



Article

Fractional Stochastic Modeling of Nonlinear Dynamical Systems: Application to an Electromechanical Process with Memory Effects

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Abstract

In this study, a comprehensive stochastic and fractional-order modeling framework is developed to investigate the dynamic behavior of a shunt DC motor under random disturbances and memory effects. The motor dynamics are formulated as a system of stochastic differential equations incorporating Gaussian noise to represent uncertainties in the electrical and mechanical subsystems. The existence, stochastic ultimate boundedness, stationary distribution, and ergodic properties of the proposed model are established. To further enhance modeling capabilities, a modified Atangana–Baleanu–Caputo (mABC) fractional operator is introduced, enabling the incorporation of nonlocal memory effects inherent in electromechanical systems. The series solution is derived using the Laplace transform and the Adomian decomposition method to handle nonlinearities. Qualitative analysis of the solution is performed through fixed-point theory, while stability assessments utilize the T-Picard method. The results of the numerical simulation indicate that the stochastic model exhibits limited variability around the operating regimes, whereas the fractional-order representation is more effective at smoothing transient responses and limiting oscillatory behavior. The study proposes a realistic and adaptable method to analyze the dynamics of shunt DC motors with uncertainty and also presents useful information for the design and control of electromechanical systems.

Keywords: random disturbances; stochastic behavior; shunt DC motor model; modified fractional operator; memory effects; system stability



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1. Introduction

Direct current (DC) motors have traditionally served as an important means of converting electrical energy into mechanical output. In earlier drive applications, they were widely preferred where accurate regulation of speed and torque was required, particularly in marine propulsion systems. Even though many conventional brushed DC drives have been superseded by alternating current (AC) drives because of their higher efficiency, commutator-based DC machines remain relevant in a number of niche marine applications. These include submarines, icebreakers, tugboats, dredgers, and research ships. Representative cases include the hybrid diesel–electric propulsion arrangement used in the Royal Navy’s Type 23 frigates and DC-based propulsion solutions such as Siemens’ SINAVY DC-Prop system adopted in Type 209 submarines [1].

DC motors continue to attract attention for their straightforward construction, reliable operation, and broad speed regulation capabilities [2–4]. These characteristics make

them suitable for use in areas such as robotics, electric transportation, cranes, and other high-precision motion-control applications [3–7]. In addition, many control approaches developed for DC motors have provided the conceptual basis for a wide range of AC machine control methods. Accurate modelling of DC motors is therefore essential, as control and drive system performance strongly depends on model fidelity, particularly for advanced techniques such as model predictive control. In addition, modern motor drive systems must comply with electromagnetic compatibility standards [8–10]. Switching power electronics introduce electromagnetic emissions over a wide frequency range [8,11,12]. These high-frequency effects can be represented as additive non-Gaussian noise in motor state equations; however, identifying such stochastic models under different discretisation schemes is beyond the scope of this paper. Dynamic modelling becomes an essential tool for studying the behaviour and performance of modern electromechanical systems [13–15]. A DC motor is an electromechanical device that converts electrical energy into mechanical rotational motion through electromagnetic interactions between the armature current and magnetic field.

The performance of DC motors in practical engineering applications is influenced by various uncertainties, such as supply voltage fluctuations, load disturbances, parameter variations, and environmental effects. These factors can significantly affect motor operation and cannot be fully represented by deterministic models. Therefore, stochastic differential equation models have attracted considerable attention because they provide a more realistic framework for describing the uncertain behavior of electromechanical systems and improving performance analysis under random operating conditions [16,17]. Many stochastic dynamical systems account for external disturbances and time-varying inputs [17,18]. For instance, Din et al. studied stochastic dynamical systems subjected to random perturbations and investigated their long-term characteristics, offering valuable insights for analyzing the reliability and stability [19].

Fractional-order operators provide a powerful mathematical framework for modeling systems with memory and hereditary properties, which cannot be adequately described using classical integer-order derivatives. By allowing differentiation and integration of non-integer order, fractional calculus captures long-range temporal dependence and complex dynamic behavior observed in many physical and engineering systems. In recent years, fractional-order models have been successfully applied to electromechanical systems, viscoelastic materials, and control theory, where they offer improved accuracy and robustness compared to integer-order formulations. Various definitions of fractional derivatives, such as the Riemann–Liouville and Caputo operators, have been widely used due to their solid theoretical foundations and suitability for physical initial conditions [20,21]. Moreover, fractional-order modeling can enhance the analysis of DC motor systems by accounting for memory effects, electromagnetic losses, and complex transient responses, leading to more accurate characterization of motor dynamics and control performance under practical operating conditions. Fractional-order models provide a more accurate representation of DC motor dynamics and control performance.

In this paper, a stochastic modeling framework is developed to analyze the dynamic behavior of a shunt DC motor under random disturbances. The motor dynamics are formulated using stochastic differential equations to account for uncertainties in the electrical and mechanical subsystems. To further enhance the modeling capability, a fractional-order operator is introduced to incorporate memory effects inherent in electromechanical systems. The proposed model is investigated through analytical considerations and numerical simulations, providing insights into system stability and dynamic response under uncertain operating conditions.

The objective of this paper is to investigate the dynamic behavior of a shunt DC motor under stochastic disturbances and fractional-order effects. In Section 2, the mathematical model of the DC motor is presented, describing the electrical and mechanical subsystems. The well-posedness of the proposed stochastic model is established in Section 3, where the existence, uniqueness, and positivity of solutions are rigorously analyzed. The long-term stochastic behavior of the system is examined in Section 4. In Section 5 discusses the Stationary Distribution. Section 6, a modified fractional-order operator is introduced to incorporate memory effects into the motor dynamics. Numerical simulations illustrating the stochastic responses and dynamic characteristics of the modified fractional system are provided in Section 7. Finally, Section 9 presents the conclusion and outlines potential directions for future research.

2. Model Description

A shunt DC motor is characterized by a parallel connection of the armature and field circuits, both of which receive power from the same DC source. Arévalo et al. [22] proposed a model for Shunt DC motor dynamics, where each circuit is supplied by its own DC voltage source, namely V_a for the armature and V_f for the field, as illustrated in Figure 1. In the equivalent circuit, the armature dynamics are characterized by the armature resistance R_a and inductance L_a , with the armature current i_a serving as the associated state variable. Likewise, the field circuit is described by the field resistance R_f and inductance L_f , and the field current i_f represents its corresponding state variable. The interaction between the electrical and mechanical subsystems gives rise to an induced armature voltage, commonly referred to as the back electromotive force (back-emf), which opposes the applied armature voltage and is expressed as:

$$E_a = L_{af}i_f\omega,$$

where L_{af} denotes the mutual inductance linking the armature and field windings, and ω represents the rotor angular speed. For a separately excited (SE) DC motor, the field winding must establish the rated magnetic flux before voltage is applied to the armature circuit. In practice, this means that the field current should be maintained at its nominal value. This precaution prevents the machine from starting under field-weakening conditions with little or no load, a situation that may cause excessive speed and potentially harm the armature winding as well as the commutator brushes. Accordingly, in the constant-torque operating region, the field current is commonly assumed to remain constant, leading to the following linearized model:

$$\begin{aligned}\frac{di_a}{dt} &= \frac{1}{L_a} \left(V_L - R_a i_a - L_{af} i_f \omega \right), \\ \frac{di_f}{dt} &= \frac{1}{L_f} \left(V_L - R_f i_f \right), \\ \frac{d\omega}{dt} &= \frac{1}{J} \left(L_{af} i_a i_f - K_L \omega \right).\end{aligned}\tag{1}$$

A description of parameters in model (1) can be seen in Table 1.

Table 1. Description of parameters in model (1).

Symbol	Type	Description
i_a	Variable	Armature current
i_f	Variable	Field current
ω	Variable	Angular velocity of the rotor
V_L	Input	Applied voltage

Table 1. Cont.

Symbol	Type	Description
R_a	Parameter	Armature resistance
R_f	Parameter	Field resistance
L_a	Parameter	Armature inductance
L_f	Parameter	Field inductance
L_{af}	Parameter	Mutual inductance between armature and field
K_L	Parameter	Load torque or friction constant
J	Parameter	Rotor moment of inertia

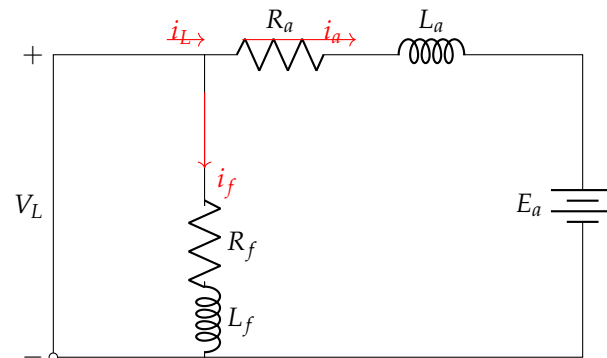


Figure 1. Circuit model of a separately excited DC motor [22].

Equilibrium Points

The equilibrium points of the system are obtained by setting

$$\frac{di_a}{dt} = \frac{di_f}{dt} = \frac{d\omega}{dt} = 0.$$

Torque-free equilibria.

Define the torque-free manifold as

$$\mathcal{E}_{\text{TFM}} = \left\{ (i_a, i_f, \omega) : \omega = 0, L_{af}i_a i_f = 0 \right\},$$

which corresponds to states where no electromagnetic torque is produced.

Imposing $i_a = i_f = 0$ under this condition yields the following no-current equilibria:

(A) Armature open (no armature current): $(i_a^*, i_f^*, \omega^*) = (0, \frac{V_L}{R_f}, 0),$

(B) Field off (no field current): $(i_a^*, i_f^*, \omega^*) = (\frac{V_L}{R_a}, 0, 0),$

(C) Both unpowered (no supply): $(i_a^*, i_f^*, \omega^*) = (0, 0, 0),$ when $V_L = 0.$

If both circuits are energized, i.e.,

$$i_a = \frac{V_L}{R_a}, \quad i_f = \frac{V_L}{R_f},$$

then $\omega = 0$ is *not* an equilibrium. This follows from

$$\dot{\omega}|_{\omega=0} = \frac{L_{af}}{J} i_a i_f > 0,$$

which indicates that the generated electromagnetic torque initiates rotor acceleration, driving the system away from the torque-free state.

To characterize the operating condition of the motor, we introduce a dimensionless performance index based on the balance between electromagnetic torque and mechanical load. Specifically, we define the *Torque Ratio Index* as

$$\mathcal{T} = \frac{L_{af} i_a^* i_f^*}{K_L \omega^*}, \quad (2)$$

where $L_{af} i_a^* i_f^*$ represents the electromagnetic torque generated by the motor, and $K_L \omega^*$ denotes the opposing mechanical load torque, including frictional effects.

This index provides a direct physical interpretation of the motor's operating regime:

- $\mathcal{T} > 1$: electromagnetic torque exceeds load torque, leading to acceleration;
- $\mathcal{T} = 1$: torque balance is achieved, corresponding to steady-state operation;
- $\mathcal{T} < 1$: load torque dominates, resulting in deceleration or possible stall.

Thus, $\mathcal{T}$ serves as a meaningful engineering indicator for assessing whether the motor can sustain stable rotation under given electrical and mechanical conditions.

Non-trivial equilibrium (operating point).

For the general case where the motor is energized and rotating, the equilibrium conditions yield

$$\begin{aligned} i_f^* &= \frac{V_L}{R_f}, \\ i_a^* &= \frac{V_L}{R_a + L_{af} \frac{i_f^*}{\omega^*}}, \\ \omega^* &= \frac{L_{af} i_a^* i_f^*}{K_L}. \end{aligned} \quad (3)$$

The equilibrium $\omega^* = 0$ corresponds to a *stall condition*, where no mechanical rotation occurs despite possible electrical excitation. In contrast, the non-trivial equilibrium $\omega^* > 0$ represents the *steady-state operating point* of the motor, where electromagnetic torque balances the mechanical load torque.

In real-world applications, the performance of a shunt DC motor is influenced by various sources of uncertainty, including fluctuations in the input voltage, variations in mechanical load, temperature-dependent changes in electrical resistance, and imperfections in system parameters. Such effects cannot be adequately captured by purely deterministic models. To account for these uncertainties, we extend the deterministic system (1) by incorporating stochastic perturbations in the form of stochastic differential equations (SDEs). In this formulation, we consider *multiplicative Gaussian noise*, where the intensity of the stochastic fluctuations depends on the current state of the system. This choice reflects the practical observation that disturbances in electrical and mechanical components often scale with their operating levels.

Accordingly, the stochastic model is given by

$$\begin{aligned} di_a(t) &= \frac{1}{L_a} \left(V_L - R_a i_a(t) - L_{af} i_f(t) \omega(t) \right) dt + \sigma_1 i_a(t) dB_1(t), \\ di_f(t) &= \frac{1}{L_f} \left(V_L - R_f i_f(t) \right) dt + \sigma_2 i_f(t) dB_2(t), \\ d\omega(t) &= \frac{1}{J} \left(L_{af} i_a(t) i_f(t) - K_L \omega(t) \right) dt + \sigma_3 \omega(t) dB_3(t), \end{aligned} \quad (4)$$

where

- σ_1 , σ_2 , and σ_3 are *noise intensity parameters* that determine the strength of stochastic perturbations in the armature current, field current, and angular velocity, respectively;
- $B_1(t)$, $B_2(t)$, and $B_3(t)$ are independent standard *Wiener processes* (Brownian motions). The multiplicative nature of the noise has a clear physical interpretation. Specifically,
- Disturbances in the armature and field circuits (e.g., voltage fluctuations, thermal effects) tend to scale with the magnitude of the currents;
- Mechanical perturbations (e.g., load variations, frictional irregularities) are proportional to the rotational speed of the motor.

