



Article

Approximate Analytical Solutions of Creeper-Type Fractional Maxwell–Faraday Systems with Piezoelectric Effects

Katica R. (Stevanović) Hedrih ^{1,2,3}

¹ Department of Mechanics, Mathematical Institute of Serbian Academy of Science and Arts, 11000 Belgrade, Serbia; katicah@mi.sanu.ac.rs or katica@masfak.ni.ac.rs or katicahedrih@gmail.com; Tel.: +381-18-42-41-663

² Faculty of Mechanical Engineering, University of Niš, 18106 Niš, Serbia

³ Serbian Scientific Society, 11000 Belgrade, Serbia

Abstract

We present newly derived approximate analytical solutions (AAS) of the motion, free and forced modes of dynamics of two viscoelastic rheological Maxwell–Faraday discrete dynamic systems (RMFDDSFTPEP), creeper type, and fractional type, with piezoelectric polarization property of the Faraday piezoelectric element. They always occur in paired rheological discrete dynamic systems depending on the order of sparse coupling of rheological basic light elements in standard light-binding structures and their connections with the rigid body and the fixed point. The research results presented in the paper AAS for the creep-flow EFM of creep-flow dynamics of a RMFDDSFTPEP, which characterizes the SLFTMF coupling set of models, which includes piezoelectric effects. To avoid the inversion of complex, fractional-order expressions of the Laplace transforms (LT), the terms of the complex expressions are expanded into power orders, based on the assumption that certain ratios are small relative to unity. The inverse LT is performed term by term to obtain AAS in the time domain.

Keywords: Rheological Maxwell–Faraday discrete dynamic system; fractional type; piezoelectric effect; differential fractional order equation; inverse Laplace transform; approximate analytical solution; creep dynamics; theorems

1. Introduction

This study continues the author's previous work on creep-flow dynamics in RMFDDSFT PEP. The author's intention is to write a purely theoretical-mathematical work and to present the methods and complete methodology for determining an approximate analytical solution and corresponding expressions for individual eigen and forced modes, which enabled the analysis of individual properties of modes in the structure of the approximate analytical solution of creep-flow dynamics in the system of fractional type and within the standard light Maxwell–Faraday model of bonding ensembles.

In the general case, determining the inverse Laplace transforms in analytical form is not possible, and here the author found two basic unique examples and showed how approximate analytical solutions can be found and based on a methodology that relies on classical methods. This is the value of these analytical results.

In the modern literature, there are more and more mathematical references, among which we single out [1–3], on the theory of fractional order differentiation. In the intro-

Academic Editor: Jordan Hristov

Received: 29 May 2026

Revised: 6 July 2026

Accepted: 7 July 2026

Published: 13 July 2026

Copyright: © 2026 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

ductory part of Reference [4], you can find a brief history of the beginnings of the refinement of fractional order differentiation.

References [5–7] present the results of the research theory and applications of fractional differential equations and the analysis of fractional differential equations, and point to the importance of modern applications of fractional calculus.

An introduction to mathematical models is presented in Reference [1] with special application to fractional calculus and waves in linear viscoelasticity (or in elastoviscous).

We particularly point out Reference [8], which contains chapters on fractional calculus with a special introduction intended for physicists, and we add that it is also very useful for researchers from other scientific fields.

The classical models of materials, and the increasing application of fractional order differentiation, were the inspiration for the formation of a new series of models of rheological materials of the fractional type, which the author compiled and published in Reference [5]. To form these complex models of fractional type materials, the author introduced a rheological model of a viscous fluid material of the fractional type with a differential constitutive relation of the fractional order and called it the rheological Newton element of the fractional type.

The first is the rheological basic complex Kelvin-Voigt elastic-viscous model of a solid material, of the fractional type.

The second is the rheological basic complex Maxwell viscoelastic model of a viscoelastic fluid material of the fractional type. The structural formula contains, in an ordered relationship, a Hooke elastic element and a Newton viscous fluid element of fractional type.

Then, the author introduced a rheological ideally elastic piezoelectric element with the effect of polarization in mechanical deformation, which she called the rheological Faraday piezoelectric element. Using this new rheological Faraday piezoelectric element, she formed a new series of rheological complex models of fractional-type materials with piezoelectric effects and polarization properties during mechanical deformation. This new series was published in Reference [9]. This series of rheological models of fractional-type materials and discrete dynamical systems with piezoelectric effects, with the introduced assumption that rheological elements are light and of negligible mass, and that they have only potential energy and mechanical energy dissipation properties of the fractional type, with piezoelectric properties, serve as connecting elements between rigid bodies, thereby forming a series of rheological discrete dynamic systems.

Two models with different orders of connection of basic rheological elements in rheological Maxwell–Faraday dynamic systems of fractional type were studied using the theoretical-analytical methodology of flow-creep dynamics. We determined approximate analytical solutions for the eigen and forced modes of flow-creep dynamics of characteristic types of connection of elements of discrete dynamic systems of fractional type. For rheological flow-creep dynamics of two models of rheological Maxwell–Faraday discrete dynamic models of discrete systems of the fractional type with piezoelectric effects, in analytical approximate form, expressions for the electric voltage and dielectric displacements of the Faraday element are determined. It is polarized due to mechanical deformation by the flow-creep dynamics of the discrete dynamic system, of which it is a constituent element.

2. Differential Constitutive Relation of the Maxwell–Faraday Standard Light Coupling Ensemble of the Fractional Type with Piezoelectric Effects

In this section, we briefly present the derivation of the differential constitutive relation of the rheological Maxwell–Faraday standard light-binding ensemble, of the fractional type with piezoelectric properties. This differential constitutive relation of the Maxwell–Faraday standard light-binding ensemble (structure) is a fractional-order differential equation of the form:

$$\frac{1}{\mathbf{E}} \mathbf{D}_t^\alpha [\sigma_z(t)] + \frac{\sigma_z(t)}{\mathbf{E}_\alpha} = \{\mathbf{D}_t^\alpha [\varepsilon_z(t)]\} \neq \text{const} \quad (1)$$

in which the following tags were introduced

$$\mathbf{E} \Rightarrow \mathbf{E}_0 + \mathbf{E}_e \quad (2)$$

and which are visible in Figure 1 and correspond to the elasticity coefficients of Hooke's ideally elastic and Faraday's piezoelectric and elastic elements, which are parallel connected in the structure and regularly connected with a Newtonian viscous element of the fractional type.

Differential Equation (1), of fractional order, is the differential constitutive relation of the fractional order of the rheological McNeill–Faraday model of a fractional type material with piezoelectric properties. This differential constitutive relation of fractional order is obtained by summing the dilations of the fractional-type rheological elements, Hooke's elastic and Faraday elastic and piezoelectric elements that are sequentially connected with the Newton viscous fractional type element. In Figure 1, as well as in the left part of Figure 2, the sum of dilations and individual parts of dilations of rheological elements in a sequential connection (for details and explanations, see References [6,7]).

Now the notation $\mathbf{D}_t^\alpha [\bullet]$ is a differential operator of Caputo type, and we write the fractional order α of differentiation in the form:

$$\mathbf{D}_t^\alpha [x(t)] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau, \quad \text{for } 0 < \alpha \leq 1 \quad (3)$$

where $\Gamma(x)$ is the Gamma function in the form:

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0 \quad (4)$$

where α , in interval $0 < \alpha < 1$, is determined experimentally. When exponent α of fractional order differentiation is $\alpha=0$, the application of fractional order derivative gives the time function $\sigma_z(t)$ which represents normal stress in the axial direction.

As the article deals with the study of the rheological creep-flow dynamics of the Maxwell–Faraday model of fractional-type materials, we limit ourselves to applications of fractional calculus, without the intention of making mathematical contributions to fractional calculus. For details on fractional-order differential operators and other details, we refer the reader to References [5–7] from the attached list or to the wider mathematical literature.

And the Laplace transform in the form:

$$\mathbf{L}\{x(t)\} = \int_0^\infty e^{-pt} x(t) dt = \frac{1}{p} X(p) \quad (5)$$

where p is a parameter, which can also be a complex number.

We also assume that:

$$\mathcal{L}\left\{\frac{d^\alpha x}{dt^\alpha}\right\} = p^\alpha \mathcal{L}\{x\} - \frac{d^{\alpha-1}x}{dt^{\alpha-1}}\bigg|_{t=0} = p^\alpha \mathcal{L}\{x\} \quad (6)$$

because we assume that the initial conditions are:

$$\frac{d^{\alpha-1}x}{dt^{\alpha-1}}\bigg|_{t=0} = 0 \quad (7)$$

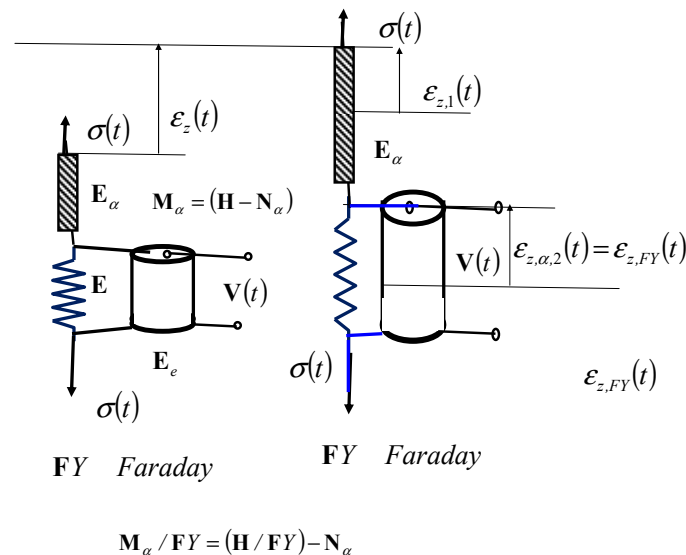


Figure 1. Structure of a rheological Maxwell–Faraday standard light-weight binding ensemble of the fractional type with piezoelectric properties, showing the component dilatations of the elements and the resulting dilatation of the ensemble.

