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Quantum Circuit Learning for Volatility Modeling: Multifractal Analysis of Realized Volatility Time Series

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Abstract

Herein, we propose a quantum circuit learning framework for modeling the realized volatility (RV) of Bitcoin and investigate the statistical properties of the predicted time series through multifractal analysis. Unlike conventional GARCH-type models, which require a pre-specified functional form for the volatility process, a parameterized quantum circuit directly approximates the volatility function from empirical data, eliminating the need for explicit model selection. Using five-minute Bitcoin price data, we construct daily RV, train a single-qubit parameterized quantum circuit, and generate a long synthetic time series from the optimized quantum circuit. Multifractal Detrended Fluctuation Analysis is applied to calculate the generalized Hurst exponent $h(q)$, the singularity spectrum $f(\alpha)$, and the multifractal scaling exponent $\tau(q)$. The predicted return series exhibits $h(2) \approx 0.5$, consistent with near-random dynamics, and both the predicted and the empirical return series display multifractality that partially persists after random shuffling. The increment series of RV shows pronounced anti-persistence with $h(2) \approx 0.05$ – 0.1 , consistent with the rough volatility hypothesis. These results demonstrate that a simple single-qubit parameterized quantum circuit captures qualitatively some observed properties in Bitcoin volatility dynamics.

Keywords: Bitcoin; generalized Hurst exponent; singularity spectrum; multifractal analysis; parameterized quantum circuit; quantum circuit learning; realized volatility; rough volatility; stylized facts

1. Introduction

Risk management is a central concern for participants in financial markets, where estimating the risk associated with asset holdings and preventing large future losses are essential tasks. Volatility is one of the most widely used measures of financial risk, and forecasting future volatility plays a crucial role in the safe and efficient management of financial assets. A common approach to volatility forecasting is to construct models that capture the dynamics of financial time series. In this context, incorporating empirical properties of financial data is a key requirement for building effective models.

Several stylized facts are known to appear universally across different classes of financial assets [1]. Among these, volatility clustering is one of the most prominent: periods of high volatility tend to be followed by high volatility, and periods of low volatility by low volatility. To capture this feature, Engle introduced the Autoregressive Conditional Heteroskedasticity (ARCH) model [2], in which volatility is modeled as an autoregressive function of past squared returns. Bollerslev later generalized this framework to the Generalized ARCH (GARCH) model [3], which has since become a standard tool in volatility modeling.



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Another important empirical property of asset returns, especially stock returns, is the leverage effect [4,5], which refers to the asymmetric response of volatility to positive and negative returns: volatility tends to increase more following negative returns than following positive ones. Since the standard GARCH model cannot capture such asymmetry, several extensions have been proposed, including the Exponential GARCH (EGARCH) [6], Threshold GARCH (TGARCH) [7], Quadratic ARCH (QARCH) [8], Rational GARCH (RGARCH) [9,10], and Asymmetric Power GARCH (APGARCH) [11] models.

A further class of models incorporating stochastic dynamics is the Stochastic Volatility (SV) [12,13], which augments the volatility equation with a stochastic disturbance term. Whereas GARCH-type models can generally be estimated via maximum likelihood, which is computationally straightforward, maximum likelihood estimation of SV models is intractable; Bayesian estimation (It is also possible to estimate the parameters of a GARCH model within a Bayesian framework. For example, see [14]) is therefore the predominant approach, implemented through Markov Chain Monte Carlo methods developed specifically for this purpose [15–17]. In lattice Quantum ChromoDynamics (QCD) simulations [18,19], physical observables are typically computed using Markov chain Monte Carlo methods, among which the standard algorithm is the Hybrid Monte Carlo (HMC) method [20]. Although HMC was originally developed (For studies on methods that improve the efficiency of the HMC algorithm in lattice QCD simulations, see, for example, References [21–24]. In realistic lattice QCD simulations, it is necessary to handle an odd number of fermion flavors, and several variants of the HMC algorithm capable of performing such simulations have been proposed, including the following [25–27]) within the context of lattice QCD computations, it has since been adopted across a wide range of fields, where it is also referred to as Hamiltonian Monte Carlo. Several studies [28,29] have applied this HMC framework to the Bayesian estimation of SV-type models.

Another stylized fact concerns the long-memory property of volatility. Although GARCH-type models imply short-memory dynamics, empirical studies have shown that volatility often exhibits long-range dependence [11,30]. To address this, long-memory models such as the Autoregressive Fractionally Integrated Moving Average (ARFIMA) [31] and Fractionally Integrated GARCH (FIGARCH) [32,33] models have been proposed.

More recently, Gatheral et al. [34] reported that the increments of realized volatility (RV) exhibit a Hurst exponent of approximately $H \approx 0.1$, implying anti-persistent behavior. This phenomenon, known as rough volatility, has been confirmed across various asset classes [35–38], and rough fractional volatility models have been shown to provide an excellent fit to empirical data [34]. The CBOE Volatility Index (VIX), which is computed from option prices on the S&P 500 index, is a widely used measure of market uncertainty and stock market volatility. Studies on the Hurst exponent of the VIX time series have shown that, similar to RV, its value is less than 0.5, indicating anti-persistence, or the rough volatility property [39,40]. Under the Mixture of Distributions Hypothesis [41–43], price fluctuations are driven by the amount of information flowing into the market, and Clark [41] employs trading volume as a proxy for information arrival. In this framework, trading volume and volatility are expected to exhibit a strong correlation (As an illustration of the strong correlation between trading volume and volatility, it has been pointed out that introducing trading volume into GARCH models leads to a reduction in the GARCH effect [44]. Subsequently, a substantial body of research has incorporated volume variables into GARCH-type models. See, for example, [45–48]), implying that the volume time series should share statistical properties similar to those of the volatility time series. Indeed, it has been shown that the Hurst exponent of trading volume time series is also less than 0.5, reflecting the rough volatility property [49].

Given the wide variety of volatility models, each producing different estimates and forecasts, selecting an appropriate model for a given dataset remains a practical challenge. Model-free measures such as RV [30,50–52], constructed from high-frequency data, provide accurate estimates of daily volatility. However, forecasting still requires a model, leading to the development of hybrid approaches such as Realized GARCH [53,54], Realized SV [55–57], Heterogeneous Autoregressive (HAR) [58], and Realized HAR GARCH [59] models.

In this study, we employ quantum circuit learning (QCL) introduced by Mitarai et al. [60], which has been proposed in the field of quantum computing, as an alternative approach to modeling volatility. Quantum computers aim to achieve computational speedup over classical methods by exploiting quantum-specific properties; in particular, their application is expected in the fields of quantum simulation, such as computational chemistry and materials science (see, e.g., [61]). QCL employs parameterized quantum circuits (PQCs) for function approximation and classification tasks. Here, we attempt to approximate the volatility function utilizing the expressive power of QCL for function approximation, not computational speedup over classical methods using quantum computers.

Although QCL offers flexibility in designing quantum circuits, it does not require specifying a functional form for volatility dynamics. Unlike traditional volatility models, the model learns a mapping from past information to future volatility directly through the optimization of quantum circuit parameters. Prior work [62] has applied QCL to synthetic time series generated by GARCH models, demonstrating that PQCs can reproduce some statistical properties of the underlying data.

Here, we extend this line of research by applying QCL to real financial data. Using an RV constructed from high-frequency Bitcoin price data, we train a single-qubit PQC to approximate the volatility time series. We then analyze the statistical properties of the time series generated by the trained quantum circuit, with a particular focus on multifractal characteristics. Specifically, we employ Multifractal Detrended Fluctuation Analysis (MF DFA) [63] to estimate generalized Hurst exponents and investigate whether the QCL-generated time series reproduces empirical features such as anti-persistence and multifractality observed in realized volatility increments. Our results provide new insights into the expressive power of quantum circuits in modeling financial time series.

The remainder of this paper is organized as follows: Section 2 reviews GARCH-type volatility models and discusses their interpretation as functional approximations to volatility dynamics. Section 3 introduces the QCL framework and describes the single-qubit parameterized quantum circuit employed to approximate the volatility function. Section 4 outlines the MF DFA methodology used to characterize the statistical properties of both empirical and model-generated time series. Section 5 describes the data construction, including the computation of RV from high-frequency Bitcoin prices, and describes the parameter optimization procedure. Section 6 presents the empirical results, focusing on the Hurst exponent, anti-persistence, and multifractal properties of the predicted return and volatility increment series. Finally, Section 7 concludes with a discussion of the main findings, their implications, and directions for future research.

