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The Existence of Mild Solutions for Hilfer Fractional Differential Equations with Infinite Delay in Orlicz Space

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Abstract

The Hilfer fractional derivative effectively captures non-locality, historical dependence, and memory effects, making it valuable for modeling real-world systems, and exponential growth can describe explosive growth phenomena in real-world problems. This paper focuses on the existence of mild solutions for infinite-delay differential equations involving Hilfer fractional derivatives, fractional Laplacian operator $(-\Delta)^\delta$, and exponentially growing functions in Orlicz spaces. First, by utilizing standard L^p - L^q estimates for strongly continuous semigroups generated by fractional Laplacian operator, the existence of global solutions in the Orlicz space $\exp L^p(\mathbb{R}^d)$ and the time-weighted $L^z(\mathbb{R}^d)$ space is established. Then, by leveraging Hölder's interpolation inequality, the existence of local solutions in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is established.

Keywords: Hilfer fractional derivative; Orlicz space; infinite delay; fixed point theorem; existence

1. Introduction

Fractional calculus arises as an extension of classical calculus, a theory first developed by Newton and Leibniz in the late 17th century. Traditional calculus, focusing primarily on integer-order derivatives and integrals, has been widely applied in physics, engineering, and economics [1–3]. However, many real-world systems exhibit complex behaviors—such as memory effects, non-locality, and nonlinearity—that cannot be effectively characterized by integer-order calculus. This has prompted scholars to explore generalizations of calculus to address more intricate phenomena.

Among these generalized forms, the Hilfer fractional derivative stands as a key extension (see Definition 3 for its precise definition). First proposed by Hilfer [4], this definition unifies the Riemann–Liouville and Caputo fractional derivatives as its special cases. Notably, the properties of the Hilfer fractional derivative vary with the parameter ν : when $\nu = 0$, it reduces to the Riemann–Liouville fractional derivative; when $\nu = 1$, it degenerates into the Caputo fractional derivative. Physically, the Riemann–Liouville fractional derivative emphasizes overall historical dependence, making it applicable to scenarios such as long-range interactions or aged polymers; the Caputo fractional derivative, by virtue of using classical integer-order initial conditions, is more suitable for traditional mechanical systems, such as the analysis of damped vibrations under external excitation. Benefiting from this hybrid property, the Hilfer fractional derivative exhibits unique advantages in multi-physics coupling problems, enabling more flexible descriptions of the



Received: 8 May 2026

Revised: 17 June 2026

Accepted: 25 June 2026

Published: 26 June 2026

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dynamic behavior of complex systems. Zhou et al. [5] examined Hilfer fractional stochastic evolution equations of order $\mu \in (0, 1)$ and type $\nu \in [0, 1]$, and discussed the existence of mild solutions under the condition that the semigroup associated with the almost sectorial operator is either compact or non-compact. Subsequently, Li and Zhou [6] investigated a class of Hilfer fractional stochastic evolution equations of order $\mu \in (1, 2)$ and type $\nu \in [0, 1]$, separately analyzing the existence of mild solutions under the compactness or non-compactness of the one-parameter family related to the operator. Other studies on the existence and stability of mild solutions for other types of equations involving Hilfer fractional derivatives are available in Refs. [7–10].

Fractional differential equations have attracted extensive scholarly attention in various fields, including fractional complex network synchronization [11], fractional wave equations [12], fractional Rayleigh–Stokes equations [13], fractional Navier–Stokes equations [14,15], fractional evolution equations [16], and fractional delay differential equations [17]. Concurrently, the widespread existence of time-delay phenomena has spurred growing research efforts in related areas. For instance, in control systems, signal transmission between sensors and actuators often involves time delays, which may induce system instability or sluggish responses [18]; in biology, cellular reactions and ecosystem dynamics are frequently influenced by time delays [19]. As research on complex systems deepens, time-delay issues have gradually garnered focused interest, driving advancements in both theory and applications. Notably, increasing attention has been paid to the existence and stability of solutions for delay differential equations involving Caputo fractional derivatives [20–22].

Analytically, the study of solution existence for differential equations with Hilfer fractional derivatives is more complex than for those with Caputo fractional derivatives. First, the initial value problem of the Hilfer fractional derivative usually involves Riemann–Liouville integrals, complicating the solution form. Second, the definition of the Hilfer fractional derivative may exhibit singularities or discontinuities near zero, so the solution domain typically excludes zero. To ensure a reasonable definition of the solution at zero, new operators are often introduced when investigating solution existence to handle these special cases.

Dynamic processes in systems often exhibit delay effects, where the current state is influenced by past states—an influence that becomes significant over longer time scales. Fractional delay differential equations are a class of partial differential equations combining fractional derivatives and time delays, usually expressed as:

$$D^\mu u(x, t) = \Delta u(x, t) + f(x, t, u(x, t - \tau)),$$

where D^μ denotes the fractional derivative, Δ the diffusion operator, τ the time delay, and f the nonlinear term. When $\mu = 1$, D^μ represents the traditional first-order derivative, describing classical diffusion phenomena such as heat conduction and gas diffusion; when $0 < \mu < 1$, it is suitable for systems with memory effects, such as cell diffusion and population dynamics in ecosystems; when $1 < \mu < 2$, the equation exhibits strong non-local characteristics, enabling better descriptions of system memory effects, such as the diffusion of complex materials and fluctuation behaviors in financial markets.

Differential equations with time delays are generally more complex than delay-free ones. Due to the introduction of time delays, the solution space becomes more intricate. Solutions of delay systems typically involve historical or delayed states, so the solution space must account for both current and past states over a preceding interval, making its structure more complex. In summary, time delays significantly increase the complexity of analyzing differential equations, making the study of solution existence more challenging and imposing higher requirements on function assumptions. In recent years, scholars have

also conducted a series of studies on fractional delay differential equations. For example, Omran et al. [23] investigated a linear finite difference Galerkin–Legendre spectral scheme for nonlinear multi-term Caputo time-fractional reaction–diffusion equations with time delays and Riesz space fractional derivatives in 2021. In 2023, Omran et al. [24] further explored nonlinear fractional space–time diffusion equations affected by time delays and provided numerical solutions. Additional relevant research on fractional delay differential equations can be found in Refs. [25–27].

For finite delay differential equations involving the Hilfer fractional derivative, Johnson et al. [28] investigated the results on the approximate controllability of Sobolev-type Hilfer fractional delay evolution equations without uniqueness of orders $\mu \in [0, 1]$ and $\nu \in (1/2, 1)$ in a Hilbert space. Subsequently, Kavitha and Vijayakumar [29] studied the existence of mild solutions for the following equation in a Hilbert space H :

$$\begin{cases} D_{0+}^{\mu,\nu} u(t) = Au(t) + A_1 u(t - \tau) + Bu(t) + f(t, u(t - \tau)), & t \in (0, \infty), \\ I_{0+}^{(1-\mu)(1-\nu)} u(t) = \varphi(t), & t \in [-\tau, 0], \end{cases}$$

where $\mu \in [0, 1]$ and $\nu \in (0, 1)$, B is a bounded operator, K is another Hilbert space, f is continuous, and there exist a positive constant $\nu_1 \in (0, \nu)$ and $m \in L^{\frac{1}{\nu_1}}(K, \mathbb{R}^+)$ such that for each $u_1, u_2 \in H$, $\|f(t, u_1) - f(t, u_2)\| \leq m t^{(1-\mu)(1-\nu)} \|u_1 - u_2\|_H$. For more research on finite delay differential equations involving the Hilfer fractional derivative, one can refer to Refs. [30–32].

For infinite delay differential equations involving the Hilfer fractional derivative, Sivasankar and Udhayakumar [33] applied the Mönch fixed point theorem in 2023 to prove the existence of mild solutions for the following Hilfer fractional neutral stochastic evolution equations with delays, via almost sectorial operators:

$$\begin{cases} D_{0+}^{\mu,\nu} [u(t) - g(t, u_t)] = \mathcal{A}u(t) + f(t, u_t) \frac{dW(t)}{dt}, & t \in \mathcal{V}' = (0, b], \\ I_{0+}^{(1-\mu)(1-\nu)} u(0) = \varphi \in L^2(\Omega, B_{\mathbb{N}}), & t \in (-\infty, 0], \end{cases}$$

where $\mu \in (0, 1)$, $\nu \in [0, 1]$, $\mathcal{A}$ denotes an almost sectorial operator, and $\{W(t)\}_{t \geq 0}$ is a $\mathcal{U}$ -valued Wiener process. Here, $f : \mathcal{V} \times B_{\mathbb{N}} \rightarrow L_2^0(\mathcal{U}, X)$ with $\mathcal{V} = [0, b]$, and $g : \mathcal{V} \times B_{\mathbb{N}} \rightarrow X$ is an X -valued function; X is a Hilbert space, $\mathcal{U}$ is another separable Hilbert space, $B_{\mathbb{N}}$ is a phase space, and $L_2^0(\mathcal{U}, X)$ denotes the space of all Q -Hilbert Schmidt operators from $\mathcal{U}$ to X (where Q is a finite trace kernel covariance operator). In 2024, Pradeesh and Vijayakumar [34] studied the existence and asymptotic stability of mild solutions for a class of Hilfer fractional neutral stochastic differential equations with infinite delays in Hilbert spaces, employing fractional calculus, stochastic analysis, semigroup theory, and the Krasnoselskii–Schaefer-type fixed point theorem. Further research on infinite delay differential equations involving Hilfer fractional derivatives can be found in Refs. [35,36].