In Figure 1, the structure of a rheological Maxwell–Faraday standard light-weighted ensemble of fractional type with piezoelectric properties is shown, with a representation of the component dilatations of the elements and the resulting dilatation of the ensemble. The constitutive differential relation of the fractional order is derived using the resulting dilatation of the fractional-type ensemble under the condition that the normal stress along each of the regularly ordered elements and the entire ensemble is the same.

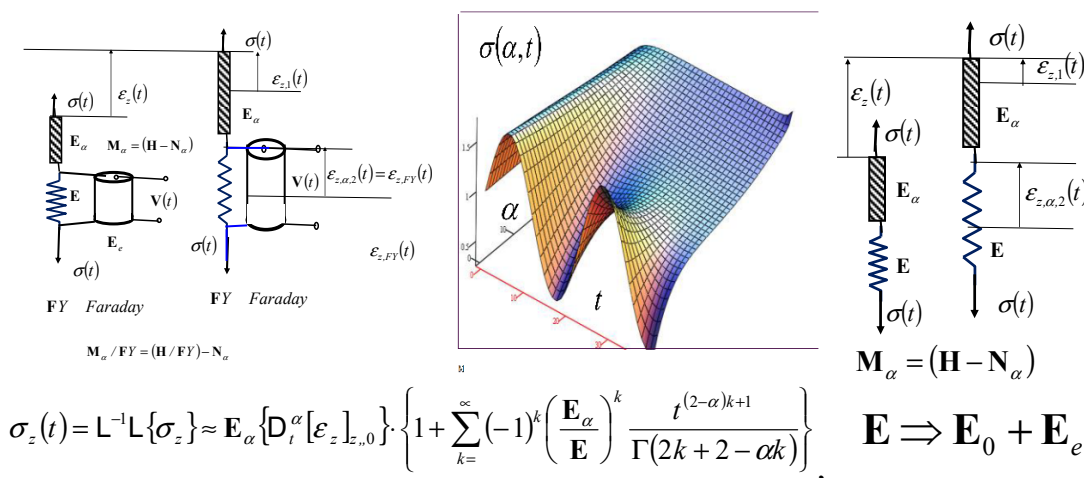


Figure 2. Relaxation surface of the normal mechanical stress of the rheological Maxwell–Faraday standard light-weight binding ensemble of the fractional type with piezoelectric properties and analogous structural models of the Maxwell–Faraday and Maxwell models.

In Figure 2, the normal mechanical stress relaxation surface of the rheological Maxwell–Faraday standard light-weighted ensemble of the fractional type with piezoelectric properties is shown. Analogous structures of the Maxwell–Faraday model of the standard light-weighted ensemble of the fractional type with piezoelectric properties and the Maxwell model of the standard light-weighted ensemble of the fractional type, whose stress relaxation surfaces are analogous, are also shown. For details, see the cited Reference [9] of the author.

Figure 2 is a composite of images of the standard light rheological Maxwell–Faraday model of the fractional-type ensemble with piezoelectric effect (left) and the standard light rheological Maxwell model of the fractional-type ensemble (right) and the stress relaxation surface as a function of the fractional-order differentiation exponent and time (center) to show that the surface is similar for these models.

We will state an approximate analytical expression that describes the regularities of the change in the normal stress in the axial direction of the flow dynamics of the normal mechanical stress of a rheological Maxwell–Faraday standard light-weight connective ensemble of the fractional type with piezoelectric properties at constant dilatation, as a function of time:

$$\sigma_z(t) = L^{-1}L\{\sigma_z\} \approx E_\alpha \{D_t^\alpha [\varepsilon_z]_{z,0}\} \cdot \left\{ 1 + \sum_{k=0}^{\infty} (-1)^k \left(\frac{E_\alpha}{E} \right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)} \right\} \quad (8)$$

The previous expression represents the functional dependence of the normal stress relaxation of standard light basic complex rheological Maxwell–Faraday mode fractional type on time and the exponential differentiation of the fractional order in the interval from zero to one, in interval $0 < \alpha < 1$.

We will also emphasize that the standard light basic complex Maxwell–Faraday model of an ideal, fractional-type, piezoelectric material possesses potential energy, a fractional-type mechanical energy dissipation property expressed by a generalized fractional-type energy dissipation function, and negligible kinetic energy.

Additionally, the standard light basic complex Maxwell–Faraday model of an ideal, fractional-type, piezoelectric property possesses the property of polarization of the Faraday element, due to mechanical deformation and the appearance of an electric current of the corresponding electric voltage, which can be identified.

3. Description of the Structure of Two Models of Rheological Maxwell–Faraday Discrete Dynamic Models of a Fractional Type System with Piezoelectric Property

In this section, we set up systems of fractional-order differential equations of the flow dynamics of two RMFDDSFT PEPs—two fractional-type creepers, with piezoelectric properties due to the polarization of the Faraday element, as a consequence of its mechanical deformation. Both RMFDDSFT PEPs have one external and one internal degree of freedom of movement.

Figure 3 shows the RMFDDSFT PEP, consisting of a single material point (rigid body in translational motion), of mass m , connected by a basic light complex standard rheological Maxwell–Faraday model of an ideal, fractional type of material, with the piezoelectric property of polarization of the rheological Faraday piezoelectric element due to mechanical deformation.

This basic, light complex standard rheological Maxwell–Faraday model of fractional type also has the property of normal stress relaxation, and is defined by the stiffnesses X . As a rheological discrete dynamic system, it has two degrees of freedom of motion-flow dynamics. One of the flow dynamics freedoms is the external and one of the internal degrees of freedom, to which correspond two independent generalized coordinates X and

$x_{u,1}$. Figure 3 shows the structure of the standard light basic complex rheological Maxwell–Faraday model of the ideal, fractional type. Figure 3 shows another rheological Maxwell–Faraday’s model of a discrete dynamic system, consisting of a single material point (rigid body in translational motion), mass m , bound by one basic light complex standard rheological Maxwell–Faraday model of the ideal, fractional type of material, with the piezoelectric property of polarization of the rheological Faraday piezoelectric element due to mechanical deformation, but with the reverse order of binding of the structure of the standard light basic complex Maxwell–Faraday model of the ideal, fractional type, in relation to the model shown in Figure 3.

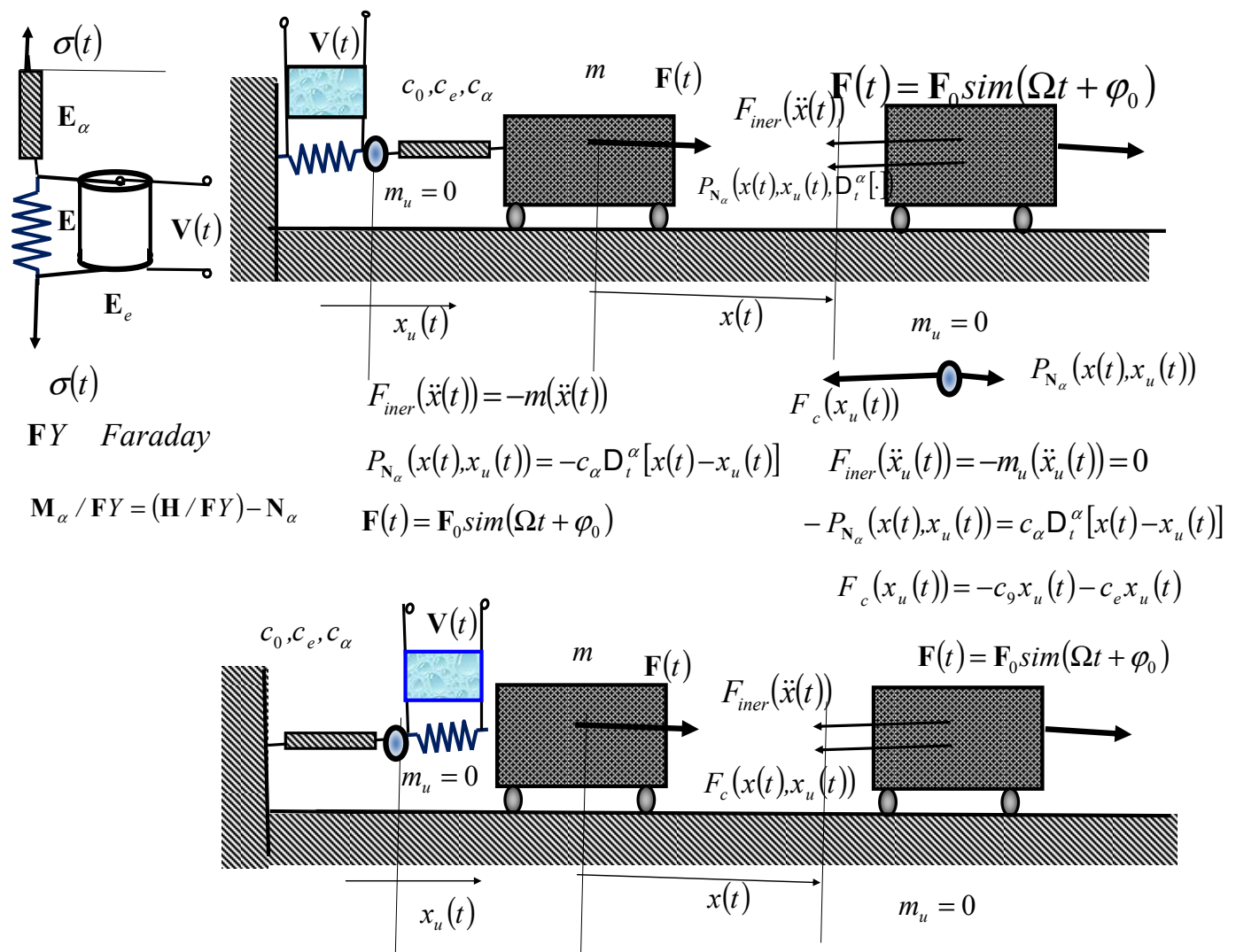


Figure 3. Two RMFDDSFT PEP models (**upper part Figure**) from a single material point (rigid body in translational motion), mass m , bound by a single light standard basic complex Maxwell–Faraday model of an ideal, fractional type, piezoelectric material; (**lower part Figure**) The structure of the standard light basic complex rheologic Maxwell–Faraday model of an ideal, fractional type, piezoelectric ensemble; (**left part Figure**) the reverse order of binding of the structure of the standard light basic complex Maxwell–Faraday model of an ideal, fractional type, piezoelectric material;.