2. GARCH Models

The GARCH model expresses the conditional volatility as a function of past returns and past volatilities, which can be interpreted as a truncated Taylor expansion of a general volatility function. Let σ_t^2 denote the conditional volatility at time t and r_t the corresponding return. Suppose the next period volatility σ_{t+1}^2 is determined by some function f as

$$\sigma_{t+1}^2 = f(\sigma_t^2, r_t), \quad (1)$$

and that the return is generated by

$$r_t = \sigma_t \epsilon_t, \quad (2)$$

where $\epsilon_t \sim \mathcal{N}(0, 1)$ is an i.i.d. standard normal random variable.

Because the true functional form of $f(\sigma_t^2, r_t)$ is unknown, for practical purposes it must be approximated by some model. Expanding $f(\sigma_t^2, r_t)$ in a Taylor series around $\sigma_t^2 = 0$ and $r_t = 0$ gives

$$f(\sigma_t^2, r_t) = f_{00} + f_{10} \sigma_t^2 + f_{01} r_t + f_{11} \sigma_t^2 r_t + \frac{1}{2} f_{20} (\sigma_t^2)^2 + \frac{1}{2} f_{02} (r_t)^2 + \dots, \quad (3)$$

where the expansion coefficients are defined as

$$f_{ij} = \left(\frac{\partial}{\partial \sigma_t^2} \right)^i \left(\frac{\partial}{\partial r_t} \right)^j f(\sigma_t^2, r_t) \Big|_{\sigma_t^2, r_t=0}. \quad (4)$$

If volatility is assumed to be symmetric with respect to the sign of r_t , the coefficients f_{01} and f_{11} vanish. Retaining only the lowest-order terms in σ_t^2 and r_t yields

$$f(\sigma_t^2, r_t) = f_{00} + f_{10} \sigma_t^2 + \frac{1}{2} f_{02} (r_t)^2, \quad (5)$$

which corresponds to the well-known GARCH(1,1) model:

$$\sigma_{t+1}^2 = \omega + \beta \sigma_t^2 + \alpha r_t^2. \quad (6)$$

Here, α , β , and ω correspond to Taylor expansion coefficients. Since their true values are unknown, they are estimated by fitting the model to historical time series data—equivalently, one is estimating the expansion coefficients from the data.

Volatility asymmetry with respect to the sign of r_t is well documented in equity price time series. Retaining the term proportional to r_t to account for this asymmetry gives

$$f(\sigma_t^2, r_t) = f_{00} + f_{10} \sigma_t^2 + f_{01} r_t + \frac{1}{2} f_{02} (r_t)^2, \quad (7)$$

which is the functional form of the QARCH model [8]. Increasing the order of the Taylor expansion improves the approximation but also increases the number of free parameters, making estimation increasingly difficult without necessarily yielding commensurate gains in accuracy. An alternative is to approximate f by a class of functions other than polynomials. Padé approximants (In the finance literature, research applying Padé approximations to the approximation of probability distributions can be found in, for example, [64,65]), for example, use rational functions and are known to outperform Taylor series in many settings. A GARCH variant exploiting this idea is the RGARCH model [9]:

$$f(\sigma_t^2, r_t) = \frac{\omega + \alpha r_t^2 + \beta \sigma_t^2}{1 + \gamma r_t}, \quad (8)$$

and its stabilized version [10]:

$$f(\sigma_t^2, r_t) = \frac{\omega + \alpha r_t^2 + \beta \sigma_t^2}{\exp(\gamma r_t)}, \quad (9)$$

where γ captures volatility asymmetry. Setting $\gamma = 0$ reduces these to the GARCH(1, 1) model in Equation (6).

Other functional forms have also been proposed [66]. The Exponential GARCH (EGARCH) model [6] operates on the logarithm of the variance, while the APARCH

model [11] treats the power exponent of volatility as a free parameter. The diversity of available models reflects the fact that each produces different volatility estimates, and no single model dominates across all data sets and asset classes.

3. Quantum Circuit Learning

To model the volatility dynamics, this study employs the Quantum Circuit Learning (QCL) approach proposed by Mitarai et al. [60]. Within this framework, a parameterized quantum circuit is trained to approximate the underlying volatility-generation process.

The objective is to construct a function that maps historical market information to future volatility. Consistent with conventional GARCH-type specifications, the explanatory variables consist of the lagged volatility and lagged return. The volatility model is therefore represented as

$$v_t^2 = v(x_{t-1}, \theta), \quad (10)$$

where $x_{t-1} = (v_{t-1}^2, r_{t-1})$ denotes the input vector and θ represents the trainable circuit parameters.

The learning task consists of adjusting θ so that the model output reproduces the observed volatility sequence. Let σ_i^2 be the target volatility value and define the input data as

$$x_i = (r_i, \sigma_i^2), \quad (11)$$

where r_i and σ_i^2 correspond to the return and volatility at time i , respectively.

Figure 1 presents the parameterized single-qubit quantum circuit adopted in this study. This model is a parsimonious specification with three parameters, and it is adopted based on comparisons with existing models. The GARCH(1,1) model, expressed in Equation (6), is a simple three-parameter model; however, it is capable of capturing key characteristics of financial time series, such as volatility clustering, and is therefore widely used in empirical financial analysis. This suggests that it may be possible to describe essential features of the financial time series without introducing a large number of parameters. In this study, the model illustrated in Figure 1 is employed as the baseline specification. It should be noted, however, that a three-parameter model has inherent limitations in terms of expressibility. Depending on the type of the financial time series under consideration, more expressive models may be required to adequately capture the underlying dynamics.

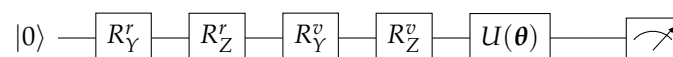


Figure 1. Single-qubit quantum circuit used for volatility modeling.

The trainable operation $U(\theta)$ is expressed by the following two-dimensional unitary matrix:

$$U(\theta) = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda} \sin(\theta/2) \\ e^{i\phi} \sin(\theta/2) & e^{i(\lambda+\phi)} \cos(\theta/2) \end{pmatrix}, \quad (12)$$

where $\theta = (\theta, \lambda, \phi)$ contains the variational parameters to be optimized during training.

The classical input variables are encoded into the quantum state through angle encoding using rotational gates R_Y and R_Z . Following the encoding scheme introduced in Ref. [60], the gate angles are defined as

$$R_Y^r = R_Y(\arcsin(r_i)), \quad (13)$$

$$R_Z^r = R_Z(\arccos(r_i^2)), \quad (14)$$

$$R_Y^v = R_Y(\arcsin(2\sigma_i^2 - 1)), \quad (15)$$

$$R_Z^v = R_Z(\arccos(\sigma_i^4)). \quad (16)$$

After application of the circuit, a measurement is performed in the computational (Z) basis. The probability of observing the state $|0\rangle$, denoted by P_0 , is interpreted as the forecasted volatility for the next period:

$$v_{i+1}^2 = P_0. \quad (17)$$

For numerical implementation, the probability P_0 is evaluated directly from the quantum state vector using Qiskit (IBM Qiskit: <https://www.ibm.com/quantum/qiskit>, accessed on 1 December 2025). Model training is carried out by optimizing the parameter vector θ so as to minimize the discrepancy between the predicted and observed volatilities. The loss function is defined as

$$\mathcal{L} = \frac{1}{N} \sum_{i=1}^N (v_i^2 - \sigma_i^2)^2, \quad (18)$$

where N denotes the total number of observations used for estimation.

4. Multifractal Analysis

To examine the multifractal characteristics of the analyzed time series, the generalized Hurst exponent is estimated using the Multifractal Detrended Fluctuation Analysis (MFDFA) framework. MFDFA, originally proposed by Kantelhardt et al. [63], has become a standard approach for detecting multifractal behavior in a wide range of financial datasets (see, for example, [67]). The main computational procedure is summarized below.

(i) Construction of the profile.

Consider a time series $\{x_t, t = 1, \dots, N\}$. The cumulative profile is obtained by integrating the demeaned observations:

$$y(k) = \sum_{t=1}^k [x_t - \langle x \rangle], \quad k = 1, \dots, N, \quad (19)$$

where $\langle x \rangle$ denotes the sample mean of the series.

(ii) Estimation of local fluctuations.