Thus, the proposed stochastic model captures state-dependent uncertainties that are consistent with the underlying electromechanical behavior of the system.

3. Well-Posedness

To establish the mathematical well-posedness of the stochastic shunt DC motor model, we first prove the existence and uniqueness of a global solution. Since the drift and diffusion coefficients are locally Lipschitz continuous, for any initial value $X(0) = (i_a(0), i_f(0), \omega(0)) \in \mathbb{R}_+^3$, there exists a unique local solution $X(t) = (i_a(t), i_f(t), \omega(t))$ on the interval $[0, \tau_e)$, where τ_e denotes the explosion time. It remains to prove that $\tau_e = \infty$ almost surely.

Theorem 1. *For any initial value $(i_a(0), i_f(0), \omega(0)) \in \mathbb{R}_+^3$, model (4) admits a unique global solution $(i_a(t), i_f(t), \omega(t))$, almost surely. Moreover, model (4) has a unique nonnegative solution.*

Proof. It is important to note that the coefficients of the model are locally Lipschitz continuous. Therefore, for a given set of initial values of the state variables $(i_a(0), i_f(0), \omega(0)) \in \mathbb{R}_+^3$, there exists a unique local solution $(i_a(t), i_f(t), \omega(t))$ on $t \in [0, \tau_e)$, where τ_e denotes the explosion time (for details, see [18,19]). To prove that this solution is also global, it is sufficient to show that $\tau_e = \infty$ a.s. Let us choose a sufficiently large real number $k_0 \geq 0$ such that all initial conditions of the problem lie between $\frac{1}{k_0}$ and k_0 . Next, we define the stopping time τ_k for every positive integer k with $k_0 \leq k$ as

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : \min\{i_a(t), i_f(t), \omega(t)\} \leq \frac{1}{k} \text{ or } \max\{i_a(t), i_f(t), \omega(t)\} \geq k \right\}. \quad (5)$$

Here, we adopt the convention that $\inf \phi = \infty$ when ϕ is the empty set. From the definition of the stopping time, it follows that τ_k is increasing as $k \rightarrow \infty$. Let $\lim_{k \rightarrow \infty} \tau_k = \tau_\infty$, which implies that $\tau_\infty \leq \tau_e$ a.s. In other words, we need to show that $\tau_\infty = \infty$ a.s. If this assertion were false, then there would exist a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{T \geq \tau_\infty\} > \epsilon. \quad (6)$$

Hence, there exists an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{T \geq \tau_k\} \geq \epsilon, \text{ for all } k_1 \leq k.$$

Define the weighted quadratic Lyapunov function

$$V(i_a, i_f, \omega) = \frac{1}{2} (ai_a^2 + bi_f^2 + c\omega^2), \quad a, b, c > 0.$$

We choose the constants a and c such that

$$\frac{a}{L_a} = \frac{c}{J}.$$

Equivalently,

$$c = \frac{aJ}{L_a}.$$

Applying Itô's formula to V , we obtain

$$dV(i_a, i_f, \omega) = LV(i_a, i_f, \omega) dt + a\sigma_1 i_a^2 dB_1(t) + b\sigma_2 i_f^2 dB_2(t) + c\sigma_3 \omega^2 dB_3(t),$$

where

$$LV = ai_a \left(\frac{V_L}{L_a} - \frac{R_a i_a}{L_a} - \frac{L_{af} i_f \omega}{L_a} \right) + bi_f \left(\frac{V_L}{L_f} - \frac{R_f i_f}{L_f} \right) + c\omega \left(\frac{L_{af} i_a i_f}{J} - \frac{K_L \omega}{J} \right) + \frac{1}{2} a\sigma_1^2 i_a^2 + \frac{1}{2} b\sigma_2^2 i_f^2 + \frac{1}{2} c\sigma_3^2 \omega^2.$$

Expanding the right-hand side gives

$$LV = \frac{aV_L}{L_a} i_a - \frac{aR_a}{L_a} i_a^2 - \frac{aL_{af}}{L_a} i_a i_f \omega + \frac{bV_L}{L_f} i_f - \frac{bR_f}{L_f} i_f^2 + \frac{cL_{af}}{J} i_a i_f \omega - \frac{cK_L}{J} \omega^2 + \frac{1}{2} a\sigma_1^2 i_a^2 + \frac{1}{2} b\sigma_2^2 i_f^2 + \frac{1}{2} c\sigma_3^2 \omega^2.$$

Since

$$\frac{a}{L_a} = \frac{c}{J},$$

we have

$$-\frac{aL_{af}}{L_a} i_a i_f \omega + \frac{cL_{af}}{J} i_a i_f \omega = 0.$$

Therefore,

$$LV = \frac{aV_L}{L_a} i_a + \frac{bV_L}{L_f} i_f - \left(\frac{aR_a}{L_a} - \frac{1}{2} a\sigma_1^2 \right) i_a^2 - \left(\frac{bR_f}{L_f} - \frac{1}{2} b\sigma_2^2 \right) i_f^2 - \left(\frac{cK_L}{J} - \frac{1}{2} c\sigma_3^2 \right) \omega^2.$$

Assume that

$$\frac{R_a}{L_a} > \frac{\sigma_1^2}{2}, \quad \frac{R_f}{L_f} > \frac{\sigma_2^2}{2}, \quad \frac{K_L}{J} > \frac{\sigma_3^2}{2}.$$

Then there exist positive constants $\alpha_1, \alpha_2, \alpha_3 > 0$ such that

$$\alpha_1 = \frac{aR_a}{L_a} - \frac{1}{2} a\sigma_1^2, \quad \alpha_2 = \frac{bR_f}{L_f} - \frac{1}{2} b\sigma_2^2, \quad \alpha_3 = \frac{cK_L}{J} - \frac{1}{2} c\sigma_3^2.$$

Hence,

$$LV = \frac{aV_L}{L_a} i_a + \frac{bV_L}{L_f} i_f - \alpha_1 i_a^2 - \alpha_2 i_f^2 - \alpha_3 \omega^2.$$

Using Young's inequality, for any $\alpha_1, \alpha_2 > 0$,

$$\frac{aV_L}{L_a} i_a \leq \frac{\alpha_1}{2} i_a^2 + \frac{a^2 V_L^2}{2\alpha_1 L_a^2},$$

and

$$\frac{bV_L}{L_f} i_f \leq \frac{\alpha_2}{2} i_f^2 + \frac{b^2 V_L^2}{2\alpha_2 L_f^2}.$$

Therefore,

$$LV \leq -\frac{\alpha_1}{2}i_a^2 - \frac{\alpha_2}{2}i_f^2 - \alpha_3\omega^2 + \frac{a^2V_L^2}{2\alpha_1L_a^2} + \frac{b^2V_L^2}{2\alpha_2L_f^2}.$$

Let

$$C = \frac{a^2V_L^2}{2\alpha_1L_a^2} + \frac{b^2V_L^2}{2\alpha_2L_f^2}.$$

Since

$$V(i_a, i_f, \omega) = \frac{1}{2}(ai_a^2 + bi_f^2 + c\omega^2),$$

there exists a constant $\lambda > 0$ such that

$$\frac{\alpha_1}{2}i_a^2 + \frac{\alpha_2}{2}i_f^2 + \alpha_3\omega^2 \geq \lambda V(i_a, i_f, \omega).$$

Suppose

$$\lambda = \min\left\{\frac{\alpha_1}{a}, \frac{\alpha_2}{b}, \frac{2\alpha_3}{c}\right\}.$$

Consequently,

$$LV(i_a, i_f, \omega) \leq -\lambda V(i_a, i_f, \omega) + C.$$

Now define the stopping time $\tau_k = \inf\{t \in [0, \tau_e) : |X(t)| \geq k\}, k \geq 1$. So $\tau_k \rightarrow \tau_\infty$ if $k \rightarrow \infty$, and $\tau_\infty \leq \tau_e$ a.s. Suppose that the assertion is false. Then there exist constants $T > 0, \varepsilon \in (0, 1)$. Applying Dynkin's formula to the Lyapunov function V , we obtain

$$\mathbb{E}[V(X(T \wedge \tau_k))] = V(X(0)) + \mathbb{E} \int_0^{T \wedge \tau_k} LV(X(s)) ds.$$

Using the estimate

$$LV(X) \leq -\lambda V(X) + C,$$

it follows that

$$\mathbb{E}[V(X(T \wedge \tau_k))] \leq V(X(0)) + CT.$$

Let

$$\Omega_k = \{\tau_k \leq T\}.$$

On the event Ω_k , we have

$$|X(\tau_k)| \geq k.$$

Therefore,

$$V(X(\tau_k)) \geq \inf_{|x| \geq k} V(x).$$

Consequently,

$$\begin{aligned} V(X(0)) + CT &\geq \mathbb{E}[V(X(T \wedge \tau_k))] \\ &\geq \mathbb{E}[\mathbf{1}_{\Omega_k} V(X(\tau_k))] \\ &\geq \mathbb{P}(\Omega_k) \inf_{|x| \geq k} V(x) \\ &\geq \varepsilon \inf_{|x| \geq k} V(x). \end{aligned}$$

Since

$$V(x) = \frac{1}{2}(ai_a^2 + bi_f^2 + c\omega^2),$$

we have

$$\inf_{|x| \geq k} V(x) \longrightarrow \infty, \quad k \rightarrow \infty.$$

Letting $k \rightarrow \infty$ in the above inequality yields

$$V(X(0)) + CT \geq \infty,$$

which is impossible since

$$V(X(0)) + CT < \infty.$$

This contradiction shows that the assumption is false. Therefore,

$$\tau_\infty = \infty \quad \text{a.s.}$$

Hence, the local solution extends uniquely to a global solution for all $t \geq 0$. Thus, the solution of system (4) is non-explosive.

Next, we prove the positivity of the proposed model (4). For the armature current equation, we have

$$di_a(t) = \left(\frac{V_L}{L_a} - \frac{R_a}{L_a} i_a(t) - \frac{L_{af}}{L_a} i_f(t) \omega(t) \right) dt + \sigma_1 i_a(t) dB_1(t).$$

For $i_a(t) > 0$, applying Itô's formula to $\ln i_a(t)$ gives

$$d \ln i_a(t) = \left[\frac{V_L}{L_a i_a(t)} - \frac{R_a}{L_a} - \frac{L_{af} i_f(t) \omega(t)}{L_a i_a(t)} - \frac{\sigma_1^2}{2} \right] dt + \sigma_1 dB_1(t).$$

This shows that $\ln i_a(t)$ is well defined up to any stopping time before $i_a(t)$ reaches zero. Since the diffusion coefficient $\sigma_1 i_a(t)$ vanishes at the boundary $i_a = 0$ and the drift at the boundary satisfies

$$\left(\frac{V_L}{L_a} - \frac{R_a}{L_a} i_a - \frac{L_{af}}{L_a} i_f \omega \right) \Big|_{i_a=0} = \frac{V_L}{L_a} > 0$$

provided $V_L > 0$, the boundary $i_a = 0$ is repelling. Hence,

$$i_a(t) > 0, \quad t \geq 0,$$

almost surely whenever $i_a(0) > 0$.

Similarly, for the field current equation,

$$di_f(t) = \left(\frac{V_L}{L_f} - \frac{R_f}{L_f} i_f(t) \right) dt + \sigma_2 i_f(t) dB_2(t).$$

For $i_f(t) > 0$, applying Itô's formula to $\ln i_f(t)$ gives

$$d \ln i_f(t) = \left[\frac{V_L}{L_f i_f(t)} - \frac{R_f}{L_f} - \frac{\sigma_2^2}{2} \right] dt + \sigma_2 dB_2(t).$$

Again, the diffusion coefficient $\sigma_2 i_f(t)$ vanishes at the boundary $i_f = 0$, while

$$\left(\frac{V_L}{L_f} - \frac{R_f}{L_f} i_f \right) \Big|_{i_f=0} = \frac{V_L}{L_f} > 0$$

provided $V_L > 0$. Hence, the boundary $i_f = 0$ is also repelling, and therefore,

$$i_f(t) > 0, \quad t \geq 0,$$

Now we prove the nonnegativity of the angular velocity $\omega(t)$. The third equation of system (4) can be written as

$$d\omega(t) = \left[\frac{L_{af}}{J} i_a(t) i_f(t) - \frac{K_L}{J} \omega(t) \right] dt + \sigma_3 \omega(t) dB_3(t).$$

Let

$$q(t) = \frac{L_{af}}{J} i_a(t) i_f(t), \quad \mu = \frac{K_L}{J}.$$

Then

$$d\omega(t) = (q(t) - \mu\omega(t))dt + \sigma_3\omega(t)dB_3(t).$$

This is a linear stochastic differential equation. Define

$$\Phi(t) = \exp \left[- \left(\mu + \frac{\sigma_3^2}{2} \right) t + \sigma_3 B_3(t) \right].$$

According to the variation-of-constants formula, its solution is

$$\omega(t) = \Phi(t) \left[\omega(0) + \int_0^t \Phi^{-1}(s) q(s) ds \right].$$

Since $\Phi(t) > 0$ for all $t \geq 0$ almost surely, if

$$\omega(0) \geq 0, \quad q(t) = \frac{L_{af}}{J} i_a(t) i_f(t) \geq 0,$$

then

$$\omega(t) \geq 0, \quad t \geq 0,$$

almost surely. In particular, if

$$i_a(t) \geq 0, \quad i_f(t) \geq 0,$$

then $q(t) \geq 0$, and hence $\omega(t)$ remains nonnegative for all $t \geq 0$ almost surely.