Figure 3 is a composite of Figures of a rheological Maxwell–Faraday model of a discrete dynamic system, of the fractional type with piezoelectric properties coupled to a SLRMFFT PEP ensemble (in the upper part) and a rheological Maxwell–Faraday model of a discrete dynamic system, of the fractional type with piezoelectric properties coupled to

the same, but oppositely coupled elements in a standard light rheological Maxwell–Faraday ensemble (in the lower part), with the aim of guiding the reader to compare the models of the differences in the resulting properties of the creep-flow dynamics. The left part of the figure shows a standard light rheological Maxwell–Faraday fractional type with piezoelectric properties, a connective ensemble as part of both the Maxwell–Faraday rheological, frictional type with piezoelectric effect, and a discrete dynamical system from the same figure.

Both have the same dynamic system parameters. They differ only in the order of binding of the basic light rheological elements.

4. Systems of Fractional Order Differential Equations of Dynamics of Two Models of Rheological Maxwell–Faraday Discrete Dynamic Models of Systems of the Fractional Type with Piezoelectric Properties

The RMFDDSFT PEP, shown in Figure 3, has one external degree of freedom of rheological flow dynamics and one internal degree of freedom of rheological flow dynamics within the basic complex standard light rheological Maxwell–Faraday model of ideal, fractional type. Let us denote by $x(t)$ the independent generalized coordinate, which corresponds to the external degree of freedom of rheological flow dynamics, and by $x_{u,1}(t)$, the internal independent generalized coordinate, which corresponds to the internal degree of freedom of flow dynamics, of the basic complex SLRMFFT PEP.

Now, at the point of internal order coupling, of simple elements of the structure of the basic complex SLRMFFTPEP, we place a fictitious material point of mass equal to zero, i.e., $m_{u,1} = 0$. That point of internal order coupling, of simple elements of the structure, introduces into the discrete dynamic system an internal degree of freedom of creep movement, whose position we have determined by an independent generalized coordinate $x_{u,1}$.

Now, let us set up a system of fractional order differential equations, assuming that the basic mass of the system is m , which describes the creep dynamics of the rheological Maxwell–Faraday discrete dynamic system for the case of coupling the system structure with a standard light rheological Maxwell–Faraday model, of fractional type, element, from the upper part of Figure 3, in the form:

$$m\ddot{x} + c_\alpha D_t^\alpha [x(t) - x_{u,1}(t)] = F_0 \sin(\Omega t + \varphi_0) \quad (9)$$

$$m_{u,1}\ddot{x}_{u,1} - c_\alpha D_t^\alpha [x(t) - x_{u,1}(t)] + (c_0 + c_e)x_{u,1}(t) = 0 \quad (10)$$

For the second case, of inverse bonding of the structural element of the standard light rheological Maxwell–Faraday bonding model, of fractional type, with piezoelectric property, from the lower part of Figure 3, the system of differential equations of fractional order, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, is of the form:

$$m\ddot{x} + (c_0 + c_e)(x(t) - x_{u,1}(t)) = F_0 \sin(\Omega t + \varphi_0) \quad (11)$$

$$m_{u,1}\ddot{x}_{u,1} - (c_0 + c_e)(x(t) - x_{u,1}(t)) + c_\alpha D_t^\alpha [x_{u,1}(t)] = 0 \quad (12)$$

Now, let us introduce the following tags:

$$\omega_0^2 = \frac{c_0 + c_e}{m} \quad \omega_\alpha^2 = \frac{c_\alpha}{m} \quad h_0 = \frac{F_0}{m} \quad (13)$$

and we rewrite the previous systems of differential equations of fractional order in the form:

* for the first case of binding structure of standard light rheological Maxwell–Faraday bonding model, of fractional type, in the following form:

$$\ddot{x} + \omega_\alpha^2 D_t^\alpha [x(t) - x_{u,1}(t)] = h_0 \sin(\Omega t + \varphi_0) \quad (14)$$

$$- \omega_\alpha^2 D_t^\alpha [x(t) - x_{u,1}(t)] + \omega_0^2 x_{u,1}(t) = 0 \quad (15)$$

* for the second case of binding structure of standard light rheological Maxwell–Faraday bonding model, of fractional type, in the following form:

$$\ddot{x} + \omega_0^2 (x(t) - x_{u,1}(t)) = h_0 \sin(\Omega t + \varphi_0) \quad (16)$$

$$- \omega_0^2 (x(t) - x_{u,1}(t)) + \omega_\alpha^2 D_t^\alpha [x_{u,1}(t)] = 0 \quad (17)$$

The previous systems of differential equations of fractional order (9)–(12) were compiled on the basis of the conditions of dynamic equilibrium of the forces, according to the D'Alembert principle, acting on the main material point of mass and the fictitious material point mass, the mass of which we introduced and the point of connection of the rheological Hooke's elastic and rheological Faraday elastic and piezoelectric element with the rheological Newtonian viscous element of the fractional type, using the constitutive equations of these rheological elements of the standard light fractional type of the Maxwell–Faraday ensemble (for details see Reference [9]).

We have introduced the assumption that at the initial moment the initial coordinate and initial velocity of the attachment point to the attachment points of Hooke's light ideally elastic element and Faraday's piezoelectric element and Newton's viscous element of fractional type are equal to zero.

4.1. Application of Laplace Transformation to the First System of Differential Equations of Fractional Order (14)–(15) and Solution Describing Creep Dynamics of Maxwell–Faraday's Discrete Dynamic System, Presented in Upper Part of Figure 3

Now let us apply Laplace's transformation to both previous systems of fractional-order differential equations.

For the first system (14)–(15):

$$L\{\ddot{x}(t)\} + \omega_\alpha^2 L\{D_t^\alpha [x(t) - x_{u,1}(t)]\} = L\{h_0 \sin(\Omega t + \varphi_0)\} \quad (18)$$

$$- \omega_\alpha^2 L\{D_t^\alpha [x(t) - x_{u,1}(t)]\} + \omega_0^2 L\{x_{u,1}(t)\} = 0 \quad (19)$$

and respectively

$$p^2 L\{x(t)\} + \omega_0^2 p^\alpha L\{x(t)\} - \omega_\alpha^2 p^\alpha L\{x_{u,1}(t)\} = \frac{h_0 \Omega}{p^2 + \Omega^2} + (px_0 + \dot{x}_0) \quad (20)$$

$$- \omega_\alpha^2 p^\alpha L\{x(t)\} + \omega_\alpha^2 p^\alpha L\{x_{u,1}(t)\} + \omega_0^2 L\{x_{u,1}(t)\} = 0 \quad (21)$$

and respectively

$$L\{x(t)\} \langle p^2 + \omega_\alpha^2 p^\alpha \rangle - \omega_\alpha^2 p^\alpha L\{x_{u,1}(t)\} = \frac{h_0 \Omega}{p^2 + \Omega^2} + (px_0 + \dot{x}_0) \quad (22)$$

$$\omega_\alpha^2 p^\alpha L\{x(t)\} - L\{x_{u,1}(t)\} \langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle = 0 \quad (23)$$

That is, from the second equation of the previous system (22) and (23) is:

$$\mathcal{L}\{x_{u,1}(t)\} = \frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \mathcal{L}\{x(t)\} \quad (24)$$

which by substituting into the first equation of the system (22) and (23) gives

$$\mathcal{L}\{x(t)\} \langle p^2 + \omega_\alpha^2 p^\alpha \rangle - \omega_\alpha^2 p^\alpha \frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \mathcal{L}\{x(t)\} = \frac{h_0 \Omega}{p^2 + \Omega^2} + (px_0 + \dot{x}_0)$$

and respectively

$$\mathcal{L}\{x(t)\} \left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle = \frac{h_0 \Omega}{p^2 + \Omega^2} + (px_0 + \dot{x}_0) \quad (25)$$

Now it is necessary to determine the inverse Laplace transformation $\mathcal{L}^{-1}\{x(t)\}$ of the external independent generalized coordinate $\{x(t)\}$, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, and move to the time domain, in the form:

$$\mathcal{L}\{x(t)\} = \frac{(px_0 + \dot{x}_0)}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} + \frac{1}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} \cdot \frac{h_0 \Omega}{p^2 + \Omega^2} \quad (26)$$

Now it is necessary to determine the inverse Laplace transformation $\mathcal{L}^{-1}\{x(t)\}$ of the external independent generalized coordinate $\{x(t)\}$, which describes the flow dynamics of that RMFDDSFTPEP, and move to the time domain, in the form:

$$x(t) = \mathcal{L}^{-1}\mathcal{L}\{x(t)\} = \mathcal{L}^{-1} \left\{ \frac{(px_0 + \dot{x}_0)}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} \cdot \left\{ \frac{h_0 \Omega}{p^2 + \Omega^2} \right\} \right\} \quad (27)$$

First, it is necessary to determine the inverse Laplace transformation $\mathcal{L}^{-1}\{x_{free}(t)\}$ of the external independent generalized coordinate $\{x_{free}(t)\}$, for free self-creeps, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, and move to the time domain, in the form:

$$x_{free}(t) = \mathcal{L}^{-1}\mathcal{L}\{x_{free}(t)\} = \mathcal{L}^{-1} \left\{ \frac{(px_0 + \dot{x}_0)}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} \right\} \quad (28)$$