The profile $y(k)$ is partitioned into $N_s = \lfloor N/s \rfloor$ consecutive non-overlapping segments, each of length s . For every segment, a polynomial trend is fitted and removed, after which the variance of the detrended profile is evaluated. For the ν -th segment ($\nu = 1, \dots, N_s$), the variance is given by

$$F^2(s, \nu) = \frac{1}{s} \sum_{i=1}^s [y((\nu-1)s + i) - p_\nu(i)]^2, \quad (20)$$

where $p_\nu(i)$ denotes the fitted polynomial representing the local trend. In the present analysis, a cubic polynomial is adopted,

$$p_\nu(i) = a + b i + c i^2 + d i^3, \quad (21)$$

with the coefficients estimated through least-squares fitting.

Since the series length N is generally not an exact multiple of s , part of the data may remain unused. To incorporate these observations, the same segmentation procedure is repeated from the opposite end of the series. For $\nu = N_s + 1, \dots, 2N_s$, the detrended variance becomes

$$F^2(s, \nu) = \frac{1}{s} \sum_{i=1}^s [y(N - (\nu - N_s - 1)s - i + 1) - p_\nu(i)]^2. \quad (22)$$

(iii) Calculation of the fluctuation function.

Using the collection of variances $\{F^2(s, \nu)\}$, the fluctuation function of order q is defined as

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} [F^2(s, \nu)]^{q/2} \right\}^{1/q}. \quad (23)$$

For a process characterized by long-range scale-invariant correlations, the fluctuation function follows the power-law relation

$$F_q(s) \propto s^{h(q)}, \quad (24)$$

where $h(q)$ denotes the generalized Hurst exponent. Its value is estimated from the slope of the linear relationship in the logarithmic representation of Equation (24). When $q = 2$, MFDFA reduces to the conventional Detrended Fluctuation Analysis (DFA) [68], and the exponent $h(2)$ corresponds to the standard Hurst exponent [69].

The expression in Equation (23) is not valid for $q = 0$. Therefore, the fluctuation function for this case is evaluated as

$$F_0(s) = \exp \left\{ \frac{1}{4N_s} \sum_{\nu=1}^{2N_s} \ln [F^2(s, \nu)] \right\}. \quad (25)$$

(iv) Estimation of the generalized Hurst exponent.

The generalized Hurst exponent $h(q)$ is obtained from the scaling behavior described in Equation (24). Throughout this study, the calculation is carried out over the interval $q \in [-5, 5]$. Large values of $|q|$ may lead to numerical instability in the estimation of moments used in the fluctuation function. Jiang et al. [67] pointed out that, for low-frequency financial data such as daily observations, it is preferable to restrict the analysis to $|q| \leq 6$. Following their recommendation, the multifractal measures were computed over the range $q = [-5, 5]$ in this study.

A time series is classified as monofractal when $h(q)$ remains independent of q , whereas variation in $h(q)$ across different values of q indicates multifractality. For an ideal Gaussian random process, $h(q) = 1/2$ for all q . Consequently, deviations from this behavior provide evidence of multifractal structure and have often been associated with departures from market efficiency.

Several additional quantities can be derived from the generalized Hurst exponent [63]. One of them is the multifractal scaling exponent,

$$\tau(q) = q h(q) - 1, \quad (26)$$

which becomes a linear function of q in the monofractal case. The singularity spectrum $f(\alpha)$ is obtained through a Legendre transformation of $\tau(q)$:

$$\alpha(q) = \frac{d\tau(q)}{dq}, \quad (27)$$

$$f(\alpha(q)) = q\alpha(q) - \tau(q). \quad (28)$$

Substituting Equation (26) into the above expressions yields

$$\alpha(q) = h(q) + q \frac{dh(q)}{dq}, \quad (29)$$

$$f(\alpha(q)) = q[\alpha(q) - h(q)] + 1. \quad (30)$$

5. Data and Parameter Optimization

In this study, daily realized volatility RV_t is constructed from Bitcoin prices traded on the Bitstamp exchange, using price data sampled at five-minute intervals. While higher sampling frequencies improve the precision of RV estimates, they also amplify the influence of microstructure noise [70–72]. The five-minute sampling interval is therefore adopted as a standard compromise between estimation accuracy and noise contamination, consistent with the recommendation in the existing literature [73]. The five-minute log-return is defined as

$$r_{t,i} = \ln P_{t,i+1} - \ln P_{t,i}, \quad (31)$$

where $P_{t,i}$ denotes the five-minute price on day t for $i = 1, \dots, 1440/5$. The daily realized volatility RV_t on day t is then given by

$$RV_t = \sum_{i=1}^n r_{t,i}^2, \quad (32)$$

where n is the number of intraday observations; for five-minute sampling, $n = 1440/5 = 288$.

The data cover the period from 31 August 2020 to 1 March 2023 (UTC) (913 days, or approximately two and a half years) obtained from the Bitstamp exchange (Bitstamp exchange: <https://www.bitstamp.net/>, accessed on 3 March 2023).

We construct a 5-min price time series from the obtained high-frequency data and compute the realized volatility according to Equation (32). If no transaction data are available within a 5-min interval, we use the price from the previous interval. Figure 2 plots (a) the daily return series R_t , computed from daily prices P_t via Equation (31), and (b) the daily realized volatility RV_t .

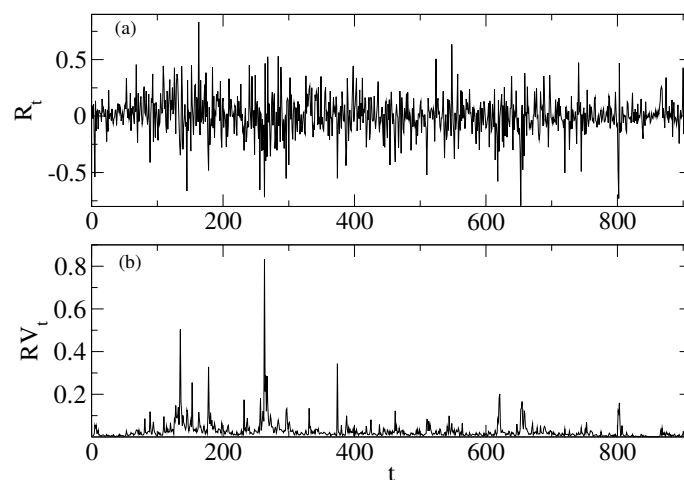


Figure 2. (a) Return time series R_t . (b) Realized volatility time series RV_t .

In this study, the model parameters are estimated using the entire realized volatility dataset to investigate its statistical and generative properties. Accordingly, we do not explicitly divide the data into training, validation, and testing subsets. The primary objective of this work is not to evaluate predictive performance, but rather to examine the model's ability to reproduce the empirical characteristics of the realized volatility.

The return series R_t and the RV series RV_t are used as inputs to the quantum circuit. Because input values are encoded as rotation angles, both series are pre-scaled: R_t is mapped to $[-1/c, 1/c]$ and RV_t to $[0, 1/c]$, with $c = 1.2$. In the formulation of Equation (11), the input data are given by $x_{t-1} = (R_{t-1}, RV_{t-1})$. The vector x_{t-1} is fed into the QCL model, and the predicted value for the next time step, $\overline{RV}_t$, is obtained as $\overline{RV}_t = P_0$ according to Equation (17). The resulting $\overline{RV}_t$ is then used to evaluate the loss function. The parameters of $U(\theta)$ are optimized by minimizing the loss function

$$\mathcal{L} = \frac{1}{T} \sum_{t=1}^T (RV_t - \overline{RV}_t)^2, \quad (33)$$

where $\overline{RV}_t$ is the quantum circuit output at time t and T is the total number of observations. Parameters are optimized using the COBYLA algorithm.

We initialized the parameters with random values and executed the COBYLA algorithm using several different random initializations; while the algorithm generally exhibited good convergence, we observed poor convergence in certain cases depending on the initial values. When convergence was achieved, the loss function converged to similar values. Figure 3 illustrates the relationship between the loss function and the number of iterations for a converged case.

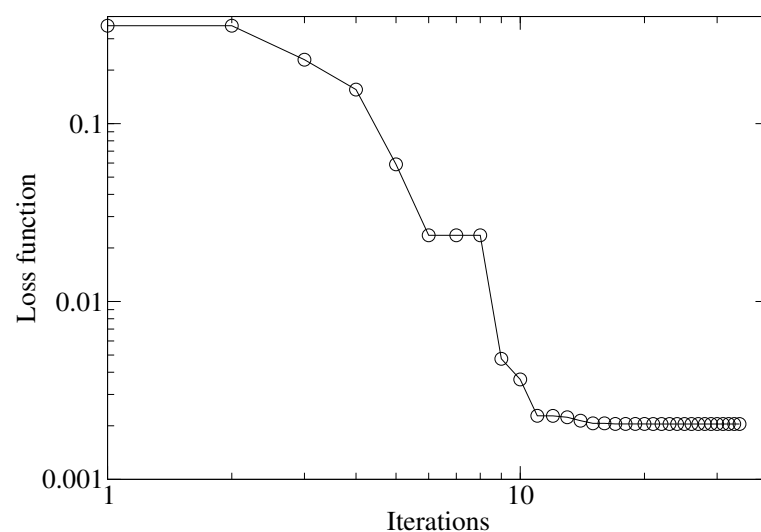


Figure 3. Loss function during optimization by the COBYLA algorithm as a function of iteration.