Additionally, functions with exponential growth properties play a key role in scientific and engineering fields, such as population growth, cell division, and radioactive decay processes. Even algorithmic time complexity in computer science may exhibit such growth. Understanding exponential growth is crucial for identifying potential risks, responding to explosive changes, and formulating effective strategies in practical problems.

The Orlicz space $\exp L^p(\mathbb{R}^d)$ is a special type of Banach space that combines the core ideas of L^p spaces and further constrains function behavior via exponential growth conditions. Compared with traditional $L^p(\mathbb{R}^d)$ spaces, Orlicz spaces impose stronger constraints through exponential growth functions, enabling them to describe functions beyond the scope of L^p spaces. For example, in physical models with abrupt changes or peak behaviors, stochastic processes, or quantum field theory, some functions exhibit very rapid growth,

and $\exp L^p(\mathbb{R}^d)$ spaces can more effectively control and analyze their properties. In recent years, scholars have investigated solution properties in Orlicz spaces: for example, Ioku et al. [37] studied semilinear heat equations with exponential nonlinearity on $\mathbb{R}^2$ and established the non-existence of local solutions and existence of global solutions in $\exp L^2(\mathbb{R}^d)$. Fino and Kirane [38] considered heat equations with fractional Laplacians and exponential nonlinearity, establishing local well-posedness results in $\exp L^p(\mathbb{R}^d)$ and deriving global solutions for small initial data. He et al. [39] examined space–time fractional diffusion equations with Caputo fractional derivatives and exponential nonlinearity, proving the existence of global and local solutions for small-norm initial data in $\exp L^p(\mathbb{R}^d)$. Further research in Orlicz spaces is available in Refs. [40–42].

Notably, the study of mild solutions for infinite-delay differential equations with Hilfer fractional derivatives in Orlicz spaces is a valuable topic worthy of further investigation. This paper studies the diffusion equation with fractional Laplacian operator and infinite delay

$$\begin{cases} D_{0+}^{\mu,\nu} u + (-\Delta)^\delta u = f(u_{r(t)}), & x \in \mathbb{R}^d, t \in (0, \infty), \\ I_{0+}^{(1-\mu)(1-\nu)} u(x, 0^+) = \phi_0(x), & x \in \mathbb{R}^d, \\ u(x, t) = \frac{t^{(\mu-1)(1-\nu)} \phi(x, t)}{\Gamma(\mu(1-\nu) + \nu)}, & x \in \mathbb{R}^d, t \in (-\infty, 0), \end{cases} \quad (1)$$

concerning the existence of mild solutions. Here, the operator $I_{0+}^{(1-\mu)(1-\nu)} u(x, 0^+)$ appearing in the initial condition is the fractional integral defined in Definition 2. Here, $d \geq 1$, and $f: \mathbb{R} \rightarrow \mathbb{R}$ is an exponentially growing function approximating $f(u_{r(t)}) \sim e^{4\pi|u_{r(t)}|^2}$ with vanishing behavior at the zero point. $u_{r(t)}(x, t) = u(x, t + r(t))$ for $t \in (0, \infty)$, ϕ is a given function, and $r: [0, \infty) \rightarrow \mathbb{R}$ is a regulating function such that $r(t) \leq t$ for all $t < \infty$. The fractional Laplacian $(-\Delta)^\delta$ ($\delta \in (0, 1)$) is defined as

$$(-\Delta)^\delta u(x) = \text{P.V.} \int_{\mathbb{R}^d} C_{\delta,d} \frac{u(x) - u(y)}{|x - y|^{d+2\delta}} dy, \quad (2)$$

where $C_{\delta,d} = -\delta 2^{2\delta} \Gamma((d + 2\delta)/2) / (\pi^{d/2} \Gamma(1 - \delta))$, and the negative sign ensures the operator is positive definite. Subsequently, we impose conditions on the function f that differ from those in [33]; herein, it is assumed that f has the asymptotic behavior $f(u_{r(t)}) \sim t^{(\mu-1)(1-\nu)} |v_{r(t)}|^\sigma$ near zero, and we set $f(0) = 0$,

$$\left| f(u_{r(t)}) - f(u'_{r(t)}) \right| \leq t^{(\mu-1)(1-\nu)} |v_{r(t)} - v'_{r(t)}| \left(|v_{r(t)}|^{\sigma-1} e^{\ell|v_{r(t)}|^\zeta} + |v'_{r(t)}|^{\sigma-1} e^{\ell|v'_{r(t)}|^\zeta} \right), \quad (3)$$

where ζ and ℓ are positive constants and $\sigma \geq 1$. Here, $v(t)$ is defined analogously to $u(t)$ via the transformation $v(t) = t^{(1-\mu)(1-\nu)} u(t)$; this transformation is used to simplify the analysis of the initial condition. Then, based on the standard $L^p - L^q$ estimates for the strongly continuous semigroup generated by the fractional Laplacian, we prove the global existence of mild solutions to this problem in the Orlicz space $\exp L^p(\mathbb{R}^d)$ and the time-weighted space $L^Z(\mathbb{R}^d)$. Subsequently and further, we apply the Hölder interpolation inequality to prove the local existence of mild solutions in the space $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \hookrightarrow \exp L^p(\mathbb{R}^d)$.

2. Preliminaries

This section mainly introduces some general notations, definitions and properties of spaces, as well as the lemmas used in this paper.

In this paper, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}^n (n \geq 1)$ denotes the n -dimensional Euclidean space, $\mathbb{R}^+$ denotes the set of positive real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\sup$ and $\inf$ denote the supremum and infimum respectively, and a.e. denotes almost everywhere. The notation $a \lesssim a'$ means $a \leq C_1 a'$, and $\sim$ means $a \leq C_1 a'$ and $a' \leq C_1 a$, where C_1 is a positive constant independent of a and a' . The meanings of the symbols $\vee$ and $\wedge$ are $a \vee a' = \max\{a, a'\}$ and $a \wedge a' = \min\{a, a'\}$ respectively.

Let $X = L^2(\Omega)$ be a Banach space with norm $\|\cdot\|$. The space consisting of all continuous functions mapping from J to X is denoted by $\mathcal{C}_J = C(J, X)$, where J is an interval and $J \subset \mathbb{R}$. For any $y \in \mathcal{C}_J$ with J being a closed interval, define the norm $\|y\|_\infty = \sup_{t \in J} \|y(t)\|$. The Banach space consisting of Lebesgue measurable functions $\omega : J \rightarrow \mathbb{R}$ can be written as $L^p(J, \mathbb{R})$ with $1 \leq p \leq \infty$.

Define

$$Y = \left\{ u \in \mathcal{C}_{(0,b]} : \lim_{t \rightarrow 0} t^{(1-\mu)(1-\nu)} u(t) \text{ is a finite constant} \right\},$$

and the corresponding norm $\|\cdot\|_Y$ is

$$\|u\|_Y = \sup_{t \in (0,b]} \left\| t^{(1-\mu)(1-\nu)} u(t) \right\|.$$

Then, it can be known that Y is a Banach space with respect to the norm $\|\cdot\|_Y$.

The Orlicz space can be expressed as

$$\exp L^p(\mathbb{R}^d) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^d) : \|u\|_{\exp L^p(\mathbb{R}^d)} < +\infty \right\},$$

with the Luxemburg norm

$$\|u\|_{\exp L^p(\mathbb{R}^d)} := \inf \left\{ \iota > 0 : \int_{\mathbb{R}^d} \left(\exp \frac{|u(x)|^p}{\iota^p} - 1 \right) dx < 1 \right\}.$$

Obviously, the Orlicz space is a Banach space.