And then, using the convolution integral and individual inverse Laplace transfor-

mations of expressions $\mathcal{L}^{-1} \left\{ \frac{1}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} \right\}$ and $\mathcal{L}^{-1} \left\{ \frac{h_0 \Omega}{p^2 + \Omega^2} \right\}$, determine external independent generalized coordinates $\mathcal{L}^{-1}\mathcal{L}\{x_{part}(t)\}$, for forced movement $\{x_{part}(t)\}$, for forced creep or pulsating forced oscillation with creep, which describes the flow dynamics of that RMFDDSFTPRP, and move to the time domain, in the form:

$$x_{part}(t) = \mathcal{L}^{-1} \mathcal{L}\{x_{part}(t)\} = \mathcal{L}^{-1} \left\{ \frac{1}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} \cdot \left\{ \frac{h_0 \Omega}{p^2 + \Omega^2} \right\} \right\} \quad (29)$$

We can, then, determine the internal independent generalized coordinate $x_{u,1}(t)$ of the displacement of the point of order of the basic parallel connected Hooke's and Faraday ideally elastic elements with Newton's ideally viscous element, fractional type, in the standard light of that rheological Maxwell–Faraday model, of fractional type, in the form:

$$x_{u,1}(t) = \mathcal{L}^{-1} \mathcal{L}\{x_{u,1}(t)\} = \mathcal{L}^{-1} \left\{ \frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \cdot \mathcal{L}\{x(t)\} \right\} \quad (30)$$

using the convolution integral and the property of three functions, $f_1(t)$, $f_2(t)$ and $f_3(t)$. In doing so, we keep in mind that when three functions are in convolution, then the Laplace transform $\mathcal{L}\{f_3(t)\} = \mathcal{L}\{x_{u,1}(t)\}$ can be written as a product of the Laplace

$$\mathcal{L}\{f_1(t)\} = \frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle}.$$

transforms of those two functions

4.2. Application of Laplace Transformation to the Second System of Differential Equations of Fractional Order (16)–(17) and Solution Describing Creep Dynamics of Maxwell–Faraday's Discrete Dynamic System, Presented in Lower Part of Figure 3

For the second case of bonding the structure of the modified standard light rheological Maxwell–Faraday's model of ideal, fractional type, differential equations of fractional order are composed in the forms (16) and (17), for describing creep dynamics of rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, presented in the lower part of Figure 3.

Now let us apply Laplace's transformation to both differential equations of fractional order, the second system of fractional-type, describing creep dynamics of the rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, presented in the lower part of Figure 3:

$$\begin{aligned} \mathcal{L}\{\ddot{x}(t)\} + \omega_0^2 \mathcal{L}\{(x(t) - x_{u,1}(t))\} &= \mathcal{L}\{h_0 \sin(\Omega t + \varphi_0)\} \\ - \omega_0^2 \mathcal{L}\{(x(t) - x_{u,1}(t))\} + \omega_\alpha^2 \mathcal{L}\{D_t^\alpha [x_{u,1}(t)]\} &= 0 \end{aligned}$$

And, respectively,

$$p^2 \mathcal{L}\{x(t)\} - \omega_0^2 \mathcal{L}\{x(t)\} - \omega_0^2 \mathcal{L}\{x_{u,1}(t)\} = \frac{h_0 \Omega}{p^2 + \Omega^2} + (px_0 + \dot{x}_0) \quad (31)$$

$$\omega_0^2 \mathcal{L}\{x(t)\} - \omega_0^2 \mathcal{L}\{x_{u,1}(t)\} - \omega_\alpha^2 p^\alpha \mathcal{L}\{x_{u,1}(t)\} = 0 \quad (32)$$

Using the same methodology as in the first case, by applying the Laplace transform to the second system (16) and (17) of fractional-order differential equations, which describes the flow dynamics in the second part of the rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the bottom of Figure 3, with the reverse order of the standard light rheological Maxwell–Faraday model, of fractional type of connections, we obtain that the Laplace transform of the independent generalized coordinate $x_{u,1}(t)$ corresponding to the internal degree of freedom of the flow, is in the form:

$$\mathcal{L}\{x_{u,1}(t)\} = \frac{\omega_0^2}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \mathcal{L}\{x(t)\} \quad (33)$$

Which replaces the first one:

$$\mathcal{L}\{x(t)\} \left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle = \frac{h_0 \Omega}{p^2 + \Omega^2} + (px_0 + \dot{x}_0) \quad (34)$$

Then, we determine the Laplace transformation $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate $\{x(t)\}$ of the rheological Maxwell–Faraday model of fluid dynamics, of fractional type, and the piezoelectric properties of the discrete dynamic system, from the lower part of Figure 3, with the reverse binding of the standard light rheological Maxwell–Faraday model, of fractional type, in the form:

$$\mathcal{L}\{x(t)\} = \frac{(px_0 + \dot{x}_0)}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} + \frac{1}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} \cdot \frac{h_0 \Omega}{p^2 + \Omega^2} \quad (35)$$

Now it is necessary to determine the inverse Laplace transformation $\mathcal{L}^{-1}\mathcal{L}\{x(t)\}$ of the external independent generalized coordinate $\{x(t)\}$, of the rheological Maxwell–Faraday model of fluid dynamics, of fractional type, and the piezoelectric properties of the discrete dynamic system, and move to the time domain, in the form:

$$x(t) = \mathcal{L}^{-1}\mathcal{L}\{x(t)\} = \mathcal{L}^{-1} \left\{ \frac{(px_0 + \dot{x}_0)}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} + \mathcal{L}^{-1} \left\{ \frac{1}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} \cdot \left\{ \frac{h_0 \Omega}{p^2 + \Omega^2} \right\} \right\} \right\} \quad (36)$$

First, it is necessary to determine the inverse Laplace transformation $\mathcal{L}^{-1}\mathcal{L}\{x_{free}(t)\}$ of the external independent generalized coordinate $\{x_{free}(t)\}$, for free self-creeps of the rheological Maxwell–Faraday model of fluid dynamics, of fractional type, and the piezoelectric properties of the discrete dynamic system, and move to the time domain, in the form:

$$x_{free}(t) = \mathcal{L}^{-1}\mathcal{L}\{x_{free}(t)\} = \mathcal{L}^{-1} \left\{ \frac{(px_0 + \dot{x}_0)}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} \right\} \quad (37)$$

And then, using the convolution integral and individual inverse Laplace transformations of expressions

$\mathcal{L}^{-1} \left\{ \frac{1}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} \right\}$ and $\mathcal{L}^{-1} \left\{ \frac{h_0 \Omega}{p^2 + \Omega^2} \right\}$, determine external independent generalized coordinates $\mathcal{L}^{-1}\mathcal{L}\{x_{part}(t)\}$, for forced movement $\{x_{part}(t)\}$, for forced creep (crawler) or pulsating forced oscillation with creep, and move to the time domain, in the form:

$$x_{paer}(t) = \mathcal{L}^{-1}\mathcal{L}\{x_{paer}(t)\} = \mathcal{L}^{-1} \left\{ \frac{1}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} \cdot \left\{ \frac{h_0 \Omega}{p^2 + \Omega^2} \right\} \right\} \quad (38)$$

We can then determine the internal independent generalized coordinate $x_{u,1}(t)$ of the displacement of the point of order of the basic parallel-connected Hooke's and Faraday's ideally elastic elements with Newton's ideally viscous elements, fractional type, in the standard light rheological Maxwell–Faraday basic complex rheological coupling model, fractional type, in the form:

$$x_{u,1}(t) = \mathcal{L}^{-1} \mathcal{L} \{x_{u,1}(t)\} = \mathcal{L}^{-1} \left\{ \frac{\omega_0^2}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \mathcal{L} \{x(t)\} \right\} \quad (39)$$

using the convolution integral and the property of three functions, $f_1(t)$, $f_2(t)$ and $f_3(t)$. In doing so, we keep in mind that when three functions are in convolution, then the Laplace transform $\mathcal{L} \{f_3(t)\} = \mathcal{L} \{x_{u,1}(t)\}$ can be written as a product of the Laplace

transforms of those two functions $\mathcal{L} \{f_1(t)\} = \frac{\omega_0^2}{\omega_0^2 + \omega_\alpha^2 p^\alpha}$ and $\mathcal{L} \{f_2(t)\} = \mathcal{L} \{x(t)\}$.

5. Comparison of the Dynamics of Two Rheological Maxwell–Faraday Discrete Dynamic Models of a Friction-Type System with Piezoelectric Properties and Series of Theorems

Comparing the obtained expressions (26)–(35) and (24)–(33), for the Laplace transforms of independent generalized coordinates and for both own and forced movements of the material point, which describes the flow dynamics of that rheological Maxwell–Faraday models, of fractional type of discrete dynamic system, represented in Figure 3, we see that these are very different dynamics. We see that the way of fitting this structure of the standard light rheological Maxwell–Faraday basic complex coupling model, of fractional type, has a greater impact on the resulting rheological creep motion.

Based on the previous comparison and conclusion in previously published References [6–8], we can formulate the following series of theorems:

Theorem 1. *The difference between rheological oscillators and rheological creepers is evident from the structural scheme of a standard light rheological binding ensemble. If a standard light rheological binding ensemble is elastoviscous type, of the fractional type with piezoelectric properties, then the rheological discrete dynamic system is a fractional type rheological oscillator with piezoelectric properties.*

Theorem 2. *The order of the structure of the standard light rheological Maxwell–Faraday basic complex model, fractional type with piezoelectric property, as well as the order of the connection with rigid bodies (material points) in translator motion of the rheological discrete dynamic system has a greater influence on the resulting rheological dynamics of flow-creep motion of the rheological Maxwell–Faraday discrete dynamic system.*

Theorem 3. *If at least one of Newton's fluid viscous elements, of fractional type, is regularly connected in the structure independently, and is not in parallel connection or is not inside a parallel connection with any of Hooke's ideally elastic elements or Faraday's ideally elastic and piezoelectric elements, then we are dealing with the viscoelastic creep dynamics of the elements of a standard light-binding rheological ensemble and also with the creep dynamics of a rheological discrete dynamic system, which is a rheological creeper.*

Theorem 4. *And if each of Newton's fluid elements, of fractional type, is connected in a parallel or intra-parallel connection with some of Hooke's ideally elastic or Faraday's ideally elastic and piezoelectric elements, then we are talking about the rheological elastoviscous dynamics of oscillations.*

tion of the elements of a standard light-binding rheological ensemble, as well as the dynamics of rheological elastoviscous oscillation of a rheological discrete dynamic system, which is a rheological oscillator.