Figure 4 compares the input series RV_t with the circuit output $\overline{RV}_t$ after optimization. The circuit output reproduces the broad temporal variation in the input RV_t .

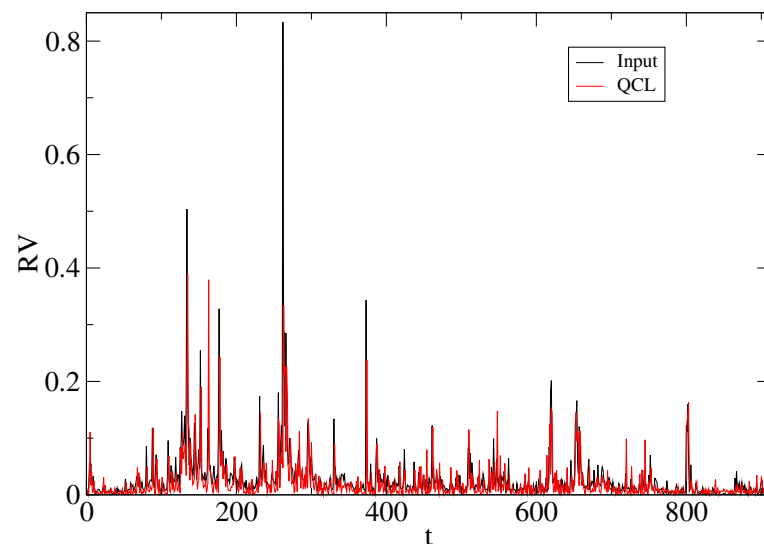


Figure 4. (Black line) Empirical time series data of RV_t used as input for the QCL. (Red line) QCL outputs $\overline{RV}_t$ after optimization.

6. Properties of the Predicted Time Series

To investigate the statistical properties of the QCL model, we generated 50,000 time steps of predicted data by the QCL model with the optimized parameters θ . The QCL model first predicts the next realized volatility $\overline{RV}_t$ using the input data from the previous period. Based on the predicted realized volatility, the corresponding return $\overline{R}_t$ is then computed as follows:

$$\overline{R}_t = \overline{RV}_t^{1/2} \epsilon_t, \quad (34)$$

where $\epsilon_t \sim \mathcal{N}(0, 1)$.

Using the predicted return $\overline{R}_t$ and realized volatility $\overline{RV}_t$ as new input data, the model proceeds to forecast the subsequent return and realized volatility.

The formulation in Equation (34) reflects the empirical observation that returns normalized by realized volatility following a Gaussian distribution [74–76]. Consequently, the fluctuations in realized volatility allow the return distribution to exhibit fat tails. For instance, if the realized volatility follows an inverse-gamma distribution (The empirical results support that the distribution of realized volatility follows an inverse gamma distribution [77]), the resulting return distribution corresponds to a Student's t-distribution.

Figure 5 shows the first 25,000 observations of the predicted return series $\overline{R}_t$ (red line), together with the 913 observations of the input data R_t (black line) used for parameter optimization. Similarly, Figure 6 displays the predicted volatility series $\overline{RV}_t$ (red line) and the input RV_t (black line).

The generalized Hurst exponent $h(q)$ is obtained as the scaling exponent of the fluctuation function, as defined in Equation (24). Figure 7 shows a log–log plot of the fluctuation function for the Bitcoin return time series as a representative example. In this study, we computed the fluctuation function for q in the range $[-5, 5]$ with a step size of $\Delta q = 0.02$. In Figure 7, for the sake of clarity, we only plot the results with a step size of $\Delta q = 0.1$, demonstrating linearity in the region of large s . We performed the linear fitting in the region $s \geq 20$ to estimate the scaling exponent $h(q)$.

The generalized Hurst exponent $h(q)$ of the 50,000-observation time series is estimated using a rolling window approach with a window size of 913 observations and a step size of 365 observations. To compare with random time series, we generated 20 surrogate time series by randomly shuffling the data within each window. We then computed the generalized Hurst exponent for each surrogate time series and averaged the results.

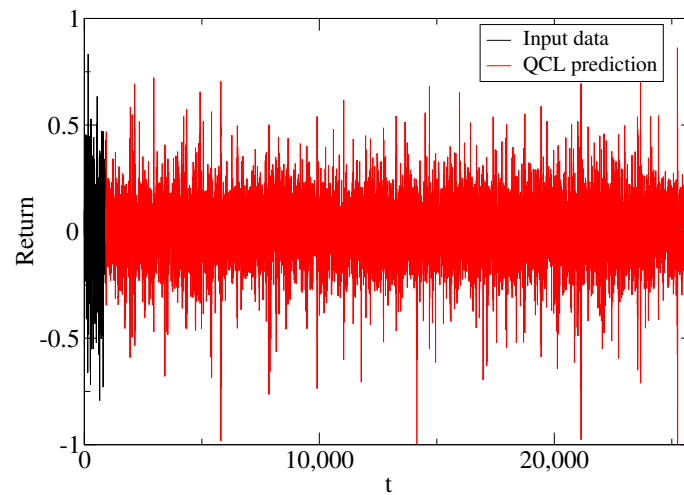


Figure 5. The predicted return series $\bar{R}_t$ (red line) and the input data R_t (black line).

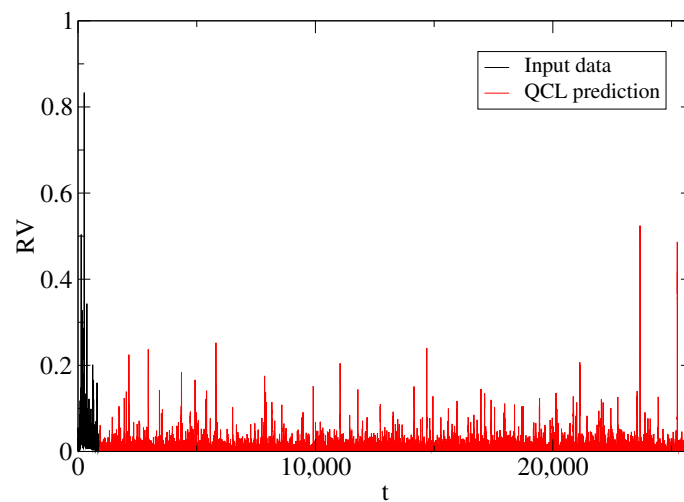


Figure 6. The predicted volatility series $\bar{RV}_t$ (red line) and the input RV_t (black line).

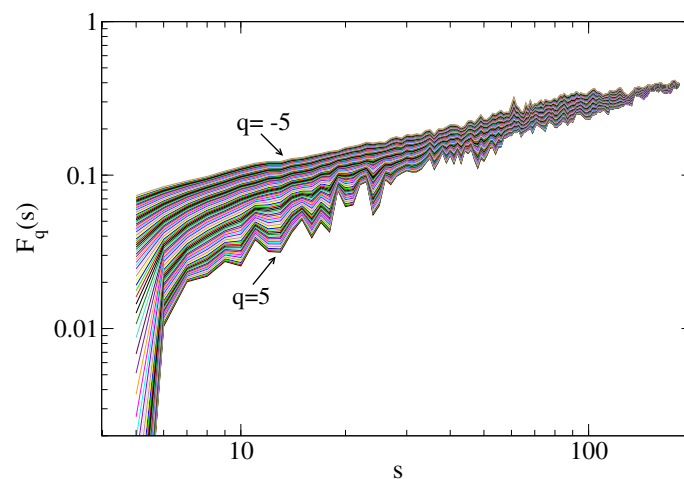


Figure 7. The fluctuation function obtained from the empirical return time series. The fluctuation functions are plotted with a step size of $\Delta q = 0.1$ in $q = [-5, 5]$.

