta(\nu, \mathbb{H}(\nu))]^{1-\omega_1-\omega_2}.
\end{aligned}$$

Thus, $\mathbb{H}$ is also a $(\lambda, \omega_1, \omega_2)$ -MICRRC, where $\lambda = \max_{\sigma=1}^{\kappa} \lambda_{\sigma}$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$. $\square$

Theorem 7. Consider the $(\lambda, \omega_1, \omega_2)$ -MICRRC, $(\mathcal{X} : \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$ on a complete b -MS $(\mathcal{X}, \delta)$ with $s \geq 1$ and the $(\lambda, \omega_1, \omega_2)$ -ICRRC $\mathbb{H} : \mathcal{X} \rightarrow \mathcal{C}(\mathcal{X})$ defined as

$$\mathbb{H}(\mu) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\mu), \forall \mu \in \mathcal{X},$$

where $\lambda = \max_{\sigma=1}^{\kappa} \lambda_{\sigma}$, $\omega_1 = \max_{\sigma=1}^{\kappa} \omega_{1\sigma}$, $\omega_2 = \max_{\sigma=1}^{\kappa} \omega_{2\sigma}$ and $\omega_1, \omega_2 \in [0, 1)$ s.t. $\omega_1 + \omega_2 < 1$. Then

$\mathbb{H}$ has a unique attractor of the IMS, say $\mu^* \in \mathcal{X}$, satisfying $\mu^* = \mathbb{H}(\mu^*) = \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(n)}(\mu^*)$ and is given by $\mu^* = \lim_{\sigma \rightarrow \infty} \mathbb{H}^*(\mu)$, for an arbitrary initial choice of $\mu \in \mathcal{X}$.

Proof. Let $\mu_0 \in \mathcal{X}$ be an arbitrary point and construct the sequence μ_n . Suppose $\mu_1 \in \mathcal{X}$ such that $\mu_1 \in \mathbb{H}\mu_0$. By using Lemma 2, we can find an $\mu_2 \in \mathbb{H}\mu_1$ such that

$$\delta(\mu_1, \mu_2) \leq \lambda [\delta(\mu_0, \mu_1)]^{\omega_2} [\delta(\mu_0, \mu_1)]^{\omega_1} [\delta(\mu_1, \mu_2)]^{1-\omega_1-\omega_2}.$$

By induction, we can construct $\mu_{n+1} \in \mathbb{H}\mu_n$, $\forall n \in \mathbb{N}$ such that

$$\begin{aligned}
\delta(\mu_{\sigma}, \mu_{\sigma+1}) &= \delta(\mathbb{H}(\mu_{\sigma-1}), \mathbb{H}(\mu_{\sigma})) \\
&\leq \lambda [\delta(\mu_{\sigma-1}, \mu_{\sigma})]^{\omega_2} [\delta(\mu_{\sigma-1}, \mathbb{H}(\mu_{\sigma-1}))]^{\omega_1} [\delta(\mu_{\sigma}, \mathbb{H}(\mu_{\sigma}))]^{1-\omega_1-\omega_2} \\
&= \lambda [\delta(\mu_{\sigma-1}, \mu_{\sigma})]^{\omega_2} [\delta(\mu_{\sigma-1}, \mu_{\sigma})]^{\omega_1} [\delta(\mu_{\sigma}, \mu_{\sigma+1})]^{1-\omega_1-\omega_2} \\
\Rightarrow [\delta(\mu_{\sigma}, \mu_{\sigma+1})]^{\omega_1+\omega_2} &\leq \lambda [\delta(\mu_{\sigma-1}, \mu_{\sigma})]^{\omega_1+\omega_2} \\
\Rightarrow \delta(\mu_{\sigma}, \mu_{\sigma+1}) &\leq \lambda^{\frac{1}{\omega_1+\omega_2}} \delta(\mu_{\sigma-1}, \mu_{\sigma}) \\
&\leq \lambda^{\frac{2}{\omega_1+\omega_2}} \delta(\mu_{\sigma-2}, \mu_{\sigma-1}) \\
&\vdots \\
&\leq \lambda^{\frac{\sigma}{\omega_1+\omega_2}} \delta(\mu_0, \mu_1).
\end{aligned} \tag{9}$$

Since $\lambda < 1$ and by taking $\sigma \rightarrow \infty$ in (9), we get $\lim_{\sigma \rightarrow \infty} \delta(\mu_{\sigma}, \mu_{\sigma+1}) = 0$.