Theorem 5. Each regular bond, of one of Newton's fluid viscous elements, of fractional type, introduces one internal degree of freedom of motion of the standard light-binding rheological ensemble, and as in the rheological dynamics of a rheological discrete dynamical system, in front of the external degrees of freedom of the rheological discrete dynamical system.

We see that the main Equation (26) for the Laplace transform of the independent generalized coordinate, which corresponds to the external degree of freedom of creep-flow motion, for the creep-flow dynamics of the rheological Maxwell–Faraday discrete dynamic system of the fractional type, with piezoelectric properties for the model from the upper part of Figure 3, as well as the corresponding Equation (35) of the rheological Maxwell–Faraday discrete dynamic system of the fractional type, with piezoelectric properties for the model from the lower part of Figure 3 are different, and the difference in the models is the inverted binding order of the standard light rheological Maxwell–Faraday ensemble, fractional tape.

6. Inverse Laplace Transform of Laplace Transforms of Independent Generalized Coordinates and Approximate Analytical Expressions of Rheological Eigen and Forced Modes of Dynamics of Rheological Two Models of Rheological Maxwell–Faraday Discrete Dynamic Models of Systems of the Fractional Type with Piezoelectric Properties

In Section 4, we determined the Laplace transforms of independent generalized coordinates (24)–(26) and (32)–(35), in the following forms:

* Expression (26) is the Laplace transform $\mathcal{L}\{x(t)\}$ for the external independent generalized coordinate $x(t)$, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, presented in the upper part of Figure 3;

* Expression (24) is the Laplace transform of the internal independent generalized coordinate $x_{u,1}(t)$, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, presented in the upper part of Figure 3;

* Expression (35) is the Laplace transform $\mathcal{L}\{x(t)\}$ for the external independent generalized coordinate $x(t)$, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, presented in the lower part of Figure 3;

* Expression (32) is the Laplace transform of the internal independent generalized coordinate $x_{u,1}(t)$, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, presented in the lower part of Figure 3;

In order to determine the approximate analytical solution of the system of fractional order differential Equations (9) and (10), as well as (11) and (12) from Section 4, for which we have only, in that Section 4, determined the expressions Laplace transforms of independent generalized coordinates (26) and (24) or (35) and (32), it is now necessary to determine the inverse Laplace transforms.

Therefore, we will divide these expressions Laplace transforms of independent generalized coordinates (26) and (24) or (35) and (32) into parts corresponding to the Laplace transforms of the own and forced modes of independent generalized coordinates, external and internal degrees of freedom of movement, which describes the flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system with piezoelectric property.

To determine the inverse Laplace transforms of independent generalized coordinates, we will expand the parts into series with respect to the Laplace transform parameter, using the approximate power series expansions presented in References [10,11]:

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \dots \quad (40)$$

$$(a+b)^k = \sum_{j=0}^k \binom{k}{j} a^{k-j} b^j \quad (41)$$

Then we use the following expression to determine the inverse Laplace transform of the power function (see the Table of Laplace transforms in Reference [12], Theory of oscillations by D.P. Rašković):

$$\frac{n!}{p^{n+1}} = \frac{n}{p} \mathcal{L}\{t^{n-1}\} \quad \text{for } n > -1 \quad (42)$$

$$\Gamma(n) = (n-1)! \quad \text{for } n = 1, 2, \dots \quad (43)$$

Now, using the previously mentioned expressions from (39) to (42) from the mentioned References [11,12] and applying them in order to the parts of the Laplace transforms of independent generalized coordinates (26) and (24) or (35) and (32) by parts corresponding to the Laplace transforms of the own and forced modes of independent generalized coordinates, we determine in the time domain approximate analytical solutions for the own and forced modes of flow dynamics of two models of the rheological Maxwell–Faraday model, of fractional type of discrete dynamic system with piezoelectric property, from Figure 3, which differ only in the order of the elements connected in order in the standard light rheological connecting ensemble Maxwell–Faraday model, of fractional type of piezoelectric property.

6.1. Now We Extract One Characteristic Expression from the First Term of the Laplace

Transform $\mathcal{L}\{x(t)\}$ of the Independent Generalized Coordinate in the Form (3), of the External Independent Generalized Coordinate, Which Describes the Flow Dynamics of That Rheological Maxwell–Faraday Model, of Fractional Type of Discrete Dynamic System, from the Upper Part in Figure 3, and After Its Transformation We Write It in the Form

$$\frac{1}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\rangle} = \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \frac{1}{\left\langle 1 + \omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \right\rangle} \quad (44)$$

Then, let us expand the previous transformed expression (26)

$$\frac{1}{\left\langle 1 + \omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \right\rangle} \quad \text{into an approximate analytically stable series, using series (40),}$$

assuming that the term $\omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha$ is much smaller than unity:
 $\omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \ll 1$. Then we obtain the following expression:

$$\frac{1}{\left\langle 1 + \omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \right\rangle} = \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(\omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \right)^k \right\rangle \quad (45)$$

Or in the following form;

$$\frac{1}{\left\langle p^2 + \omega_\alpha^2 p^\alpha - \frac{\omega_\alpha^4 p^{2\alpha}}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} = \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \quad (46)$$

Now we extract two expressions from the first term of the Laplace transform $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate in the form (26), of the external independent generalized coordinate $x(t)$ which describes the free eigen creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the upper part in Figure 3. We will introduce these two separate terms into the expressions of Laplace transforms of eigenmodes similar to sine, or cosine, and write them in the form:

$$\mathcal{L}\{x_{free,like sin}(t)\} = \dot{x}_0 \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \quad (47)$$

$$\mathcal{L}\{x_{free,like cos}(t)\} = x_0 p \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \quad (48)$$

Then, we extract the last term from the Laplace transform $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate in the form (26), of the external independent generalized coordinate $x(t)$ which describes the forced creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the upper part in Figure 3. We will call that separate term the Laplace transform expression of the forced mode similar to sine, because it is an external forced force of sinusoidal properties, or similar to cosine, when the external forced force is of the cosine type of properties, and we write it in the form:

$$\mathcal{L}\{x_{Forced,like sin}(t)\} = \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \cdot \frac{\Omega}{p^2 + \Omega^2} \quad (49)$$

Next, it is necessary to expand the binomial, which appears in the previous expressions from the Laplace transform $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate in the form (35), of the external independent generalized coordinate $x(t)$ in its own and

forced modes into powers of the order $(a+b)^k = \sum_{j=0}^k \binom{k}{j} a^{k-j} b^j$, using Formula (41), so that we obtain that this term is expanded into powers of the order of the form:

$$\left(\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k = \sum_{j=0}^k \binom{k}{j} (\omega_\alpha^2 p^{-2})^{k-j} \left(\frac{\omega_\alpha^2}{\omega_0^2} \right)^j \quad (50)$$

Now, the previous expressions (47), (48) and (49) of the Laplace transforms of the own and forced modes of the external independent generalized coordinate $x(t)$ which describes the free and forced modes of creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from upper part in Figure 3, with piezoelectric property, we write it in the form:

* Laplace transform expression for the eigenmode “like sin” of the external generalized coordinate:

$$\mathcal{L}\{x_{free,like\sin}(t)\} = \dot{x}_0 \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k p^{\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\rangle \quad (51)$$

Or in the form of:

$$\begin{aligned} \mathcal{L}\{x_{free,like\sin}(t)\} = & \dot{x}_0 \left(p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-2} \right) + \dot{x}_0 \sum_{k=1}^{\infty} (-1)^k p^{\alpha k-2} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} + \\ & + \dot{x}_0 \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha(k+1)-2} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \end{aligned} \quad (52)$$

* Laplace transform expression for the eigenmode “like-skew” of the external generalized coordinate

$$\mathcal{L}\{x_{free,like\cos}(t)\} = x_0 \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k p^{\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k-j}}{\omega_0^{2j}} p^{-2(k-j)} \right\rangle \quad (53)$$

Or in the form of:

$$\begin{aligned} \mathcal{L}\{x_{free,like\cos}(t)\} = & x_0 \left(p^{-1} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-1} \right) + x_0 p^{-1} \sum_{k=1}^{\infty} (-1)^k p^{\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k-j}}{\omega_0^{2j}} p^{-2(k-j)} + \\ & + x_0 \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-1} \sum_{k=1}^{\infty} (-1)^k p^{\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k-j}}{\omega_0^{2j}} p^{-2(k-j)} \end{aligned} \quad (54)$$

The previous expressions of the Laplace transform of the eigenmodes (52) and (54) of the external generalized coordinates $x(t)$ we transformed into the form of a power series by the parameter of the Laplace transform, suitable for determining the inverse Laplace transforms of power functions by expressions (40) and (41). Now, by applying these Formulas (40) and (41) to the Laplace transforms (52) and (54), we determine their inverse Laplace transformations, and thus we translate these expressions into the time domain. Thus, we obtain approximate analytical expressions for the eigenmodes of independent generalized coordinates $x(t)$ in the time domain, in the form:

* The inverse Laplace transform of its own mode, of the “sinus-like” type of rheological creep dynamics of that rheological Maxwell–Faraday model, of a fractional-type discrete dynamic system, from the upper part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} x_{free,like\sin}(t) = & \mathcal{L}^{-1} \mathcal{L}\{x_{free,like\sin}(t)\} = \dot{x}_0 \mathcal{L}^{-1} \left\{ \left(p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-2} \right) \right\} + \\ & + \dot{x}_0 \mathcal{L}^{-1} \left\{ \sum_{k=1}^{\infty} (-1)^k p^{\alpha k-2} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\} + \\ & + \dot{x}_0 \mathcal{L}^{-1} \left\{ \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha(k+1)-2} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\} \end{aligned} \quad (55)$$