Figure 8 shows the time variation in the Hurst exponent $h(2)$ for the predicted and shuffled return series, analyzed by the rolling window method. The values fluctuate around 0.5, indicating that the predicted return series is approximately random (Empirical studies have shown that, in most developed markets, the Hurst exponent of asset returns is close to 0.5, corresponding to a random time series; however, it has also been reported that some

developed markets exhibit anti-persistent behavior, with Hurst exponents below 0.5 [78]), consistent with the empirical behavior of Bitcoin returns reported in recent results [79]. The $h(2)$ value of the empirical return series used as input data is also close to 0.5 (Table 1). It should be noted that, since the return in the QCL model is defined by Equation (34), it is expected that the Hurst exponent takes a value close to 0.5; therefore, the present result simply confirms the generation of return time-series data based on Equation (34).

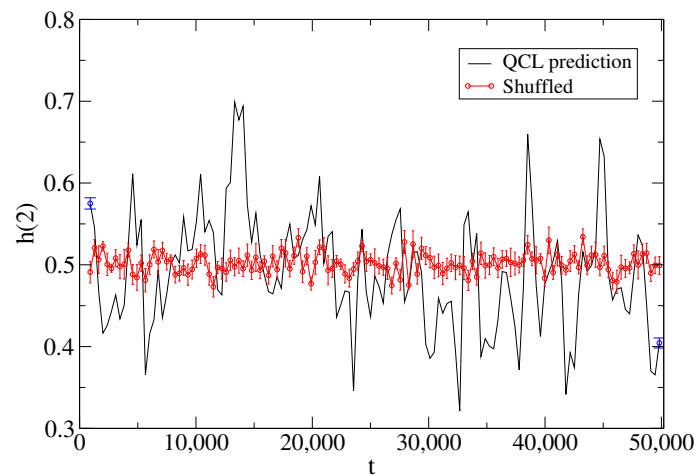


Figure 8. Time variation in the Hurst exponent $h(2)$ for the predicted and shuffled return series. The blue symbols with error bars in the figure indicate the typical magnitude of the fitting errors.

Table 1. $h(2)$ for empirical (input) return, IRV, and IAR time series.

	$h(2)$
Return	0.504(5)
Return (Shuffled)	0.498(11)
IRV	0.155(2)
IRV (Shuffled)	0.516(11)
IAR	0.030(1)
IAR (Shuffled)	0.494(14)

Figures 9–11 display $h(q)$, $\alpha(q)$, and $\tau(q)$ for the QCL-predicted return series in a representative window. The variation in $h(q)$ with q seems to show multifractality in the predicted return series, and after shuffling, the q -dependence of $h(q)$ is reduced but not eliminated. Both $f(\alpha)$ and $\tau(q)$ also exhibit similar multifractal properties.

However, it is known that finite-size effects in the time series of finite length can lead to the emergence of spurious multifractality [80–82]. To estimate the magnitude of these effects for the time series length used in this study, we generated 200 Gaussian time series with a length of $n = 913$, computed their $h(q)$, $\alpha(q)$, and $\tau(q)$ values, and plotted the results in Figures 12–14. For comparison, the scales in these figures were set to be identical to those in Figures 9–11. It is observed that while Gaussian time series are inherently monofractal, they exhibit multifractality due to finite-size effects. Comparing Figures 9–11 with Figures 12–14, the degree of multifractality in the original time series is smaller in the Gaussian case, suggesting that the results presented in Figures 9–11 indicate the possible presence of multifractality beyond what can be attributed to finite-size effects.

It has been reported that the fat-tailed nature of the distribution in a time series can act as a source of multifractality [63] (Table 2 shows the kurtosis of return, IRV and IAR time series analyzed in this study). Although shuffling a time series eliminates temporal correlations and is expected to allow for the separation of contributions from temporal

correlations and distribution shapes as the origin of multifractality, recent studies have reported that true multifractality arises solely from non-linear temporal correlations [83,84]. Consequently, it is difficult to discuss the origin of multifractality based solely on the shuffling process. Therefore, this study only reports the results for the original and shuffled time series.

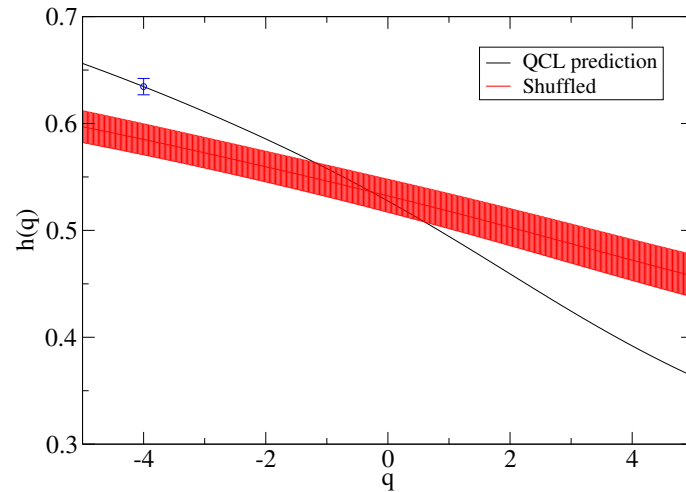


Figure 9. $h(q)$ of the QCL-predicted and shuffled return time series in a representative window. The blue symbol with error bars in the figure indicates the typical magnitude of the fitting error.

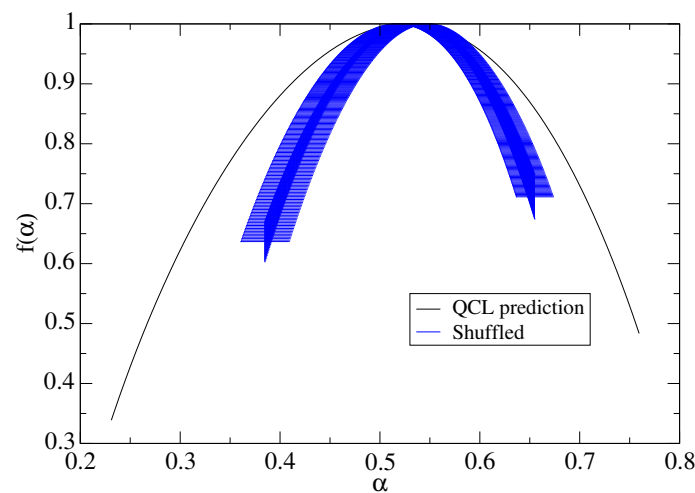


Figure 10. $f(\alpha)$ of the QCL-predicted and shuffled return time series in a representative window.

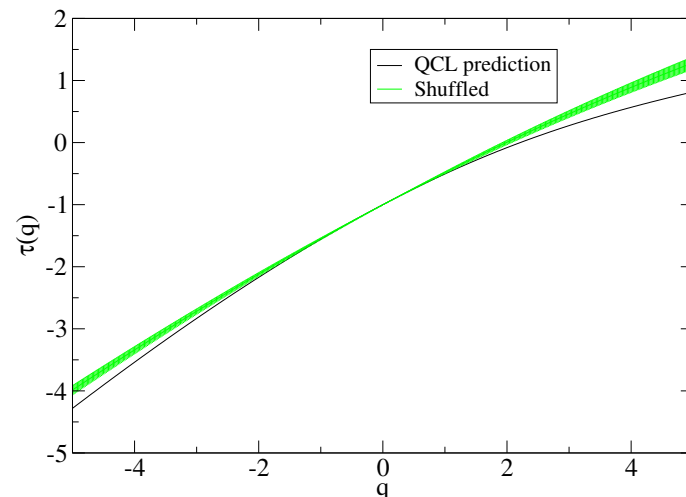


Figure 11. $\tau(q)$ of the predicted return time series in a representative window.

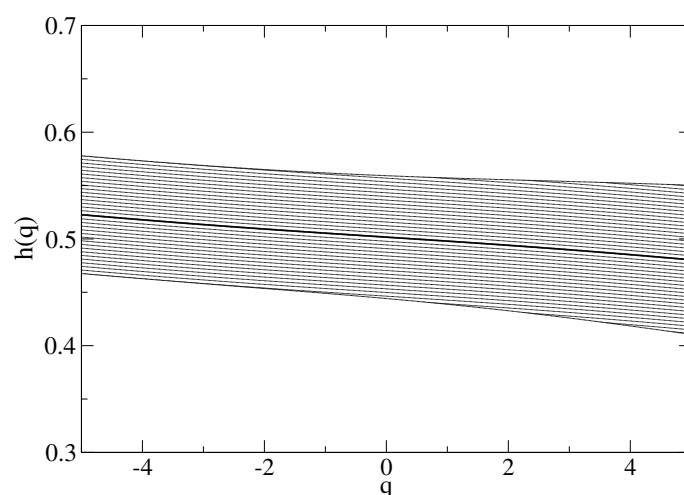


Figure 12. $h(q)$ of Gaussian random time series from 200 samples. The width in the figure represents the standard deviation.

