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High-Harmonic Spectroscopy as a Probe of Fractional Quantum Dynamics

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Abstract

We investigate high-order harmonic generation (HHG) driven by strong laser fields in a reduced-dimensionality time-dependent Schrödinger equation (TDSE) model generalized through a fractional kinetic-energy operator. By fixing the ionization potential, we isolate the effect of the fractional continuum dispersion from trivial changes in the binding energy. A classical three-step analysis is first extended to the fractional case, showing that the maximum return energy remains proportional to a fractional ponderomotive scale, but with a cutoff coefficient that decreases systematically as the fractional exponent is reduced. We then benchmark these predictions with TDSE simulations for different values of the fractional exponent α . The computed HHG spectra exhibit a progressive reduction in the accessible cutoff region and pronounced modifications of the plateau interference structure. Time-frequency Gabor analysis of the dipole acceleration further reveals that the fractional kinetic operator reshapes the recollision dynamics, modifying the emission branches and enhancing interference features associated with multiple quantum pathways and returns. Finally, we analyze cutoff scaling laws with respect to laser intensity and wavelength. At fixed wavelength, the shifted cutoff harmonic follows an effective fractional intensity scaling, while at fixed intensity the cutoff energy shift follows a fractional wavelength scaling. Although the cutoff extracted from finite-pulse TDSE spectra should be regarded as an empirical measure, the combined spectral, scaling, and time-frequency analyses demonstrate, within the present model, that non-quadratic continuum dispersion can produce distinctive signatures in strong-field dynamics.

Keywords: fractional quantum mechanics; strong-field physics; time-dependent simulations



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1. Introduction

High-harmonic generation (HHG) has evolved from a nonlinear frequency up-conversion process into a precision spectroscopic method capable of probing electronic structure and ultrafast dynamics with attosecond and angstrom resolution [1–4]. In atoms and molecules, the harmonic spectrum encodes the strong-field ionization step, laser-driven continuum propagation, and recombination with the parent ion, forming the basis of the semiclassical three-step model [1]. This recollision sensitivity has enabled orbital tomography [5], channel-resolved molecular spectroscopy [6,7], and multidimensional high-harmonic spectroscopy [8]. In solids, HHG provides complementary access to band

structure, interband polarization, and ultrafast electron–hole dynamics under extreme fields [9,10].

A central observable in HHG spectroscopy is the cutoff energy. In conventional strong-field physics, the cutoff follows the well-known semiclassical law

$$\hbar\omega_{\text{cut}} = I_p + 3.17U_p, \quad (1)$$

where I_p is the ionization potential and U_p is the ponderomotive energy. This result is not simply an empirical scaling rule; it is a direct consequence of the quadratic kinetic-energy dispersion of ordinary quantum mechanics and of the associated classical electron trajectories in the laser field. The classical cutoff therefore provides a particularly stringent diagnostic of the kinetic law governing continuum propagation. Any modification of the electron dispersion is expected to alter the mapping between ionization time, return time, and recombination energy, and hence to modify the cutoff scaling with laser intensity, wavelength, and field amplitude.

A fundamentally different scenario emerges in fractional quantum mechanics, where the Laplacian kinetic operator is replaced by a fractional Riesz operator motivated by path integrals over Lévy trajectories rather than Brownian paths [11,12]. The resulting fractional Schrödinger equation modifies the dispersion from the conventional quadratic form $p^2/2m$ to a nonlocal form proportional to $|p|^\alpha$, with $1 < \alpha \leq 2$. This modification preserves Hermiticity but changes the relation between momentum, velocity, phase accumulation, and wave-packet spreading. In the strong-field context, these changes directly affect the three essential stages of HHG: tunneling ionization, continuum acceleration, and recombination.

The present work focuses on the continuum stage and asks whether the HHG cutoff can serve as a clear signature of fractional quantum dynamics. To this end, we develop a fractional classical recollision model derived from the generalized dispersion relation [13]

$$\mathcal{E}_\alpha(p) = D_\alpha |p|^\alpha, \quad (2)$$

where D_α is the fractional kinetic coefficient and the conventional limit is recovered at $\alpha = 2$. In this model, the electron momentum is still driven by the laser field through the classical force equation, but the velocity is no longer proportional to the momentum. Instead, the velocity follows from the fractional group velocity,

$$v_\alpha(p) = \frac{\partial \mathcal{E}_\alpha}{\partial p} = \alpha D_\alpha |p|^{\alpha-1} \text{sgn}(p). \quad (3)$$

As a result, the return condition, excursion time, and maximum recombination energy are all modified. The usual cutoff law $I_p + 3.17U_p$ is then replaced by a fractional cutoff scaling law whose dependence on laser intensity and wavelength is controlled by α .

Strong-field phenomena are uniquely sensitive to such changes. Tunneling ionization depends exponentially on the kinetic operator through barrier penetration, while continuum propagation determines the recollision timing and return energy. Recombination amplitudes depend on the near-core structure of the continuum wave function, which is also reshaped by fractional dynamics. HHG therefore provides simultaneous constraints on all three stages of the process. Unlike observables that probe only bound-state spectra, harmonic emission measures real-time continuum motion under intense driving, where nonlocal kinetic effects can accumulate over sub-cycle trajectories.

A first step in this direction has recently been established for the ionization stage alone. In static-field tunneling ionization within space-fractional quantum mechanics, the conventional ADK/PPT exponent can be systematically deformed by replacing the quadratic dispersion with $\mathcal{E}_\alpha(p)$. For the convention $D_\alpha = 1/2$, the resulting fractional-ADK exponent changes the familiar $I_p^{3/2}$ dependence into an $I_p^{1+1/\alpha}$ scaling and introduces a characteristic complex-branch factor $\sin(\pi/\alpha)$ while recovering the standard ADK result in the limit $\alpha \rightarrow 2$ [14]. Time-dependent simulations of the one-dimensional fractional Schrödinger equation further show that ionization remains exponentially tunneling-like, but with a reduced effective action and enhanced ionization probability as α decreases. This static-field result provides a controlled benchmark for the first step of HHG and motivates the next question addressed here: whether the same fractional kinetic law also leaves a measurable imprint on the continuum recollision stage, in particular on the cutoff energy and its scaling with laser parameters.

We propose HHG spectroscopy as a potential probe of fractional quantum dynamics. Fractional propagation predicts an altered mapping between return time and emission frequency, a modified cutoff scaling with intensity and wavelength, and changes in the interference between short and long trajectories. Within the present theoretical framework, the fractional cutoff law provides a means of distinguishing the effects of non-quadratic continuum dispersion from a simple rescaling of the laser parameters. Because the cutoff, trajectory structure, and harmonic-yield scaling are interdependent, HHG observables offer multiple, complementary signatures of fractional kinetic behavior. Establishing the generality of these signatures in realistic physical systems, however, requires further theoretical and experimental investigation.

The use of HHG spectroscopy to impose multiple independent constraints on electron dynamics is naturally connected to broader theoretical developments in fractional quantum evolution. In particular, non-Brownian quantum processes governed by Lévy statistics provide a path-integral route to nonlocal evolution equations. Turgeman, Carmi, and Barkai derived a fractional Feynman–Kac equation for functionals of Lévy paths, leading to forward and backward fractional Schrödinger-type equations involving fractional substantial derivatives [15]. Although formulated from a statistical perspective, this framework reinforces the idea that non-Gaussian path ensembles can generate effective quantum dynamics beyond the standard Brownian/quadratic-dispersion paradigm. Related developments in fractional Hamiltonian wave dynamics, including integrable fractional nonlinear Schrödinger systems that reduce to fractional Schrödinger-type equations in the linear limit [16], further illustrate how fractional derivatives can enter conservative wave evolution. Together with Laskin’s formulation of fractional quantum mechanics based on Lévy path integrals [11–13], these results provide the theoretical background for considering kinetic operators whose dispersion is nonquadratic and nonlocal. In the strong-field setting, HHG offers a particularly sensitive arena to test such modified transport laws, because tunneling, continuum acceleration, and recombination all depend on the same underlying dispersion relation.

By leveraging the conceptual infrastructure of attosecond HHG spectroscopy [3], semiclassical recollision physics [1,2], and molecular orbital reconstruction [5], the present framework recasts HHG from a source of ultrafast light to a probe of fundamental quantum transport laws. In this sense, the HHG cutoff is not only a spectral boundary, but also a dynamical fingerprint of the underlying kinetic operator. Establishing how this boundary scales in fractional quantum mechanics opens a path toward experimentally accessible tests of nonlocal quantum dynamics in strong-field regimes available with current laser technology. Atomic units are used unless otherwise stated.