And what are the inverse Laplace transforms of the following terms in order:

$$\mathcal{L}^{-1}\left\{\left(p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-2}\right)\right\} = t + \frac{\omega_\alpha^2}{\omega_0^2} \frac{(2-\alpha)}{(2-\alpha)!} t^{1-\alpha} = t + \frac{\omega_\alpha^2}{\omega_0^2} \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \quad (56)$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\sum_{k=1}^{\infty} (-1)^k p^{\alpha k-2} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)}\right\} = \\ = \sum_{k=1}^{\infty} (-1)^k t^{2k+1-\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \end{aligned} \quad (57)$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha(k+1)-2} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)}\right\} = \\ = \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k t^{2k-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j+1)-\alpha(k+1))} \end{aligned} \quad (58)$$

that is, in the final approximate analytical form in the form of an expression as a function of time we write the final approximate analytical expression of its own mode, of the “sinus-like” type of rheological creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from upper part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} x_{free,like\sin}(t) = \dot{x}_0 t + \dot{x}_0 \frac{\omega_\alpha^2}{\omega_0^2} \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + \\ + \dot{x}_0 \sum_{k=1}^{\infty} (-1)^k t^{2k+1-\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} + \\ + \dot{x}_0 \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k t^{2k-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j+1)-\alpha(k+1))} \end{aligned} \quad (59)$$

* The inverse Laplace transform of its own mode, of the “cosinus-like” type of rheological creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the upper part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} x_{free,like\cos}(t) = \mathcal{L}^{-1}\mathcal{L}\{x_{free,like\cos}(t)\} = x_0 \mathcal{L}^{-1}\left\{\left(p^{-1} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-1}\right)\right\} + \\ + x_0 \mathcal{L}^{-1}\left\{\sum_{k=1}^{\infty} (-1)^k p^{\alpha k-1} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k-j}}{\omega_0^{2j}} p^{-2(k-j)}\right\} + \\ + x_0 \mathcal{L}^{-1}\left\{\frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha(k+1)-1} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k-j}}{\omega_0^{2j}} p^{-2(k-j)}\right\} \end{aligned} \quad (60)$$

And what are the inverse Laplace transforms of the following terms in order;

$$\mathcal{L}^{-1}\left\{\left(p^{-1} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-1}\right)\right\} = 1 + \frac{\omega_\alpha^2}{\omega_0^2} \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} \quad (61)$$

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \sum_{k=1}^{\infty} (-1)^k p^{\alpha k-1} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} p^{-2(k-j)} \right\} = \\ = \sum_{k=1}^{\infty} (-1)^k t^{-\alpha k+2k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j)-\alpha k)} \end{aligned} \quad (62)$$

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{\omega_{\alpha}^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha(k+1)-1} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} p^{-2(k-j)} \right\} = \\ = \frac{\omega_{\alpha}^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k t^{2k-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j)-\alpha(k+1)+1)} \end{aligned} \quad (63)$$

that is, in the final approximate analytical form in the form of an expression as a function of time we write the final approximate analytical expression of its own mode, of the “simus-like” type of rheological creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from upper part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} x_{free,like\cos}(t) = x_0 + x_0 \frac{\omega_{\alpha}^2}{\omega_0^2} \frac{1}{\Gamma(1-\alpha)} t^{-\alpha} + \\ + x_0 \sum_{k=1}^{\infty} (-1)^k t^{-\alpha k+2k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j)-\alpha k)} + \\ + x_0 \frac{\omega_{\alpha}^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k t^{2k-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j)-\alpha(k+1)+1)} \end{aligned} \quad (64)$$

What is the Laplace transform of a forced mode under the action of an external periodic sinusoidal force of the form:

$$\mathcal{L}\{x_{Forced,like\sin}(t)\} = \frac{(\omega_0^2 + \omega_{\alpha}^2 p^{\alpha})}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(\omega_{\alpha}^2 p^{-2} + \frac{\omega_{\alpha}^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \cdot \frac{\Omega}{p^2 + \Omega^2} \quad (65)$$

To obtain its inverse Laplace transform, we can apply the Convolution Theorem.

Now, based on the previous expression (65) by applying the Convolution Theorem, in the final approximate analytical form, in the form of an expression as a function of time, we write the final approximate analytical expressions of the forced modes, of the type “to sinus-like” or “cosinus-like” for the rheological forced creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from upper part in Figure 3, with piezoelectric property, under the action of an external forced periodic sinusoidal, or cosine force we write it in the forms:

$$\begin{aligned} x_{Forced,like\sin}(t) = \int_0^t \sin \Omega(t-\tau) \left\langle \tau + \frac{\omega_{\alpha}^2}{\omega_0^2} \frac{1}{\Gamma(2-\alpha)} \tau^{1-\alpha} \right\rangle d\tau + \\ + \int_0^t \sin \Omega(t-\tau) \left\langle \sum_{k=1}^{\infty} (-1)^k \tau^{2k+1-\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right\rangle d\tau + \\ + \int_0^t \sin \Omega(t-\tau) \left\langle \frac{\omega_{\alpha}^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k \tau^{2k-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\omega_{\alpha}^{2k-j}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha(k+1))} \right\rangle d\tau \end{aligned} \quad (66)$$

$$\begin{aligned}
x_{\text{Forced, like cos}}(t) = & \int_0^t \cos \Omega(t-\tau) \left\langle \tau + \frac{\omega_\alpha^2}{\omega_0^2} \frac{1}{\Gamma(2-\alpha)} \tau^{1-\alpha} \right\rangle d\tau + \\
& + \int_0^t \cos \Omega(t-\tau) \left\langle \sum_{k=1}^{\infty} (-1)^k \tau^{2k+1-\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k-j}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right\rangle d\tau + \\
& + \int_0^t \cos \Omega(t-\tau) \left\langle \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k \tau^{2k+1-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k-j}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha(k+1))} \right\rangle d\tau
\end{aligned} \quad (67)$$

6.2. We Now Turn to Determining Approximate Analytical Solutions of the Independent Generalized Coordinates as a Function of Time of the System of Fractional-Order Differential Equations (16) and (17), Which Describe the Rheological Creep-Flow Dynamics of the Flow of the Second Model of RMFDDS, Shown in the Lower Part of Figure 3, with the Reverse Order of Binding of the SLRMF M Binding Ensemble

* For the second case of binding structure of the standard light rheological Maxwell–Faraday bonding model of ensemble, of fractional type, in the following form: (16) and (17) is presented in the lower part in Figure 3.

Now, using the previously mentioned expressions from (39) to (42) from the mentioned References [8–10] and applying them in order to the parts of the Laplace transforms of independent generalized coordinates (35) and (33) by parts corresponding to the Laplace transforms of the own and forced modes of independent generalized coordinates, we determine in the time domain approximate analytical solutions for the own and forced modes of creep-flow dynamics of two models of the rheological Maxwell–Faraday models, of fractional type of discrete dynamic system with piezoelectric property, from lower part Figure 3, which differ only in the order of the elements connected in order in the standard light rheological connecting ensemble Maxwell–Faraday model, of fractional type of piezoelectric property.

Now, we extract one characteristic expression from the first term of the Laplace transform $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate in the form (35), of the external independent generalized coordinate, which describes the creep-flow dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the lower part in Figure 3, and after its transformation, we write it in the form:

$$\frac{1}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} = \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \frac{1}{\left\langle 1 - \omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \right\rangle} \quad (68)$$

Then, let us expand the previous transformed expression (68), into an approximate analytically stable series, using series (40), assuming that the term $-\omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha$ is much smaller than unity: $-\omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \ll 1$. Then we obtain the following expression:

$$\frac{1}{\left\langle p^2 - \omega_\alpha^2 p^\alpha + \frac{\omega_\alpha^4 p^{2\alpha}}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} = \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(-\omega_\alpha^2 p^{\alpha-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \right)^k p^{\alpha k} \right\rangle \quad (69)$$

Now we extract two expression terms from the first term of the Laplace transform $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate in the form (35), of the external inde-

pendent generalized coordinate which describes the free eigen creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the lower part in Figure 3. We will introduce these two separate terms into the expressions of Laplace transforms of eigenmodes similar to sine, or cosine, and write them in the form:

$$\mathcal{L}\{x_{free,like\sin}(t)\} = \dot{x}_0 \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(-\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \quad (70)$$

$$\mathcal{L}\{x_{free,like\cos}(t)\} = x_0 \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(-\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \quad (71)$$

Then, we extract the last term from the Laplace transform $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate in the form (35), of the external independent generalized coordinate $\mathcal{L}\{x(t)\}$ which describes the forced creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the lower part in Figure 3. We will call that separate term the Laplace transform expression of the forced mode similar to sine, because it is an external forced force of sinusoidal properties, or similar to cosine, when the external forced force is of the cosine type of properties, and we write it in the form:

$$\mathcal{L}\{x_{Forced,like\sin}(t)\} = \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(-\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \cdot \frac{\Omega}{p^2 + \Omega^2} \quad (72)$$

Next, it is necessary to expand the binomial, which appears in the previous expressions from the Laplace transform $\mathcal{L}\{x(t)\}$ of the independent generalized coordinate in the form (35), of the external independent generalized coordinate $x(t)$ in its own and

$$(a+b)^k = \sum_{j=0}^k \binom{k}{j} a^{k-j} b^j$$

forced modes into powers of the order j , using Formula (41), so that we obtain that this term is expanded into powers of the order of the form:

$$\left(-\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k = \sum_{j=0}^k \binom{k}{j} \left(-\omega_\alpha^2 p^{-2} \right)^{k-j} \left(\frac{\omega_\alpha^2}{\omega_0^2} \right)^j \quad (73)$$