Next, we prove that $\{\mu_\sigma\}_{\sigma=0}^\infty$ is a Cauchy in $(\mathcal{X}, \delta)$. For $r > \sigma$, we get

$$\begin{aligned} \delta(\mu_\sigma, \mu_{\sigma+r}) &\leq s\delta(\mu_\sigma, \mu_{\sigma+1}) + s^2\delta(\mu_{\sigma+1}, \mu_{\sigma+2}) + \cdots + s^r\delta(\mu_{\sigma+r-1}, \mu_{\sigma+r}) \\ &\leq s\lambda^{\frac{\sigma}{\omega_1+\omega_2}}\delta(\mu_0, \mu_1) + s^2\lambda^{\frac{\sigma+1}{\omega_1+\omega_2}}\delta(\mu_0, \mu_1) + \cdots + s^r\lambda^{\frac{\sigma+r-1}{\omega_1+\omega_2}}\delta(\mu_0, \mu_1) \\ &= s\lambda^{\frac{\sigma}{\omega_1+\omega_2}}\delta(\mu_0, \mu_1) \left(1 + s\lambda^{\frac{1}{\omega_1+\omega_2}} + s^2\lambda^{\frac{2}{\omega_1+\omega_2}} + \cdots + s^{r-1}\lambda^{\frac{r-1}{\omega_1+\omega_2}} \right) \\ &= s\lambda^{\frac{\sigma}{\omega_1+\omega_2}}\delta(\mu_0, \mu_1) \left(\frac{1 - \left(s\lambda^{\frac{1}{\omega_1+\omega_2}} \right)^r}{1 - s\lambda^{\frac{1}{\omega_1+\omega_2}}} \right) \rightarrow 0 \text{ as } \sigma \rightarrow \infty. \end{aligned}$$

Therefore, $\{\mu_\sigma\}$ is a Cauchy and by completeness of $\mathcal{X}$, we can find $\mu^* \in \mathcal{X}$ such that $\mu_\sigma \rightarrow \mu^*$ as $\sigma \rightarrow \infty$. Next, we show $\mu^* \in \text{Fix}(\mathbb{H})$. Using $(\lambda, \omega_1, \omega_2)$ -ICRRC, we obtain

$$\begin{aligned} \mathcal{H}_\delta(\mu_{\sigma+1}, \mathbb{H}(\mu^*)) &= \mathcal{H}_\delta(\mathbb{H}(\mu_\sigma), \mathbb{H}(\mu^*)) \\ &\leq \lambda[\delta(\mu_\sigma, \mu^*)]^{\omega_2}[\delta(\mu_\sigma, \mathbb{H}(\mu_\sigma))]^{\omega_1}[\delta(\mu^*, \mathbb{H}(\mu^*))]^{1-\omega_1-\omega_2}. \end{aligned}$$

Taking $\sigma \rightarrow \infty$, we obtain $\mathcal{H}_\delta(\mu^*, \mathbb{H}(\mu^*)) = 0$, so $\mu^* \in \text{Fix}(\mathbb{H})$, and μ^* is an attractor of the ICRR-IMS. $\square$

Example 2. Let $\mathcal{X} = [0, \infty)$ and $\delta(\hbar, \ell) = (\hbar - \ell)^2$. Consider two multivalued mappings defined as

$$\begin{aligned} \mathbb{J}^{(1)}(\hbar) &= \begin{cases} \{0\}, & \text{if } \hbar \in [0, 1) \\ \{\frac{\hbar}{4}, \hbar + 1\}, & \text{if } \hbar \geq 1 \end{cases} \\ \text{and, } \mathbb{J}^{(2)}(\hbar) &= \begin{cases} \{0\}, & \text{if } \hbar \in [0, 2) \\ \{\frac{\hbar}{8}, \hbar + 2\}, & \text{if } \hbar \geq 2. \end{cases} \end{aligned}$$

Then both mappings are MICRRC mappings with contraction triple $(\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ respectively. Then the corresponding $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ -ICRRC-Hutchinson-like operator is defined as

$$\mathbb{H}(z) = \mathbb{J}^{(1)}(z) \cup \mathbb{J}^{(2)}(z), \quad \forall z \in \mathcal{X}.$$

The attractor of the ICRR-IO is the set $\{0\}$.