2. Classical Model: Hamilton Equations and Three-Step Trajectories

To establish a transparent reference for the cutoff scaling, we reformulate the classical recollision model for an electron driven by a linearly polarized laser field in the fractional scenario. The electric field is taken as

$$E(t) = E_0 \sin(\omega t), \quad (4)$$

where E_0 is the peak field amplitude and ω is the laser angular frequency. In the calculations presented below, we use $\omega = 0.057$ a.u. (corresponding to a laser wavelength $\lambda = 800$ nm) and $E_0 = 0.053$ a.u. (corresponding to a laser intensity $I = 1 \times 10^{14}$ W/cm²). The associated optical period is $T = \frac{2\pi}{\omega} \sim 110$ a.u. (~ 2.7 fs).

The conventional three-step model assumes that the electron is released into the continuum at an ionization time t_i , propagates solely under the laser field, the binding potential is neglected, and recombines with the parent ion when it returns to its initial position [1]. The initial conditions at ionization are therefore

$$x(t_i) = 0, \quad p(t_i) = 0, \quad (5)$$

and the recombination, or recollision, time t_r is determined by the return condition

$$x(t_r; t_i) = 0, \quad t_r > t_i. \quad (6)$$

For the conventional case, the continuum Hamiltonian is

$$H_2(x, p, t) = \frac{p^2}{2} + xE(t), \quad (7)$$

which gives the Hamilton equations

$$\dot{x} = p, \quad \dot{p} = -E(t). \quad (8)$$

Using Equation (4) together with the initial conditions (5), the canonical momentum is obtained analytically as

$$p(t; t_i) = \frac{E_0}{\omega} [\cos(\omega t) - \cos(\omega t_i)]. \quad (9)$$

Thus, the corresponding trajectory is

$$x(t; t_i) = \frac{E_0}{\omega^2} [\sin(\omega t) - \sin(\omega t_i) - \omega(t - t_i) \cos(\omega t_i)]. \quad (10)$$

The return time is then obtained by solving Equation (6). The kinetic energy at recombination is

$$K_2(t_r; t_i) = \frac{p^2(t_r; t_i)}{2}. \quad (11)$$

Maximizing this quantity over all first-return trajectories gives the standard semiclassical cutoff

$$K_{2,\max} \simeq 3.17 U_p, \quad (12)$$

where

$$U_p = \frac{E_0^2}{4\omega^2} \quad (13)$$

is the conventional ponderomotive energy.

We now generalize this construction to the fractional classical model. The fractional continuum dispersion is written as [13]

$$\mathcal{E}_\alpha(p) = D_\alpha |p|^\alpha, \quad 1 < \alpha \leq 2, \quad (14)$$

where D_α fixes the kinetic-energy scale. In the present calculations, we use $D_\alpha = \frac{1}{2}$, so that the conventional dispersion is recovered smoothly at $\alpha = 2$.

The corresponding time-dependent Hamiltonian is

$$H_\alpha(x, p, t) = D_\alpha |p|^\alpha + xE(t). \quad (15)$$

Therefore, Hamilton's equations become

$$\dot{x} = \frac{\partial H_\alpha}{\partial p} = \alpha D_\alpha |p|^{\alpha-1} \text{sgn}(p), \quad \dot{p} = -\frac{\partial H_\alpha}{\partial x} = -E(t). \quad (16)$$

Let us note that the laser-driven canonical momentum is unchanged with respect to the conventional case, because the force equation remains $\dot{p} = -E(t)$. Therefore, Equation (9) remains valid for all α . The fractional modification enters through the velocity,

$$v_\alpha(p) = \dot{x} = \alpha D_\alpha |p|^{\alpha-1} \text{sgn}(p), \quad (17)$$

which is no longer proportional to p when $\alpha \neq 2$. The fractional trajectory must therefore be obtained from

$$x_\alpha(t; t_i) = \int_{t_i}^t \alpha D_\alpha |p(t'; t_i)|^{\alpha-1} \text{sgn}[p(t'; t_i)] dt', \quad (18)$$

with $p(t; t_i)$ given by Equation (9). The first-return time is defined by as in the conventional case, i.e.,

$$x_\alpha(t_r; t_i) = 0, \quad t_r > t_i. \quad (19)$$

The fractional kinetic energy carried by the electron at recombination can be obtained as

$$K_\alpha(t_r; t_i) = D_\alpha |p(t_r; t_i)|^\alpha. \quad (20)$$

To compare different values of α on a common scale, we introduce a fractional ponderomotive energy. Since the kinetic energy is no longer quadratic in the canonical momentum, the conventional expression $U_p = E_0^2/(4\omega^2)$ is not the natural expression for $\alpha \neq 2$. We therefore define

$$U_{p,\alpha} = D_\alpha \left(\frac{E_0}{\omega} \right)^\alpha \langle |\cos(\omega t)|^\alpha \rangle, \quad (21)$$

where the cycle average results

$$\langle |\cos(\omega t)|^\alpha \rangle = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\alpha+2}{2}\right)}. \quad (22)$$

For $\alpha = 2$ and $D_\alpha = 1/2$, this definition reduces exactly to the standard ponderomotive energy:

$$U_{p,2} = \frac{1}{2} \left(\frac{E_0}{\omega} \right)^2 \langle \cos^2(\omega t) \rangle = \frac{E_0^2}{4\omega^2} = U_p. \quad (23)$$

The fractional three-step procedure is then implemented as follows. For each ionization time in the interval

$$0 \leq \frac{t_i}{T} \leq 0.5, \quad (24)$$

the momentum is computed from Equation (9). The fractional displacement is obtained from Equation (18), and the first root satisfying Equation (19) is identified within the recombination window

$$0.5 \leq \frac{t_r}{T} \leq 1.25. \quad (25)$$

The return energy is then evaluated from Equation (20) and normalized by $U_{p,\alpha}$. The fractional cutoff is defined as

$$K_{\alpha,\max} = \max_{t_i} K_{\alpha}(t_r; t_i), \quad (26)$$

where t_r is the first-return solution associated with t_i . In addition to the cutoff energy, it is useful to track the excursion time of the cutoff trajectory $\tau = t_r - t_i$. This quantity measures the time spent by the electron in the continuum between ionization and recombination.

Table 1 summarizes the cutoff data for selected values of α . The conventional limit reproduces the standard cutoff value $K_{\max}/U_p \simeq 3.17$. The definition of the fractional ponderomotive energy, Equation (21), allows us to make the fractional case also independent of the laser intensity, i.e., the maximum kinetic energy fraction at recollision is constant, irrespective of the laser intensity. Reducing α lowers the normalized cutoff energy and shifts the ionization time associated with the cutoff to later phases of the cycle. The recombination time changes only weakly, remaining near $t_r/T \simeq 0.95$, while the excursion time decreases monotonically.

Table 1. Classical fractional three-step cutoff data. The energy is normalized to the fractional ponderomotive scale $U_{p,\alpha}$ defined in Equation (21).

α	$C_{\alpha} = K_{\alpha,\max}/U_{p,\alpha}$	t_i/T	t_r/T	τ/T
2.00	3.17310	0.29967	0.95078	0.65111
1.80	3.04109	0.30551	0.94889	0.64338
1.40	2.75378	0.32053	0.95013	0.62960
1.01	2.42757	0.34641	0.95761	0.61120

The monotonic reduction in τ/T with decreasing α provides an additional dynamical signature of the fractional three-step model. The return-energy cutoff is modified together with the subcycle timing of the trajectory, indicating that fractional dynamics changes the mapping between ionization time, recombination time, and emitted photon energy.

We solve numerically the Hamilton equations, Equation (16), for the fractional classical three-step model and construct the corresponding recollision map shown in Figure 1. For each value of the fractional order α , the ionization time t_i is scanned in the interval $0 \leq t_i/T \leq 0.5$, and the first return satisfying $x_{\alpha}(t_r; t_i) = 0$ is searched within the recombination window $0.5 \leq t_r/T \leq 1.25$. The canonical momentum is obtained from the laser force equation $\dot{p} = -E(t)$, while the position is propagated using the fractional velocity, Equation (17). At each return, the kinetic energy is evaluated using Equation (20), and normalized to the fractional ponderomotive scale $U_{p,\alpha}$, Equation (21).

Figure 1 displays the resulting fractional recollision map. The horizontal axis is the time in optical cycles, t/T . The interval $0 \leq t/T \leq 0.5$ corresponds to the ionization times t_i , while the interval $0.5 \leq t/T \leq 1.25$ corresponds to the associated recombination times t_r . For each ionization time, the trajectory is propagated according to Equation (18), and the first return is selected from the condition Equation (19). The vertical axis shows the return kinetic energy computed from Equation (20) and normalized by the fractional ponderomotive scale defined in Equation (21).