Now, the previous expressions (70), (71) and (72) of the Laplace transforms of the own and forced modes of the external independent generalized coordinate which describes the free and forced modes of creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from lower part in Figure 3, with piezoelectric property, we write it in the form:

* Laplace transform expression for the eigenmode “like sin” of the external generalized coordinate:

$$\mathcal{L}\{x_{free,like\sin}(t)\} = \dot{x}_0 \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p^2} \left\langle 1 + \sum_{k=1}^{\infty} p^{\alpha k} \sum_{j=0}^k \binom{k}{j} (-1)^j \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\rangle \quad (74)$$

Or in the form of:

$$\begin{aligned} \mathcal{L}\{x_{free,like\sin}(t)\} = & \dot{x}_0 \left(p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-2} \right) + \dot{x}_0 p^{-2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} + \\ & + \dot{x}_0 \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \end{aligned} \quad (75)$$

* Laplace transform expression for the eigenmode “like-cos” of the external generalized coordinate

$$\mathcal{L}\{x_{free,like\cos}(t)\} = x_0 \frac{(\omega_0^2 + \omega_\alpha^2 p^\alpha)}{\omega_0^2 p} \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(-\omega_\alpha^2 p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} \right)^k p^{\alpha k} \right\rangle \quad (76)$$

Or in the form of:

$$\begin{aligned} \mathcal{L}\{x_{free,like\cos}(t)\} = & \left(p^{-1} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-1} \right) + \sum_{k=1}^{\infty} (-1)^k p^{\alpha k-1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} + \\ & + \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha(k+1)-1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \end{aligned} \quad (77)$$

The previous expressions of the Laplace transform of the eigenmodes (75) and (77) of the external generalized coordinates, we transformed into the form of power series by the parameter of the Laplace transform, suitable for determining the inverse Laplace transforms of power functions by expressions (40) and (41). Now, by applying these formulas (42) and (43) to the Laplace transforms (75) and (77), we determine their inverse Laplace transformations, and thus we translate these expressions into the time domain. And thus we obtain approximate analytical expressions for the eigenmodes of independent generalized coordinates $x(t)$ in the time domain, in the form:

* The inverse Laplace transform of its own mode, of the “sinus-like” type of rheological creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from upper part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} x_{free,like\sin}(t) = & \mathcal{L}^{-1} \mathcal{L}\{x_{free,like\sin}(t)\} = \dot{x}_0 \mathcal{L}^{-1} \left\{ p^{-2} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-2} \right\} + \\ & + \dot{x}_0 \mathcal{L}^{-1} \left\{ \sum_{k=1}^{\infty} (-1)^k p^{\alpha k-2} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\} + \\ & + \dot{x}_0 \mathcal{L}^{-1} \left\{ \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k p^{\alpha(k+1)-2} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\} \end{aligned} \quad (78)$$

As in the previous Section 6.1, we solved the inverse Laplace transforms of the three terms in the previous expression (78), and in the final approximate analytical form, in the form of an expression as a function of time, we write the final approximate analytical expression of the own mode, of the “sinusoidal” type of the rheological creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from lower part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} x_{free,like\sin}(t) = & \dot{x}_0 t + \dot{x}_0 \frac{\omega_\alpha^2}{\omega_0^2} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} t^{\alpha-2} + \\ & + \dot{x}_0 \sum_{k=1}^{\infty} (-1)^k t^{(2-\alpha)k+1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \\ & + \dot{x}_0 \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k t^{2k+1-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \end{aligned} \quad (79)$$

* The inverse Laplace transform of its own mode, of the “cosines-like” type of rheological creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from the lower part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} \mathcal{L}\{x_{free,like cos}(t)\} = & \left(p^{-1} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-1} \right) + \sum_{k=0}^{\infty} (-1)^k p^{\alpha k-1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} + \\ & + \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=0}^{\infty} (-1)^k p^{\alpha(k+1)-1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \end{aligned}$$

and in rewritten form

$$\begin{aligned} x_{free,like cos}(t) = & \mathcal{L}^{-1} \mathcal{L}\{x_{free,like cos}(t)\} = \mathcal{L}^{-1} \left\{ p^{-1} + \frac{\omega_\alpha^2}{\omega_0^2} p^{\alpha-1} \right\} + \\ & + \mathcal{L}^{-1} \left\{ \sum_{k=0}^{\infty} (-1)^k p^{\alpha k-1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\} + \\ & + \mathcal{L}^{-1} \left\{ \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=0}^{\infty} (-1)^k p^{\alpha(k+1)-1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} p^{-2(k-j)} \right\} \end{aligned} \quad (80)$$

As in the previous Section 6.1, we solved the inverse Laplace transforms of the three terms in the previous expression (80), and in the final approximate analytical form, in the form of an expression as a function of time, we write the final approximate analytical expression of the own mode, of the “sinusoidal” type of the rheological creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from lower part in Figure 3, with piezoelectric property, we write it in the form:

$$\begin{aligned} x_{free,like cos}(t) = & 1 + \frac{\omega_\alpha^2}{\omega_0^2} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \\ & + \sum_{k=0}^{\infty} (-1)^k t^{k(\alpha-2)-4} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{2j}}{\Gamma(k(\alpha-2)-1+2j)} \\ & + \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=0}^{\infty} (-1)^k t^{k(\alpha-2)+\alpha-2} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{t^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \end{aligned} \quad (81)$$

Which is the Laplace transform of the forced mode under the action of an external periodic sinusoidal force of the form:

$$\mathcal{L}\{x_{Forced,like sin}(t)\} = \frac{1}{\left\langle p^2 - \omega_0^2 + \frac{\omega_0^4}{\omega_0^2 + \omega_\alpha^2 p^\alpha} \right\rangle} \cdot \frac{\Omega}{p^2 + \Omega^2} \quad (82)$$

to obtain its inverse Laplace transform, we can apply the Convolution Theorem.

Now, based on the previous (79) expression by applying the Convolution Theorem, in the final approximate analytical form, in the form of an expression as a function of time, we write the final approximate analytical expressions of the forced modes, of the “sinus-like” or “cosine-like” type for the rheological forced creep dynamics of that rheological Maxwell–Faraday model, of fractional type of discrete dynamic system, from lower part in Figure 3, with piezoelectric property, under the action of an external forced periodic sinusoidal, or cosine force we write it in the form:

$$\begin{aligned}
x_{\text{Forced, like sin}}(t) &= \int_0^t \sin \Omega(t-\tau) \left\langle \tau + \frac{\omega_\alpha^2}{\omega_0^2} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \tau^{\alpha-2} \right\rangle d\tau + \\
&+ \int_0^t \sin \Omega(t-\tau) \left\langle \sum_{k=1}^{\infty} (-1)^k \tau^{(2-\alpha)k+1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right\rangle d\tau + \\
&+ \int_0^t \sin \Omega(t-\tau) \left\langle \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k \tau^{2k+1-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right\rangle d\tau
\end{aligned} \quad (83)$$

$$\begin{aligned}
x_{\text{Forced, like sin}}(t) &= \int_0^t \sin \Omega(t-\tau) \left\langle \tau + \frac{\omega_\alpha^2}{\omega_0^2} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \tau^{\alpha-2} \right\rangle d\tau + \\
&+ \int_0^t \sin \Omega(t-\tau) \left\langle \sum_{k=1}^{\infty} (-1)^k \tau^{(2-\alpha)k+1} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right\rangle d\tau + \\
&+ \int_0^t \sin \Omega(t-\tau) \left\langle \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k \tau^{2k+1-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right\rangle d\tau
\end{aligned} \quad (84)$$

6.3. Approximate Analytical Expressions for the Modes of Independent Generalized Coordinates, Which Correspond to Each of the Internal Degrees of Freedom of the Flow Dynamics, Each of the Two Rheological Maxwell–Faraday Models, of Fractional Type of Discrete Dynamic Systems, from Upper and Lower Parts in Figure 3, with Piezoelectric Property, Which Are Inversely Related to the Standard Light Rheological Maxwell–Faraday Model, of Fractional Type with Piezoelectric Property

Based on the previously determined relationships: relationship (24) from Section 4.1, and relationship (33) from Section 4.2, respectively, give the relationships of independent generalized coordinates corresponding to the Laplace transforms of the coordinates that correspond to the internal and external degrees of freedom of the flow dynamics in each of the two rheological Maxwell–Faraday models, of fractional type with piezoelectric properties, discrete dynamic systems, which are shown in the upper part and lower part of Figure 3, respectively.

In order to apply the convolution theorem

$$\mathcal{L}^{-1} \mathcal{L} \{x_{u,1, \text{free, like sin}}(t)\} = \mathcal{L}^{-1} \left\{ \frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} \right\} * \mathcal{L}^{-1} \mathcal{L} \{x_{\text{free, like sin}}(t)\} \quad (85)$$

to apply the convolution theorem, we need to determine the inverse Laplace transforms of the following factors:

$$\frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle} = \frac{\omega_\alpha^2}{\omega_0^2} p^\alpha \sum_{k=0}^{\infty} (-1)^k \frac{\omega_\alpha^{2k}}{\omega_0^{2k}} p^{\alpha k} \quad (86)$$

$$\frac{\omega_0^2}{\omega_0^2 + \omega_\alpha^2 p^\alpha} = \frac{\omega_0^2}{\omega_0^2} \sum_{n=0}^{\infty} (-1)^n \frac{\omega_0^{2n}}{\omega_0^{2n}} p^{\alpha n} \quad (87)$$