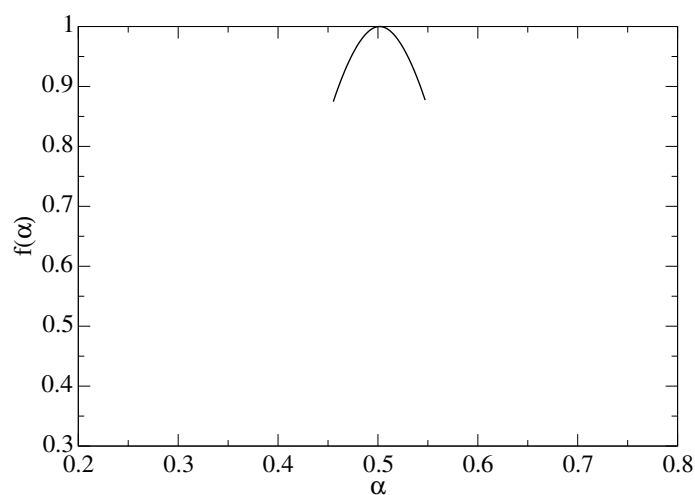


Figure 13. $f(\alpha)$ of Gaussian random time series. $f(\alpha)$ was obtained from $h(q)$ shown in Figure 12 numerically.

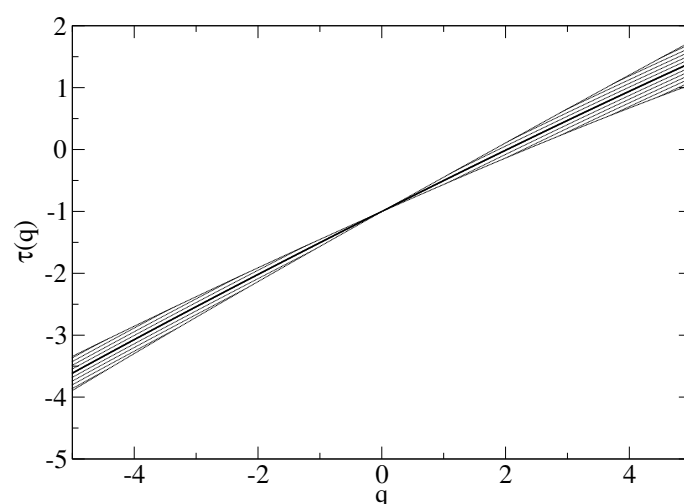


Figure 14. $\tau(q)$ of Gaussian random time series from 200 samples. The width in the figure represents the standard deviation.

From Table 2, we observe that the kurtosis of both the empirical return data and the QCL model predictions exceeds 3, the value for a Gaussian distribution, indicating the presence of fat-tailed behavior. However, the kurtosis of the QCL model is larger than

that of the empirical data. This discrepancy suggests that the single-qubit QCL model employed in this study may not fully capture certain properties of the empirical data and may overestimate the fat-tailed behavior.

Table 2. Kurtosis of return, IRV, and IAR time series.

	Input (Empirical) Data	QCL Prediction
Return	5.79(60)	15.7(33)
IRV	4.43(87)	4.55(7)
IAR	4.40(42)	4.35(10)

Figures 15–17 display $h(q)$, $f(\alpha)$, and $\tau(q)$ for the empirical return time series used as input. The multifractal characteristics of the empirical data closely resemble those of the time series generated by the QCL model, demonstrating that the model reproduces the qualitatively similar multifractal features of empirical returns.

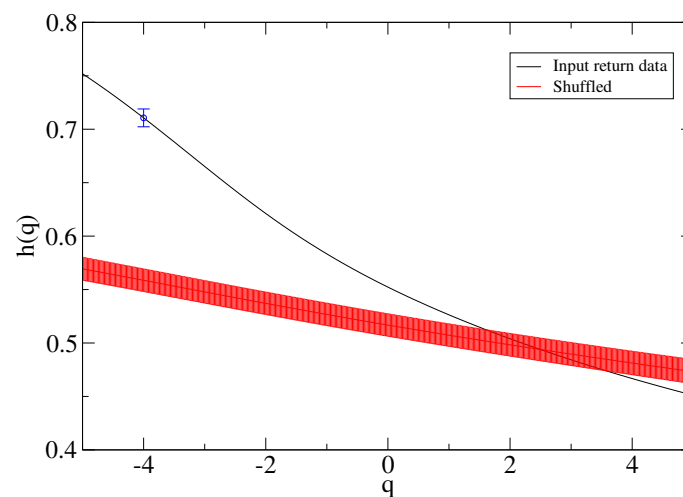


Figure 15. $h(q)$ of the empirical (input) return time series. The blue symbol with error bars in the figure indicates the typical magnitude of the fitting error.

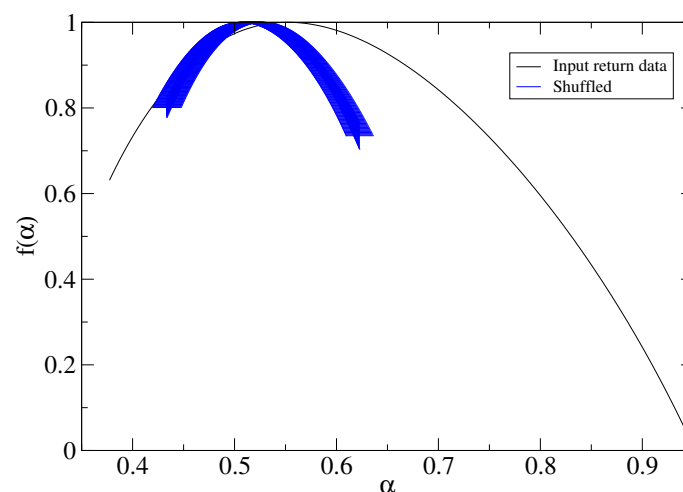


Figure 16. $f(\alpha)$ of the empirical (input) return time series.

Next, the same multifractal analysis is applied to the increment series of realized volatility (IRV), defined as

$$\text{IRV}_t = \ln \text{RV}_t - \ln \text{RV}_{t-1}. \quad (35)$$

Figure 18 shows the Hurst exponent $h(2)$ of the IRV series, with values in the range $h(2) \approx 0.05\text{--}0.1$ (on average ~ 0.07), well below the random benchmark of 0.5. This clearly confirms the presence of anti-persistence, which is consistent with the rough-volatility characteristics observed in the real financial data of Bitcoin [85]. After shuffling, $h(2)$ reverts to approximately 0.5, demonstrating that the anti-persistence is attributable to the temporal correlation structure of the time series.

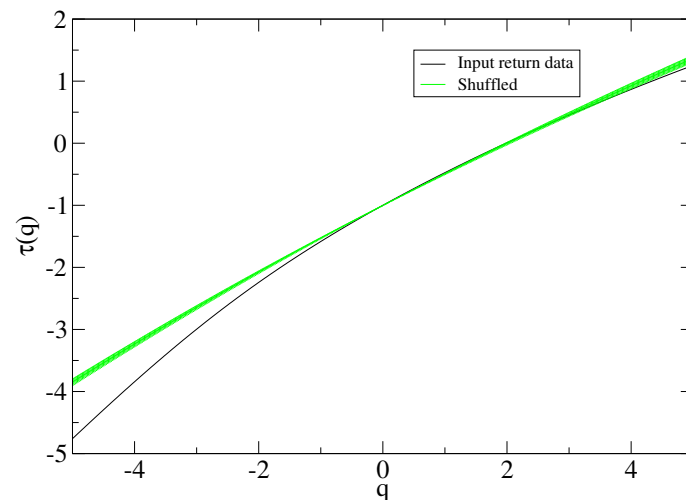


Figure 17. $\tau(q)$ of the empirical (input) return time series.