Definition 1 ([43]). The Riemann–Liouville derivative of a real-valued function f is defined as

$$D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, \quad n = [\alpha] + 1,$$

where $\alpha > 0$ and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2 ([4]). The fractional integral of a real-valued function f is defined as

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,$$

where $\alpha > 0$ and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 3 (Hilfer fractional derivative, see [4]). When each term on the right-hand side is reasonably defined within its applicable scope, the generalized fractional Riemann–Liouville derivative of order $\mu, \nu \in (0, 1)$ with lower limit 0 is defined as

$$D_{0+}^{\mu, \nu} f(t) = I_{0+}^{\mu(1-\nu)} \frac{d}{dt} I_{0+}^{(1-\mu)(1-\nu)} f(t).$$

Lemma 1 ([44]). Let (X, d) be a complete metric space. Suppose $T : X \rightarrow X$ is a contraction, meaning there exists a constant $k' \in [0, 1)$ such that

$$d(T(x), T(y)) \leq k' \cdot d(x, y),$$

holds for every pair $x, y \in X$. Then, T admits a unique fixed point $x^* \in X$ satisfying $T(x^*) = x^*$, and for any initial guess $x_0 \in X$, the sequence $\{T^n(x_0)\}$ of iterates converges to x^* as $n \rightarrow \infty$.

Lemma 2 ([45]). Hölder's Interpolation Inequality: Let f and g be measurable functions, and let $0 < p < z < q < \infty$. Then, for any θ satisfying $0 < \theta < 1$,

$$\|f\|_z \leq \|f\|_p^{1-\theta} \|f\|_q^\theta,$$

where $1/z = (1 - \theta)/p + \theta/q$.

Definition 4. A function $x : [\varkappa, \omega] \rightarrow \exp L^p(\mathbb{R}^d)$ is called a regulated function if for all $t \in (\varkappa, \omega]$, the left limit

$$x(t^-) = \lim_{s \rightarrow t^-} x(s) \in \exp L^p(\mathbb{R}^d)$$

exists, and for all $t \in [\varkappa, \omega)$, the right limit

$$x(t^+) = \lim_{s \rightarrow t^+} x(s) \in \exp L^p(\mathbb{R}^d)$$

exists. For more properties of regulated functions, one can refer to Ref. [46].

Lemma 3 ([40]). For each $1 \leq p \leq q < +\infty$, the embedding $\exp L^p(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ holds. Moreover,

$$\|u\|_{L^q(\mathbb{R}^d)} \leq \left(\Gamma\left(\frac{q}{p} + 1\right) \right)^{1/q} \|u\|_{\exp L^p(\mathbb{R}^d)}.$$

Lemma 4 ([38]). For each $1 \leq q \leq p < +\infty$, the embedding $L^q(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \hookrightarrow \exp L^p(\mathbb{R}^d)$ holds. Moreover,

$$\|u\|_{\exp L^p(\mathbb{R}^d)} \leq \ln^{-1/p} 2 \left(\|u\|_{L^q(\mathbb{R}^d)} + \|u\|_{L^\infty(\mathbb{R}^d)} \right).$$

To describe the type of equation studied in this paper and determine the research results, it is necessary to specify an appropriate phase space. Here, the phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ to $\exp L^p(\mathbb{R}^d)$, endowed with the norm $\|\cdot\|_{\mathcal{B}}$. It is assumed that $\mathcal{B}$ satisfies the following condition:

If $F : (-\infty, \infty) \rightarrow \exp L^p(\mathbb{R}^d)$ is continuous on $(0, \infty)$, and there exists an operator $G_r : \mathcal{B} \rightarrow \mathcal{B}$ such that $G_r(F(v(t))) = F(v_{r(t)}(t))$, then there exists a constant $\hat{K} > 0$ satisfying

$$\|G_r(F(v(t)))\|_{\mathcal{B}} \leq \hat{K} \min \left\{ \max_{0 \leq \eta \leq t} \|F(v(\eta))\|_{L^p(\mathbb{R}^d)}, \max_{0 \leq \eta \leq t} \|F(v(\eta))\|_{L^l(\mathbb{R}^d)} \right\},$$

where $p \geq 1$ and $1/l < 2\delta/d + 1/z$.

The space $\mathcal{B}_\tau$ consists of all functions $\phi \in \mathcal{B}$ such that $\phi_s \in \mathcal{B}$ for all $\tau \leq s \leq 0$, and the function $[\tau, 0] \rightarrow \mathcal{B}$ defined by $s \mapsto \phi_s$ is continuous. The norm of $\mathcal{B}_\tau$ is defined as

$$\|\phi\|_\tau = \sup_{\tau \leq s \leq 0} \|\phi_s\|_{\mathcal{B}}, \quad \phi \in \mathcal{B}_\tau,$$

where $\tau = \inf_{0 \leq t \leq \infty} (r(t) - t)$.

Set the Banach space

$$\mathcal{D} = \left\{ v : (-\infty, +\infty) \rightarrow \exp L^p(\mathbb{R}^d) : \frac{\phi(t)}{\Gamma(\mu(1-\nu) + \nu)} \in \mathcal{B}_\tau \text{ and } v|_{(0,\infty)} \text{ is continuous} \right\},$$

with the equivalent norm

$$\|v\|_{\mathcal{D}} = \left\| \frac{\phi(t)}{\Gamma(\mu(1-\nu) + \nu)} \right\|_\tau + \sup_{t \in (0,\infty)} \|v(t)\|_{\exp L^p(\mathbb{R}^d)}. \quad (4)$$

It is well-known that the fractional Laplacian $(-\Delta)^\delta$ can generate a strongly continuous semigroup $T_\delta(t) = \exp(-t(-\Delta)^\delta)$ on $L^p(\mathbb{R}^d)$ for $p \geq 1$ and $d \geq 1$, and its Fourier transform is given by $(-\Delta)^\delta u = \mathcal{F}^{-1}(|\xi|^{2\delta} \mathcal{F}(u))$. Moreover, for $t > 0$ and all $\psi \in L^q(\mathbb{R}^d)$ with $q \geq 1$ (see Ref. [47]), the space-time estimates of this operator are given by

$$\|T_\delta(t)\psi\|_{L^p(\mathbb{R}^d)} \lesssim t^{d(1/p-1/q)/(2\delta)} \|\psi\|_{L^q(\mathbb{R}^d)}. \quad (5)$$

Following Ref. [48], we present the definition of mild solutions to (1).

Definition 5. A function $u \in C((0, \infty); \exp L^p(\mathbb{R}^d))$ is defined as a mild solution of Equation (1) if u satisfies

$$u(t) = \begin{cases} S'_{\mu,\nu}(t)\phi_0 + \int_0^t K'_\nu(t-s)f(u_{r(s)}(s))ds, & t \in (0, \infty), \\ \frac{t^{(\mu-1)(1-\nu)}\phi(t)}{\Gamma(\mu(1-\nu) + \nu)}, & t \in (-\infty, 0). \end{cases} \quad (6)$$

Here,

$$S'_{\mu,\nu}(t) = I_{0+}^{\mu(1-\nu)} K'_\nu(t), \quad K'_\nu(t) = t^{\nu-1} P'_\nu(t),$$

where

$$P'_\nu(t) = \int_0^\infty \nu \theta M_\nu(\theta) T_\delta(t^\nu \theta) d\theta.$$

The function $M_\nu(\theta)$, where $\nu \in (0, 1)$ and $\theta \in \mathbb{C}$, is defined as the infinite series

$$M_\nu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)! \Gamma(1-\nu n)},$$

and satisfies the integral identity

$$\int_0^\infty \theta^q M_\nu(\theta) d\theta = \frac{\Gamma(1+q)}{\Gamma(1+\nu q)}, \quad \theta \geq 0, \quad q > -1.$$

Let $u(t) = t^{(\mu-1)(1-\nu)}v(t)$. Since the limit of $t^{(1-\mu)(1-\nu)}S'_{\mu,\nu}(t)\phi_0$ as $t \rightarrow 0^+$ is $\frac{\phi_0}{\Gamma(\mu(1-\nu)+\nu)}$, for any $v \in \mathcal{D}$, set

$$\mathcal{Q}(v)(t) = \begin{cases} t^{(1-\mu)(1-\nu)} \left[S'_{\mu,\nu}(t)\phi_0 + \int_0^t K'_\nu(t-s)f(u_{r(s)}(s))ds \right], & t \in (0, \infty), \\ \frac{\phi(t)}{\Gamma(\mu(1-\nu) + \nu)}, & t \in (-\infty, 0]. \end{cases} \quad (7)$$

Obviously, u is a mild solution of Equation (1) if and only if $\mathcal{Q}$ has a fixed point in the space $\mathcal{D}$.

