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On the Averaging Principle for Fuzzy Fractional Stochastic Differential Equations

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Abstract

This paper aims to extend the averaging principle for first order fuzzy stochastic differential equations to Riemann–Liouville fuzzy fractional stochastic differential equations, addressing the research gap in this field. This extension enables the averaging principle for fuzzy stochastic systems to cover fractional scenarios with memory, providing a simplified analytical tool for practical systems involving both “randomness” and “fuzziness” uncertainties.

Keywords: Riemann–Liouville; fuzzy fractional stochastic differential equations; averaging principle; Gronwall inequality

MSC: 39B72; 93B05

1. Introduction

The averaging principle is a powerful tool for studying stochastic dynamical systems, which has received extensive attention and research over the past few decades. The core idea of the averaging method is to transform the original complex equation into an “averaged equation” with a simpler structure, and then use the solution of the averaged equation to approximate the solution of the original equation. This provides a relatively simple approach to solve differential equations, so it is widely applied in research across various fields, laying a solid foundation for the averaging processes of complex equations in control theory, engineering mechanics, and mathematics [1,2].

The averaging principle for ordinary stochastic differential equations (SDEs) has achieved relatively systematic development. Guo et al. [3] considered the averaging principle for a class of stochastic differential equations under weak averaging conditions. Xu et al. investigated various types of two-time-scale stochastic differential equations in [4,5], and found that under appropriate conditions, the slow equation always converges to the corresponding averaged equation. Xu et al. [6] established the averaging principle for stochastic differential equations with non-Gaussian Lévy noise, and used the solution of the averaged stochastic differential equation to approximate the solution of the stochastic differential equation with Lévy noise. In the research on the averaging principle, most scholars have adopted the global Lipschitz condition. However, Guo et al. [7] discussed the averaging principle for a class of nonlinear stochastic differential equations that only satisfy local Lipschitz and monotonicity conditions. Xu et al. [8] discussed a class of stochastic differential equations coupled with Lévy noise and non-Lipschitz coefficients, and derived their strong convergence. You et al. [9] developed a two-time-scale stochastic dengue model driven by Brownian motion, proved the existence of an ergodic invariant measure



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for the mosquito subsystem, and established the strong convergence of the human subsystem to the solution of the averaged equations via the averaging principle for stochastic differential equations.

Compared with classical SDEs, fractional stochastic differential equations (FSDEs) can better characterize the memory and long-term dependence in real-world systems. Therefore, FSDEs have a wide range of applications in signal processing, biomedicine, finance, and physics [10–13], and many scholars have begun to explore the averaging principle for fractional stochastic differential equations. Xu et al. focused on the averaging principle for fractional stochastic differential equations in [14–16]. By constructing an “averaged equation”, they simplified the complex stochastic systems with fractional characteristics, and proved that the solution of the original equation can be approximated by the solution of the averaged equation in the mean-square sense. However, their research scenarios focused on “fractional types”, “noise types”, and “fractional intervals” respectively, which collectively improved the averaging principle for fractional stochastic systems. Liu et al. [17] proposed the averaging principle under non-Lipschitz conditions for impulse SDEs, which solves the limitation of the traditional cycle averaging method in dealing with impulse terms by an improved estimation method. Luo et al. derived the averaging principle for a class of stochastic fractional delay differential equations through Laplace transform and inverse Laplace transform in [18]. Luo et al. researched a class of stochastic Hilfer-type fractional differential equations with non-Lipschitz coefficients in [19], obtained the implicit solution of the equation by means of Laplace transform and its inverse, and proved its averaging principle. Guo et al. researched a class of stochastic Caputo fractional differential equations where the nonlinear term satisfies the non-Lipschitz condition in [20]. They derived the existence and uniqueness of the solution through the generalized Gronwall inequality and Picard iteration method, and proved its averaging principle. Xiao et al. [21] enhanced the estimation approach by incorporating the growth condition of the nonlinear term and put forward a novel averaging condition. Zou et al. explored a class of Caputo-type fractional delay stochastic differential equations driven by Brownian motion in [22], and discussed the relevant properties of the solution of equations as well as the averaging principle. Yang et al. studied a class of Caputo-type generalized proportional fractional delay stochastic differential equations with non-Lipschitz coefficients in [23], and investigated the relevant properties of the equation’s solution and the averaging principle. Huang et al. [24] improved the processing method of singular kernel integrals, and proved the averaging principle of a class of variable-order FSDEs. In [25], Huang et al. investigated a class of generalized Caputo–Katugampola fractional stochastic neutral differential equations driven by multiplicative fractional Brownian motion, under certain conditions. Firstly, they proved the existence and uniqueness of the solution by using the Banach fixed point theorem. Then, they demonstrated the L^p convergence of the solution of the averaged equation to the solution of the original equation through Hölder inequality and Jensen inequality. Moualkia et al. [26] researched novel results on the averaging principle for a class of time-varying fractional-order Caputo neutral stochastic systems with variable delays, driven by Markovian switching and Lévy noise.

To more accurately characterize the dual uncertainties of “randomness” and “fuzziness” that coexist in real-world systems, and make up for the limitations of ordinary stochastic differential equations, research on fuzzy stochastic differential equations has become particularly important. For instance, in real-world hybrid dynamic systems, evolution processes are often driven by both stochastic fluctuations (e.g., fractional Brownian motion) and data imprecision expressed by fuzzy sets [27]. In physical and biological systems, such as population dynamics, models frequently suffer from environmental stochastic perturbations while their inherent parameters possess natural fuzziness [28].