Combining the local existence and uniqueness, the non-explosion result, and the positivity of $i_a(t)$, $\omega(t)$, and $i_f(t)$, we conclude that model (4) admits a unique global solution for all $t \geq 0$ almost surely, with

$$i_a(t) > 0, \quad i_f(t) > 0, \quad \omega(t) > 0, \quad t \geq 0,$$

almost surely. $\square$

4. Stochastic Ultimate Boundedness

For stochastic dynamical systems with multiplicative noise and constant external forcing, the notion of extinction, understood as almost sure convergence of all state variables to zero, is generally not appropriate. In the stochastic shunt DC motor model (4), the applied voltage V_L continuously drives the electrical subsystem, and therefore, the origin does not represent a meaningful operating state even in the absence of stochastic perturbations. Furthermore, the multiplicative noise terms cause the trajectories to fluctuate around the deterministic operating regime. Consequently, the relevant long-term behavior of system (4) is characterized by stochastic boundedness rather than extinction. In particular, the armature current, field current, and angular velocity are expected to remain within a bounded region of the state space and oscillate around a stable operating regime determined by the system parameters and external voltage source.

The following theorem establishes the stochastic ultimate boundedness of system (4).

Theorem 2. Assume that

$$\frac{R_a}{L_a} > \frac{\sigma_1^2}{2}, \quad \frac{R_f}{L_f} > \frac{\sigma_2^2}{2}, \quad \frac{K_L}{J} > \frac{\sigma_3^2}{2}.$$

Then the solution of system (4) is stochastically ultimately bounded. More precisely,

$$\limsup_{t \rightarrow \infty} \mathbb{E}[V(X_t)] \leq \frac{C}{\lambda},$$

where

$$V(X) = \frac{1}{2} (ai_a^2 + bi_f^2 + c\omega^2),$$

is the Lyapunov function introduced in Theorem 1, and $\lambda, C > 0$ are constants satisfying

$$\mathcal{L}V(X) \leq -\lambda V(X) + C.$$

Proof. From Theorem 1, the solution of system (4) exists globally for all $t \geq 0$ almost surely. Recall that the Lyapunov function

$$V(X) = \frac{1}{2} (ai_a^2 + bi_f^2 + c\omega^2),$$

satisfies

$$\mathcal{L}V(X) \leq -\lambda V(X) + C, \quad (7)$$

for some positive constants λ and C . Applying Itô's formula to $V(X_t)$ yields

$$dV(X_t) = \mathcal{L}V(X_t) dt + dM_t,$$

where

$$M_t = \int_0^t a\sigma_1 i_a^2(s) dB_1(s) + \int_0^t b\sigma_2 i_f^2(s) dB_2(s) + \int_0^t c\sigma_3 \omega^2(s) dB_3(s),$$

is a local martingale. Taking expectations on both sides gives

$$\mathbb{E}[V(X_t)] = V(X_0) + \mathbb{E} \int_0^t \mathcal{L}V(X_s) ds.$$

Using (7), we obtain

$$\mathbb{E}[V(X_t)] \leq V(X_0) + \int_0^t (-\lambda \mathbb{E}[V(X_s)] + C) ds.$$

Define

$$y(t) = \mathbb{E}[V(X_t)].$$

Then

$$y(t) \leq y(0) + \int_0^t (-\lambda y(s) + C) ds.$$

According to Grönwall's inequality,

$$y(t) \leq y(0)e^{-\lambda t} + \frac{C}{\lambda} (1 - e^{-\lambda t}).$$

Consequently,

$$\mathbb{E}[V(X_t)] \leq V(X_0)e^{-\lambda t} + \frac{C}{\lambda} (1 - e^{-\lambda t}).$$

Taking the upper limit as $t \rightarrow \infty$, we obtain

$$\limsup_{t \rightarrow \infty} \mathbb{E}[V(X_t)] \leq \frac{C}{\lambda}.$$

Therefore, the second moment of the solution remains uniformly bounded for sufficiently large time. Hence, the armature current, field current, and angular velocity remain bounded in mean square and fluctuate around a stable operating regime rather than converging to zero. This completes the proof. $\square$

5. Existence and Uniqueness of a Stationary Distribution

In stochastic dynamical systems subject to continuous external excitation and random perturbations, the long-term dynamics are generally characterized by convergence in distribution rather than convergence to a deterministic equilibrium state. For the stochastic shunt DC motor model (4), the applied voltage V_L continuously supplies energy to the system, while multiplicative noise prevents the state variables from approaching a fixed operating point. Therefore, it is natural to study invariant (stationary) probability measures that describe the asymptotic probabilistic behavior of the motor currents and rotor speed.

Lemma 1. *Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^d$ with diffusion matrix $A(x)$. Assume that there exists a bounded domain $U \subset \mathbb{R}^d$ such that*

1. *The smallest eigenvalue of $A(x)$ is bounded away from zero in U ;*
2. *For every $x \in \mathbb{R}^d \setminus U$, the expected hitting time*

$$\tau_U = \inf\{t > 0 : X(t) \in U\}$$

is finite and locally bounded.

Then $X(t)$ admits a stationary distribution. Moreover, if the process is irreducible, the stationary distribution is unique and ergodic.

Theorem 3. *Assume that the conditions of Lemma 1 are satisfied. Then system (4) admits a stationary distribution $\pi(\cdot)$. Furthermore, if the associated Markov process*

$$X(t) = (i_a(t), i_f(t), \omega(t))^T$$

is irreducible, then the stationary distribution is unique and any solution of system (4) possesses the ergodic property.

Proof. The main objective of this theorem is to verify the second condition of Lemma 1. To this end, we first construct a positive $\mathcal{C}^2$ -function $\mathcal{H}_1 : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ defined by

$$\mathcal{H}_1 = i_a + i_f + \omega - a_1 \ln i_a - a_2 \ln i_f - a_3 \ln \omega,$$

where $a_1, a_2, a_3 > 0$ are constants to be determined later. Applying Itô's formula to system (4), we obtain

$$\begin{aligned}
\mathcal{L}(i_a + i_f + \omega) &= \frac{V_L - R_a i_a - L_a i_f \omega}{L_a} + \frac{V_L - R_f i_f}{L_f} + \frac{L_a i_a i_f - K_L \omega}{J}, \\
\mathcal{L}(-\ln i_a) &= -\frac{V_L - R_a i_a - L_a i_f \omega}{L_a i_a} + \frac{\sigma_1^2}{2}, \\
\mathcal{L}(-\ln i_f) &= -\frac{V_L - R_f i_f}{L_f i_f} + \frac{\sigma_2^2}{2}, \\
\mathcal{L}(-\ln \omega) &= -\frac{L_a i_a i_f - K_L \omega}{J \omega} + \frac{\sigma_3^2}{2}.
\end{aligned} \tag{8}$$

Consequently,

$$\mathcal{L}\mathcal{H}_1 = \mathcal{L}(i_a + i_f + \omega) + a_1 \mathcal{L}(-\ln i_a) + a_2 \mathcal{L}(-\ln i_f) + a_3 \mathcal{L}(-\ln \omega). \tag{9}$$

By choosing suitable positive constants a_1, a_2 and a_3 , it follows that there exist positive constants c_1 and c_2 such that

$$\mathcal{L}\mathcal{H}_1 \leq -c_1 + c_2, \quad (i_a, i_f, \omega) \in \mathbb{R}_+^3. \tag{10}$$

$$\begin{aligned}
\mathcal{L}\mathcal{H}_1 &= \frac{V_L - R_a i_a - L_a i_f \omega}{L_a} + \frac{V_L - R_f i_f}{L_f} + \frac{L_a i_a i_f - K_L \omega}{J} \\
&+ a_1 \left(-\frac{V_L - R_a i_a - L_a i_f \omega}{L_a i_a} + \frac{\sigma_1^2}{2} \right) + a_2 \left(-\frac{V_L - R_f i_f}{L_f i_f} + \frac{\sigma_2^2}{2} \right) \\
&+ a_3 \left(-\frac{L_a i_a i_f - K_L \omega}{J \omega} + \frac{\sigma_3^2}{2} \right).
\end{aligned}$$

Expanding and rearranging the above expression, we obtain

$$\begin{aligned}
\mathcal{L}\mathcal{H}_1 &= -\frac{a_1 V_L}{L_a i_a} - \frac{a_2 V_L}{L_f i_f} - \frac{a_3 L_a i_a i_f}{J \omega} - \frac{R_a}{L_a} i_a - \frac{R_f}{L_f} i_f - \frac{K_L}{J} \omega \\
&- \frac{L_a i_f \omega}{L_a} + \frac{L_a i_a i_f}{J} + \frac{a_1 L_a i_f \omega}{L_a i_a} + \frac{V_L}{L_a} + \frac{V_L}{L_f} + \frac{a_1 R_a}{L_a} + \frac{a_2 R_f}{L_f} + \frac{a_3 K_L}{J} \\
&+ \frac{a_1 \sigma_1^2}{2} + \frac{a_2 \sigma_2^2}{2} + \frac{a_3 \sigma_3^2}{2}.
\end{aligned}$$

Consequently, from the previous estimate, there exists a positive constant C_1 such that

$$\mathcal{L}\mathcal{H}_1 \leq -\frac{a_1 V_L}{L_a i_a} - \frac{a_2 V_L}{L_f i_f} - \frac{a_3 L_a i_a i_f}{J \omega} - \frac{R_a}{L_a} i_a - \frac{R_f}{L_f} i_f - \frac{K_L}{J} \omega + C_1,$$

where

$$C_1 = \frac{V_L}{L_a} + \frac{V_L}{L_f} + \frac{a_1 R_a}{L_a} + \frac{a_2 R_f}{L_f} + \frac{a_3 K_L}{J} + \frac{a_1 \sigma_1^2}{2} + \frac{a_2 \sigma_2^2}{2} + \frac{a_3 \sigma_3^2}{2}.$$

Thus,

$$\mathcal{L}\mathcal{H}_1 \leq -C_2 + C_1,$$

where

$$C_2 = \frac{a_1 V_L}{L_a i_a} + \frac{a_2 V_L}{L_f i_f} + \frac{a_3 L_a i_a i_f}{J \omega} + \frac{R_a}{L_a} i_a + \frac{R_f}{L_f} i_f + \frac{K_L}{J} \omega.$$

Furthermore, we define another positive $\mathcal{C}^2$ -function $\mathcal{H}_2 : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ by

$$\mathcal{H}_2 = a_4 \mathcal{H}_1 + i_a + i_f + \omega,$$

where $a_4 > 0$ is a constant to be determined later. Therefore,

$$\mathcal{H}_2 = (a_4 + 1)(i_a + i_f + \omega) - a_4 a_1 \ln i_a - a_4 a_2 \ln i_f - a_4 a_3 \ln \omega.$$

A few direct calculations yield

$$\liminf_{(i_a, i_f, \omega) \in \mathbb{R}_+^3 \setminus U_k} \mathcal{H}_2(i_a, i_f, \omega) = +\infty, \quad k \rightarrow \infty, \quad (11)$$

where

$$U_k = \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right) \times \left(\frac{1}{k}, k\right).$$

In parallel, we show that the function $\mathcal{H}_2$ admits a unique minimum in $\mathbb{R}_+^3$. By considering the partial derivatives of $\mathcal{H}_2$ with respect to the state variables, we obtain

$$\begin{aligned} \frac{\partial \mathcal{H}_2}{\partial i_a} &= (a_4 + 1) - \frac{a_4 a_1}{i_a}, \\ \frac{\partial \mathcal{H}_2}{\partial i_f} &= (a_4 + 1) - \frac{a_4 a_2}{i_f}, \\ \frac{\partial \mathcal{H}_2}{\partial \omega} &= (a_4 + 1) - \frac{a_4 a_3}{\omega}. \end{aligned}$$

Thus, the unique critical point is given by

$$(i_a^*, i_f^*, \omega^*) = \left(\frac{a_4 a_1}{a_4 + 1}, \frac{a_4 a_2}{a_4 + 1}, \frac{a_4 a_3}{a_4 + 1} \right).$$

Moreover, the Hessian matrix corresponding to $\mathcal{H}_2$ is

$$D^2 \mathcal{H}_2 = \begin{pmatrix} \frac{a_4 a_1}{i_a^2} & 0 & 0 \\ 0 & \frac{a_4 a_2}{i_f^2} & 0 \\ 0 & 0 & \frac{a_4 a_3}{\omega^2} \end{pmatrix}.$$

Since $a_1, a_2, a_3, a_4 > 0$, the Hessian matrix is positive definite on $\mathbb{R}_+^3$. Hence, $\mathcal{H}_2$ attains its unique minimum at (i_a^*, i_f^*, ω^*) .

It can be easily verified that the function $\mathcal{H}_2$ has a unique critical point given by

$$(i_a^*, i_f^*, \omega^*) = \left(\frac{a_4 a_1}{1 + a_4}, \frac{a_4 a_2}{1 + a_4}, \frac{a_4 a_3}{1 + a_4} \right).$$

Furthermore, corresponding to the function $\mathcal{H}_2$, the Hessian matrix takes the form

$$\mathbf{B} = \begin{bmatrix} \frac{a_4 a_1}{i_a^2} & 0 & 0 \\ 0 & \frac{a_4 a_2}{i_f^2} & 0 \\ 0 & 0 & \frac{a_4 a_3}{\omega^2} \end{bmatrix}.$$

Since $a_1, a_2, a_3, a_4 > 0$, the matrix $\mathbf{B}$ is positive definite on $\mathbb{R}_+^3$. Consequently, the function $\mathcal{H}_2$ attains its unique minimum in $\mathbb{R}_+^3$. Referring to relation (11) and using the continuity of $\mathcal{H}_2$, we conclude that $\mathcal{H}_2$ is a proper Lyapunov function on the positive state space.