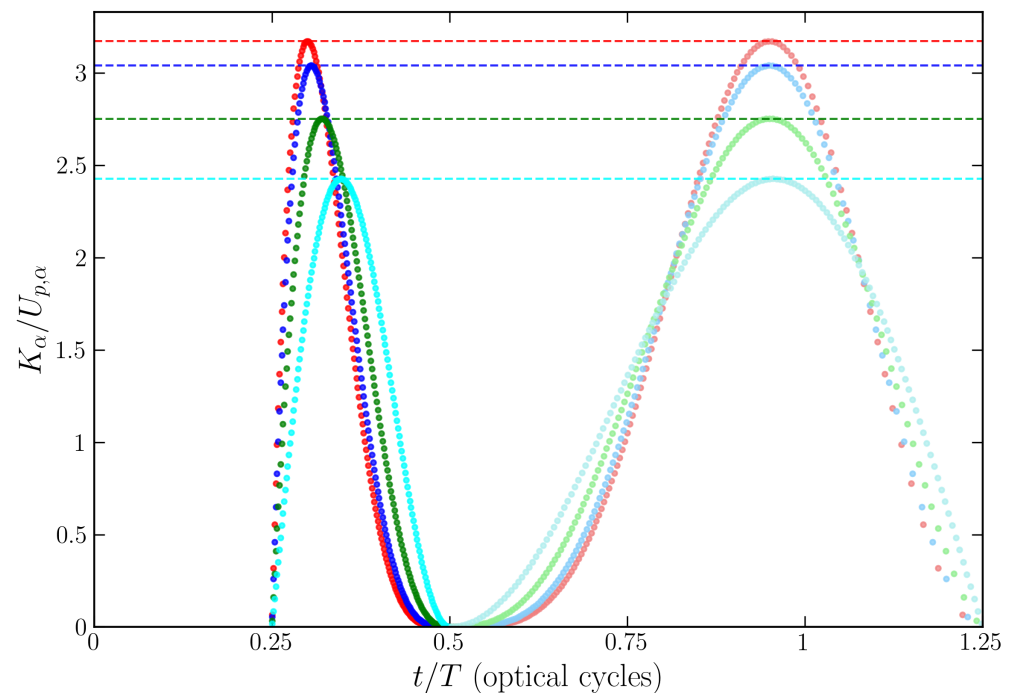


Figure 1. Fractional classical recollision map obtained from the Hamilton equations associated with Equation (15). The horizontal axis shows time in optical cycles. Points in the interval $0 \leq t/T \leq 0.5$ represent ionization times t_i , while points in the interval $0.5 \leq t/T \leq 1.25$ represent the corresponding recombination times t_r . The return condition is $x_\alpha(t_r; t_i) = 0$, Equation (19). The vertical axis shows the return kinetic energy $K_\alpha = D_\alpha |p(t_r; t_i)|^\alpha$ normalized to $U_{p,\alpha}$, Equations (20) and (21). Solid colors denote ionization times and lighter colors denote recombination times. Horizontal dashed lines mark the cutoff energies $K_{\alpha,\max}/U_{p,\alpha}$ for the four values of α listed in Table 1.

Each returning trajectory is represented by two points at the same value of $K_\alpha/U_{p,\alpha}$: one point at the ionization time t_i and one point at the recombination time t_r . Solid colors denote the ionization branch, while the lighter colors denote the corresponding recombination branch. The four color pairs correspond to the four values of α reported in Table 1, namely red color $\alpha = 2$ (conventional case), blue color $\alpha = 1.8$, green color $\alpha = 1.4$ and cyan color $\alpha = 1.01$. The horizontal dashed lines indicate the cutoff energies $K_{\alpha,\max}/U_{p,\alpha}$, as defined in Equation (26).

The map makes clear that decreasing α modifies both the cutoff energy and the subcycle timing of the cutoff trajectory. In the conventional limit, $\alpha = 2$, the dashed cutoff line lies close to the standard value $K_{2,\max} \simeq 3.17U_p$, consistent with Equation (12). For smaller α , the normalized cutoff decreases, in agreement with Table 1. In the conventional case, the cutoff trajectory has an excursion time of approximately $0.65T$, consistent with the usual excursion trajectory that produces the maximum return energy. As α decreases, we observe that the cutoff ionization time shifts to later phases of the optical cycle, while the recombination time remains close to $t_r/T \simeq 0.95$. Consequently, the cutoff excursion time decreases systematically. This behavior shows that the fractional kinetic law modifies the continuum propagation itself: the cutoff is not produced by a simple rescaling of the conventional trajectory but by a shifted ionization phase and a shorter effective continuum excursion.

3. One-Dimensional Time-Dependent Schrödinger Equation Model

To complement the classical cutoff analysis, we solve a one-dimensional time-dependent Schrödinger equation (1D-TDSE) model for a hydrogen-like atom driven by a linearly polarized laser field. The purpose of this calculation is not to reproduce the full

three-dimensional Coulomb problem but rather to provide a controlled quantum benchmark in which the ionization potential is fixed, the driving field is varied systematically, and the resulting high-order harmonic generation (HHG) spectra can be compared with the fractional classical cutoff predictions.

The field-free atomic Hamiltonian is written as [13]

$$\hat{H}_0^{(\alpha)} = D_\alpha \left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} + V(x), \quad (27)$$

where $1 < \alpha \leq 2$ is the fractional kinetic exponent and D_α is the generalized kinetic prefactor. In the present calculations, we set $D_\alpha = \frac{1}{2}$ for all the studied cases. The fractional kinetic operator is evaluated spectrally. Namely, if $\tilde{\psi}(k, t)$ is the Fourier transform of $\psi(x, t)$, then

$$\left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} \psi(x, t) \longleftrightarrow |k|^\alpha \tilde{\psi}(k, t). \quad (28)$$

Thus, the kinetic-energy dispersion is

$$T_\alpha(k) = D_\alpha |k|^\alpha. \quad (29)$$

The binding potential is modeled by a soft-core Coulomb potential,

$$V(x) = -\frac{Z}{\sqrt{x^2 + a_\alpha^2}}, \quad (30)$$

with $Z = 1$. For each value of α , the soft-core parameter a_α is chosen so that the ground-state ionization potential is fixed at $I_p = 0.5$ a.u., corresponding to the hydrogen ionization potential, for all the α values. This fixed- I_p protocol is essential: it ensures that the changes observed in the cutoff position arise from the modified kinetic dispersion and not from a trivial change of the binding energy [14].

The laser-driven 1D-TDSE is solved in the length gauge,

$$i \frac{\partial}{\partial t} \psi(x, t) = \left[D_\alpha \left(-\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} + V(x) + xE(t) \right] \psi(x, t), \quad (31)$$

using a laser field of the form

$$E(t) = E_0 f(t) \sin(\omega_0 t + \varphi_{\text{CEP}}), \quad (32)$$

where E_0 is the laser electric field peak amplitude, ω_0 is the carrier frequency, φ_{CEP} the carrier envelope phase, and $f(t)$ is a finite pulse envelope. The simulations discussed here are performed at fixed ionization potential and compare four representative values of the fractional exponent: $\alpha = 2.00$, corresponding to the conventional Schrödinger case, and $\alpha = 1.80, 1.40$, and 1.01 , which progressively depart from the quadratic dispersion limit. This set spans the standard case, the strongly fractional near- $\alpha = 1$ regime, and two intermediate values, allowing us to track the continuous deformation of the HHG cutoff and time-frequency structure as the dispersion law is modified.

For the laser field, we set $E_0 = 0.0755$ a.u. (corresponding to a laser intensity $I = 2 \times 10^{14}$ W/cm²), $\omega_0 = 0.057$ a.u. ($\lambda = 800$ nm) and $f(t)$ as a trapezoidal-shaped laser pulse with one cycle ramp up (down) and six constant cycles (total time duration ~ 21 fs).

The ground state is obtained by imaginary-time propagation of the field-free Hamiltonian. Starting from a localized trial wavepacket, the wavefunction is propagated with $t \rightarrow -i\tau$ and renormalized after each step until convergence of the ground-state energy.

The real-time propagation is then performed using a split-operator method. In Fourier space, the kinetic propagator is diagonal,

$$\exp[-iD_\alpha|k|^\alpha\Delta t], \quad (33)$$

whereas the potential and laser-interaction terms are diagonal in coordinate space. Absorbing boundaries are included near the edges of the numerical box in order to suppress unphysical reflections of the ionized wavepacket.

The emitted harmonic radiation is computed from the dipole acceleration. In the acceleration form, we use

$$a(t) = \left\langle \psi(t) \left| -\frac{dV}{dx} \right| \psi(t) \right\rangle - E(t), \quad (34)$$

with

$$-\frac{dV}{dx} = -\frac{Zx}{(x^2 + a_\alpha^2)^{3/2}}. \quad (35)$$

The HHG spectrum is obtained from the Fourier transform of the windowed acceleration,

$$S(\omega) = \left| \int dt w(t) a(t) e^{-i\omega t} \right|^2, \quad (36)$$

where $w(t)$ is a smooth temporal window.