According to the following estimates, the operators $t^{(1-\mu)(1-\nu)}K'_\nu(t)$ and $t^{(1-\mu)(1-\nu)}S'_{\mu,\nu}(t)$ are well-defined. We first obtain $\|T_\delta(t^\nu\theta)x\|_{L^p} \leq \|x\|_{L^p}$ via (5), and combining the analysis in [49], we obtain that

$$\begin{aligned}\|P'_\nu(t)x\|_{L^p(\mathbb{R}^d)} &\leq \int_0^\infty \nu\theta M_\nu(\theta) \|T_\delta(t^\nu\theta)x\|_{L^p(\mathbb{R}^d)} d\theta \\ &\leq \|x\|_{L^p(\mathbb{R}^d)} \int_0^\infty \nu\theta M_\nu(\theta) d\theta \\ &= \|x\|_{L^p(\mathbb{R}^d)} \cdot \frac{\nu\Gamma(2)}{\Gamma(1+\nu)} \\ &\lesssim \frac{1}{\Gamma(\nu)} \|x\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

and

$$\begin{aligned}\|t^{(1-\mu)(1-\nu)}K'_\nu(t)x\|_{L^p(\mathbb{R}^d)} &= \|t^{(1-\mu)(1-\nu)}t^{\nu-1}P'_\nu(t)x\|_{L^p(\mathbb{R}^d)} \\ &\lesssim \frac{t^{\mu(\nu-1)}}{\Gamma(\nu)} \|x\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

then,

$$\begin{aligned}\|t^{(1-\mu)(1-\nu)}S'_{\mu,\nu}(t)x\|_{L^p(\mathbb{R}^d)} &= \|t^{(1-\mu)(1-\nu)}I_{0+}^{\mu(1-\nu)}K'_\nu(t)x\|_{L^p(\mathbb{R}^d)} \\ &= \left\| t^{(1-\mu)(1-\nu)} \frac{1}{\Gamma(\mu(1-\nu))} \int_0^t (t-s)^{\mu(1-\nu)-1} K'_\nu(s)x ds \right\|_{L^p(\mathbb{R}^d)} \\ &= \left\| \frac{1}{\Gamma(\mu(1-\nu))} \int_0^1 (1-s)^{\mu(1-\nu)-1} s^{\nu-1} P_\nu(ts)x ds \right\|_{L^p(\mathbb{R}^d)} \\ &\lesssim \frac{1}{\Gamma(\mu(1-\nu))} \int_0^\infty \nu\theta M_\nu(\theta) d\theta \int_0^1 (1-s)^{\mu(1-\nu)-1} s^{\nu-1} ds \|x\|_{L^p(\mathbb{R}^d)} \\ &= \frac{1}{\Gamma(\mu(1-\nu)+\nu)} \|x\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Lemma 5. Let $\nu \in (0, 1)$, $\delta \in (0, 1)$, and for $1 \leq p \leq q \leq +\infty$, let $d(1/p - 1/q)/(2\delta) < 1$. Then, for $t > 0$, the following hold:

- (i) $\|P'_\nu(t)f\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\nu d/(2\delta)(1/p-1/q)} \|f\|_{L^p(\mathbb{R}^d)}$;
- (ii) For $f \in \exp L^p(\mathbb{R}^d)$, $\|P'_\nu(t)f\|_{\exp L^p(\mathbb{R}^d)} \lesssim \|f\|_{\exp L^p(\mathbb{R}^d)}$;
- (iii) For $f \in L^p(\mathbb{R}^d)$, $\|P'_\nu(t)f\|_{\exp L^q(\mathbb{R}^d)} \lesssim t^{-\nu d/(2\delta p)} \ln^{-1/q} \left(t^{-\nu d/(2\delta)} + 1 \right) \|f\|_{L^p(\mathbb{R}^d)}$.

Proof. (i) By the $L^p - L^q$ estimate of the semigroup $T_\delta(t)$, for the operator $P'_\nu(t)$ and $t > 0$, we have

$$\|P'_\nu(t)f\|_{L^q(\mathbb{R}^d)} \leq \int_0^\infty \nu\theta M_\nu(\theta) \|T_\delta(t^\nu\theta)f\|_{L^q(\mathbb{R}^d)} d\theta.$$

Deriving from (5), we get the semigroup estimate $\|T_\delta(t^\nu\theta)f\|_{L^q} \lesssim (t^\nu\theta)^{-\frac{\nu d}{2\delta}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}$ and obtain

$$\|P'_\nu(t)f\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{\nu d}{2\delta}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)} \int_0^\infty \nu\theta^{1-\frac{d}{2\delta}(\frac{1}{p}-\frac{1}{q})} M_\nu(\theta) d\theta.$$

The integral converges because the moment generating function $\int_0^\infty \theta^q M_\nu(\theta) d\theta = \Gamma(1+q)/\Gamma(1+\nu q)$ is finite for $q > -1$, See [49]. Here, $q = 1 - \frac{d}{2\delta}(\frac{1}{p} - \frac{1}{q}) > -1$ under our

assumptions. Denoting $1/z = 1/p - 1/q$, we simplify the exponent to $-vd/(2\delta z)$. Thus, the estimate holds.

(ii) For any $\iota > 0$ and $t > 0$, using the Taylor expansion and combining with (i), we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\exp \left(\frac{P'_v(t)f}{\iota} \right)^p - 1 \right) dx \\ &= \sum_{k=1}^{\infty} \frac{\|P'_v(t)f\|_{L^{pk}(\mathbb{R}^d)}^{pk}}{k! \iota^{pk}} \lesssim \sum_{k=1}^{\infty} \frac{\|f\|_{L^{pk}(\mathbb{R}^d)}^{pk}}{k! \iota^{pk}} \lesssim \int_{\mathbb{R}^d} \left(\exp \left(\frac{f}{\iota} \right)^p - 1 \right) dx, \end{aligned}$$

which implies that for $f \in \exp L^p(\mathbb{R}^d)$,

$$\|P'_v(t)f\|_{\exp L^p(\mathbb{R}^d)} \lesssim \|f\|_{\exp L^p(\mathbb{R}^d)}.$$

(iii) According to (i), for $t > 0$, we can obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\exp \left(\frac{P'_v(t)f}{\iota} \right)^q - 1 \right) dx \\ & \leq \sum_{k=1}^{\infty} \frac{\|P'_v(t)f\|_{L^{qk}(\mathbb{R}^d)}^{qk}}{k! \iota^{qk}} \lesssim \sum_{k=1}^{\infty} \frac{t^{-vd/(2\delta)(1/p-1/(qk))qk} \|f\|_{L^{pk}(\mathbb{R}^d)}^{qk}}{k! \iota^{qk}}. \end{aligned}$$

Then, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\exp \left(\frac{P'_v(t)f}{\iota} \right)^q - 1 \right) dx \\ & \leq t^{vd/(2\delta)} \left(\exp \left(\frac{C_1 t^{-vd/(2\delta p)} \|f\|_{L^p(\mathbb{R}^d)}}{\iota} \right)^q - 1 \right), \end{aligned}$$

and thus,

$$\|P'_v(t)f\|_{\exp L^q(\mathbb{R}^d)} \lesssim t^{-vd/(2\delta p)} \ln^{-1/q} \left(t^{-vd/(2\delta)} + 1 \right) \|f\|_{L^p(\mathbb{R}^d)}.$$

□

3. Main Results

3.1. Global Existence

Theorem 1. Let $1 \leq d < 2\delta p$, $p \geq 1$, $0 < \delta < 1$, and $\sigma \geq 1 + 2\delta p/d$. If there exists a $\chi > 0$ such that for all $\phi_0 \in \exp L^p(\mathbb{R}^d)$ with $\|\phi_0\|_{\exp L^p(\mathbb{R}^d)} \leq \Gamma(\mu(1-\nu))\chi$, then the Cauchy problem (1) has a unique mild solution. Moreover, the following decay estimate holds for some $z > 2\delta p^2/d + p$:

$$\|v(t)\|_{L^z(\mathbb{R}^d)} \lesssim t^{-vd/(2\delta)(1/p-1/z)} \|\phi_0\|_{\exp L^p(\mathbb{R}^d)}.$$

Proof. Let $\rho = vd/(2\delta)(1/p - 1/z)$. For any $\epsilon > 0$, define a complete metric space

$$\mathcal{D}_\epsilon := \left\{ v \in \mathcal{D} : \sup_{t>0} t^\rho \max_{0 \leq \eta \leq t} \|v(\eta)\|_{L^z(\mathbb{R}^d)} + \max_{0 \leq \eta \leq t} \|v(\eta)\|_{L^\infty(\exp L^p(\mathbb{R}^d))} \leq \epsilon \right\},$$

endowed with the distance $d(v, v') = \sup_{t>0} t^\rho \max_{0 \leq \eta \leq t} \|v(\eta) - v'(\eta)\|_{L^z(\mathbb{R}^d)}$.

Next, the existence of mild solutions for the equation with exponential growth terms in this chapter is proved using the contraction mapping method in two steps.