To move forward, define another positive $\mathcal{C}^2$ -function

$$\mathcal{H} : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$$

as follows:

$$\mathcal{H}(i_a(t), i_f(t), \omega(t)) = \mathcal{H}_2(i_a(t), i_f(t), \omega(t)) - \mathcal{H}_2(i_a^*, i_f^*, \omega^*).$$

With the help of system (4) and an application of Itô's formula, we obtain

$$\begin{aligned} \mathcal{L}\mathcal{H} &= \mathcal{L}\mathcal{H}_2 \\ &= a_4 \mathcal{L}\mathcal{H}_1 + \mathcal{L}(i_a + i_f + \omega). \end{aligned} \quad (12)$$

Using the previous estimate for $\mathcal{L}\mathcal{H}_1$, we have

$$\begin{aligned} \mathcal{L}\mathcal{H} &\leq -a_4 \left(\frac{a_1 V_L}{L_a i_a} + \frac{a_2 V_L}{L_f i_f} + \frac{a_3 L_{af} i_a i_f}{J \omega} + \frac{R_a}{L_a} i_a + \frac{R_f}{L_f} i_f + \frac{K_L}{J} \omega \right) \\ &\quad + a_4 C_1 + \frac{V_L - R_a i_a - L_{af} i_f \omega}{L_a} + \frac{V_L - R_f i_f}{L_f} + \frac{L_{af} i_a i_f - K_L \omega}{J}. \end{aligned} \quad (13)$$

Equivalently,

$$\begin{aligned} \mathcal{L}\mathcal{H} &\leq -\frac{a_4 a_1 V_L}{L_a i_a} - \frac{a_4 a_2 V_L}{L_f i_f} - \frac{a_4 a_3 L_{af} i_a i_f}{J \omega} \\ &\quad - \left(\frac{a_4 R_a}{L_a} + \frac{R_a}{L_a} \right) i_a - \left(\frac{a_4 R_f}{L_f} + \frac{R_f}{L_f} \right) i_f - \left(\frac{a_4 K_L}{J} + \frac{K_L}{J} \right) \omega \\ &\quad - \frac{L_{af}}{L_a} i_f \omega + \frac{L_{af}}{J} i_a i_f + a_4 C_1 + \frac{V_L}{L_a} + \frac{V_L}{L_f}. \end{aligned}$$

The subsequent step is to define a bounded domain of the form

$$\mathcal{D} = \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : \epsilon_1 < i_a < \frac{1}{\epsilon_4}, \quad \epsilon_2 < i_f < \frac{1}{\epsilon_5}, \quad \epsilon_3 < \omega < \frac{1}{\epsilon_6} \right\},$$

where $\epsilon_i > 0$ ($i = 1, 2, \dots, 6$) are sufficiently small constants to be determined later. To deal with the whole domain, we split $\mathbb{R}_+^3 \setminus \mathcal{D}$ into the following subdomains:

$$\begin{aligned} \mathcal{D}_1 &= \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : 0 < i_a \leq \epsilon_1 \right\}, \\ \mathcal{D}_2 &= \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : 0 < i_f \leq \epsilon_2 \right\}, \\ \mathcal{D}_3 &= \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : 0 < \omega \leq \epsilon_3 \right\}, \\ \mathcal{D}_4 &= \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : i_a \geq \frac{1}{\epsilon_4} \right\}, \\ \mathcal{D}_5 &= \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : i_f \geq \frac{1}{\epsilon_5} \right\}, \\ \mathcal{D}_6 &= \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : \omega \geq \frac{1}{\epsilon_6} \right\}. \end{aligned}$$

In the following, our main task is to prove that $\mathcal{LH} < 0$ on the whole domain $\mathbb{R}_+^3 \setminus \mathcal{D}$. To this end, define

$$\mathcal{H}(i_a, i_f, \omega) = \mathcal{H}_2(i_a, i_f, \omega) + qV(i_a, i_f, \omega),$$

where $q > 0$ is sufficiently large and

$$V(i_a, i_f, \omega) = \frac{1}{2} (ai_a^2 + bi_f^2 + c\omega^2).$$

From the stochastic ultimate boundedness result, we have

$$\mathcal{L}V \leq -\lambda V + C,$$

where $\lambda > 0$ and $C > 0$. Therefore,

$$\begin{aligned} \mathcal{LH} &= \mathcal{LH}_2 + q\mathcal{L}V \\ &\leq \mathcal{LH}_2 - q\lambda V + qC. \end{aligned} \quad (14)$$

Let

$$\mathcal{D} = \left\{ (i_a, i_f, \omega) \in \mathbb{R}_+^3 : \epsilon < i_a < R, \quad \epsilon < i_f < R, \quad \epsilon < \omega < R \right\},$$

where $0 < \epsilon < 1$ is sufficiently small and $R > 1$ is sufficiently large. Then $\mathbb{R}_+^3 \setminus \mathcal{D}$ can be treated through the following two cases.

Case 1. Assume that

$$(i_a, i_f, \omega) \in \mathbb{R}_+^3 \setminus \mathcal{D}, \quad \min\{i_a, i_f, \omega\} \leq \epsilon.$$

Using the logarithmic terms in $\mathcal{H}_2$ and the estimate obtained for $\mathcal{LH}_1$, we have

$$\mathcal{LH} \leq -\frac{a_4 a_1 V_L}{L_a i_a} - \frac{a_4 a_2 V_L}{L_f i_f} - \frac{a_4 a_3 L_a f i_a i_f}{J \omega} + K_1,$$

where $K_1 > 0$ is a constant independent of ϵ . Hence, by choosing $\epsilon > 0$ sufficiently small, we obtain

$$\mathcal{LH} \leq -\mathcal{K}, \quad \text{for all } (i_a, i_f, \omega) \in \mathbb{R}_+^3 \setminus \mathcal{D} \text{ with } \min\{i_a, i_f, \omega\} \leq \epsilon,$$

where $\mathcal{K} > 0$.

Case 2. Assume that

$$(i_a, i_f, \omega) \in \mathbb{R}_+^3 \setminus \mathcal{D}, \quad \max\{i_a, i_f, \omega\} \geq R.$$

From relation (14), and since

$$V(i_a, i_f, \omega) \rightarrow \infty \quad \text{as} \quad |(i_a, i_f, \omega)| \rightarrow \infty,$$

we can choose sufficiently large $R > 1$ and sufficiently large $q > 0$ such that

$$\mathcal{LH} \leq -\mathcal{K}, \quad \text{for all } (i_a, i_f, \omega) \in \mathbb{R}_+^3 \setminus \mathcal{D} \text{ with } \max\{i_a, i_f, \omega\} \geq R.$$

Similarly, the result can be established for ω .

Combining Case 1 and Case 2, we conclude that

$$\mathcal{LH}(i_a, i_f, \omega) < -\mathcal{K} < 0, \quad \text{for all } (i_a, i_f, \omega) \in \mathbb{R}_+^3 \setminus \mathcal{D}.$$

Let $x = (x_1, x_2, x_3) \in \mathbb{R}_+^3 \setminus \mathcal{D}$ and define

$$\tau^x = \inf\{t \geq 0 : X(t) \in \mathcal{D}\}, \quad \tau_n = \inf\{t \geq 0 : |X(t)| \geq n\},$$

and

$$\tau^{(n)}(t) = \tau^x \wedge t \wedge \tau_n.$$

Applying Dynkin's formula to $\mathcal{H}$ over the interval $[0, \tau^{(n)}(t)]$, we obtain

$$\begin{aligned} & \mathbb{E}\mathcal{H}(X(\tau^{(n)}(t))) - \mathcal{H}(x) \\ &= \mathbb{E} \int_0^{\tau^{(n)}(t)} \mathcal{LH}(X(s)) ds \\ &\leq -\mathcal{K} \mathbb{E}\tau^{(n)}(t). \end{aligned} \tag{15}$$

Since $\mathcal{H}$ is non-negative, it follows that

$$\mathbb{E}\tau^{(n)}(t) \leq \frac{\mathcal{H}(x)}{\mathcal{K}}.$$

Letting $n \rightarrow \infty$ and then $t \rightarrow \infty$, Fatou's lemma yields

$$\mathbb{E}\tau^x \leq \frac{\mathcal{H}(x)}{\mathcal{K}} < \infty.$$

Therefore, for every compact set $K \subset \mathbb{R}_+^3$,

$$\sup_{x \in K} \mathbb{E}\tau^x < \infty.$$

This verifies condition 2 of Lemma 1.

Moreover, the diffusion matrix corresponding to system (4) is

$$\mathbf{A}(i_a, i_f, \omega) = \begin{bmatrix} \sigma_1^2 i_a^2 & 0 & 0 \\ 0 & \sigma_2^2 i_f^2 & 0 \\ 0 & 0 & \sigma_3^2 \omega^2 \end{bmatrix}.$$

For any $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$ and $(i_a, i_f, \omega) \in \overline{\mathcal{D}}$, we have

$$\begin{aligned} \sum_{i,j=1}^3 a_{ij}(i_a, i_f, \omega) \zeta_i \zeta_j &= \sigma_1^2 i_a^2 \zeta_1^2 + \sigma_2^2 i_f^2 \zeta_2^2 + \sigma_3^2 \omega^2 \zeta_3^2 \\ &\geq M |\zeta|^2, \end{aligned}$$

where

$$M = \min\{\sigma_1^2 \epsilon^2, \sigma_2^2 \epsilon^2, \sigma_3^2 \epsilon^2\} > 0.$$

Thus, condition 1 of Lemma 1 is verified. Therefore, according to Lemma 1, system (4) admits a unique stationary distribution $\pi(\cdot)$ and possesses the ergodic property. $\square$

6. Modified Fractional Operation

Fractional calculus, characterized by its fractional-order framework that integrates differentiation and integration, provides an advanced approach for addressing complex real-world challenges [20,23–25]. The concept of fractional derivatives, originally introduced by Riemann–Liouville through the power law, has recently been revisited and critically examined by researchers.

Experimental evidence indicates that relying on a single operator, including Caputo, Caputo–Fabrizio, or Atangana–Baleanu operators, may not adequately represent the intricate phenomena observed in science and engineering [26–28]. In [29], a model was developed using the newly introduced Constant Proportional Caputo operator to represent generalized memory effects.

In the meantime, the development of a new recursive algorithm to solve particular initial value problems (IVPs) of fractional differential equations is reported in [30]. Various analytical and semi-analytical methods, such as the Adomian Decomposition Method (ADM), have been widely employed to investigate fractional-order dynamical systems. These techniques provide efficient approximate solutions for complex nonlinear models and have demonstrated considerable potential in the analysis of electromechanical systems, including fractional-order DC motor models [31–33].

The modified Atangana–Baleanu–Caputo (mABC) derivative, which incorporates this kernel, has demonstrated improved performance over standard fractional operators, particularly due to its ability to handle integrable singularities at the origin. In a related development [34], a finite difference scheme was proposed for solving fractional differential equations involving the mABC derivative. Furthermore, the combination of the mABC operator with the Mittag–Leffler kernel provides a flexible framework for modeling systems with non-local memory effects. The definition of modified Atangana–Baleanu–Caputo (mABC) derivative

$${}^{MABC}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \left[f(t) - E_\alpha \left(-\frac{\alpha}{1-\alpha} t^\alpha \right) f(0) - \frac{\alpha}{1-\alpha} \int_0^t E_{\alpha,\alpha} \left(-\frac{\alpha}{1-\alpha} (t-\tau)^\alpha \right) f(\tau) d\tau \right]. \quad (16)$$

Such characteristics are particularly relevant in electromechanical systems, where memory effects arise due to magnetic hysteresis, energy dissipation, and transient electromagnetic interactions. Therefore, fractional-order modeling provides a more realistic representation of the dynamic behavior of shunt DC motors under non-ideal operating conditions.

This study investigates a nonlinear dynamic model of a shunt DC motor, incorporating the coupling between electrical and mechanical subsystems through electromagnetic torque. In contrast to the classical integer-order model System (1), the proposed formulation employs the modified Atangana–Baleanu fractional operator to capture these memory effects.