For comparison between the different spectra, we also compute the fractional classical cutoff. The harmonic cutoff is written as

$$q_c^{(\alpha)} = \frac{I_p + K_{\alpha,\max}}{\omega_0}, \quad (37)$$

where $K_{\alpha,\max}$ can be extracted from Table 1. For $\alpha = 2$, we recover the usual three-step result

$$q_c^{(2)} = \frac{I_p + 3.17U_p}{\omega_0}. \quad (38)$$

In the fractional cases, the numerical classical trajectory calculation gives the values (extracted from Table 1) $K_{1.8,\max} = 3.04109U_{p,1.8}$, $K_{1.4,\max} = 2.75378U_{p,1.4}$, and $K_{1.01,\max} = 2.42757U_{p,1.01}$. This trend can be understood as a direct consequence of the fractional dispersion law. As α is reduced, the continuum electron no longer follows the standard quadratic energy–momentum relation. The laser-driven momentum therefore maps into velocity, excursion time, and return energy differently. In practice, the returning electron accumulates a smaller maximum kinetic energy when measured in units of the corresponding fractional ponderomotive scale. This explains why the cutoff coefficient decreases systematically from the standard value near 3.17 for $\alpha = 2$ to lower values as the dynamics becomes increasingly fractional.

Figure 2 shows the resulting HHG spectra. Each panel corresponds to a different value of the fractional exponent. The $\alpha = 2$ case provides the standard Schrödinger benchmark, while decreasing α modifies the continuum dispersion and shifts the effective cutoff. In all the plots, the dashed vertical lines indicate the classical cutoff prediction from Equation (37). Each spectrum is normalized independently, so the figure clearly highlights the displacement of the cutoff, rather than comparing absolute harmonic yields across different values of α that are not an adequate observable in this single-atom simulations. Likewise, the interference structure within the plateau reflects the coherent superposition of different quantum-emission pathways. In the standard HHG picture, plateau harmonics receive contributions from electron wave packets that ionize at different times and recombine with

the parent core along distinct short and long trajectories. Their relative phases determine the oscillatory modulation of the spectral yield across the plateau. In the fractional case, the modified dispersion law changes the phase accumulated during the continuum excursion and therefore alters both the position and contrast of these interference fringes. Thus, the plateau modulations provide an additional diagnostic of how the fractional kinetic operator reshapes the electron dynamics, beyond the overall displacement of the cutoff. The HHG spectrum provides the global spectral content of the emitted radiation, but it does not directly indicate when, during the laser pulse, the different harmonic components are generated. This temporal information is particularly important in the present problem, because the fractional dispersion modifies the relation between momentum, velocity, excursion time, and return energy. As a result, changes in the cutoff are expected to be accompanied by changes in the time-frequency structure of the emission.

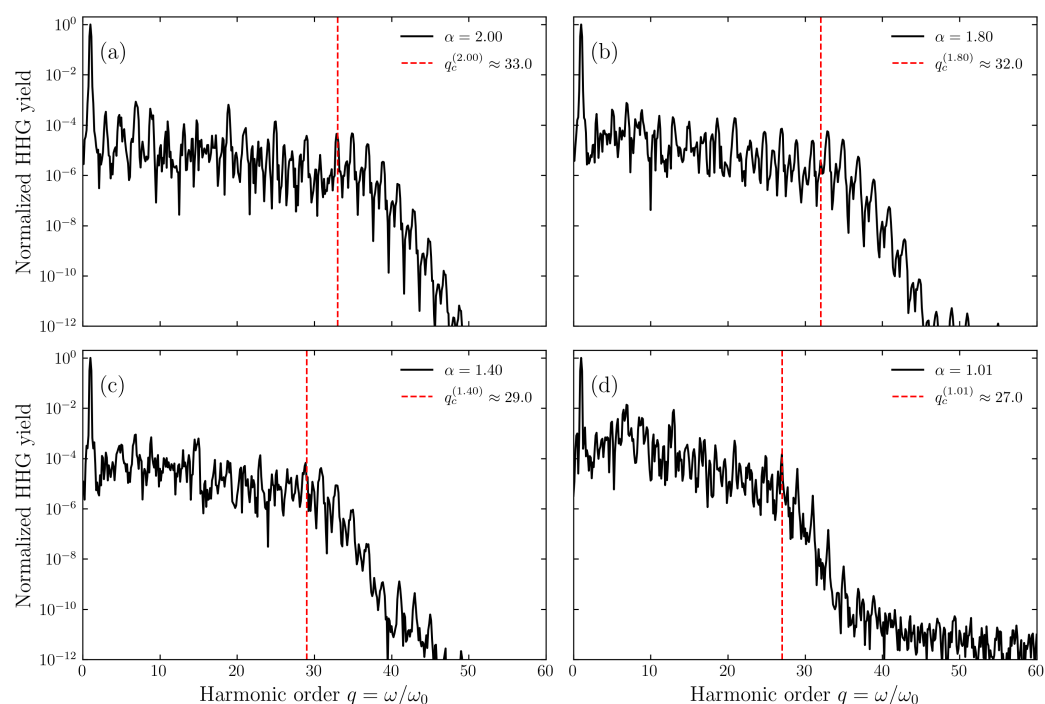


Figure 2. HHG spectra obtained from the 1D-TDSE simulations for four values of the fractional exponent: (a) $\alpha = 2.00$, (b) $\alpha = 1.80$, (c) $\alpha = 1.40$, and (d) $\alpha = 1.01$. The spectra are shown as normalized harmonic yields on a logarithmic scale as a function of the harmonic order $q = \omega/\omega_0$. The vertical dashed lines indicate the corresponding classical cutoff positions, $q_c^{(\alpha)} = (I_p + K_{\alpha, \max})/\omega_0$, with $K_{\alpha, \max} = C_\alpha U_{p, \alpha}$ (see Table 1). The cutoff coefficients C_α are obtained from the fractional classical trajectory calculation. The $\alpha = 2$ panel corresponds to the conventional Schrödinger case, while decreasing α illustrates the systematic modification of the cutoff structure induced by the fractional dispersion law.

To resolve these features, we compute a Gabor transform of the dipole acceleration,

$$G(t, \omega) = \int dt a(\tau) \exp \left[-\frac{\sigma^2 (\tau - t)^2}{2} \right] e^{-i\omega\tau}, \quad (39)$$

and compute the power $|G(t, \omega)|^2$ as a function of the emission time t/T and the harmonic order $q = \omega/\omega_0$, where $T = 2\pi/\omega_0$. In the calculations, we use a Gaussian time window with a width parameter of the order of a few carrier frequencies, typically $\sigma = 3\omega_0$, which provides a compromise between temporal localization of the recollision bursts and spectral resolution of the harmonic ridges.

Figure 3 displays the Gabor power maps, $|G(t, \omega)|^2$, for the same four values of α presented in Figure 2. These time-frequency distributions resolve the sub-cycle emission bursts associated with the return of the electron wave packet to the core. In the conventional $\alpha = 2$ case, the dominant emission branches are consistent with the familiar short- and long-trajectory structure of the three-step model. As α is reduced, the fractional dispersion modifies the relation between momentum, velocity, excursion time, and recombination energy. Consequently, the emission branches are reshaped and the cutoff region is progressively reduced. In addition, the Gabor maps reveal a much richer structure in the middle part of the pulse and throughout the plateau region for the fractional cases. This structure can be associated with the coherent interference of several emission pathways, including contributions from short and long trajectories as well as from multiple-return events. In the fractional cases, the modified dispersion changes the phase accumulated during the continuum excursion and alters the timing of the returning wave packets. As a result, the different contributions overlap and interfere in a modified way, producing additional modulations and branch-like features within the plateau. This richer time-frequency pattern therefore provides further evidence that the fractional kinetic operator does not merely shift the cutoff but reorganizes the underlying recollision dynamics. This effect is more pronounced for the strongly fractional near- $\alpha = 1$ case. Figures 2 and 3 therefore provide complementary views of the same physics: the HHG spectra quantify the cutoff shift and plateau interference structure, while the Gabor maps reveal how the associated emission times and trajectory branches are reorganized by the fractional kinetic operator.