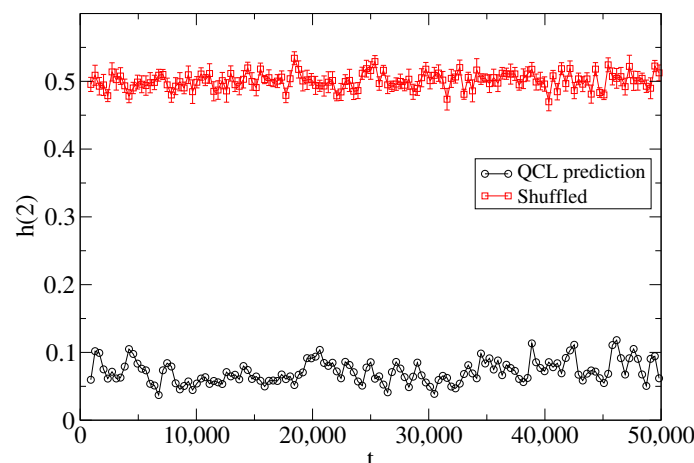


Figure 18. Hurst exponent $h(2)$ of the IRV time series and those of the shuffled time series. For QCL predictions, the typical magnitude of the fitting error is on the order of the symbol size or smaller.

RV requires high-frequency data for accurate construction; however, when such data are unavailable, the absolute value of daily returns is often used as a proxy for volatility. In relation to RV, the square of the absolute daily return corresponds to the case in which RV is constructed using only a single intraday observation. Let $|R_t|$ denote the absolute daily return. The increment series of absolute returns, denoted IAR_t , is defined as

$$IAR_t = \ln |R_t| - \ln |R_{t-1}|. \quad (36)$$

Figure 19 plots the Hurst exponent of the increment series IAR . The estimated values fall within the range of $h(2) \approx 0.04\text{--}0.07$ (on average ~ 0.05), indicating anti-persistent behavior. These values are smaller than those obtained for the IRV (see also Figure 20). Prior studies [38,86] examining RV constructed with varying sampling frequencies report that the estimated Hurst exponent decreases as the number of intraday observations

used in RV construction becomes smaller. The present results are consistent with this empirical regularity.

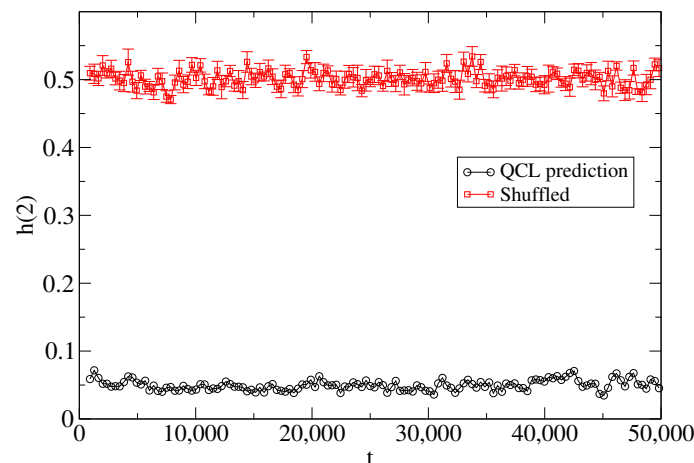


Figure 19. Hurst exponent $h(2)$ of the IAR time series and those of the shuffled time series. For QCL predictions, the typical magnitude of the fitting error is on the order of the symbol size or smaller.

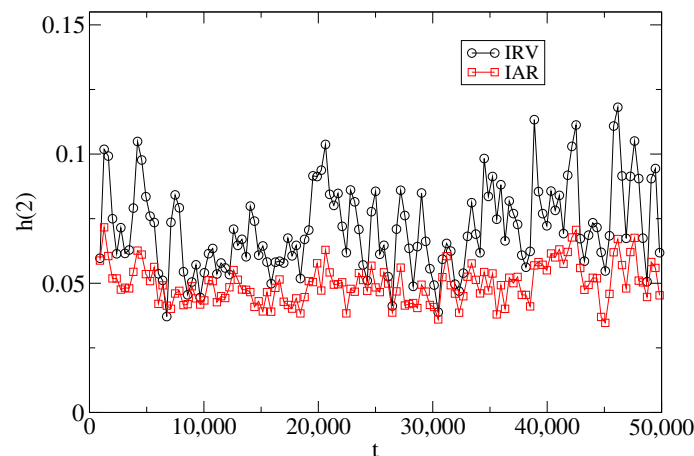


Figure 20. Comparison of the Hurst exponent $h(2)$ computed for the IRV and IAR time series. The typical magnitude of the fitting error is on the order of the symbol size or smaller.

Next, we investigate the multifractal properties of the IRV series. Figure 21 displays the generalized Hurst exponent $h(q)$ for both the QCL-predicted IRV series and its shuffled counterpart within a representative window. The IRV series exhibits a monotonically decreasing $h(q)$ as q increases, demonstrating clear multifractality. Although the shuffled series shows $h(2)$ values shifted toward 0.5, consistent with random behavior, its $h(q)$ still decreases monotonically with q , indicating that multifractality persists even after temporal correlations are removed. Comparing the variation range of $h(q)$ between the IRV and shuffled series reveals that the width of the $h(q)$ spectrum remains largely unchanged.

Figure 22 presents the singularity spectra $f(\alpha)$ computed for the IRV time series and its shuffled counterpart within a representative window. Both $f(\alpha)$ exhibit a finite width in α , indicating that each series possesses multifractal characteristics.

A similar pattern is observed for $\tau(q)$: in both the original and shuffled series, $\tau(q)$ remains nonlinear, further confirming the presence of multifractality (Figure 23). After shuffling, the position of the peak in $f(\alpha)$ shifts. This is because the value of $h(0)$, which corresponds to the peak position, increases after the shuffling process. Similarly, the slope of $\tau(q)$ becomes steeper after shuffling; this is attributed to the fact that the values of $h(q)$, which approximately represent the magnitude of the slope of $\tau(q)$, increase after shuffling, as shown in Figure 23.

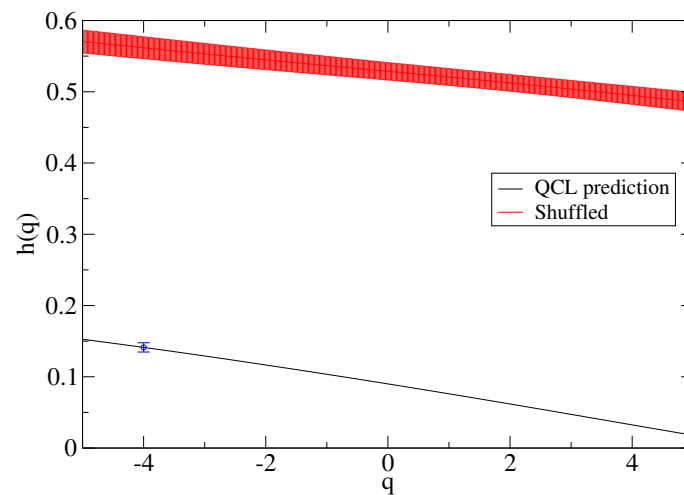


Figure 21. $h(q)$ of the IRV time series in a representative window. The blue symbol with error bars in the figure indicates the typical magnitude of the fitting error.

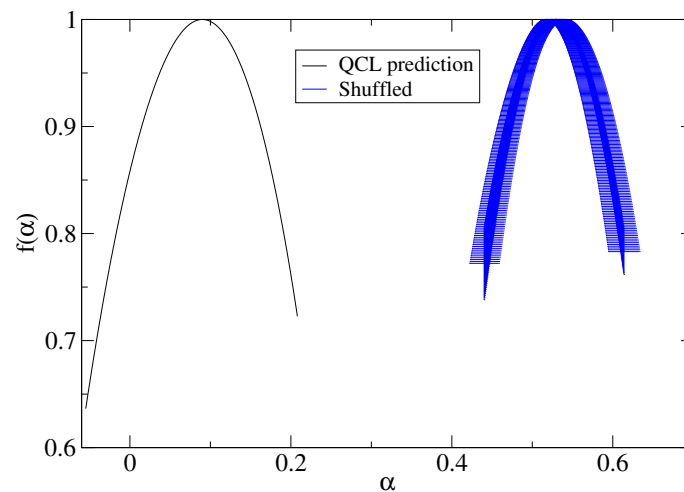


Figure 22. $f(\alpha)$ of the IRV time series in a representative window.

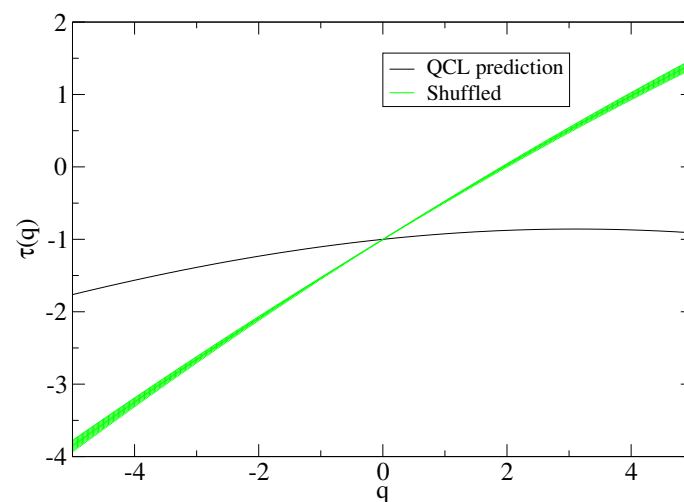


Figure 23. $\tau(q)$ of the IRV time series in a representative window.