To extend the conventional model into the fractional-order domain, we reformulate System (1) using the modified Atangana–Baleanu–Caputo (mABC) derivative as follows:

$$\begin{aligned} {}^{mABC}D_t^\zeta i_a &= \frac{1}{L_a} (V_L - R_a i_a - L_{af} i_f \omega), \\ {}^{mABC}D_t^\zeta i_f &= \frac{1}{L_f} (V_L - R_f i_f), \\ {}^{mABC}D_t^\zeta \omega &= \frac{1}{J} (L_{af} i_a i_f - K_L \omega), \end{aligned} \quad (17)$$

subject to the initial conditions

$$i_a(0) = 0, \quad i_f(0) = 0, \quad \omega(0) = 0.$$

6.1. Analysis of the Model

To establish the presence of a meaningful solution to the homogeneous fractional initial value problem for the proposed motor dynamics model, the subsequent expressions will be employed:

$$\mathcal{L}[\mathbf{F}_\varsigma(\hbar t^\varsigma)] = \frac{s^{\varsigma-1}}{s^\varsigma - \hbar} \left| \frac{\hbar}{s^\varsigma} \right| < 1, \quad (18)$$

$$\mathcal{L}[t^{\varsigma-1} \mathbf{F}_{\varsigma,\varsigma}(\hbar t^\varsigma)] = \frac{1}{s^\varsigma - \hbar} \left| \frac{\hbar}{s^\varsigma} \right| < 1. \quad (19)$$

Lemma 2. [33] Consider the following fractional initial value problem:

$$\begin{aligned} {}^{\text{mABC}}\mathbf{D}_0^\varsigma i_a(t) &= \mathcal{U} i_a, \quad t > 0, i_a(0) = i_{a0}, \\ {}^{\text{mABC}}\mathbf{D}_0^\varsigma i_f(t) &= \mathcal{U} i_f, \quad t > 0, i_f(0) = i_{f0}, \\ {}^{\text{mABC}}\mathbf{D}_0^\varsigma \omega(t) &= \mathcal{U} \omega, \quad t > 0, \omega(0) = \omega_0. \end{aligned} \quad (20)$$

(1). For $\mathcal{U} = \frac{\mathbf{N}(\varsigma)}{1-\varsigma}$, the solution is:

$$\begin{aligned} i_a(t) &= i_{a0} \begin{cases} -\frac{t^{-\varsigma}}{\mu_\varsigma \Gamma(1-\varsigma)}, & t \neq 0, \\ 1, & t = 0 \end{cases}, \\ i_f(t) &= i_{f0} \begin{cases} -\frac{t^{-\varsigma}}{\mu_\varsigma \Gamma(1-\varsigma)}, & t \neq 0, \\ 1, & t = 0 \end{cases}, \\ \omega(t) &= \omega_0 \begin{cases} -\frac{t^{-\varsigma}}{\mu_\varsigma \Gamma(1-\varsigma)}, & t \neq 0, \\ 1, & t = 0 \end{cases}. \end{aligned}$$

(2). For $\mathcal{U} \neq \frac{\mathbf{N}(\varsigma)}{1-\varsigma}$, the solution is:

$$\begin{aligned} i_a(t) &= i_{a0} \cdot \frac{\mathbf{F}_\varsigma\left(\mu_\varsigma \cdot \frac{\tau_\varsigma}{1-\tau_\varsigma} t^\varsigma\right)}{1 - \tau_\varsigma}, \\ i_f(t) &= i_{f0} \cdot \frac{\mathbf{F}_\varsigma\left(\mu_\varsigma \cdot \frac{\tau_\varsigma}{1-\tau_\varsigma} t^\varsigma\right)}{1 - \tau_\varsigma}, \\ \omega(t) &= \omega_0 \cdot \frac{\mathbf{F}_\varsigma\left(\mu_\varsigma \cdot \frac{\tau_\varsigma}{1-\tau_\varsigma} t^\varsigma\right)}{1 - \tau_\varsigma}, \end{aligned}$$

where $\tau_\varsigma = \frac{\mathcal{U}(1-\varsigma)}{\mathbf{N}(\varsigma)}$.

Proof. (1) Given that

$$\int_0^t (t-y)^{\varsigma-1} \mathbf{F}_{\varsigma,\varsigma}(-\mu_\varsigma(1-y)^\varsigma) y^{-\varsigma} dy = \Gamma(1-\varsigma) \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma), \quad (21)$$

for $0 < t$, we have

$$\begin{aligned}
{}^{\text{mABC}}\mathbf{D}_0^\varsigma i_a(t) &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} i_a(t) - \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) i_{a0} - \mu_\varsigma \int_0^t (t-y)^{\varsigma-1} \mathbf{F}_{\varsigma,\varsigma}(-\mu_\varsigma(t-y)^\varsigma) \\
&\quad \times \left(-\frac{i_{a0}}{\mu_\varsigma \Gamma(1-\mathbf{F}_{\varsigma,\varsigma})} y^{-\varsigma} \right) dy, \\
{}^{\text{mABC}}\mathbf{D}_0^\varsigma i_f(t) &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} i_f(t) - \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) i_{f0} - \mu_\varsigma \int_0^t (t-y)^{\varsigma-1} \mathbf{F}_{\varsigma,\varsigma}(-\mu_\varsigma(t-y)^\varsigma) \\
&\quad \times \left(-\frac{i_{f0}}{\mu_\varsigma \Gamma(1-\mathbf{F}_{\varsigma,\varsigma})} y^{-\varsigma} \right) dy, \\
{}^{\text{mABC}}\mathbf{D}_0^\varsigma \omega(t) &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \omega(t) - \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) \omega_0 - \mu_\varsigma \int_0^t (t-y)^{\varsigma-1} \mathbf{F}_{\varsigma,\varsigma}(-\mu_\varsigma(t-y)^\varsigma) \\
&\quad \times \left(-\frac{\omega_0}{\mu_\varsigma \Gamma(1-\mathbf{F}_{\varsigma,\varsigma})} y^{-\varsigma} \right) dy,
\end{aligned} \tag{22}$$

$$\begin{aligned}
{}^{\text{mABC}}\mathbf{D}_0^\varsigma i_a(t) &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} (i_a(t) - \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) i_{a0} + \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) i_{a0}) = \mathcal{U} i_a(t), \\
{}^{\text{mABC}}\mathbf{D}_0^\varsigma i_f(t) &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} (i_f(t) - \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) i_{f0} + \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) i_{f0}) = \mathcal{U} i_f(t), \\
{}^{\text{mABC}}\mathbf{D}_0^\varsigma \omega(t) &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} (\omega(t) - \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) \omega_0 + \mathbf{F}_\varsigma(-\mu_\varsigma t^\varsigma) \omega_0) = \mathcal{U} \omega(t),
\end{aligned} \tag{23}$$

which completes the proof.

Using Equations (18) and (19) for $t > 0$,

$$\begin{aligned}
\mathcal{L}[{}^{\text{mABC}}\mathbf{D}_0^\varsigma i_a; s] &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \frac{1}{s^\varsigma + \mu_\varsigma} \left(\frac{i_{a0} s^\varsigma}{1-\tau_\varsigma} \times \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}} - i_{a0} s^{\varsigma-1} \right), \\
\mathcal{L}[{}^{\text{mABC}}\mathbf{D}_0^\varsigma i_f; s] &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \frac{1}{s^\varsigma + \mu_\varsigma} \left(\frac{i_{f0} s^\varsigma}{1-\tau_\varsigma} \times \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}} - i_{f0} s^{\varsigma-1} \right), \\
\mathcal{L}[{}^{\text{mABC}}\mathbf{D}_0^\varsigma \omega; s] &= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \frac{1}{s^\varsigma + \mu_\varsigma} \left(\frac{\omega_0 s^\varsigma}{1-\tau_\varsigma} \times \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}} - \omega_0 s^{\varsigma-1} \right).
\end{aligned}$$

Simplifying these equations yields

$$\begin{aligned}
&= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} i_{a0} \frac{\tau_\varsigma}{1-\tau_\varsigma} \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}} = \mathcal{U} \frac{i_{a0}}{1-\tau_\varsigma} \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}}, \\
&= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} i_{f0} \frac{\tau_\varsigma}{1-\tau_\varsigma} \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}} = \mathcal{U} \frac{i_{f0}}{1-\tau_\varsigma} \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}}, \\
&= \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \omega_0 \frac{\tau_\varsigma}{1-\tau_\varsigma} \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}} = \mathcal{U} \frac{\omega_0}{1-\tau_\varsigma} \frac{s^{\varsigma-1}}{s^\varsigma - \mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&= \mathcal{U} \frac{i_{a0}}{1-\tau_\varsigma} \mathcal{L} \left[\mathbf{F}_\varsigma \left(\mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma} t^\varsigma \right) \right], \\
&= \mathcal{U} \frac{i_{f0}}{1-\tau_\varsigma} \mathcal{L} \left[\mathbf{F}_\varsigma \left(\mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma} t^\varsigma \right) \right], \\
&= \mathcal{U} \frac{\omega_0}{1-\tau_\varsigma} \mathcal{L} \left[\mathbf{F}_\varsigma \left(\mu_\varsigma \frac{\tau_\varsigma}{1-\tau_\varsigma} t^\varsigma \right) \right].
\end{aligned}$$

Thus, the proof is complete. $\square$

Lemma 3. [33] Consider the non-homogeneous fractional system:

$$\begin{aligned} {}^{\text{mABC}}\mathbf{D}_0^\varsigma i_a(t) + \mathfrak{U}i_a &= \mathcal{X}_1(t), \quad i_a(0) = i_{a0}, \\ {}^{\text{mABC}}\mathbf{D}_0^\varsigma i_f(t) + \mathfrak{U}i_f &= \mathcal{X}_2(t), \quad i_f(0) = i_{f0}, \\ {}^{\text{mABC}}\mathbf{D}_0^\varsigma \omega(t) + \mathfrak{U}\omega &= \mathcal{X}_3(t), \quad \omega(0) = \omega_0. \end{aligned}$$

where $0 < \varsigma < 1$ and $\mathfrak{U} \neq -\frac{\mathbf{N}(\varsigma)}{1-\varsigma}$. The solution of the above initial value system is given by

$$\begin{aligned} i_a(t) &= \begin{cases} \tilde{i}_a, & t \neq 0, \\ i_{a0}, & t = 0, \end{cases} \\ i_f(t) &= \begin{cases} \tilde{i}_f, & t \neq 0, \\ i_{f0}, & t = 0, \end{cases} \\ \omega(t) &= \begin{cases} \tilde{\omega}, & t \neq 0, \\ \omega_0, & t = 0. \end{cases} \end{aligned} \quad (24)$$

Then, the solution is given by:

$$\begin{aligned} i_a(t) &= \tilde{i}_a = i_{a0} \frac{\mathbf{N}(\varsigma)}{z_\varsigma} \mathbf{F}_\varsigma \left(-\frac{\mathfrak{U}\varsigma}{z_\varsigma} t^\varsigma \right) + \frac{1-\varsigma}{z_\varsigma} \mathcal{X}_1(t) + \frac{1-\varsigma}{z_\varsigma} \left(\mu_\varsigma - \frac{\mathfrak{U}\varsigma}{z_\varsigma} \right) t^{\varsigma-1} \mathbf{F}_{\varsigma, \varsigma} \left(-\frac{\mathfrak{U}\varsigma}{z_\varsigma} t^\varsigma \right) \mathcal{X}_1(0), \\ i_f(t) &= \tilde{i}_f = i_{f0} \frac{\mathbf{N}(\varsigma)}{z_\varsigma} \mathbf{F}_\varsigma \left(-\frac{\mathfrak{U}\varsigma}{z_\varsigma} t^\varsigma \right) + \frac{1-\varsigma}{z_\varsigma} \mathcal{X}_2(t) + \frac{1-\varsigma}{z_\varsigma} \left(\mu_\varsigma - \frac{\mathfrak{U}\varsigma}{z_\varsigma} \right) t^{\varsigma-1} \mathbf{F}_{\varsigma, \varsigma} \left(-\frac{\mathfrak{U}\varsigma}{z_\varsigma} t^\varsigma \right) \mathcal{X}_2(0), \\ \omega(t) &= \tilde{\omega} = \omega_0 \frac{\mathbf{N}(\varsigma)}{z_\varsigma} \mathbf{F}_\varsigma \left(-\frac{\mathfrak{U}\varsigma}{z_\varsigma} t^\varsigma \right) + \frac{1-\varsigma}{z_\varsigma} \mathcal{X}_3(t) + \frac{1-\varsigma}{z_\varsigma} \left(\mu_\varsigma - \frac{\mathfrak{U}\varsigma}{z_\varsigma} \right) t^{\varsigma-1} \mathbf{F}_{\varsigma, \varsigma} \left(-\frac{\mathfrak{U}\varsigma}{z_\varsigma} t^\varsigma \right) \mathcal{X}_3(0), \end{aligned}$$

where $z_\varsigma = \mathbf{N}(\varsigma) + \mathfrak{U}(1-\varsigma)$.

Proof. By using Equations (18) and (19), the following can be verified for the system variables i_a, i_f, ω :

$$\begin{cases} \mathcal{L}[\tilde{i}_a; s] = \frac{i_{a0}\mathbf{N}(\varsigma)s^{\varsigma-1} + (1-\varsigma)(s^\varsigma + \mu_\varsigma)\mathcal{L}[\mathcal{X}_1; s]}{z_\varsigma s^\varsigma + \mathfrak{U}\varsigma}, \\ \mathcal{L}[\tilde{i}_f; s] = \frac{i_{f0}\mathbf{N}(\varsigma)s^{\varsigma-1} + (1-\varsigma)(s^\varsigma + \mu_\varsigma)\mathcal{L}[\mathcal{X}_2; s]}{z_\varsigma s^\varsigma + \mathfrak{U}\varsigma}, \\ \mathcal{L}[\tilde{\omega}; s] = \frac{\omega_0\mathbf{N}(\varsigma)s^{\varsigma-1} + (1-\varsigma)(s^\varsigma + \mu_\varsigma)\mathcal{L}[\mathcal{X}_3; s]}{z_\varsigma s^\varsigma + \mathfrak{U}\varsigma}. \end{cases} \quad (25)$$

Then:

$$\begin{cases} \mathcal{L}[\text{mABC}\mathbf{D}_0^\varsigma i_a + \mathfrak{U}i_a; s] = \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \cdot \frac{s^\varsigma \mathcal{L}[\tilde{i}_a; s] - s^{\varsigma-1}i_{a0}}{s^\varsigma + \mu_\varsigma} + \mathcal{L}[\tilde{i}_a; s], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\varsigma i_f + \mathfrak{U}i_f; s] = \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \cdot \frac{s^\varsigma \mathcal{L}[\tilde{i}_f; s] - s^{\varsigma-1}i_{f0}}{s^\varsigma + \mu_\varsigma} + \mathcal{L}[\tilde{i}_f; s], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\varsigma \omega + \mathfrak{U}\omega; s] = \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \cdot \frac{s^\varsigma \mathcal{L}[\tilde{\omega}; s] - s^{\varsigma-1}\omega_0}{s^\varsigma + \mu_\varsigma} + \mathcal{L}[\tilde{\omega}; s]. \end{cases} \quad (26)$$