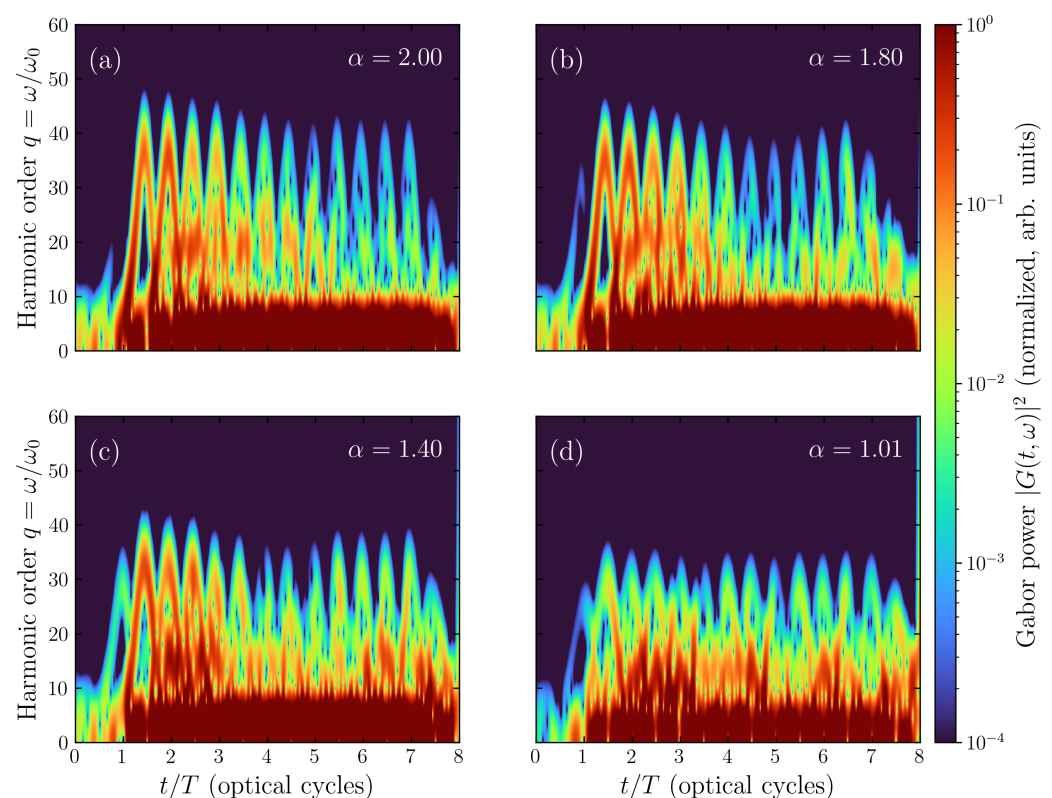


Figure 3. Gabor time-frequency analysis of the TDSE dipole acceleration for (a) $\alpha = 2.00$, (b) $\alpha = 1.80$, (c) $\alpha = 1.40$, and (d) $\alpha = 1.01$. The horizontal axis gives the emission time in optical cycles, t/T , while the vertical axis gives the harmonic order $q = \omega/\omega_0$. The color scale represents the normalized Gabor power, $|G(t, \omega)|^2$, plotted on a logarithmic scale in arbitrary units. The maps resolve the sub-cycle emission bursts associated with the recolliding electron wave packet and show how the time-frequency structure of the plateau and cutoff regions is modified as the fractional exponent is reduced.

4. Cutoff Scaling Laws from the TDSE Spectra

The fractional classical analysis predicts that the HHG cutoff should obey a modified scaling law with both laser intensity and driving wavelength. We therefore use the 1D-TDSE spectra to extract an operational cutoff position and test whether its dependence on the laser parameters follows the expected fractional trend. The goal of this analysis is not only to compare the absolute cutoff positions but also to determine whether the cutoff displacement carries a measurable signature of the fractional dispersion relation.

4.1. Intensity Scaling at Fixed Wavelength

First, we test the cutoff scaling as a function of intensity at fixed wavelength. In this case, ω_0 is fixed, while the laser field amplitude satisfies

$$E_0 = \sqrt{\frac{I}{I_{\text{au}}}}, \quad I_{\text{au}} = 3.50944506 \times 10^{16} \text{ W/cm}^2. \quad (40)$$

From Equation (21), one obtains

$$U_{p,\alpha} \propto E_0^\alpha \propto I^{\alpha/2}. \quad (41)$$

Therefore, the shifted harmonic cutoff obeys

$$q_c^{(\alpha)} - \frac{I_p}{\omega_0} \propto I^{\alpha/2}. \quad (42)$$

This shifted quantity is the adequate object to fit. Fitting $q_c^{(\alpha)}$ directly is generally problematic because it contains the intensity-independent offset I_p/ω_0 , which can be comparable to the cutoff displacement, especially at moderate intensities.

For the intensity-scaling calculations, we fix $\omega_0 = 0.04556$ a.u., corresponding to $\lambda = 1000$ nm, and use the intensity set $I = \{0.5, 0.6, 0.8, 1.0, 1.2, 1.5, 1.8\} \times 10^{14} \text{ W/cm}^2$. For each value of α , the soft-core parameter a_α is tuned so that the field-free ground state has $I_p = 0.5$ a.u. The 1D-TDSE is then propagated with all numerical parameters fixed except for the peak intensity. The dipole acceleration is recorded, and the HHG spectrum is computed from the Fourier transform of the windowed acceleration, as was already explained in Section 3. Importantly, the spectra used for the scaling analysis are not normalized by their individual maxima, because the absolute spectral level is needed to define a consistent cutoff threshold.

The cutoff is extracted from each spectrum using a spectral-envelope criterion. We first construct an upper envelope of $\log_{10} S(q)$ by dividing the spectrum into harmonic-order bins and retaining the maximum value in each bin. This avoids identifying artificial cutoffs at minima between neighboring harmonic peaks. The envelope is then weakly smoothed. A plateau reference level is estimated from the interval

$$0.35q_c^{\text{cl}} \leq q \leq 0.65q_c^{\text{cl}}, \quad (43)$$

where q_c^{cl} is the corresponding fractional classical cutoff from Equation (37). The TDSE cutoff q_c^{TDSE} is defined as the first robust point, beyond the plateau region, where the spectral envelope falls below the plateau level by a fixed number of decades. In the intensity-scaling analysis, we typically use a drop of two decades, together with a requirement that the envelope remain below threshold for several consecutive harmonic bins. This stabilizes the extraction against isolated spectral minima.

The extracted cutoffs are then fitted to

$$q_c^{\text{TDSE}}(I) - \frac{I_p}{\omega_0} = A_\alpha I^{p_\alpha}. \quad (44)$$

The ideal fractional classical prediction is

$$p_\alpha^{\text{cl}} = \frac{\alpha}{2}. \quad (45)$$

Thus, the expected exponents are

α	$p_\alpha^{\text{cl}} = \alpha/2$	
2.00	1.000	
1.80	0.900	
1.40	0.700	(46)
1.01	0.505	

Figure 4 shows the cutoff scaling extracted from the 1D-TDSE HHG spectra as a function of laser intensity for four values of the fractional order, $\alpha = 2.0, 1.8, 1.4$, and 1.01 . The wavelength is fixed at $\lambda = 1000$ nm, and the intensity is scanned from 0.5×10^{14} to 1.8×10^{14} W/cm². In each panel, the black symbols denote the cutoff harmonic $q_c^{(\alpha)}$ extracted from the 1D-TDSE spectrum using the plateau-drop criterion described previously, the dashed gray line gives the corresponding fractional classical cutoff estimate, the red solid line is a free power-law fit, and the blue dotted line is a reference fit with the exponent fixed to the fractional prediction $p = \alpha/2$. The fitted law is

$$q_c^{(\alpha)}(I) - \frac{I_p}{\omega_0} = AI^p, \quad (47)$$

where I_p/ω_0 accounts for the ionization threshold contribution to the cutoff harmonic. Here, A is a fitted prefactor that sets the overall magnitude of the intensity-dependent cutoff contribution, while p is the extracted scaling exponent.

The conventional case, shown in Figure 4a, corresponds to $\alpha = 2$ and therefore has the expected classical exponent $p = \alpha/2 = 1$. The extracted 1D-TDSE cutoffs increase approximately linear from $q_c^{(2)} \simeq 29.7$ at 0.5×10^{14} W/cm² to $q_c^{(2)} \simeq 62.6$ at 1.8×10^{14} W/cm². The free fit gives $p = 0.79$, smaller than the ideal linear-in-intensity scaling. This deviation is expected in finite-pulse 1D-TDSE spectra. The classical law $K_{\text{max}} = 3.17U_p$ is derived for an idealized trajectory in a monochromatic field with a well-defined and constant peak amplitude, whereas the 1D-TDSE spectrum is produced by a finite pulse and by a coherent superposition of emission bursts occurring throughout the pulse envelope. Consequently, the measured spectral edge is determined not solely by the maximum possible return energy at the peak field but also by the temporal weight of the corresponding trajectories. At high intensity, ionization depletion becomes important: population is removed from the bound state before and around the pulse maximum, so the strongest-field half cycles do not contribute with the same weight as they would in an undepleted classical ensemble. This suppresses the contribution of the highest-energy returns and can shift the operationally extracted cutoff below the ideal classical expectation. In the present data, this effect is reflected by the appreciable reduction of the final norm as the peak intensity is increased, indicating non-negligible ionization depletion at the highest intensities; correspondingly, the effective cutoff scaling is sublinear, with $p = 0.79$ rather than the ideal value $p = 1$.