The first step is to prove that $\mathcal{Q}$ is contractive on $\mathcal{D}_\epsilon$. For any $v, v' \in \mathcal{D}_\epsilon$, by Lemma 5(i), we have

$$\begin{aligned} & \|\mathcal{Q}(v)(t) - \mathcal{Q}(v')(t)\|_{L^z(\mathbb{R}^d)} \\ &= t^{(1-\mu)(1-\nu)} \left\| \int_0^t K'_\nu(t-s) [f(u_{r(s)}) - f(u'_{r(s)})] ds \right\|_{L^z(\mathbb{R}^d)} \\ &= t^{(1-\mu)(1-\nu)} \left\| \int_0^t (t-s)^{\nu-1} P_\nu(t-s) [f(u_{r(s)}) - f(u'_{r(s)})] ds \right\|_{L^z(\mathbb{R}^d)} \\ &\lesssim t^{(1-\mu)(1-\nu)} \int_0^t (t-s)^{\nu-1-\nu d/(2p\delta)} \|f(u_{r(s)}) - f(u'_{r(s)})\|_{L^l(\mathbb{R}^d)} ds, \end{aligned}$$

where $d(1/l - 1/z) < 2\delta$. By the assumptions on f and condition (A), combined with the Taylor expansion,

$$\begin{aligned} & t^{(1-\mu)(1-\nu)} \int_0^t (t-s)^{\nu-1-\nu d/(2p\delta)} \|f(u_{r(s)}) - f(u'_{r(s)})\|_{L^l(\mathbb{R}^d)} ds \\ &\lesssim \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \int_0^t (t-s)^{\nu-1-\nu d/(2p\delta)} \|v_{r(s)} - v'_{r(s)}\|_{\mathcal{B}} (|v_{r(s)}|^{\sigma-1+\zeta k} + |v'_{r(s)}|^{\sigma-1+\zeta k}) ds \\ &\lesssim \hat{K} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \int_0^t (t-s)^{\nu-1-\nu d/(2p\delta)} \\ &\quad \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{\mathcal{B}} (|v(\eta)|^{\sigma-1+\zeta k} + |v'(\eta)|^{\sigma-1+\zeta k}) \|_{L^l(\mathbb{R}^d)} ds. \end{aligned}$$

By Hölder's and Minkowski's inequalities, for $1/l = 1/z + 1/p$,

$$\begin{aligned} & \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{\mathcal{B}} (|v(\eta)|^{\sigma-1+\zeta k} + |v'(\eta)|^{\sigma-1+\zeta k}) \|_{L^l(\mathbb{R}^d)} \\ &\lesssim \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{L^z(\mathbb{R}^d)} \max_{0 \leq \eta \leq s} (|v(\eta)|^{\sigma-1+\zeta k} + |v'(\eta)|^{\sigma-1+\zeta k}) \|_{L^p(\mathbb{R}^d)} \\ &\lesssim \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{L^z(\mathbb{R}^d)} \\ &\quad \times \left(\max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{p(\sigma-1+\zeta k)}(\mathbb{R}^d)}^{\sigma-1+\zeta k} + \max_{0 \leq \eta \leq s} \|v'(\eta)\|_{L^{p(\sigma-1+\zeta k)}(\mathbb{R}^d)}^{\sigma-1+\zeta k} \right). \end{aligned}$$

For some $0 < c < (\sigma-1)(z-p)/(\delta p z) \wedge 2(\sigma-2)/(d(\sigma-1))$, set

$$\varsigma = \frac{(1-\nu)\delta p^2 z c}{v(z-p-\delta p^2 c)}, \quad v = \frac{\delta p z c}{(\sigma-1+\zeta k)(z-p)}, \quad \frac{1}{\delta p} = \frac{2}{d} - c.$$

Then, for each $k \in \mathbb{N} \cup \{0\}$ and with $\nu - \nu d/(2\delta p) - \rho(\sigma-1+\zeta k)v = 0$, we have $\varsigma \geq p$ and $0 < v \leq 1$. Thus, by Hölder's interpolation inequality,

$$\max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{p(\sigma-1+\zeta k)}(\mathbb{R}^d)} \leq \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^z(\mathbb{R}^d)}^v \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^\varsigma(\mathbb{R}^d)}^{1-v},$$

where

$$\frac{1}{p(\sigma-1+\zeta k)} = \frac{v}{z} + \frac{1-v}{\varsigma}.$$

In addition, for any $y \in \exp L^p(\mathbb{R}^d)$, by Lemma 3,

$$\|y\|_{L^\varsigma(\mathbb{R}^d)}^{(\sigma-1+\zeta k)(1-v)} \lesssim \left(\Gamma\left(\frac{\varsigma}{p} + 1\right) \right)^{(\sigma-1+\zeta k)(1-v)/\varsigma} \|y\|_{\exp L^p(\mathbb{R}^d)}^{(\sigma-1+\zeta k)(1-v)}.$$

By the property $\Gamma(x+1) \leq C_1 x^{x+1/2}$ for all $x \geq 1$ and some constant $C_1 > 0$, using Stirling's formula and the inequality $(\sigma - 1 + \zeta k)(1 - v) \leq \zeta$,

$$\left(\Gamma\left(\frac{\zeta}{p} + 1\right) \right)^{(\sigma-1+\zeta k)(1-v)/\zeta} \leq C_1^k \Gamma(k+1).$$

Moreover, by the fact that

$$\int_0^t (t-s)^{m_1-1} s^{m_2-1} ds = \frac{\Gamma(m_1)\Gamma(m_2)}{\Gamma(m_1+m_2)} t^{m_1+m_2-1}, \quad m_1, m_2 > 0, \quad t > 0,$$

combined with $d < 2p\delta$, $0 < \mu, \nu < 1$, and $\rho + \rho(\sigma - 1 + \zeta k)v < 1$, we obtain

$$\begin{aligned} & \|Q(v)(t) - Q(v')(t)\|_{L^z(\mathbb{R}^d)} \\ & \lesssim \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k \int_0^t (t-s)^{\nu-1-\nu d/(2p\delta)} \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{L^z(\mathbb{R}^d)} \\ & \quad \left(\max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^z(\mathbb{R}^d)}^{(\sigma-1+\zeta k)v} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{\exp L^p(\mathbb{R}^d)}^{(\sigma-1+\zeta k)(1-v)} \right. \\ & \quad \left. + \max_{0 \leq \eta \leq s} \|v'(\eta)\|_{L^z(\mathbb{R}^d)}^{(\sigma-1+\zeta k)v} \max_{0 \leq \eta \leq s} \|v'(\eta)\|_{\exp L^p(\mathbb{R}^d)}^{(\sigma-1+\zeta k)(1-v)} \right) ds \\ & \lesssim \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k e^{\sigma-1+\zeta k} \int_0^t (t-s)^{\nu-1-\nu d/(2p\delta)} s^{-\rho-\rho(\sigma-1+\zeta k)v} ds d(v, v') \\ & \lesssim \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k e^{\sigma-1+\zeta k} t^{-\rho} d(v, v'). \end{aligned}$$

Thus, for any $v, v' \in \mathcal{D}_\epsilon$, there exists a constant $C_1 > 0$ such that

$$d(Q(v), Q(v')) \leq C_1 \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k e^{\sigma-1+\zeta k} d(v, v').$$

For sufficiently small $\epsilon > 0$ satisfying

$$C_1 \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k e^{\sigma-1+\zeta k} \leq \frac{1}{4},$$

it follows that Q is contractive on $\mathcal{D}_\epsilon$.

The second step is to prove that Q maps $\mathcal{D}_\epsilon$ into itself. The continuity of Q follows from the continuity of $P'_\nu(t)$ and the strong continuity of the semigroup $T_\delta(t)$ for all $t \geq 0$. Since $g(x) = \ln(x+1) - x/2 = 0$ has two zeros, let $m_1 \in (2, 4)$, where $m_1/2 = \ln(m_1+1)$. Then, for $0 \leq s \leq t - m_1^{-2\delta/(vd)}$, $\ln^{-1/p} \left((t-s)^{-\nu d/(2\delta)} + 1 \right) \leq 2^{1/p} (t-s)^{\nu d/(2\delta p)}$, and for $t - m_1^{-2\delta/(vd)} \leq s \leq t$, $\ln^{-1/p} \left((t-s)^{-\nu d/(2\delta)} + 1 \right) \leq 1$. Thus, by Lemma 5(iii), for some $1 \leq d < 2p\delta$,

$$\begin{aligned}
& t^{(1-\mu)(1-\nu)} \left\| \int_0^t K'_\nu(t-s) f(u_{r(s)}) ds \right\|_{\exp L^p(\mathbb{R}^d)} \\
& \lesssim t^{(1-\mu)(1-\nu)} \int_0^t (t-s)^{\nu-1-\nu d/(2p\delta)} \ln^{-1/p} \left((t-s)^{-\nu d/(2\delta)} + 1 \right) \left\| f(u_{r(s)}) \right\|_{L^p(\mathbb{R}^d)} ds \\
& \lesssim t^{(1-\mu)(1-\nu)} \int_0^{t-m_1^{-2\delta/(\nu d)}} (t-s)^{\nu-1} \left\| f(u_{r(s)}) \right\|_{L^p(\mathbb{R}^d)} ds \\
& \quad + t^{(1-\mu)(1-\nu)} \int_{t-m_1^{-2\delta/(\nu d)}}^t (t-s)^{\nu-1-\nu d/(2p\delta)} \left\| f(u_{r(s)}) \right\|_{L^p(\mathbb{R}^d)} ds \\
& \lesssim t^{(1-\mu)(1-\nu)} \int_0^t (t-s)^{\nu-1} \left\| f(u_{r(s)}(s)) \right\|_{L^p(\mathbb{R}^d)} ds + t^{(1-\mu)(1-\nu)} \sup_{s>0} \left\| f(u_{r(s)}(s)) \right\|_{L^p(\mathbb{R}^d)} \\
& =: \text{I} + \text{II}.
\end{aligned}$$