After calculations, these yield:

$$\begin{cases} \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta i_a + \mathcal{U}i_a; s] = \frac{1}{(1-\zeta)(s^\zeta + \mu_\zeta)} \left[(z_\zeta s^\zeta + \mathcal{U}\zeta) \mathcal{L}[\tilde{i}_a; s] - \mathbf{N}(\zeta) s^{\zeta-1} i_{a0} \right], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta i_f + \mathcal{U}i_f; s] = \frac{1}{(1-\zeta)(s^\zeta + \mu_\zeta)} \left[(z_\zeta s^\zeta + \mathcal{U}\zeta) \mathcal{L}[\tilde{i}_f; s] - \mathbf{N}(\zeta) s^{\zeta-1} i_{f0} \right], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta \omega + \mathcal{U}\omega; s] = \frac{1}{(1-\zeta)(s^\zeta + \mu_\zeta)} \left[(z_\zeta s^\zeta + \mathcal{U}\zeta) \mathcal{L}[\tilde{\omega}; s] - \mathbf{N}(\zeta) s^{\zeta-1} \omega_0 \right]. \end{cases} \quad (27)$$

Now substituting Equation (25) into Equation (27), we get:

$$\begin{cases} \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta i_a + \mathcal{U}i_a; s] = \frac{1}{(1-\zeta)(s^\zeta + \mu_\zeta)} \left[i_{a0} \mathbf{N}(\zeta) s^{\zeta-1} + (1-\zeta)(s^\zeta + \mu_\zeta) \mathcal{L}[\mathcal{X}_1; s] \right. \\ \quad \left. - \mathbf{N}(\zeta) s^{\zeta-1} i_{a0} \right], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta i_f + \mathcal{U}i_f; s] = \frac{1}{(1-\zeta)(s^\zeta + \mu_\zeta)} \left[i_{f0} \mathbf{N}(\zeta) s^{\zeta-1} + (1-\zeta)(s^\zeta + \mu_\zeta) \mathcal{L}[\mathcal{X}_2; s] \right. \\ \quad \left. - \mathbf{N}(\zeta) s^{\zeta-1} i_{f0} \right], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta \omega + \mathcal{U}\omega; s] = \frac{1}{(1-\zeta)(s^\zeta + \mu_\zeta)} \left[\omega_0 \mathbf{N}(\zeta) s^{\zeta-1} + (1-\zeta)(s^\zeta + \mu_\zeta) \mathcal{L}[\mathcal{X}_3; s] \right. \\ \quad \left. - \mathbf{N}(\zeta) s^{\zeta-1} \omega_0 \right]. \end{cases} \quad (28)$$

Which simplifies to:

$$\begin{cases} \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta i_a + \mathcal{U}i_a; s] = \mathcal{L}[\mathcal{X}_1; s], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta i_f + \mathcal{U}i_f; s] = \mathcal{L}[\mathcal{X}_2; s], \\ \mathcal{L}[\text{mABC}\mathbf{D}_0^\zeta \omega + \mathcal{U}\omega; s] = \mathcal{L}[\mathcal{X}_3; s]. \end{cases} \quad (29)$$

We finish the proof. $\square$

Remark 1. If $\mathcal{X}_i \in C[0, T]$ and

$$\mathcal{U}i_{a0} = \mathcal{X}_1(0), \quad \mathcal{U}i_{f0} = \mathcal{X}_2(0), \quad \mathcal{U}\omega_0 = \mathcal{X}_3(0), \quad (30)$$

then we have

$$\tilde{i}_a = i_{a0}, \quad \tilde{i}_f = i_{f0}, \quad \tilde{\omega} = \omega_0.$$

Consequently, the solution provided in Equation (24) satisfies the continuity condition. Based on these conditions, a solution exists for the given model.

6.2. Series Solution by LADM

We apply the Laplace transform of the modified version of ABC, as mentioned in [23,25], to system (17) under consideration:

$$\begin{aligned} \mathcal{L}[\text{mABC}D_t^\zeta i_a(t)] &= \mathcal{L}\left[\frac{1}{L_a} (V_L - R_a i_a - L_{af} i_f \omega)\right], \\ \mathcal{L}[\text{mABC}D_t^\zeta i_f(t)] &= \mathcal{L}\left[\frac{1}{L_f} (V_L - R_f i_f)\right], \\ \mathcal{L}[\text{mABC}D_t^\zeta \omega(t)] &= \mathcal{L}\left[\frac{1}{J} (L_{af} i_a i_f - K_L \omega)\right]. \end{aligned} \quad (31)$$

Consequently, we get

$$\begin{aligned}\frac{\mathbf{N}(\varsigma)}{1-\varsigma} \times \frac{s^\varsigma \mathcal{L}\{i_a(t)\} - s^{\varsigma-1}i_a(0)}{s^\varsigma + \mu_\varsigma} &= \mathcal{L}\left[\frac{1}{L_a}\left(V_L - R_a i_a(t) - L_{af} i_f(t)\omega(t)\right)\right], \\ \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \times \frac{s^\varsigma \mathcal{L}\{i_f(t)\} - s^{\varsigma-1}i_f(0)}{s^\varsigma + \mu_\varsigma} &= \mathcal{L}\left[\frac{1}{L_f}\left(V_L - R_f i_f(t)\right)\right], \\ \frac{\mathbf{N}(\varsigma)}{1-\varsigma} \times \frac{s^\varsigma \mathcal{L}\{\omega(t)\} - s^{\varsigma-1}\omega(0)}{s^\varsigma + \mu_\varsigma} &= \mathcal{L}\left[\frac{1}{J}\left(L_{af} i_a(t)i_f(t) - K_L \omega(t)\right)\right].\end{aligned}\quad (32)$$

On using the initial condition, we obtain

$$\begin{aligned}\mathcal{L}\{i_a(t)\} &= \frac{i_a(0)}{s} + \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L}\left[\frac{1}{L_a}\left(V_L - R_a i_a(t) - L_{af} i_f(t)\omega(t)\right)\right], \\ \mathcal{L}\{i_f(t)\} &= \frac{i_f(0)}{s} + \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L}\left[\frac{1}{L_f}\left(V_L - R_f i_f(t)\right)\right], \\ \mathcal{L}\{\omega(t)\} &= \frac{\omega(0)}{s} + \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L}\left[\frac{1}{J}\left(L_{af} i_a(t)i_f(t) - K_L \omega(t)\right)\right].\end{aligned}\quad (33)$$

We decompose each state variable as

$$\begin{aligned}i_a(t) &= \sum_{n=0}^{\infty} A_n(t), \\ i_f(t) &= \sum_{n=0}^{\infty} F_n(t), \\ \omega(t) &= \sum_{n=0}^{\infty} W_n(t).\end{aligned}$$

For the nonlinear terms, we define the Adomian polynomials

$$\begin{aligned}i_f(t)\omega(t) &= \sum_{n=0}^{\infty} FW_n(t), \\ i_a(t)i_f(t) &= \sum_{n=0}^{\infty} AF_n(t),\end{aligned}$$

where

$$\begin{aligned}FW_n(t) &= \sum_{k=0}^n F_k(t)W_{n-k}(t), \\ AF_n(t) &= \sum_{k=0}^n A_k(t)F_{n-k}(t).\end{aligned}$$

On comparing both sides, the recursive relations are obtained as

$$\begin{aligned}A_0(t) &= i_a(0), \\ F_0(t) &= i_f(0), \\ W_0(t) &= \omega(0),\end{aligned}\quad (34)$$

and for $n \geq 0$,

$$\begin{aligned}
A_{n+1}(t) &= \mathcal{L}^{-1} \left\{ \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_a} (V_L \delta_n - R_a A_n - L_{af} F W_n) \right] \right\}, \\
F_{n+1}(t) &= \mathcal{L}^{-1} \left\{ \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_f} (V_L \delta_n - R_f F_n) \right] \right\}, \\
W_{n+1}(t) &= \mathcal{L}^{-1} \left\{ \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{J} (L_{af} A F_n - K_L W_n) \right] \right\}.
\end{aligned} \tag{35}$$

$$\begin{aligned}
\mathcal{L}\{A_0\} &= \frac{i_a(0)}{s}, \\
\mathcal{L}\{A_1\} &= \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_a} (V_L - R_a A_0 - L_{af} F W_0) \right], \\
\mathcal{L}\{A_2\} &= \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_a} (-R_a A_1 - L_{af} F W_1) \right], \\
&\vdots \\
\mathcal{L}\{A_{n+1}\} &= \frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_a} (-R_a A_n - L_{af} F W_n) \right].
\end{aligned} \tag{36}$$

Using the inverse Laplace transform and taking into account the first three terms, we obtain

$$\begin{aligned}
A_0 &= i_a(0) = i_{a0}, \\
A_1(t) &= \Theta(t) \frac{1}{L_a} (V_L - R_a A_0 - L_{af} F W_0), \\
A_2(t) &= \Theta(t) \frac{1}{L_a} (-R_a A_1 - L_{af} F W_1).
\end{aligned} \tag{37}$$

$$\begin{aligned}
F_0 &= i_f(0) = i_{f0}, \\
F_1(t) &= \Theta(t) \frac{1}{L_f} (V_L - R_f F_0), \\
F_2(t) &= \Theta(t) \frac{1}{L_f} (-R_f F_1).
\end{aligned} \tag{38}$$

$$\begin{aligned}
W_0 &= \omega(0) = \omega_0, \\
W_1(t) &= \Theta(t) \frac{1}{J} (L_{af} A F_0 - K_L W_0), \\
W_2(t) &= \Theta(t) \frac{1}{J} (L_{af} A F_1 - K_L W_1).
\end{aligned} \tag{39}$$

The corresponding Adomian polynomials are given by

$$\begin{aligned}
A F_0 &= A_0 F_0, \\
A F_1 &= A_0 F_1 + A_1 F_0, \\
A F_2 &= A_0 F_2 + A_1 F_1 + A_2 F_0,
\end{aligned}$$

and

$$\begin{aligned}FW_0 &= F_0W_0, \\FW_1 &= F_0W_1 + F_1W_0, \\FW_2 &= F_0W_2 + F_1W_1 + F_2W_0.\end{aligned}$$

Consequently, the approximate series solutions of the fractional DC motor model are

$$\begin{aligned}i_a(t) &= A_0 + A_1(t) + A_2(t) + \cdots, \\i_f(t) &= F_0 + F_1(t) + F_2(t) + \cdots, \\\omega(t) &= W_0 + W_1(t) + W_2(t) + \cdots.\end{aligned}$$

Hence, the truncated second-order approximations can be written as

$$\begin{aligned}i_a(t) &\approx A_0 + A_1(t) + A_2(t), \\i_f(t) &\approx F_0 + F_1(t) + F_2(t), \\\omega(t) &\approx W_0 + W_1(t) + W_2(t).\end{aligned}$$

6.3. Analysis for Stability and Uniqueness

Theorem 4 (Existence and Uniqueness). *Let $(\mathbf{B}, \|\cdot\|)$ be a Banach space of continuous real-valued functions defined on $[0, T]$, and suppose the nonlinear operator $\mathbf{K} : \mathbf{B} \rightarrow \mathbf{B}$ is defined for the fractional motor model (17) as follows:*

$$\begin{aligned}\mathbf{K}[i_a](t) &= i_a(0) + \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_a} (V_L - R_a i_a(t) - L_{af} i_f(t) \omega(t)) \right] \right], \\ \mathbf{K}[i_f](t) &= i_f(0) + \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_f} (V_L - R_f i_f(t)) \right] \right], \\ \mathbf{K}[\omega](t) &= \omega(0) + \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{J} (L_{af} i_a(t) i_f(t) - K_L \omega(t)) \right] \right].\end{aligned}\quad (40)$$

Assume the nonlinear terms satisfy the following Lipschitz conditions:

$$\begin{aligned}\left| \frac{1}{L_a} (V_L - R_a i_{a1} - L_{af} i_{f1} \omega_1) - \frac{1}{L_a} (V_L - R_a i_{a2} - L_{af} i_{f2} \omega_2) \right| &\leq L_1 (\|i_{a1} - i_{a2}\| \\ &\quad + \|i_{f1} - i_{f2}\| + \|\omega_1 - \omega_2\|), \\ \left| \frac{1}{L_f} (V_L - R_f i_{f1}) - \frac{1}{L_f} (V_L - R_f i_{f2}) \right| &\leq L_2 \|i_{f1} - i_{f2}\|, \\ \left| \frac{1}{J} (L_{af} i_{a1} i_{f1} - K_L \omega_1) - \frac{1}{J} (L_{af} i_{a2} i_{f2} - K_L \omega_2) \right| &\leq L_3 (\|i_{a1} - i_{a2}\| \\ &\quad + \|i_{f1} - i_{f2}\| + \|\omega_1 - \omega_2\|).\end{aligned}\quad (41)$$

If $L = \max\{L_1, L_2, L_3\}$ and

$$\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \cdot L < 1,$$

it follows that the operator $\mathbf{K}$ is a contraction. Consequently, by the Banach Fixed Point Theorem, system (17) admits a unique solution $(i_a(t), i_f(t), \omega(t))$ on $[0, T]$.