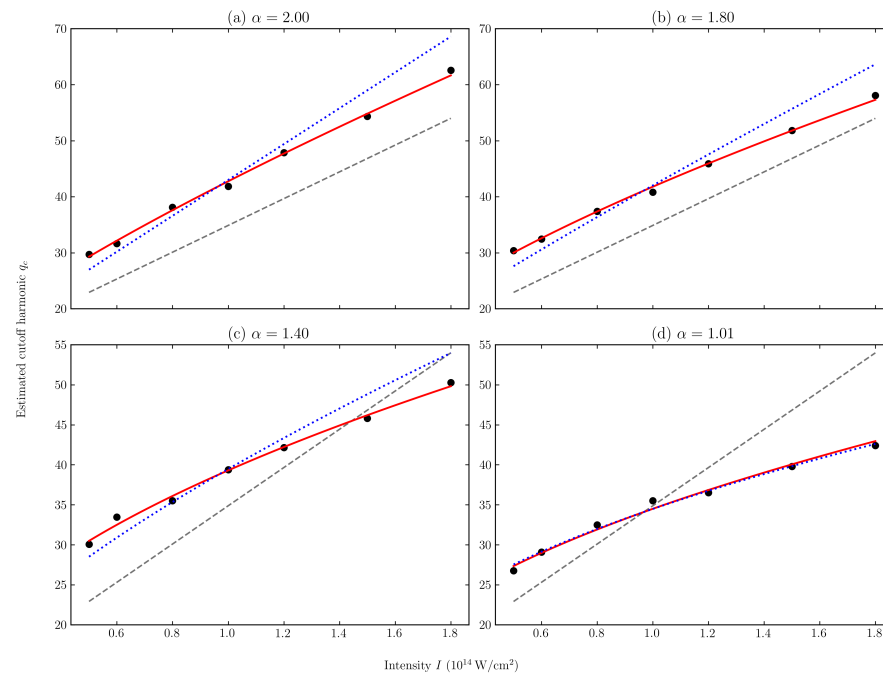


Figure 4. Intensity scaling of the HHG cutoff extracted from 1D-TDSE spectra at $\lambda = 1000$ nm for (a) $\alpha = 2.0$, (b) $\alpha = 1.8$, (c) $\alpha = 1.4$, and (d) $\alpha = 1.01$. Black symbols show the TDSE-estimated cutoff harmonic $q_c^{(\alpha)}$, dashed gray lines show the fractional classical cutoff estimate, red solid lines are free power-law fits of $q_c^{(\alpha)} - I_p/\omega_0 = AI^p$, and blue dotted lines show reference fits with the exponent fixed to $p = \alpha/2$. Here, A is a fitted prefactor that sets the overall magnitude of the intensity-dependent cutoff contribution, while p is the extracted scaling exponent. The fitted exponents are $p = 0.79, 0.69, 0.54$, and 0.52 , respectively, compared with the fractional reference values $1.00, 0.90, 0.70$, and 0.505 . The systematic decrease in the fitted exponent with decreasing α indicates a modified cutoff scaling induced by the fractional kinetic dispersion.

For $\alpha = 1.8$, Figure 4b, the expected fractional exponent is $p = \alpha/2 = 0.9$, whereas the free fit yields $p = 0.69$. The 1D-TDSE cutoff again grows monotonically with intensity, from $q_c^{(1.8)} \simeq 30.4$ to $q_c^{(1.8)} \simeq 58.1$. The trend is similar to the conventional case, but the extracted scaling exponent is reduced. The final norm of the electron wavefunction remains larger than in the $\alpha = 2$ case at the largest intensity, indicating weaker depletion over the same scan.

For $\alpha = 1.4$, Figure 4c, the expected exponent is $p = 0.7$, while the fitted TDSE exponent is $p = 0.54$. The extracted cutoff increases from $q_c^{(1.4)} \simeq 30.1$ to $q_c^{(1.4)} \simeq 50.3$ over the same intensity interval. In this case, the 1D-TDSE cutoffs lie above the classical estimate at low and intermediate intensities but fall below it at the largest intensities. Such a crossing should not be interpreted as a separate physical threshold. Rather, it indicates that the spectral cutoff extracted from the finite-pulse 1D-TDSE spectra and the idealized fractional classical cutoff are not tracking exactly the same quantity across the full intensity range. The important point is the global scaling trend, namely the reduced exponent relative to the $\alpha = 2$ case.

Finally, for $\alpha = 1.01$, Figure 4d, the expected fractional exponent is $p = \alpha/2 = 0.505$, and the extracted value is $p = 0.523$. This is the closest agreement between the 1D-TDSE scaling and the fractional classical prediction among the four cases. The cutoff increases from $q_c^{(1.01)} \simeq 26.8$ to $q_c^{(1.01)} \simeq 42.4$, substantially more slowly than in the conventional case. The final norm remains essentially unity throughout the scan, showing that the reduced cutoff growth is not caused by depletion but is instead associated with the modified fractional kinetic law.

Overall, Figure 4 demonstrates that the cutoff growth with intensity becomes progressively weaker as α decreases. The free-fit exponent decreases from $p = 0.79$ for $\alpha = 2$ to $p = 0.69$ for $\alpha = 1.8$, $p = 0.54$ for $\alpha = 1.4$, and $p = 0.52$ for $\alpha = 1.01$. Although finite-pulse effects, depletion, spectral broadening, and the cutoff extraction criterion lead to quantitative deviations from the ideal classical law, the systematic reduction in the fitted exponent provides a clear spectroscopic signature of fractional continuum dynamics. The comparison also shows that crossings between TDSE-extracted cutoffs, the classical estimate, and the fitted scaling curves can occur when the fitted exponent differs from the reference exponent. Therefore, the physically meaningful observable is not a single crossing point but the global intensity-scaling exponent and its systematic dependence on α .

4.2. Wavelength Scaling at Fixed Intensity

The second scaling test is performed at fixed intensity while changing the driving wavelength. In this case, the laser electric field peak amplitude E_0 is constant, while the carrier frequency ω_0 is calculated varying the associated laser wavelength λ . The cutoff energy shift is

$$K_{\text{cut}}^{(\alpha)} = q_c^{(\alpha)} \omega_0 - I_p = C_\alpha U_{p,\alpha}. \quad (48)$$

Since $U_{p,\alpha} \propto \omega_0^{-\alpha}$, and $\omega_0 \propto \lambda^{-1}$, one obtains

$$K_{\text{cut}}^{(\alpha)} = q_c^{(\alpha)} \omega_0 - I_p \propto \lambda^\alpha. \quad (49)$$

Equivalently, the shifted harmonic-order cutoff obeys

$$q_c^{(\alpha)} - \frac{I_p}{\omega_0} \propto \lambda^{\alpha+1}. \quad (50)$$

The energy-shift form in Equation (49) is the cleaner scaling law, because it removes the trivial conversion from photon energy to harmonic order. By contrast, fitting $q_c^{(\alpha)}$ directly is not recommended, since $q_c^{(\alpha)}$ contains both the bound-state offset $I_p/\omega_0 \propto \lambda$ and the continuum-energy contribution proportional to $\lambda^{\alpha+1}$.

For the wavelength-scaling calculations, we fix a laser intensity $I = 1.2 \times 10^{14}$ W/cm² (corresponding to a laser electric field peak strength $E_0 \approx 0.0585$ a.u.) and use $\lambda = 800, 900, 1000, 1100, 1200, 1300, 1400, 1500, 1600$ nm. For each wavelength, the 1D-TDSE is propagated using the same soft-core potential, the same fixed I_p , and the same pulse shape in optical cycles (see Section 3 for details). The spectra are again analyzed using the same envelope-drop procedure but over a wider harmonic range because the harmonic cutoff increases substantially with wavelength.

The extracted cutoff is used to compute

$$K_{\text{cut}}^{\text{TDSE}}(\lambda) = q_c^{\text{TDSE}}(\lambda) \omega_0(\lambda) - I_p, \quad (51)$$

and the fit is performed as

$$K_{\text{cut}}^{\text{TDSE}}(\lambda) = B_\alpha \lambda^{s_\alpha}. \quad (52)$$

The ideal classical prediction is $s_\alpha^{\text{cl}} = \alpha$. Equivalently, if the shifted harmonic order is fitted instead, the expected exponent is $s_{\Delta q}^{\text{cl}} = \alpha + 1$. Thus,

α	s_α^{cl}	$s_{\Delta q}^{\text{cl}}$
2.00	2.00	3.00
1.80	1.80	2.80
1.40	1.40	2.40
1.01	1.01	2.01

(53)

Figure 5 shows the wavelength scaling of the HHG cutoff extracted from the 1D-TDSE spectra for four fractional orders, $\alpha = 2.0, 1.8, 1.4$, and 1.01 . In each panel, black symbols denote the TDSE-estimated cutoff harmonic $q_c^{(\alpha)}$, the dashed gray line gives the corresponding fractional classical cutoff, and the red solid line is a power-law fit performed on the kinetic cutoff energy, Equation (52).