By the Taylor expansion,

$$|f(u_{r(t)})| \leq t^{(\mu-1)(1-\nu)} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} |v_{r(t)}|^{\zeta k + \sigma},$$

then,

$$\begin{aligned}
\text{I} & \leq \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \int_0^t (t-s)^{\nu-1} \left\| |v_{r(s)}(s)|^{\zeta k + \sigma} \right\|_{\mathcal{B}} ds \\
& \leq \hat{K} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \| |v(\eta)|^{\zeta k + \sigma} \|_{L^p(\mathbb{R}^d)} ds.
\end{aligned}$$

For $z > 2\delta p^2/d + p$ and $\sigma \geq 1 + 2\delta p/d$, set

$$\theta = \frac{2\delta p z}{d(z-p)(\zeta k + \sigma)}, \quad \omega = \frac{2\delta p^2 z(1-\theta)}{\theta(d(z-p) - 2\delta p^2)}.$$

Clearly, for each $k \in \mathbb{N} \cup \{0\}$, $\theta \in (0, 1]$ and $\omega = (\zeta k + \sigma)\theta\rho$. By the property of Hölder's interpolation inequality,

$$\max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^{p(\zeta k + \sigma)}(\mathbb{R}^d)}^{\zeta k + \sigma} \lesssim \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^z(\mathbb{R}^d)}^{\theta(\zeta k + \sigma)} \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^\omega(\mathbb{R}^d)}^{(1-\theta)(\zeta k + \sigma)},$$

where

$$\frac{1}{p(\zeta k + \sigma)} = \frac{\theta}{z} + \frac{1-\theta}{\omega}.$$

Similar to the proof in the first step, for $v \in \mathcal{D}_\epsilon$ and $\Gamma(\omega/p + 1) \leq C_1^k \Gamma(k+1)$ (since $(1-\theta)(\zeta k + \sigma) \leq \omega$), by Lemma 3,

$$\begin{aligned}
& \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^{p(\zeta k + \sigma)}(\mathbb{R}^d)}^{\zeta k + \sigma} ds \\
& \lesssim \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^z(\mathbb{R}^d)}^{\theta(\zeta k + \sigma)} \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^\omega(\mathbb{R}^d)}^{(1-\theta)(\zeta k + \sigma)} ds \\
& \lesssim \left(\Gamma\left(\frac{\omega}{p} + 1\right) \right)^{(1-\theta)(\zeta k + \sigma)/\omega} \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^z(\mathbb{R}^d)}^{\theta(\zeta k + \sigma)} \\
& \quad \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{\exp L^p(\mathbb{R}^d)}^{(1-\theta)(\zeta k + \sigma)} ds \\
& \lesssim C_1^k \Gamma(k+1) \epsilon^{\zeta k + \sigma}.
\end{aligned}$$

Thus, by

$$\max_{0 \leq \eta \leq s} \| |v(\eta)|^{\zeta k + \sigma} \|_{L^p(\mathbb{R}^d)} = \max_{0 \leq \eta \leq s} \| |v(\eta)| \|_{L^{p(\zeta k + \sigma)}(\mathbb{R}^d)}^{\zeta k + \sigma},$$

we get

$$I \lesssim \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k + \sigma}. \quad (8)$$

Next, we prove the second term. In fact, by the assumptions on f and Hölder's inequality, for some constants $a_1 \geq 1 \vee 1/\zeta$ and $a_2 \geq 1$,

$$\begin{aligned} \|f(u_{r(s)})\|_{L^p(\mathbb{R}^d)} &\lesssim t^{(\mu-1)(1-\nu)} \left\| |v_{r(s)}|^{\sigma-1} e^{\ell |v_{r(s)}|^{\zeta}} \right\|_{\mathcal{B}} \\ &\lesssim t^{(\mu-1)(1-\nu)} \hat{K} \max_{0 \leq \eta \leq s} \| |v(\eta)|^{\sigma-1} e^{\ell |v(\eta)|^{\zeta}} \|_{L^p(\mathbb{R}^d)} \\ &\lesssim t^{(\mu-1)(1-\nu)} \hat{K} \left[\max_{0 \leq \eta \leq s} \| e^{\ell |v(\eta)|^{\zeta}} - 1 \|_{L^{a_1 p}(\mathbb{R}^d)} \right. \\ &\quad \left. + \max_{0 \leq \eta \leq s} \| v(\eta) \|_{L^{\sigma a_2 p}(\mathbb{R}^d)}^{\sigma} + \max_{0 \leq \eta \leq s} \| v(\eta) \|_{L^{\sigma p}(\mathbb{R}^d)}^{\sigma} \right], \end{aligned}$$

where $1/p = 1/(a_1 p) + 1/(a_2 p)$. For $a_1 p \geq 1$ and $\Gamma(x+1) \leq C_1 x^{(x+1)/2}$, by Stirling's formula,

$$\begin{aligned} &\max_{0 \leq \eta \leq s} \| e^{\ell |v(\eta)|^{\zeta}} - 1 \|_{L^{a_1 p}(\mathbb{R}^d)} \\ &\leq \sum_{k=1}^{\infty} \frac{\ell^k}{k!} \max_{0 \leq \eta \leq s} \| v(\eta) \|_{L^{\zeta k a_1 p}(\mathbb{R}^d)}^{\zeta k} \\ &\leq \sum_{k=1}^{\infty} \frac{\ell^k}{k!} (\Gamma(\zeta k a_1 + 1))^{1/(a_1 p)} \max_{0 \leq \eta \leq s} \| v(\eta) \|_{\exp L^p(\mathbb{R}^d)}^{\zeta k} \\ &\lesssim \sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k}. \end{aligned}$$

In addition,

$$\begin{aligned} II &\lesssim \hat{K} \left(\sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k} \max_{0 \leq \eta \leq s} \| v(\eta) \|_{\exp L^p(\mathbb{R}^d)}^{\sigma} + \max_{0 \leq \eta \leq s} \| v(\eta) \|_{\exp L^p(\mathbb{R}^d)}^{\sigma} \right) \\ &\lesssim \hat{K} \left(\sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k + \sigma} + \epsilon^{\sigma} \right). \end{aligned} \quad (9)$$

By Lemma 5(ii), we have

$$\begin{aligned} &t^{(1-\mu)(1-\nu)} \| S'_{\mu, \nu}(t) \phi_0 \|_{\exp L^p(\mathbb{R}^d)} \\ &= t^{(1-\mu)(1-\nu)} \| I_{0+}^{\mu(1-\nu)} K'_{\nu}(t) \phi_0 \|_{\exp L^p(\mathbb{R}^d)} \\ &= t^{(1-\mu)(1-\nu)} \left\| \frac{1}{\Gamma(\mu(1-\nu))} \int_0^t (t-s)^{\mu(1-\nu)-1} K'_{\nu}(s) \phi_0 ds \right\|_{\exp L^p(\mathbb{R}^d)} \\ &= t^{(1-\mu)(1-\nu)} \left\| \frac{t^{(\mu-1)(1-\nu)}}{\Gamma(\mu(1-\nu))} \int_0^1 (1-s)^{\mu(1-\nu)-1} s^{\nu-1} P_{\nu}(ts) \phi_0 ds \right\|_{\exp L^p(\mathbb{R}^d)} \\ &\lesssim \frac{1}{\Gamma(\mu(1-\nu))} \int_0^1 (1-s)^{\mu(1-\nu)-1} s^{\nu-1} ds \| \phi_0 \|_{\exp L^p(\mathbb{R}^d)} \\ &= \frac{1}{\Gamma(\mu(1-\nu))} \| \phi_0 \|_{\exp L^p(\mathbb{R}^d)}, \end{aligned} \quad (10)$$

Then, by (8), (9), and (10),

$$\|\mathcal{Q}(v)(t)\|_{\exp L^p(\mathbb{R}^d)} \lesssim \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k + \sigma} + \hat{K} \epsilon^{\sigma} + \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{\exp L^p(\mathbb{R}^d)}. \quad (11)$$