Proof. To establish that the operator $\mathbf{K}$ has a fixed point, we consider a pair $(q, p) \in \mathbb{N} \times \mathbb{N}$. Then, we have:

$$\begin{aligned}
\mathbf{K}[i_{a,q}](t) - \mathbf{K}[i_{a,p}](t) &= i_{a,q} - i_{a,p} \\
&+ \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left(\frac{1}{L_a} (V_L - R_a i_{a,q} - L_{af} i_{f,q} \omega_q) \right) \right] \\
&- \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left(\frac{1}{L_a} (V_L - R_a i_{a,p} - L_{af} i_{f,p} \omega_p) \right) \right], \\
\mathbf{K}[i_{f,q}](t) - \mathbf{K}[i_{f,p}](t) &= i_{f,q} - i_{f,p} + \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left(\frac{1}{L_f} (V_L - R_f i_{f,q}) \right) \right] \\
&- \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left(\frac{1}{L_f} (V_L - R_f i_{f,p}) \right) \right], \\
\mathbf{K}[\omega_q](t) - \mathbf{K}[\omega_p](t) &= \omega_q - \omega_p + \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left(\frac{1}{J} (L_{af} i_{a,q} i_{f,q} - K_L \omega_q) \right) \right] \\
&- \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left(\frac{1}{J} (L_{af} i_{a,p} i_{f,p} - K_L \omega_p) \right) \right].
\end{aligned} \tag{42}$$

By taking the first equation from Equation (17), then figuring out the norm for each side, we obtain:

$$\begin{aligned}
&\|\mathbf{K}[i_{a,q}(t)] - \mathbf{K}[i_{a,p}(t)]\| \\
&= \left\| i_{a,q} - i_{a,p} + \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_a} (V_L - R_a i_{a,q} - L_{af} i_{f,q} \omega_q) \right] \right] \right. \\
&\quad \left. - \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\frac{1}{L_a} (V_L - R_a i_{a,p} - L_{af} i_{f,p} \omega_p) \right] \right] \right\|.
\end{aligned} \tag{43}$$

Applying the triangle inequality to Equation (43) and simplifying the resulting expression yields:

$$\begin{aligned}
&\|\mathbf{K}[i_{a,q}(t)] - \mathbf{K}[i_{a,p}(t)]\| \\
&\leq \|i_{a,q} - i_{a,p}\| \\
&+ \mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L} \left[\left\| \frac{-R_a}{L_a} (i_{a,q} - i_{a,p}) - \frac{L_{af}}{L_a} (i_{f,q} \omega_q - i_{f,p} \omega_p) \right\| \right] \right].
\end{aligned} \tag{44}$$

By letting both relations follow

$$\|i_{a,q}(t) - i_{a,p}(t)\| \cong \|i_{f,q}(t) - i_{f,p}(t)\| \cong \|\omega_q(t) - \omega_p(t)\|, \tag{45}$$

and assuming the boundedness condition

$$\|i_{a,p}\| < \mathbf{N}_1, \quad \|i_{f,p}\| < \mathbf{N}_2, \quad \|\omega_p\| < \mathbf{N}_3, \quad (q, p) \in \mathbb{N} \times \mathbb{N}, \tag{46}$$

we substitute these into Equation (44) to get

$$\|\mathbf{K}[i_{a,q}(t)] - \mathbf{K}[i_{a,p}(t)]\| \leq \left(1 - \frac{R_a}{L_a} w_1(\theta) - \frac{L_{af}}{L_a} (\mathbf{N}_2 + \mathbf{N}_3) w_2(\theta) \right) \|i_{a,q} - i_{a,p}\|. \tag{47}$$

Here, $w_1(\theta)$ and $w_2(\theta)$ denote the fractional kernel functions derived from

$$\mathcal{L}^{-1} \left[\frac{(1-\varsigma)(s^\varsigma + \mu_\varsigma)}{\mathbf{N}(\varsigma)s^\varsigma} \mathcal{L}[\cdot] \right].$$

Similarly, applying the same procedure to the remaining components i_f and ω , we obtain:

$$\|\mathbf{K}[i_{fq}(t)] - \mathbf{K}[i_{fp}(t)]\| \leq \left(1 - \frac{R_f}{L_f} w_3(\theta)\right) \|i_{fq} - i_{fp}\|, \quad (48)$$

$$\|\mathbf{K}[\omega_q(t)] - \mathbf{K}[\omega_p(t)]\| \leq \left(1 + \frac{L_{af}}{J} (N_1 + N_2) w_4(\theta) - \frac{K_L}{J} w_5(\theta)\right) \|\omega_q - \omega_p\|.$$

Therefore, there exists a fixed point for $\mathbf{K}$ if the following set of inequalities holds:

$$\Psi = \begin{cases} 1 - \frac{R_a}{L_a} w_1(\theta) - \frac{L_{af}}{L_a} (N_2 + N_3) w_2(\theta) < 1, \\ 1 - \frac{R_f}{L_f} w_3(\theta) < 1, \\ 1 + \frac{L_{af}}{J} (N_1 + N_2) w_4(\theta) - \frac{K_L}{J} w_5(\theta) < 1. \end{cases} \quad (49)$$

Consequently, the proof is complete since the conditions of Theorem 1 are satisfied. $\square$

Theorem 5. The fractional-order DC motor model (17) admits a unique solution on $[0, T]$ for any given initial conditions $i_a(0) = i_{a0}$, $i_f(0) = i_{f0}$, and $\omega(0) = \omega_0$.

Proof. Define

$$X(t) = \begin{pmatrix} i_a(t) \\ i_f(t) \\ \omega(t) \end{pmatrix},$$

and

$$F(X) = \begin{pmatrix} \frac{1}{L_a} (V_L - R_a i_a - L_{af} i_f \omega) \\ \frac{1}{L_f} (V_L - R_f i_f) \\ \frac{1}{J} (L_{af} i_a i_f - K_L \omega) \end{pmatrix}.$$

The function $F(X)$ is continuous and continuously differentiable with respect to (i_a, i_f, ω) . Hence, F is locally Lipschitz on every bounded subset of $\mathbb{R}^3$. Therefore, based on the standard existence and uniqueness theorem for fractional differential equations with the modified Atangana–Baleanu Caputo derivative, there exists a unique local solution of system (17) satisfying the prescribed initial conditions. Moreover, since the nonlinear terms are polynomial and the system parameters are finite positive constants, the solution remains bounded on every finite interval $[0, T]$. Consequently, the local solution can be extended uniquely to the whole interval $[0, T]$.

Thus, the fractional DC motor model possesses a unique solution. $\square$

7. Numerical Simulation

To illustrate the stochastic behaviour of the proposed shunt DC motor model, numerical simulations of the stochastic differential equation system (4) are carried out using the Euler–Maruyama scheme. The simulations are performed directly in terms of the physical state variables of the system, namely the armature current $i_a(t)$, the field current $i_f(t)$, and

the angular velocity $\omega(t)$. This ensures consistency with the original DC motor model and avoids the introduction of artificial variables.

$$\begin{aligned} i_{a,i+1} &= i_{a,i} + \frac{1}{L_a} \left(V_L - R_a i_{a,i} - L_{af} i_{f,i} \omega_i \right) \Delta t + \sigma_1 i_{a,i} \sqrt{\Delta t} \eta_{1,i}, \\ i_{f,i+1} &= i_{f,i} + \frac{1}{L_f} \left(V_L - R_f i_{f,i} \right) \Delta t + \sigma_2 i_{f,i} \sqrt{\Delta t} \eta_{2,i}, \\ \omega_{i+1} &= \omega_i + \frac{1}{J} \left(L_{af} i_{a,i} i_{f,i} - K_L \omega_i \right) \Delta t + \sigma_3 \omega_i \sqrt{\Delta t} \eta_{3,i}. \end{aligned} \quad (50)$$

where $\eta_{k,i} \sim \mathcal{N}(0,1)$, $k = 1, 2, 3$, are mutually independent standard Gaussian random variables, so that the Wiener increments are given by

$$\Delta B_{k,i} = \sqrt{\Delta t} \eta_{k,i}.$$

The stochastic terms are multiplicative because the diffusion coefficients depend on the instantaneous state variables. This reflects the physical behavior of the shunt DC motor, where electrical disturbances scale with the current levels and mechanical perturbations scale with the rotational speed.

The parameter values for the stochastic shunt DC motor are listed in Table 2.

Table 2. Model parameter values for the stochastic shunt DC motor model (4).

Symbol	Set 1	Set 2
V_L	220 V	220 V
R_a	1.20 Ω	1.50 Ω
L_a	0.05 H	0.50 H
R_f	110 Ω	150 Ω
L_f	1.00 H	1.10 H
L_{af}	0.08 H	0.80 H
K_L	0.02 N m s	0.02 N m s
J	0.01 kg m ²	0.10 kg m ²
σ_1	0.15	0.12
σ_2	0.05	0.02
σ_3	0.15	0.10

7.1. Numerical Illustrations of Stochastic Ultimate Boundedness

Figure 2 depicts the stochastic response of the armature current in the proposed stochastic model (4). For comparison, both the deterministic solution (1) and the stochastic solution (4) are displayed for the armature current, field current, and angular velocity. As the rotor speed increases, the back electromotive force counteracts the applied voltage, stabilising the armature current. Throughout the simulation, all trajectories remain bounded and physically admissible, and the parameter is taken from Table 2 Set 1 and the initial conditions are chosen as $(i_a(0), i_f(0), \omega(0)) = (0.01, \frac{V_L}{R_f}, 0.01)$. Figure 2a shows that the armature current rapidly approaches its operating regime and remains bounded throughout the simulation interval. Similarly, Figure 2b demonstrates that the field current converges toward its operating value and remains stable in the presence of stochastic perturbations. The evolution of the angular velocity is depicted in Figure 2c. As expected, the rotor speed increases from its initial state and approaches the motor's operating regime. The confined stochastic orbit illustrates how random fluctuations in the armature current are directly transmitted to the mechanical subsystem through electromagnetic interaction.

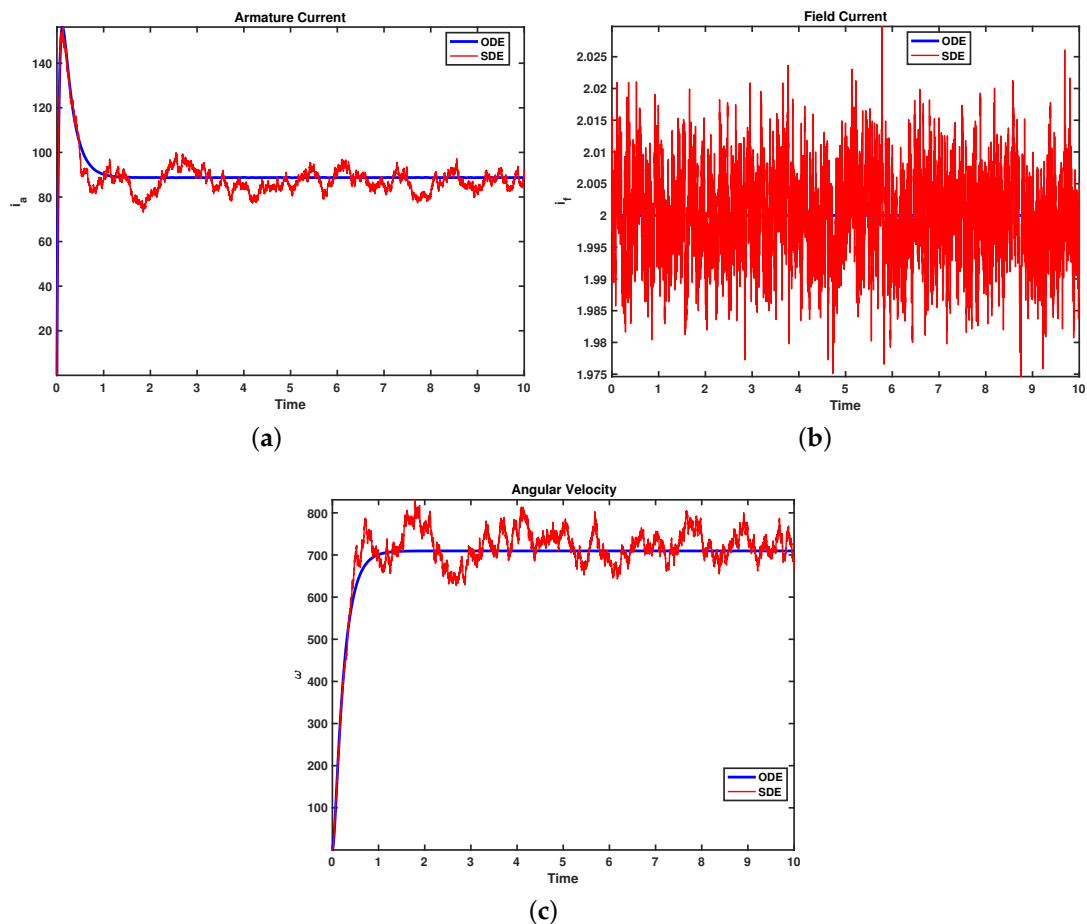


Figure 2. Comparison of deterministic (1) and stochastic (4) solutions: (a) armature current i_a ; (b) field current i_f ; (c) angular velocity ω .

Figure 3 summarises the Long-Term behavior of the stochastic shunt DC Motor model (4). Figure 3a,c,e illustrate the long-term behaviour of the stochastic shunt DC motor model (4) under multiplicative perturbations. The deterministic and stochastic trajectories demonstrate that the state variables remain bounded and fluctuate around their corresponding operating regimes. The empirical probability density functions obtained from the stationary portions of the stochastic trajectories further provide evidence of stationary behaviour. As shown in Figure 3b,d,f, the probability densities remain concentrated around the operating values of the corresponding state variables, indicating that the system spends most of its time in a bounded neighbourhood of the deterministic operating regime. The above result validates Theorem 3. Throughout the simulation, all trajectories remain bounded and physically admissible, and the parameter is taken from Table 2 Set 2 and the initial conditions are chosen as $(i_a(0), i_f(0), \omega(0)) = (0.01, \frac{V_L}{R_f}, 0.01)$. Therefore, the numerical simulations support the theoretical analysis and suggest that the stochastic shunt DC motor model admits a stationary regime characterised by bounded stochastic oscillations around the deterministic equilibrium.