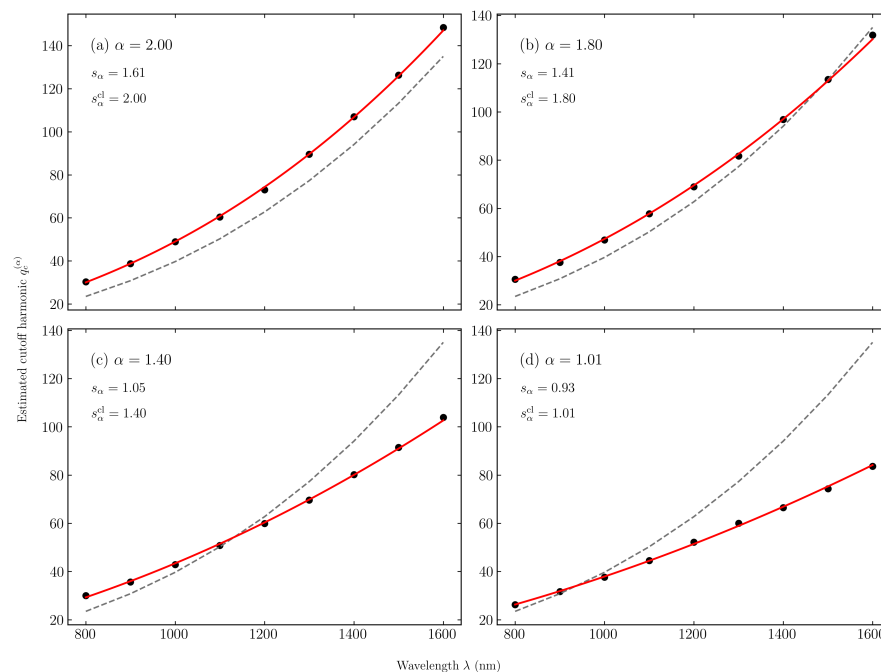


Figure 5. Wavelength scaling of the HHG cutoff extracted from 1D-TDSE spectra at fixed intensity $I = 1.2 \times 10^{14} \text{ W/cm}^2$ for (a) $\alpha = 2.0$, (b) $\alpha = 1.8$, (c) $\alpha = 1.4$, and (d) $\alpha = 1.01$. Black symbols show the TDSE-estimated cutoff harmonic $q_c^{(\alpha)}$, dashed gray lines show the fractional classical cutoff, and red solid lines are power-law fits of the kinetic cutoff energy $K_{\text{cut}}^{\text{TDSE}}(\lambda) = q_c^{\text{TDSE}}(\lambda)\omega_0(\lambda) - I_p = B_\alpha \lambda^{s_\alpha}$. The fitted exponents are $s_\alpha = 1.61, 1.41, 1.05$, and 0.93 , respectively, compared with the fractional classical expectations $s_\alpha^{\text{cl}} = \alpha = 2.0, 1.8, 1.4$, and 1.01 . The systematic decrease of s_α with decreasing α indicates a modified wavelength scaling of the HHG cutoff induced by fractional kinetic dispersion.

For the conventional case, $\alpha = 2$, shown in Figure 5a, the 1D-TDSE cutoff increases from $q_c^{(2.0)} \simeq 30.3$ at $\lambda = 800 \text{ nm}$ to $q_c^{(2.0)} \simeq 148.5$ at $\lambda = 1600 \text{ nm}$. The fitted kinetic-energy exponent is $s_\alpha = 1.61$, below the ideal classical value $s_\alpha^{\text{cl}} = 2$. This reduction is consistent with the finite-pulse nature of the TDSE spectra and with increasing ionization depletion at longer wavelengths. Thus, the trajectories that would contribute to the largest return energies are increasingly suppressed at the longest wavelengths.

For $\alpha = 1.8$, Figure 5b, the fitted exponent is $s_\alpha = 1.41$, compared with the fractional classical expectation $s_\alpha^{\text{cl}} = 1.8$. The extracted cutoff grows from $q_c^{(1.8)} \simeq 30.6$ at 800 nm to $q_c^{(1.8)} \simeq 131.9$ at 1600 nm . The 1D-TDSE cutoff follows the classical trend reasonably well over the central part of the wavelength scan, but begins to fall below the fractional classical estimate at the largest wavelengths. This behavior again reflects the fact that the spectral cutoff obtained from a finite-pulse 1D-TDSE calculation is affected by depletion, wave-packet spreading, and the empirical cutoff criterion, whereas the classical curve represents the maximum return energy of an idealized trajectory.

For $\alpha = 1.4$, Figure 5c, the fitted exponent is further reduced to $s_\alpha = 1.05$, compared with the expected $s_\alpha^{\text{cl}} = 1.4$. The cutoff increases from $q_c^{(1.4)} \simeq 30.0$ to $q_c^{(1.4)} \simeq 103.9$ across the wavelength interval. The 1D-TDSE points lie close to the classical prediction around intermediate wavelengths but become substantially smaller than the classical estimate at longer wavelengths. This indicates that the simple fractional classical cutoff overestimates

the spectral edge in this regime, where the actual HHG spectrum is controlled not only by the maximum allowed return energy but also by the strength of the recombination signal and the spectral visibility of the cutoff region.

Finally, for $\alpha = 1.01$, the smallest value considered in this work, we observe in Figure 5d that the extracted exponent is $s_\alpha = 0.93$, in relatively close agreement with the fractional expectation $s_\alpha^{\text{cl}} = 1.01$. The cutoff grows more slowly with wavelength than in the larger- α cases, increasing from $q_c^{(1.01)} \simeq 26.2$ at 800 nm to $q_c^{(1.01)} \simeq 83.6$ at 1600 nm. In this case, the final norm remains essentially unity throughout the scan, indicating that depletion is negligible. The reduced wavelength scaling is therefore mainly associated with the fractional kinetic dispersion rather than with loss of population.

Overall, Figure 5 shows that the wavelength dependence of the cutoff weakens systematically as α decreases. The fitted kinetic-energy exponents are $s_\alpha = 1.61, 1.41, 1.05$, and 0.93 for $\alpha = 2.0, 1.8, 1.4$, and 1.01 , respectively, compared with the fractional classical expectations $s_\alpha^{\text{cl}} = 2.0, 1.8, 1.4$, and 1.01 . Although the fitted exponents are generally smaller than the ideal classical values, their monotonic reduction with decreasing α provides a clear signature of fractional cutoff scaling. Equivalently, the shifted harmonic cutoff $\Delta q = q_c^{(\alpha)} - I_p/\omega_0$ scales with fitted exponents $2.61, 2.41, 2.05$, and 1.93 , close to the corresponding expectation $\Delta q \propto \lambda^{\alpha+1}$, especially in the strongly fractional case.

To assess the robustness of the cutoff extraction procedure, we repeated the complete scaling analysis for all values of α using different cutoff thresholds (including both two- and three-decade drops below the plateau) for both the intensity and wavelength scaling. Although the fitted exponents indeed change slightly, the variations remain within the fitting uncertainties and the overall dependence on α is unchanged. This confirms that the conclusions presented here are robust and do not depend sensitively on the particular cutoff criterion adopted.

4.3. Practical Difficulties and Limitations

Several practical issues must be considered when interpreting these scaling laws. First, the TDSE spectrum, both in their reduced and full dimensionality versions, does not possess an infinitely sharp cutoff. The classical cutoff is a kinematic boundary, whereas the TDSE cutoff is an operational spectral edge whose position depends on the visibility of the high-energy emission. Consequently, the extracted exponent can differ from the ideal classical value, especially when the plateau is weak, when the cutoff region is broadened, or when the spectrum contains strong interference fringes.

Second, the extraction depends on the threshold criterion. A cutoff defined by a two-decade drop and one defined by a three-decade drop can yield slightly different values of q_c^{TDSE} . For this reason, the exponent should be tested for robustness by varying the drop threshold, the amount of envelope smoothing, and the plateau reference window. A stable trend under these variations is more meaningful than a single fitted exponent.

Third, finite-pulse effects are important. The simulations use a finite pulse with a fixed number of optical cycles. The harmonic spectrum therefore contains contributions from different half cycles with different instantaneous field strengths, especially during the turn-on and turn-off portions of the pulse. This temporal inhomogeneity broadens the cutoff and can shift the operational cutoff below the ideal classical value.

Fourth, ionization depletion can affect the highest intensities. At larger field strengths, the ground-state population may be substantially depleted before the maximum-return-energy trajectories recombine. This can reduce the spectral weight near the classical cutoff and lead to an apparent cutoff that scales more slowly than the ideal prediction. The final norm of the wavefunction is therefore monitored for every run, and strongly depleted points should be excluded from the scaling fit or at least identified separately.