Furthermore, by applying Lemma 5(i) and Lemma 3 and setting $f(u'_{r(t)}) = 0$ for $u'_{r(t)} = 0$, as in the first step, for $t > 0$, we obtain

$$\begin{aligned} & \|\mathcal{Q}(v)(t)\|_{L^z(\mathbb{R}^d)} \\ & \leq t^{(1-\mu)(1-\nu)} \left\| S'_{\mu,\nu}(t) \phi_0 \right\|_{L^z(\mathbb{R}^d)} \\ & \quad + t^{(1-\mu)(1-\nu)} \left\| \int_0^t K'_\nu(t-s) f(u_{r(s)}) ds \right\|_{L^z(\mathbb{R}^d)} \\ & \lesssim t^{-\rho} \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{\exp L^p(\mathbb{R}^d)} + t^{-\rho} \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k + \sigma}. \end{aligned} \quad (12)$$

Finally, from (11) and (12), there exists a constant $C_1 > 0$ such that

$$\|\mathcal{Q}(v)\|_{\mathcal{D}_\epsilon} \leq C_1 \left(\frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{\exp L^p(\mathbb{R}^d)} + \hat{K} \sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k + \sigma} + \hat{K} \epsilon^{\sigma} \right).$$

If we take $\epsilon = 4C_1\chi$, where χ is sufficiently small such that $C_1\hat{K}\epsilon^{\sigma} < \epsilon/4$ and $C_1\hat{K}\sum_{k=0}^{\infty} C_1^k \ell^k \epsilon^{\zeta k} < 1$, then $\mathcal{Q}$ is a contraction from $\mathcal{D}_\epsilon$ to $\mathcal{D}_\epsilon$. Therefore, by the contraction mapping principle, the Cauchy problem (1) has a unique mild solution. $\square$

3.2. Local Existence

For each $d \geq 1$, let $E = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $C((0, b], E)$ be the space of continuous functions from $(0, b]$ to E endowed with the norm $\|\cdot\|_C$. For a small initial value $\phi_0 \in E$, the local solution of the Cauchy problem (1) can be obtained. This section focuses on the existence and uniqueness of local mild solutions for Equation (1). Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$, by Lemma 3, it is known that $E \hookrightarrow \exp L_0^p(\mathbb{R}^d)$ for all $p \geq 1$, where $\exp L_0^p(\mathbb{R}^d)$ is the closure of $C_0^\infty(\mathbb{R}^d)$ in $\exp L^p(\mathbb{R}^d)$ with respect to the same norm; see, e.g., [40]. Thus, it is natural to consider local solutions in the space E without the Orlicz space.

Remark 1. The definition of mild solutions for the Cauchy problem (1) in this section is similar to Definition 5 in the previous section, except that the space to which u belongs is changed, requiring $u \in C((0, b]; E)$.

Consider the Banach space

$$\mathcal{D}' = \left\{ v : (-\infty, b] \rightarrow E : \frac{\phi(t)}{\Gamma(\mu(1-\nu) + \nu)} \in \mathcal{B}_\tau \text{ and } v|_{(0,b]} \text{ is continuous} \right\},$$

where the norm $\|v\|_{\mathcal{D}'}$ is similar to that in (4).

Theorem 2. Let $\phi_0 \in E$ and $\sigma > 3/2$. Then, the Cauchy problem (1) has a unique mild solution.

Proof. For any $v \in \mathcal{D}'$, the operator $\mathcal{Q}$ is defined similarly to (7). By the definition of $\mathcal{Q}$, the existence of local mild solutions should be proved using the fixed point theorem. For given $b > 0$ and $R > 0$, a ball in the Banach space $\mathcal{D}'$ is defined as

$$\mathcal{D}'_R = \left\{ v \in \mathcal{D}' : \max_{0 \leq \eta \leq t} \|v(\eta)\|_* \leq R \right\},$$

where $\max_{0 \leq \eta \leq t} \|v(\eta)\|_* = \max_{0 \leq \eta \leq t} \|v(\eta)\|_{L^\infty(E)}$.

First, it needs to be proved that $\mathcal{Q}(\mathcal{D}'_R) \subset \mathcal{D}'_R$. In fact, for $\sigma > 3/2$ and each $k \in \mathbb{N} \cup \{0\}$, let $\theta = 1/(\zeta k + \sigma)$. This θ serves as a Hölder interpolation parameter, which is distinct from the argument of the Wright function. The positive constant $\zeta > 0$ corresponds to the exponent of the exponential nonlinearity $e^{\ell|v|^\zeta}$ given in (3). Clearly, $\theta \in (0, 1)$. By Hölder's interpolation inequality,

$$\max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{\zeta k + \sigma}(\mathbb{R}^d)} \lesssim \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^1(\mathbb{R}^d)}^\theta \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^\infty(\mathbb{R}^d)}^{1-\theta}.$$

For any $v \in \mathcal{D}'_R$,

$$\begin{aligned} & \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{\zeta k + \sigma}(\mathbb{R}^d)}^{\zeta k + \sigma} ds \\ & \lesssim \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^1(\mathbb{R}^d)}^{\theta(\zeta k + \sigma)} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^\infty(\mathbb{R}^d)}^{(1-\theta)(\zeta k + \sigma)} ds \\ & \lesssim b^\nu R^{\zeta k + \sigma}, \end{aligned}$$

and by Lemma 5(i),

$$\begin{aligned} & \|\mathcal{Q}(v)(t)\|_{L^1(\mathbb{R}^d)} \\ & \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{L^1(\mathbb{R}^d)} + t^{(1-\mu)(1-\nu)} \int_0^t (t-s)^{\nu-1} \left\| f(u_r(s))(s) \right\|_{L^1(\mathbb{R}^d)} ds \\ & \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \sum_{k=0}^{\infty} \int_0^t (t-s)^{\nu-1} \left\| |v_r(s)|^{\zeta k + \sigma} \right\|_B ds \\ & \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \hat{K} \sum_{k=0}^{\infty} \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^1(\mathbb{R}^d)}^{\zeta k + \sigma} ds \\ & \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \hat{K} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \int_0^t (t-s)^{\nu-1} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{\zeta k + \sigma}(\mathbb{R}^d)}^{\zeta k + \sigma} ds. \end{aligned}$$

Thus, for any $v \in \mathcal{D}'_R$,

$$\begin{aligned} \|\mathcal{Q}(v)(t)\|_{L^1(\mathbb{R}^d)} & \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \hat{K} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} b^\nu R^{\zeta k + \sigma} \\ & \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \hat{K} b^\nu R^\sigma e^{\ell R^\zeta}. \end{aligned}$$

In addition, for any $p \in (d/(2\delta) \vee 1, \infty)$,

$$\begin{aligned}
& \|\mathcal{Q}(v)(t)\|_{L^\infty(\mathbb{R}^d)} \\
& \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{L^\infty(\mathbb{R}^d)} \\
& \quad + t^{(1-\mu)(1-\nu)} \int_0^t (t-s)^{\nu-1-\nu d/(2\delta p)} \left\| f(u_{r(s)})(s) \right\|_{L^p(\mathbb{R}^d)} ds \\
& \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{L^\infty(\mathbb{R}^d)} \\
& \quad + \sum_{k=0}^{\infty} \int_0^t (t-s)^{\nu-1-\nu d/(2\delta p)} \left\| |v_{r(s)}|^{\zeta k + \sigma} \right\|_{\mathcal{B}} ds \\
& \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{L^\infty(\mathbb{R}^d)} \\
& \quad + \hat{K} \sum_{k=0}^{\infty} \int_0^t (t-s)^{\nu-1-\nu d/(2\delta p)} \max_{0 \leq \eta \leq s} \left\| |v(\eta)|^{\zeta k + \sigma} \right\|_{L^p(\mathbb{R}^d)} ds \\
& \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_{L^\infty(\mathbb{R}^d)} \\
& \quad + \hat{K} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \int_0^t (t-s)^{\nu-1-\nu d/(2\delta p)} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{(\zeta k + \sigma)p}(\mathbb{R}^d)}^{\zeta k + \sigma} ds.
\end{aligned}$$