Theorem 8 (Generalized Collage Theorem). Let $(\mathcal{X}, \delta)$ be a complete b -MS with $s \geq 1$, $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \dots)$ a $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRR-IFS and $\mathbb{H}$ be the corresponding $(\lambda, \omega_1, \omega_2)$ -ICRRC mapping with attractor $\Omega^* \in C(\mathcal{X})$. If $\Theta \in C(\mathcal{X})$ and $\epsilon \geq 0$ such that

$$\mathcal{H}_\delta\left(\Theta, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta)\right) \leq \epsilon \quad (10)$$

then

$$\mathcal{H}_\delta(\Omega^*, \Theta) \leq \frac{\epsilon}{1 - s\lambda^{\frac{1}{\omega_1+\omega_2}}} \text{ s.t. } s\lambda^{\frac{1}{\omega_1+\omega_2}} < 1.$$

Or equivalently,

$$\mathcal{H}_\delta(\Omega^*, \Theta) \leq \frac{1}{1 - s\lambda^{\frac{1}{\omega_1+\omega_2}}} \cdot \mathcal{H}_\delta\left(\Theta, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta)\right) \text{ s.t. } s\lambda^{\frac{1}{\omega_1+\omega_2}} < 1.$$

Proof. By Theorem 5 and (10), we have

$$\begin{aligned}
 \mathcal{H}_\delta(\Omega^*, \Theta) &= \mathcal{H}_\delta(\Theta, \Omega^*) \\
 &\leq s\mathcal{H}_\delta(\Theta, \mathbb{H}(\Theta)) + s^2\mathcal{H}_\delta(\mathbb{H}(\Theta), \mathbb{H}^2(\Theta)) + s^3\mathcal{H}_\delta(\mathbb{H}^2(\Theta), \mathbb{H}^3(\Theta)) + \dots \\
 &\leq s\mathcal{H}_\delta(\Theta, \mathbb{H}(\Theta)) + s^2\lambda^{\frac{1}{\omega_1+\omega_2}}\mathcal{H}_\delta(\Theta, \mathbb{H}(\Theta)) + s^3\lambda^{\frac{2}{\omega_1+\omega_2}}\mathcal{H}_\delta(\Theta, \mathbb{H}(\Theta)) + \dots \\
 &= \left(1 + s\lambda^{\frac{1}{\omega_1+\omega_2}} + s^2\lambda^{\frac{2}{\omega_1+\omega_2}} + \dots\right)\mathcal{H}_\delta(\Theta, \mathbb{H}(\Theta)) \\
 &= \frac{1}{1 - s\lambda^{\frac{1}{\omega_1+\omega_2}}}s\mathcal{H}_\delta(\Theta, \mathbb{H}(\Theta)) \\
 &= \frac{1}{1 - s\lambda^{\frac{1}{\omega_1+\omega_2}}}s\mathcal{H}_\delta\left(\Theta, \bigcup_{\sigma=1}^{\kappa} \mathbb{J}^{(\sigma)}(\Theta)\right).
 \end{aligned}$$

Or equivalently,

$$\mathcal{H}_\delta(\Omega^*, \Theta) \leq \frac{\epsilon}{1 - s\lambda^{\frac{1}{\omega_1+\omega_2}}} \text{ s.t. } s\lambda^{\frac{1}{\omega_1+\omega_2}} < 1.$$

□

3. Application in Well-Posedness of the ICRRC-Hutchinson Problem

Let $(\mathcal{X}, \delta)$ be a b -MS and $\mathbb{H}: C(\mathcal{X}) \rightarrow C(\mathcal{X})$ be an ICRRC-HO related to ICRRC-IFS $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$. Then the ICRRC-IFS problem is called well-posed if

- (i) $\text{Fix}(\mathbb{H}) = \{\Omega^*\}$;
- (ii) For a sequence $\Omega_{\sigma+1} = \mathbb{H}(\Omega_\sigma)$, for $\sigma \in \mathbb{N}$ and $\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma)) \rightarrow 0$ as $\sigma \rightarrow \infty$ implies $\mathcal{H}_\delta(\Omega_\sigma, \Omega^*) \rightarrow 0$ as $\sigma \rightarrow \infty$.