Figures 24–26 present the multifractal analysis results of $h(q)$, $f(\alpha)$, and $\tau(q)$ for the IRV time series constructed from the RV data used as input to the QCL model. The input data exhibit multifractal behavior similar to that observed in the QCL-generated time series

(Figures 21–23), demonstrating that the multifractal characteristics of the empirical IRV series are similar to those generated by the QCL model.

Overall, our results demonstrate that the QCL-generated time series qualitatively reproduces empirical properties of Bitcoin volatility, including anti-persistence and multifractality in IRV.

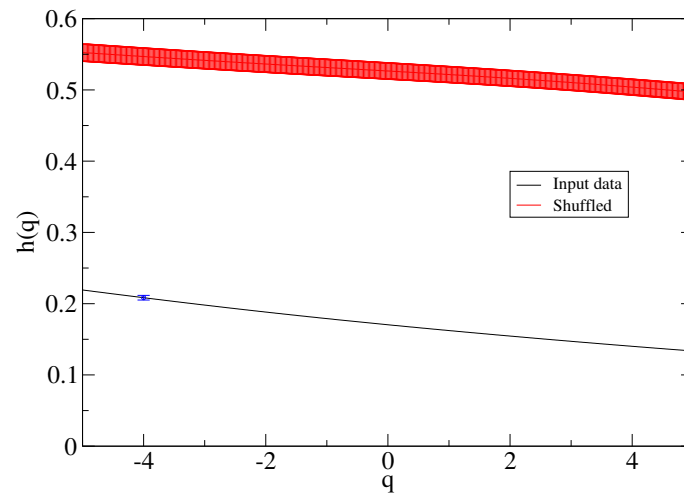


Figure 24. $h(q)$ of the IRV time series computed from the input data. The blue symbol with error bars in the figure indicates the typical magnitude of the fitting error.

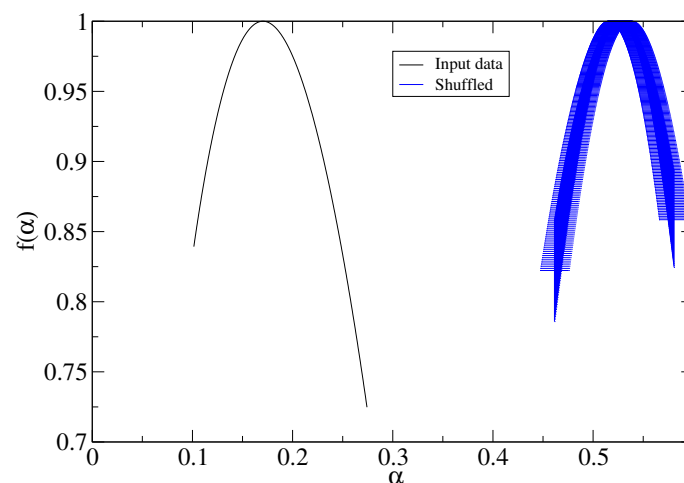


Figure 25. $f(\alpha)$ of the IRV time series computed from the input data.

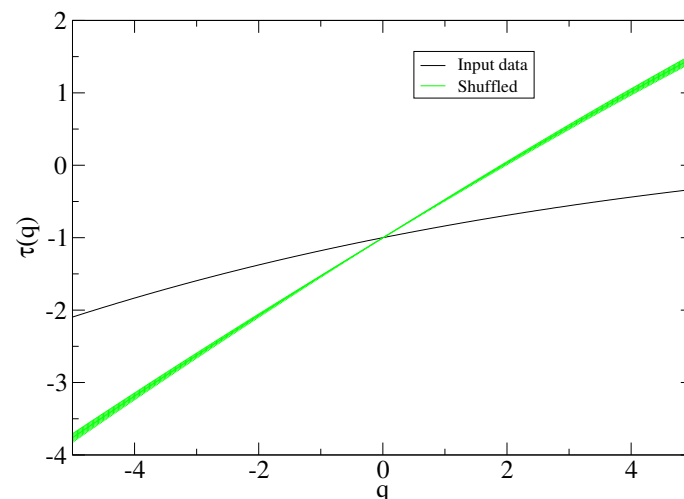


Figure 26. $\tau(q)$ of the IRV time series computed from the input data.

7. Discussion and Conclusions

In this study, the QCL was applied to approximate the RV time series of Bitcoin, and the resulting predicted series is subjected to multifractal analysis. We discuss the significance and interpretation of the results below.

7.1. Effectiveness of QCL for Volatility Approximation

As shown in Figure 4, the optimized quantum circuit reproduces the broad temporal variation in the empirical RV series. Unlike GARCH-type models, which require a pre-specified functional form, the present approach directly approximates the volatility function from data, eliminating the need for model selection. A limitation, however, is that the single-qubit, three-parameter circuit employed here has limited expressive power; deeper multi-qubit circuits or the data re-uploading technique [87,88] may improve the approximation accuracy.

7.2. Multifractality of the Return Time Series

Multifractality in $h(q)$ is observed for the predicted return series (Figure 9), and partial multifractality persists even after shuffling (Figures 9–11). However, analysis based on finite-length Gaussian random time series (Figures 12–14) indicates the presence of multifractality arising from finite-size effects, suggesting that the partial multifractality observed after shuffling can be attributed to such finite-size effects. Analogous results are obtained for the original data (Figures 15–17), confirming that the QCL model reproduces the multifractal structure of the empirical return time series.

7.3. Anti-Persistence of IRV

The IRV series exhibits $h(2) \approx 0.05\text{--}0.1$ (Figure 18), which is consistent with the rough volatility findings of Gatheral et al. [34]. This anti-persistence is reproduced by the QCL model, and the reversion of $h(2)$ to approximately 0.5 after shuffling (Figure 18) clearly attributes the anti-persistence to temporal correlations. The smaller $h(2)$ values of the IAR time series relative to IRV (Figure 20) are consistent with the finding that the RV from fewer intraday samples produces lower Hurst exponents [38]. The randomness observed in the return series is expected from the model definition; however, the emergence of anti-persistence in volatility is a non-trivial issue. The fact that the volatility time series predicted by the QCL model exhibits anti-persistence can be regarded as evidence that the model successfully captures the essential characteristics of volatility dynamics in real data.

7.4. Comparison and Future Directions

The QCL model qualitatively reproduces the major multifractal properties of the empirical Bitcoin RV series: near-random return dynamics ($h(2) \approx 0.5$), multifractality of returns, and anti-persistence of IRV ($h(2) \approx 0.05\text{--}0.1$).

It is noted that the present model adopts a single-qubit parameterized quantum circuit with a highly simplified structure, inspired by classical parsimonious models such as GARCH(1, 1), which successfully capture some stylized facts of financial markets using only a small number of parameters. The primary objective of this study is to examine whether such a minimal QCL framework can reproduce the essential statistical properties of financial time series. However, in doing so, it may constrain the expressive power of the model. Therefore, extensions to deeper architectures and multi-qubit circuits are expected to be necessary for enhancing representational capacity, which we leave for future investigations. Furthermore, demonstrating the practical utility of the proposed approach requires systematic benchmarking against established models, including GARCH, HAR, and Realized GARCH, as well as evaluation in tasks such as volatility forecasting. Such

analyses are beyond the scope of the present study and are reserved for future work. Accordingly, this study should be regarded as an initial step toward assessing the capability of QCL to capture statistical properties of financial time series, while comprehensive performance comparisons and applications to forecasting and risk management remain important directions for future research.

Additional future directions include the confirmation of empirical facts observed in financial asset prices—such as the Taylor effect [89–91], the leverage effect [92–94], the Zumbach effect [95,96], and the inverse cubic law in return distributions [97]—as well as applications to other asset classes (equities, foreign exchange, and commodities) to assess the generality of the approach.

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Data Availability Statement: The original data presented in the study are openly available at [<https://github.com/takaishi-physics/QCL-volatility>], accessed on 23 June 2026.

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Conflicts of Interest: The author declares no conflicts of interest.

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