The inverse Laplace transforms of the previous expressions are:

$$\mathcal{L}^{-1}\left\{\frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle}\right\} = \frac{\omega_\alpha^2}{\omega_0^2} \mathcal{L}^{-1}\left\{\sum_{k=0}^{\infty} (-1)^k \frac{\omega_\alpha^{2k}}{\omega_0^{2k}} p^{\alpha(k+1)}\right\} \quad (88)$$

$$\mathcal{L}^{-1}\left\{\frac{\omega_\alpha^2 p^\alpha}{\langle \omega_0^2 + \omega_\alpha^2 p^\alpha \rangle}\right\} = \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=0}^{\infty} (-1)^k \frac{\omega_\alpha^{2k}}{\omega_0^{2k}} \frac{t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}$$

$$\mathcal{L}^{-1}\left\{\frac{\omega_0^2}{\omega_0^2 + \omega_\alpha^2 p^\alpha}\right\} = \frac{\omega_0^2}{\omega_0^2} \sum_{n=9}^{\infty} (-1)^n \frac{\omega_0^{2n}}{\omega_0^{2n}} \frac{t^{\alpha n-1}}{\Gamma(n\alpha)} \quad (89)$$

By applying the Convolution Theorem, taking into account the previously solved inverse Laplace transformations (88) and (89) of expressions (86) and (87), as well as the solved approximate analytical expressions for the modes of independent generalized coordinates, which correspond to the internal degree of freedom of the creep-flow dynamics of each of the two rheological Maxwell–Faraday models, of fractional type with piezoelectric properties, discrete dynamic systems, which are shown in the upper part or lower part of Figure 3. Given the large expressions, we will only give in the text one approximate analytical expression for the modes of “sine-like” or for “cosine-like” generalized coordinates of the internal degree of freedom of the creep-flow dynamics of the rheological Maxwell–Faraday discrete systems, of fractional type from Figure 3:

* Approximate analytical expression of the “sine-like” eigenmode of the internal generalized coordinate of the creep-flow dynamics of the rheological Maxwell–Faraday discrete systems, of fractional type, for the model from the upper part of Figure 3 is in the form:

$$x_{u,1,free,like\sin}(t) = \frac{\omega_\alpha^2}{\omega_0^2} \sum_{n=0}^{\infty} (-1)^n \frac{\omega_\alpha^{2n}}{\omega_0^{2n}} \int_0^t \frac{(t-\tau)^{\alpha(n+1)-1}}{\Gamma(\alpha(n+1))} \left(\tau + \frac{\omega_\alpha^2}{\omega_0^2} \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \right) d\tau +$$

$$+ \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=0}^{\infty} (-1)^k \frac{\omega_\alpha^{2k}}{\omega_0^{2k}} \int_0^t \frac{(t-\tau)^{\alpha(n+1)-1}}{\Gamma(\alpha(n+1))} \left(\sum_{k=1}^{\infty} (-1)^k \tau^{2k+1-\alpha k} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right) d\tau$$

$$+ \frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=0}^{\infty} (-1)^k \frac{\omega_\alpha^{2k}}{\omega_0^{2k}} \int_0^t \frac{(t-\tau)^{\alpha(n+1)-1}}{\Gamma(\alpha(n+1))} \left(\frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=1}^{\infty} (-1)^k \tau^{2k-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha(k+1))} \right) d\tau \quad (90)$$

* Approximate analytical expression of the eigenmode “like cosine” of the internal generalized coordinate of the creep-flow dynamics of the rheological Maxwell–Faraday discrete dynamical systems, of fractional type model from the bottom of Figure 3:

$$x_{u,1,free,likesib}(t) = \frac{\omega_0^2}{\omega_0^2} \sum_{n=9}^{\infty} (-1)^n \frac{\omega_0^{2n}}{\omega_0^{2n}} \int_0^t \frac{(t-\tau)^{\alpha n-1}}{\Gamma(n\alpha)} \left(\tau + \frac{\omega_\alpha^2}{\omega_0^2} \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \right) d\tau +$$

$$+ \frac{\omega_0^2}{\omega_0^2} \sum_{n=9}^{\infty} (-1)^n \frac{\omega_0^{2n}}{\omega_0^{2n}} \int_0^t \frac{(t-\tau)^{\alpha n-1}}{\Gamma(n\alpha)} \left(\sum_{k=0}^{\infty} (-1)^k \tau^{2k-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right) d\tau +$$

$$+ \frac{\omega_0^2}{\omega_0^2} \sum_{n=9}^{\infty} (-1)^n \frac{\omega_0^{2n}}{\omega_0^{2n}} \int_0^t \frac{(t-\tau)^{\alpha n-1}}{\Gamma(n\alpha)} \left(\frac{\omega_\alpha^2}{\omega_0^2} \sum_{k=0}^{\infty} (-1)^k \tau^{2k+1-\alpha(k+1)} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\omega_\alpha^{2k}}{\omega_0^{2j}} \frac{\tau^{-2j}}{\Gamma(2(k-j+1)-\alpha k)} \right) d\tau \quad (91)$$

6.4. Short Comment on Convergence

The question is raised about the mathematical convergence of the approximate analytical solutions of the expressions described in this paper. The answer to the question about the proof of the convergence of the obtained approximate analytical solutions and approximate analytical expressions for the eigen and forced modes of creep dynamics in the rheological Maxwell–Faraday discrete dynamic system is also proved to be dynamic by the property of normal stress relaxation, which is quite visible from Figure 2 and the

surface of the relaxation of the normal stress over time and in the interval from zero to one of the exponents of the fractional order of the derivative.

7. Final Considerations

The paper contains newly derived approximate analytical expressions that describe the eigen and forced modes of the “sine-like” or “cosine-like” type of the external and internal generalized coordinates of the creep-flow dynamics of the rheological Maxwell–Faraday discrete systems, of fractional type models from the upper and lower part of Figure 3:

The complete methodology and the mentioned methods with explanations for determining the inverse Laplace transforms are presented. In this case, it is shown that the solutions of the system of two differential equations of fractional reference can be solved in approximate analytical form. The work is a high theoretical-analytical approach that opens up possibilities for specialized numerical research.

In the tasks, which contain the determination of the inverse Laplace transforms, in the general case it is not possible to reach approximate analytical solutions, but the tasks must be solved by numerical methods.

Therefore, we emphasize the importance and originality of the author’s approximate analytical results, which are presented in this work.

These analytical results are not only of mathematical significance, but also have practical significance in terms of the analytical study of the dynamics of viscoelastic and piezoelectric materials, as well as dynamics in discrete cores and continuous media, and open new directions of research, especially in the dynamics of biomaterials.

In the concluding comments, we clearly emphasize that the main scientific contributions of this study are the obtained approximate analytical expressions of the eigen and forced modes of creep-flow dynamics in RMFDDSFTPEP.

Based on the results of this paper devoted to the study of fractional-order Maxwell–Faraday rheological discrete dynamical systems with piezoelectric properties, new approximate analytical solutions have been developed to describe creep-flow processes and the dynamic behavior of viscoelastic properties in the rheological Maxwell–Faraday discrete dynamical systems. The paper considers two-degree-of-freedom models, taking into account both mechanical and electromechanical effects arising from the piezoelectric polarization of structural elements. The results obtained allow us to determine the influence of the connection order of rheological elements on the nature of motion, stress relaxation, and the formation of the electrical response of the system. The relevance of this research is driven by the growing interest in materials and structures with hereditary properties described by fractional calculus, as well as the need to improve the accuracy of mathematical models for intelligent and multifunctional materials. The practical significance of this work lies in the development of methods for the analysis and design of modern viscoelastic and piezoelectric systems used in engineering, mechatronics, and next-generation materials.

The results are approximately analytical and divided into descriptions of natural and forced creep modes of the reaction type.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article. The author declares and confirms that all data and elements from the content of the article, and in general, may be available to the public and interested researchers and that they may be used with appropriate citation of the source.

Acknowledgments: Author would like to emphasize that the content of this article is presented as an invited lecture in the program of the successfully implemented The 1st International Online Conference on Fractal and Fractional: Theoretical Foundations and Interdisciplinary Applications (IOCF 2026), which will be held online by the MDPI open access journal Fractal and Fractional (ISSN: 2504-3110; Impact Factor: 3.3) from 13 to 15 April 2026. <https://sciforum.net/event/IOCF2026> (accessed on 15 April 2026). Also, author would like to thank the Editorial Board of the Journal “Fractal and Fractional”, and especially the Assistant Editors for their permanent support and assistance in publishing this article.

Conflicts of Interest: The author confirms that there are no elements of conflict of interest in the content of the article and in general.

References

1. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*; Imperial College Press: London, UK, 2010. <https://doi.org/10.1142/9781848163300>.
2. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science Publishers: Amsterdam, The Netherlands, 1993.
3. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Pitman Research Notes in Mathematics Series; Longman Scientific & Technical: Harlow, UK, 1994; Volume 301.
4. Hedrih (Stevanović), K.; Milovanović, G.V. Elements of mathematical phenomenology and analogies of electrical and mechanical oscillators of the fractional type with finite number of degrees of freedom of oscillations: Linear and nonlinear modes. *Commun. Anal. Mech.* **2024**, *16*, 738–785. <https://doi.org/10.3934/cam.2024033>.
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
6. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
7. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer-Verlag: Berlin, Germany, 2010. <https://doi.org/10.1007/978-3-642-14574-2>.
8. Herrmann, R. *Fractional Calculus: An Introduction for Physicists*; World Scientific: Singapore, 2014. <https://doi.org/10.1142/8934>.
9. Hedrih (Stevanović), K.R. Rheologic fractional oscillators or creepers. *Fractal Fract.* **2025**, *9*, 552. <https://doi.org/10.3390/fractalfract9080552>.
10. Bronstein, I.N.; Semendjajew, K.A. *Taschenbuch der Mathematik*; B.G. Teubner Verlagsgesellschaft: Leipzig, Germany, 1958; p. 176.
11. Mitrović, D.S.; Đoković, D.Ž. *Special Functions (Specijalne Funkcije)*; Građevinska knjiga: Beograd, Yugoslavia, 1964; p. 267.
12. Rašković, D.P. *Teorija Oscilacija (Theory of Oscillations)*; 1952 (1st ed.); 1965 (2nd ed.); Naučna knjiga: Beograd, Yugoslavia, 1950. Available online: <http://elibrary.matf.bg.ac.rs/handle/123456789/4754> (accessed on 15 April 2026)

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.