Furthermore, let $z \in (1, \sigma p)$. For each $k \in \mathbb{N} \cup \{0\}$, $v = z/((\zeta k + \sigma)p) \in (0, 1)$. Thus, by Hölder's interpolation inequality,

$$\begin{aligned}
\max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{((\zeta k + \sigma)p)(\mathbb{R}^d)}} & \lesssim \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^z(\mathbb{R}^d)}^v \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^\infty(\mathbb{R}^d)}^{1-v} \\
& \lesssim \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^1(\mathbb{R}^d)}^{\omega v} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^\infty(\mathbb{R}^d)}^{1-\omega v},
\end{aligned}$$

where $\omega = 1/z \in (0, 1)$. This implies that

$$\begin{aligned}
& \|\mathcal{Q}(v)(t)\|_{L^\infty(\mathbb{R}^d)} \\
& \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \hat{K} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \int_0^t (t-s)^{\nu-1-\nu d/(2\delta p)} \\
& \quad \max_{0 \leq \eta \leq s} \|v(\eta)\|_E^{(\zeta k + \sigma)\omega v} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^\infty(\mathbb{R}^d)}^{(\zeta k + \sigma)(1-\omega v)} ds \\
& \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \hat{K} \sum_{k=0}^{\infty} \frac{\ell^k}{k!} b^{\nu-\nu d/(2\delta p)} R^{\zeta k + \sigma} \\
& \lesssim \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + \hat{K} b^{\nu-\nu d/(2\delta p)} R^\sigma e^{\ell R^\zeta}.
\end{aligned}$$

Thus, there exists a constant $C_1 > 0$ such that

$$\|\mathcal{Q}(v)(t)\|_E \leq C_1 \frac{1}{\Gamma(\mu(1-\nu))} \|\phi_0\|_E + C_1 \hat{K} b^{\nu-\nu d/(2\delta p)} R^\sigma e^{\ell R^\zeta} + C_1 \hat{K} b^\nu R^\sigma e^{\ell R^\zeta},$$

Then, let $R = 2C_1/\Gamma(\mu(1-\nu))\|\phi_0\|_E$ and choose a sufficiently small $b > 0$ such that

$$C_1 \hat{K} (b^{\nu-\nu d/(2\delta p)} + b^\nu) R^{\sigma-1} e^{\ell R^\zeta} \leq \frac{1}{2}. \quad (13)$$

Thus, it can be inferred that $\|\mathcal{Q}(v)\|_E \leq R$ and $\mathcal{Q}(v) \in \mathcal{D}'_R$ for any $v \in \mathcal{D}'_R$.

Next, it needs to be proved that $\mathcal{Q}$ is a contraction mapping. Let $\beta = 1/(2(\zeta k + \sigma - 1))$ where $\sigma > 3/2$ and $k \in \mathbb{N} \cup \{0\}$, and clearly $\beta \in (0, 1)$. Then, by Hölder's interpolation inequality, for any $v \in \mathcal{D}'_R$,

$$\begin{aligned} & \max_{0 \leq \eta \leq s} \left\| |v(\eta)|^{\sigma-1} e^{\ell|v(\eta)|^\zeta} \right\|_{L^2(\mathbb{R}^d)} \\ &= \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^{2(\zeta k + \sigma - 1)}(\mathbb{R}^d)}^{\zeta k + \sigma - 1} \\ &\lesssim \sum_{k=0}^{\infty} \frac{\ell^k}{k!} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^1(\mathbb{R}^d)}^{\beta(\zeta k + \sigma - 1)} \max_{0 \leq \eta \leq s} \|v(\eta)\|_{L^\infty(\mathbb{R}^d)}^{(1-\beta)(\zeta k + \sigma - 1)} \\ &\lesssim R^{\sigma-1} e^{\ell R^\zeta}. \end{aligned}$$

Let $v, v' \in \mathcal{D}'_R$, then

$$\begin{aligned} & \left\| \mathcal{Q}(v)(t) - \mathcal{Q}(v')(t) \right\|_{L^1(\mathbb{R}^d)} \\ &\leq t^{(1-\mu)(1-\nu)} \int_0^t (t-s)^{\nu-1} \left\| f(u_{r(s)})(s) - f(u'_{r(s)})(s) \right\|_{L^1(\mathbb{R}^d)} ds, \end{aligned}$$

Then, by Lemma 5(i) and Hölder's inequality,

$$\begin{aligned} & \left\| f(u_{r(s)})(s) - f(u'_{r(s)})(s) \right\|_{L^1(\mathbb{R}^d)} \\ &\lesssim t^{(\mu-1)(1-\nu)} \left\| |v_{r(s)} - v'_{r(s)}| \left(|v_{r(s)}|^{\sigma-1} e^{\ell|v_{r(s)}|^\zeta} + |v'_{r(s)}|^{\sigma-1} e^{\ell|v'_{r(s)}|^\zeta} \right) \right\|_{\mathcal{B}} \\ &\lesssim t^{(\mu-1)(1-\nu)} \hat{K} \max_{0 \leq \eta \leq s} \left\| |v(\eta) - v'(\eta)| \left(|v(\eta)|^{\sigma-1} e^{\ell|v(\eta)|^\zeta} + |v'(\eta)|^{\sigma-1} e^{\ell|v'(\eta)|^\zeta} \right) \right\|_{L^1(\mathbb{R}^d)} \\ &\lesssim t^{(\mu-1)(1-\nu)} \hat{K} \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{L^2(\mathbb{R}^d)} \\ &\quad \max_{0 \leq \eta \leq s} \left\| |v(\eta)|^{\sigma-1} e^{\ell|v(\eta)|^\zeta} + |v'(\eta)|^{\sigma-1} e^{\ell|v'(\eta)|^\zeta} \right\|_{L^2(\mathbb{R}^d)} \\ &\lesssim t^{(\mu-1)(1-\nu)} \hat{K} R^{\sigma-1} e^{\ell R^\zeta} \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{L^1(\mathbb{R}^d)}^{1/2} \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_{L^\infty(\mathbb{R}^d)}^{1/2} \\ &\lesssim t^{(\mu-1)(1-\nu)} \hat{K} R^{\sigma-1} e^{\ell R^\zeta} \max_{0 \leq \eta \leq s} \|v(\eta) - v'(\eta)\|_E. \end{aligned}$$

Thus, we have

$$\left\| \mathcal{Q}(v)(t) - \mathcal{Q}(v')(t) \right\|_{L^\infty(\mathbb{R}^d)} \lesssim b^{\nu-\nu d/(2\delta p)} \hat{K} R^{\sigma-1} e^{\ell R^\zeta} \|v - v'\|_C.$$

Then, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} & \left\| \mathcal{Q}(v)(t) - \mathcal{Q}(v')(t) \right\|_E \\ &\leq \left(C_1 b^\nu \hat{K} R^{\sigma-1} e^{\ell R^\zeta} + C_1 b^{\nu-\nu d/(2\delta p)} \hat{K} R^{\sigma-1} e^{\ell R^\zeta} \right) \|v - v'\|_C. \end{aligned}$$

According to (13), we know that $C_1 b^\nu \hat{K} R^{\sigma-1} e^{\ell R^\zeta} + C_1 b^{\nu-\nu d/(2\delta p)} \hat{K} R^{\sigma-1} e^{\ell R^\zeta} \leq 1/2$, so $\mathcal{Q}$ is contractive on $\mathcal{D}'_R$. Since $(-\Delta)^\delta$ generates a strongly continuous semigroup $T_\delta(t)$ on $L^1(\mathbb{R}^d)$, it is easy to prove that $\mathcal{Q}$ is continuous. Therefore, by the Banach fixed point theorem, the Cauchy problem (1) has a unique mild solution. $\square$

4. Conclusions

The aim of this paper is to investigate the existence of mild solutions for Hilfer fractional infinite delay differential equations, which, unlike equations involving the Riemann–

Liouville and Caputo derivatives, are rooted in a novel theoretical framework of fractional calculus. It provides a more comprehensive and unified formulation, thereby allowing a more accurate description of the dynamics of complex systems. Consequently, exploring the existence of mild solutions to such delay differential equations fosters a deeper understanding of their behavior and yields precise mathematical models for addressing practical problems, rendering this research academically significant for optimizing engineering designs, predicting system outcomes, and controlling natural processes.

This study focuses on the existence of mild solutions for time-dependent delay differential equations involving the fractional Laplacian operator with infinite delay. First, we establish the definition of a mild solution to the equation. Next, by utilizing the contraction mapping principle, we prove the existence of a global solution for initial data with small norms in the space $\exp L^p(\mathbb{R}^d)$ and obtain the corresponding decay estimates of the solution. Finally, we further demonstrate the existence of a local solution in the subspace $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ of the Orlicz space.

Author Contributions: Conceptualization, R.S. and J.M.; methodology, R.S. and Y.J.; formal analysis, Y.W. and Y.J.; writing—original draft, R.S. and Y.J.; writing—review and editing, J.M. and L.G.; supervision, J.M.; funding acquisition, J.M. and L.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Fundamental Research Funds for the Central Universities (Nos. 31920250035, 31920260101), the Scientific Research Project of the Education Department of Gansu Province (No. 2026QB-009), the Innovation Team of Ecosystem Restoration Modeling Theory and Application of Northwest Minzu University, and National Natural Science Foundation of China (62366048).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors gratefully acknowledge all funding bodies that supported this research. The authors would like to express their sincere gratitude to the editors and reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflicts of interest.

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