Theorem 9. The ICRRC-Hutchinson problem related to ICRRC-IFS $(\mathcal{X}; \mathbb{J}^{(\sigma)}, \sigma = 1, 2, \dots, \kappa)$ is well posed.

Proof. Since $\mathbb{H}$ is a $(\lambda, \omega_1, \omega_2)$ -ICRRC-HO by Theorem 5, we get $\text{Fix}(\mathbb{H}) = \{\Omega^*\}$. For the sequence $\Omega_{\sigma+1} = \mathbb{H}(\Omega_\sigma)$, for $\sigma \in \mathbb{N}$,

$$\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma)) \rightarrow 0 \text{ as } \sigma \rightarrow \infty.$$

Then, we obtain

$$\begin{aligned}
 \mathcal{H}_\delta(\Omega_\sigma, \Omega^*) &\leq s\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma)) + s\mathcal{H}_\delta(\mathbb{H}(\Omega_\sigma), \Omega^*) \\
 &= s\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma)) + s\mathcal{H}_\delta(\mathbb{H}(\Omega_\sigma), \mathbb{H}(\Omega^*)) \\
 &\leq s\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma)) + s\lambda[\mathcal{H}_\delta(\Omega_\sigma, \Omega^*)]^{\omega_2}[\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma))]^{\omega_1}[\mathcal{H}_\delta(\Omega^*, \mathbb{H}(\Omega^*))]^{1-\omega_1-\omega_2} \\
 &= s\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma)) + s\lambda[\mathcal{H}_\delta(\Omega_\sigma, \Omega^*)]^{\omega_2}[\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma))]^{\omega_1}[\mathcal{H}_\delta(\Omega^*, \Omega^*)]^{1-\omega_1-\omega_2} \\
 \Rightarrow \mathcal{H}_\delta(\Omega_\sigma, \Omega^*) &\leq s\mathcal{H}_\delta(\Omega_\sigma, \mathbb{H}(\Omega_\sigma)) \rightarrow 0 \text{ as } \sigma \rightarrow \infty.
 \end{aligned}$$

Hence, the ICRRC-IFS-Hutchinson problem is well-posed. □

4. Conclusions

Firstly, we introduced the concept of $(\lambda_\sigma, \omega_{1\sigma}, \omega_{2\sigma})$ -ICRRC-IFS and IMS using ICRRC operators which is expanding the traditional CRRC mappings and more flexible. This outcome was presented the initial finding and sets the stage for further analysis in Theorem 3. The objective of this paper is to construct fractals using the HO and HLO involving ICRRC mappings, which extend the class of classical contractions, in the framework of

b-MS. The use of ICRRC guarantees unique fractal attractors, thereby playing a vital role in the analysis of geometric structures and their wide-range applications in scientific and engineering fields. We verified our results with the help of examples. Using the above new definitions, we established the well-known Collage theorem in the context of IFSs involving ICRRC. Further, we studied the well-posedness of the new ICRRC-IFS-Hutchinson problem. Our findings unify, generalize, and extend various earlier results reported in the literature. The present results provide a foundation for future investigations aimed to establishing new existence conditions for solutions of integral and differential equations. Additionally, the operators developed in this study may be utilized to analyze the existence of FPs in different generalized spaces.

Author Contributions: Conceptualization, L.M.S., Y.R., T.B., N.S. and M.Ö. Formal analysis, Y.R. and N.S. Investigation, L.M.S., Y.R., T.B., N.S. and M.Ö. Writing, original draft preparation, L.M.S., Y.R., T.B., N.S. and M.Ö. Writing, review and editing, L.M.S., Y.R., T.B., N.S. and M.Ö. All authors have read and agreed to the published version of the manuscript.

Funding: This study received no external funding.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study. No data sets were generated or analyzed during the current study.

Conflicts of Interest: Author Mahpeyker Öztürk was employed by the company Picode Software, Education Training Consultancy Research and Development and Trade Co., Ltd. This affiliation had no role in the study design, analysis, interpretation, or preparation of the manuscript. The remaining authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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