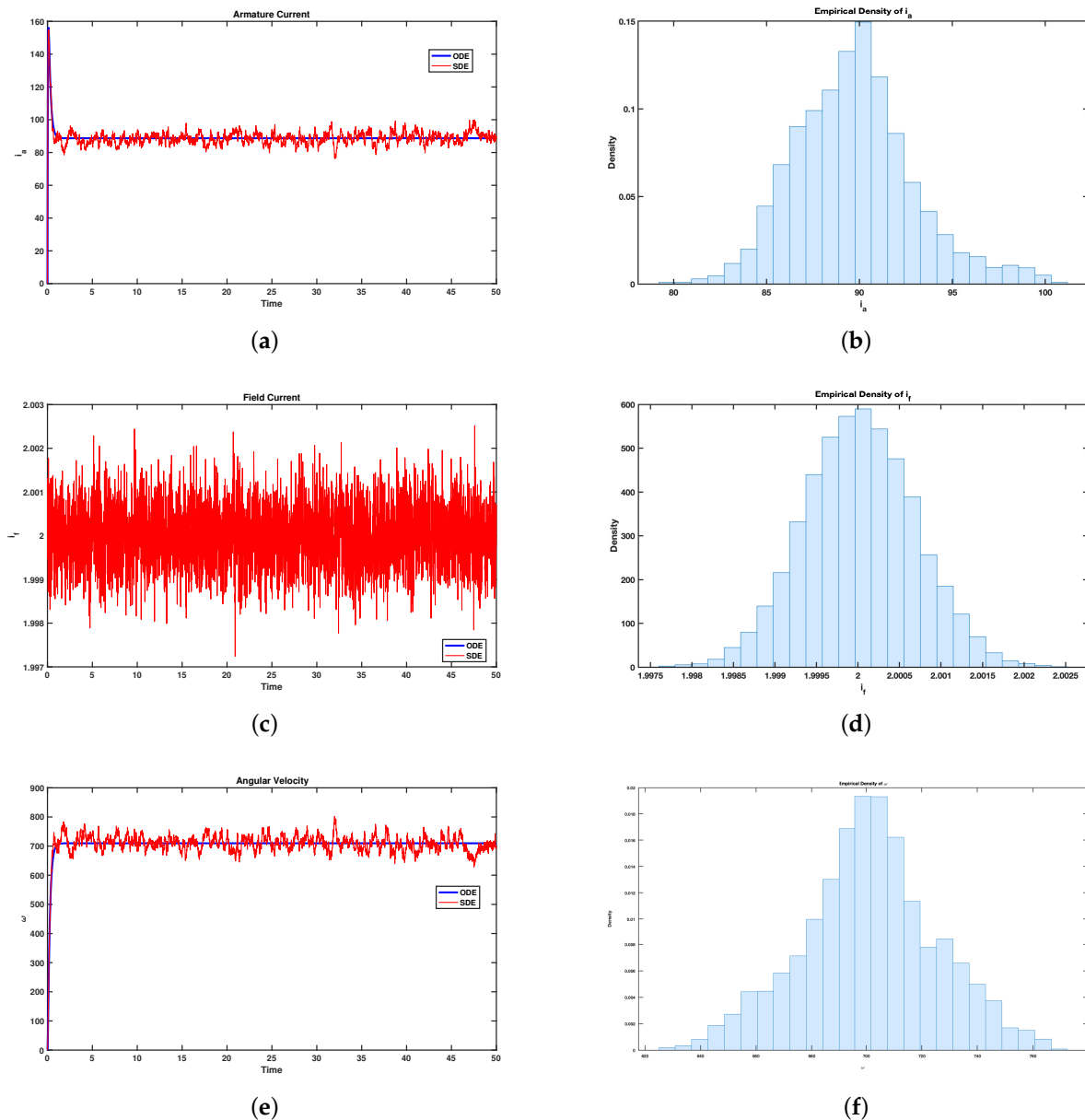


Figure 3. Deterministic (1) and stochastic (4) solutions with their stationary probability distributions: (a) armature current i_a ; (b) empirical density of i_a ; (c) field current i_f ; (d) empirical density of i_f ; (e) angular velocity ω ; (f) empirical density of ω .

8. Numerical Simulation for Modified Fractional Operator

This section investigates the influence of the modified fractional-order operator on the dynamical behavior of the shunt DC motor model. In particular, Figures 4 and 5 are used to illustrate how memory effects introduced by the modified fractional derivative affect the transient response, stability, and long-term behavior of the system. Figure 4 illustrates the time evolution of the state variables under the modified fractional-order operator for different values of the fractional order $\varsigma \in (0, 1)$, and the parameter is taken from Table 2 Set 2, and the initial conditions are $(i_a(0), i_f(0), \omega(0)) = (190, 20, 10000)$. The results clearly demonstrate that the fractional order plays a significant role in shaping the system dynamics through nonlocal memory effects. As the fractional order ς decreases, the system exhibits a slower transient response and smoother trajectories. This is inherent in the memory property of the modified fractional operator, in which the present state is dependent upon the previous state of the system, as well as the present state. This results

in quick changes in the armature current $i_a(t)$, field current $i_f(t)$, and angular velocity $\omega(t)$ being effectively dampened, resulting in the improvement of damping properties. In the case where the value of ζ takes the value of unity, the system response tends to the value of the classical integer-order model, which validates the consistency of the fractional formulation with the standard model as a limiting case. Conversely, lower values of ζ are also smaller, which causes stronger memory effects and consequently, slower convergence and dampened oscillations. The latter features are especially applicable to electromechanical systems, in which inductive and mechanical components naturally have a hereditary behaviour.

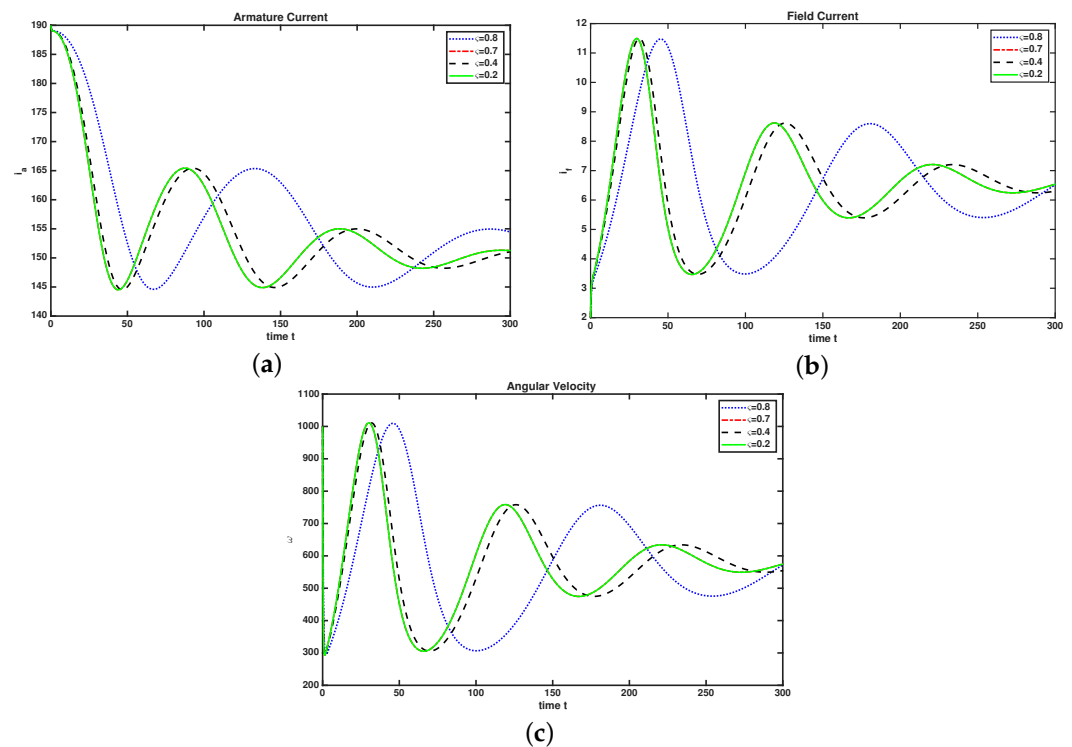


Figure 4. Effect of the modified fractional-order operator on the dynamics of the shunt DC motor for different fractional orders ζ : (a) armature current i_a ; (b) field current i_f ; (c) angular velocity ω .

Figure 5 below gives a direct comparison between the classical integer-order model and the fractional-order modified model with the same initial conditions and the same values of the parameters of Figure 4. The number underlines the stabilizing effect of the fractional operator on the system reaction. It is noted that the fractional-order model shows less overshoot with less transient behavior and better stability properties as compared to the convergent operating regimes of both models. These advantages are due to the fact that the modified fractional derivative is nonlocal in nature and the effect of the past states spreads out with time and the sharp changes in the system variables are avoided. Specifically, the angular velocity $\omega(t)$ in the fractional model, or more specifically, the angular velocity, is operated by a less pronounced acceleration of the model and removes the unrealistic spikes, which can be found in the integer order formulation. Equally, the armature and field currents exhibit reduced oscillations, which signifies increased strength to the variations in the parameters and external perturbations.

Collectively, Figures 4 and 5 prove that the adjusted fractional-order model is doing a better job of capturing the shunt DC motor dynamics and is much more flexible. The fractional order ζ serves as a tuning parameter is used to adjust the strength of memory effects and enables an ad hoc shift between classical integer-order behavior and very history-dependent dynamics. These results substantiate the theoretical contributions in

this section and reason why modified fractional operators are used to model and analyze complex electromechanical systems. Moreover, the fractional formulation adds to the stochastic analysis to improve the stability of a system and relaxes the dynamic response of the system by smoothing of the uncertain operating conditions.

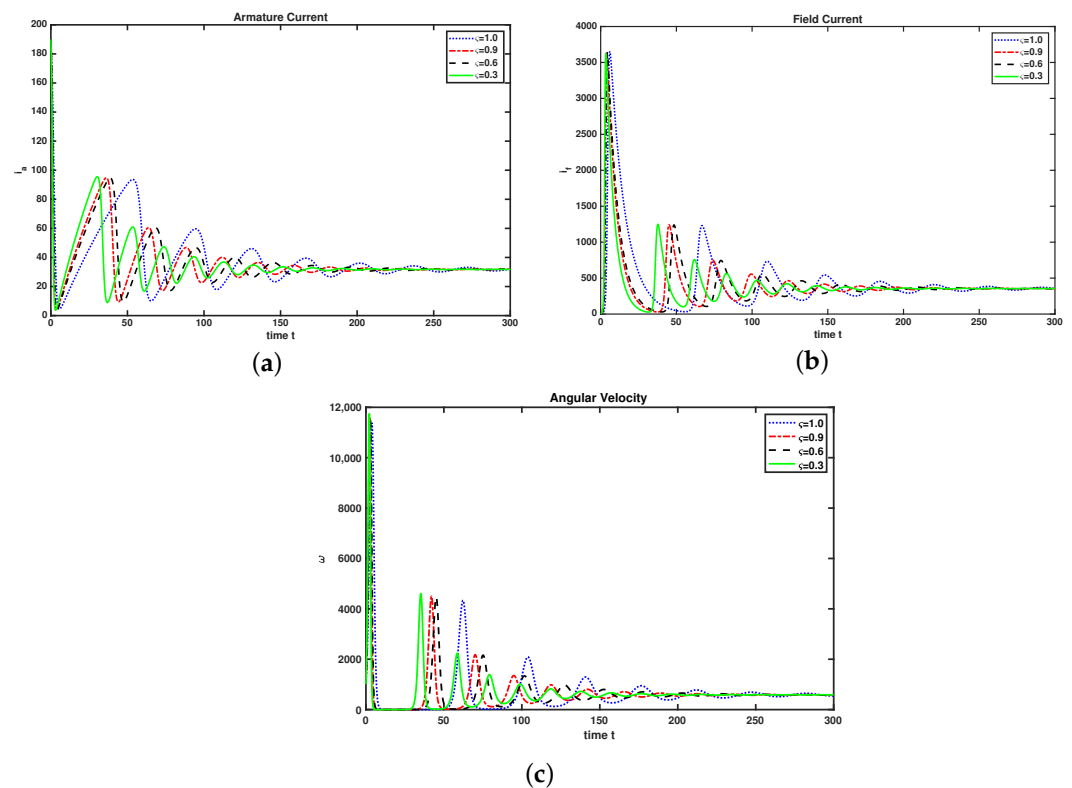


Figure 5. Comparison of the classical integer-order model and the modified fractional-order model (17) for ς : (a) armature current i_a ; (b) field current i_f ; (c) angular velocity ω .

9. Conclusions

In this work, a stochastic and fractional-order modelling framework has been proposed to analyse the dynamic behaviour of a shunt DC motor under uncertainty and memory effects. The motor dynamics were formulated as a system of stochastic differential equations, with random disturbances in the electrical and mechanical subsystems modelled by additive Gaussian noise. Numerical simulations based on the Euler–Maruyama scheme demonstrated that all state variables remain bounded and fluctuate around stable operating regimes, confirming the robustness of the stochastic model. To further enhance the realism of the model, a modified fractional-order operator was introduced to incorporate hereditary and nonlocal memory effects inherent in electromechanical systems. The fractional formulation provided an additional degree of freedom through the fractional order parameter, enabling smooth transitions between classical integer-order dynamics and strongly memory-dependent behaviour. Numerical results showed that the fractional-order model significantly smooths transient responses, reduces overshoot, and improves stability compared to its integer-order counterpart. Comparative analysis between the stochastic and modified fractional-order models highlighted the complementary roles of randomness and memory effects in shaping system dynamics. While stochastic perturbations account for uncertainty and environmental disturbances, the fractional operator effectively mitigates rapid fluctuations and enhances long-term stability. The proposed modelling approach offers a flexible, realistic framework for analysing the dynamics of shunt DC motors under uncertain, complex operating conditions. The results provide valuable insights for the

design, control, and optimisation of electromechanical systems, and the framework can be readily extended to other motor types and engineering systems exhibiting stochastic behaviour and memory effects.

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