Fifth, long-wavelength simulations are numerically more demanding. The electron excursion length grows rapidly with wavelength, requiring a sufficiently large spatial box and an absorber placed far enough from the origin to avoid suppressing physically relevant returning wave packets. At the same time, the harmonic cutoff increases strongly with wavelength, requiring adequate temporal resolution and sufficiently long propagation to resolve the relevant harmonic frequencies.

Finally, the one-dimensional soft-core model has intrinsic limitations. While it captures the essential strong-field ionization, propagation, and recollision dynamics and provides a controlled framework for comparing different dispersion laws, it does not incorporate the full three-dimensional Coulomb potential, angular-momentum channels, or transverse wave-packet spreading. Consequently, quantitative features such as harmonic yields, interference contrast, and the relative contributions of different trajectories may differ in a full three-dimensional treatment. By contrast, the central result of this work—the systematic reduction in the HHG cutoff with decreasing fractional exponent α —originates from the modified momentum–velocity relation imposed by the fractional kinetic operator and is therefore expected to persist beyond the one-dimensional model. Thus, the most robust conclusion is not the precise numerical value of a fitted exponent, but the consistent trend in the cutoff scaling with α , obtained under identical physical conditions and analysis procedures.

Within these limitations, the cutoff-scaling analysis provides a direct spectroscopic diagnostic of the fractional dispersion law. At fixed wavelength, the shifted harmonic cutoff is expected to scale as $I^{\alpha/2}$. At fixed intensity, the cutoff energy shift is expected to scale as λ^{α} . These two complementary tests isolate the fractional dependence on field strength and driving frequency, and together provide a stringent benchmark for the modified continuum dynamics underlying the fractional HHG response.

5. Conclusions and Outlook

We investigated HHG in a dimensionally reduced time-dependent Schrödinger equation model generalized through a fractional kinetic-energy operator. The main goal was to understand how a non-quadratic continuum dispersion law modifies the recollision dynamics, the HHG cutoff, and the time-frequency structure of the emitted radiation. To make the comparison meaningful, all cases were treated at fixed ionization potential, corresponding to a hydrogen-like reference system. This allowed us to isolate the effect of the fractional kinetic operator itself, rather than mixing it with changes in the binding energy.

The classical trajectory analysis shows that the usual cutoff picture is preserved in structure but quantitatively modified by the fractional dispersion. In the conventional case, the standard cutoff coefficient is recovered. As the fractional exponent is reduced, the cutoff coefficient decreases systematically. This behavior reflects the fact that the electron no longer follows the usual quadratic relation between energy and momentum. Consequently, the same laser-driven momentum history leads to a different velocity, excursion time, and return energy. The maximum kinetic energy acquired by the returning electron is therefore reduced when measured relative to the corresponding fractional ponderomotive scale.

The 1D-TDSE simulations support this interpretation. The HHG spectra show a clear reduction in the cutoff as the fractional exponent is reduced. At the same time, the plateau region displays pronounced interference modulations, which arise from the coherent superposition of different emission pathways. In the conventional HHG picture, these pathways are associated with short and long trajectories, as well as with wave packets returning to the core at different times. In the fractional cases, the modified dispersion changes the phase accumulated during the continuum excursion and alters the timing of the returning wave packets. This produces a richer interference pattern within the plateau and a systematic reduction in the accessible cutoff region.

The Gabor power maps provide a complementary view of the same physics. While the HHG spectra reveal the global frequency-domain response, the time-frequency analysis shows when the different harmonic components are emitted during the pulse. In the conventional case, the dominant branches follow the familiar structure expected from the three-step model. As the fractional exponent is reduced, these branches are reshaped, the maximum accessible recombination energy decreases, and additional modulations appear in the middle of the pulse and throughout the plateau. These features indicate that the fractional kinetic operator does not merely shift the cutoff but reorganizes the underlying recollision dynamics.

We also analyzed cutoff scaling laws with respect to both laser intensity and driving wavelength. At fixed wavelength, the shifted cutoff harmonic is expected to follow a fractional intensity power law determined by the kinetic exponent. At fixed intensity, the cutoff energy shift is expected to follow a corresponding fractional wavelength power law. These two complementary tests provide practical spectroscopic routes to identifying the effect of fractional dispersion in HHG. In the TDSE-based spectra, however, the cutoff is not an infinitely sharp boundary. It must be extracted operationally from the spectral envelope, and the resulting exponent can depend on the threshold criterion, finite-pulse effects, depletion, and the visibility of the high-energy part of the plateau. For this reason, the extracted exponents should be interpreted as effective TDSE-based cutoff exponents rather than exact classical values.

Several limitations should be kept in mind. The present calculations are one-dimensional and use a soft-core binding potential. This model captures the essential longitudinal recollision physics, but it does not include the full three-dimensional Coulomb geometry, angular momentum, transverse spreading, or macroscopic propagation. The cutoff extraction is also delicate, especially in the strongly fractional cases, where the plateau can be strongly modulated and weak above-cutoff structures may complicate the definition of the spectral edge. Systematic convergence tests with respect to box size, absorber position, pulse duration, spectral windowing, and cutoff-threshold definition are therefore essential.

Future work should extend this analysis beyond the present one-dimensional setting. A natural direction is the development of a three-dimensional fractional TDSE or semiclassical model including Coulomb and angular-momentum effects. It would also be important to test whether the same cutoff-scaling signatures survive in molecules, solids, or engineered systems with anomalous dispersion. From a spectroscopic perspective, the combination of cutoff scaling, plateau interference, and time-frequency trajectory analysis appears especially promising. Together, these observables may provide a robust way to identify non-quadratic or fractional continuum dynamics through strong-field HHG measurements.

A closely related direction is above-threshold ionization (ATI). In the standard strong-field picture, ATI provides direct access to the laser-driven continuum through the positions of the direct and rescattered electron cutoffs, the plateau structure, and the interference patterns in momentum space. A fractional kinetic term would be expected to change the relation between drift momentum, excursion time, and return energy, thereby producing modified cutoff scalings, altered rescattering plateaus, and potentially characteristic distortions of ATI rings or holographic interference fringes. Since ATI measures the continuum electron more directly than HHG, a combined analysis of harmonic spectra and photoelectron momentum distributions could provide a stringent cross-check of the fractional model. In this sense, ATI spectroscopy offers a complementary route to test whether the non-quadratic dispersion inferred from HHG also appears in the strong-field ionization channel, strengthening the case for fractional or anomalous continuum dynamics.

Overall, our results suggest that HHG can act as a sensitive probe not only of ionization and recombination dynamics but also of the underlying dispersion law governing the motion of the laser-driven electron. The fractional kinetic operator leaves simultaneous signatures in the cutoff position, the plateau interference structure, and the time-frequency emission pattern, making HHG spectroscopy a natural diagnostic tool for exploring fractional strong-field dynamics.

Although no electronic material is presently known to realize a genuine fractional kinetic-energy operator, several theoretical proposals suggest possible routes toward its implementation. A notable example is the *Lévy crystal*, proposed by Stickler, in which engineered long-range hopping produces the same low-energy dispersion as the space-fractional Schrödinger equation [17]. More recently, fractional Schrödinger dynamics has been experimentally realized in programmable photonic systems through engineered fractional dispersion, demonstrating that fractional continuum dynamics can be faithfully implemented in laboratory settings [18].

Although these optical experiments do not involve electronic strong-field processes, they establish the experimental feasibility of fractional wave dynamics. From the perspective of strong-field physics, artificial systems with tunable nonlocal hopping or engineered dispersion—including designer lattices, cold-atom optical lattices, photonic lattices, metamaterials, and moiré superlattices—appear to be promising future platforms for investigating the strong-field phenomena predicted in this work. In particular, engineered condensed-matter systems capable of realizing effective fractional continuum dispersion could enable direct experimental tests of the modified ionization dynamics and high-harmonic generation spectra presented here.

While the realization of electronic systems exhibiting genuine fractional continuum dispersion remains an open challenge, rapid advances in engineered quantum materials and synthetic lattices suggest that such platforms may become experimentally accessible in the near future. The present work therefore provides theoretical predictions that may guide future strong-field experiments once these systems are realized.

It should also be emphasized that values of α approaching unity are considered primarily to explore the full parameter space of the fractional Schrödinger equation rather than to represent a specific experimentally established material. As $\alpha \rightarrow 1$, the dispersion becomes nearly linear and the group velocity

$$v(p) = \frac{\partial E}{\partial p} \propto |p|^{\alpha-1}, \quad (54)$$

becomes only weakly dependent on momentum. This behavior represents an extreme departure from the conventional quadratic continuum and provides a useful benchmark for identifying signatures of fractional continuum dynamics. Nevertheless, the semiclassical picture underlying strong-field processes remains valid, since the electron trajectories continue to follow the Hamiltonian equations associated with the fractional dispersion relation, leading to modified trajectories, return energies, and recollision conditions rather than a breakdown of the three-step mechanism. At present, the physically accessible range of α remains an open question, further motivating theoretical investigations across the full interval $1 < \alpha \leq 2$.

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