Furthermore, the framework of fuzzy fractional stochastic differential equations (FFSDEs) is highly suited for modeling modern control systems, which require the simultaneous handling of unpredictable random impulses and fuzzy state variables [29]. Specifically, in networked control theory and electronic circuits, fractional-order systems must often maintain consensus while defending against unpredictable random external perturbations (such as cyber-attacks) under fuzzy environments [30]. Thus, extending analytical tools like the averaging principle to these practical scenarios with dual uncertainties is of great significance. However, up to now, there are relatively few research studies on the averaging principle for fuzzy SDEs. In [31], the authors proposed a method for defining fuzzy stochastic integrals, which can embed clear stochastic integrals into the fuzzy space, and thereby constructed fuzzy random variables. In [32], the authors researched the averaging principle for a class of first-order fuzzy SDEs.

While the existing literature has extensively explored the averaging principle for Caputo-type (e.g., [20,22,23]) and Hilfer-type (e.g., [19]) FSDEs, these studies are primarily confined to crisp (non-fuzzy) environments. To clearly position the novelty of our work relative to these prior studies, it should be emphasized that this paper specifically focuses on Riemann–Liouville fuzzy fractional stochastic systems. Unlike classical FSDEs, our state space is the fuzzy number space equipped with the Hausdorff metric. This transition from crisp to fuzzy fractional frameworks is not a trivial extension; the theoretical proofs require a novel analytical approach to simultaneously handle fractional singular kernels, stochastic integrals, and the unique algebraic complexities of fuzzy arithmetic. Thus, our work fundamentally distinguishes itself from existing Caputo-type, Hilfer-type, and non-fuzzy fractional stochastic averaging results.

This paper aims to extend the averaging principle for fuzzy stochastic differential equations in [32] to FFSDEs. The main contributions of this paper can be divided into the following aspects:

- (1) The averaging principle for first order fuzzy SDEs was investigated in [32], and we extend it to Riemann–Liouville FFSDEs, filling the gap in this field. This extension enables the averaging principle for fuzzy stochastic systems to cover fractional scenarios with memory, providing a simplified analysis tool for practical systems with dual uncertainties (randomness and fuzziness) and clarifying the application value of the results.
- (2) The integral form of the solution to the original equation is obtained through Laplace transform and inverse Laplace transform, and the solution can be expressed using the Mittag–Leffler function. This expression of the solution provides a key technical foundation for the subsequent proof of the convergence of the averaging principle, and also offers a theoretical basis for numerical simulations, highlighting the role of the technical contribution.

The main content of this paper is as follows. In Section 2, some basic knowledge and the knowledge required for subsequent proofs are provided. In Section 3, the averaging principle for FFSDEs under specific conditions is investigated. In Section 4, an example is given to verify our results. In Section 5, our conclusions are presented.

2. Preliminaries

To facilitate subsequent work, we briefly present the preparatory work in this section.

Throughout this paper, $C(I, R_F)$ denotes the space of all continuous functions from I into R_F .

Definition 1. Denote

$$R_F = \{v : \mathbb{R} \rightarrow [0, 1] \mid v \text{ satisfies conditions (1)–(4)}\}$$

as the space of fuzzy numbers, where:

- (1) v is normal (i.e., $\exists s_0 \in \mathbb{R}$ s.t. $v(s_0) = 1$),
- (2) v is convex fuzzy set (i.e., $v(\xi s + (1 - \xi)r) \geq \min\{v(s), v(r)\}$) for all $s, r \in \mathbb{R}$ and $\xi \in [0, 1]$,
- (3) v is upper semi-continuous on $\mathbb{R}$,
- (4) $[v]^0 = \overline{\{s \in \mathbb{R} : v(s) > 0\}}$ is compact.

Let $\gamma \in (0, 1]$. Consider the γ -level set of $v \in R_F$ by $[v]^\gamma = \{s \in \mathbb{R} \mid v(s) \geq \gamma\}$, which is a nonempty compact interval for all $\gamma \in (0, 1]$. We use $[v]^\gamma = [\underline{v}^\gamma, \overline{v}^\gamma]$ to denote explicitly the γ -level set of v . We call $\underline{v}^\gamma$ and $\overline{v}^\gamma$ the lower and upper branches of v , respectively. We use the notation $\text{diam}([v]^\gamma) = \overline{v}^\gamma - \underline{v}^\gamma$ to denote the length of v .

For any $\gamma \in [0, 1]$, $u, v \in R_F$ and $k \in \mathbb{R}$, we define the sum $u + v$, the subtraction $u - v$ and the product ku , with the following forms:

- (1)

$$[u + v]^\gamma = [u]^\gamma + [v]^\gamma = [\underline{u}^\gamma + \underline{v}^\gamma, \overline{u}^\gamma + \overline{v}^\gamma],$$
- (2)

$$[k \odot u]^\gamma = k \odot [u]^\gamma = \begin{cases} [k\underline{u}^\gamma, k\overline{u}^\gamma] & k > 0, \\ [k\overline{u}^\gamma, k\underline{u}^\gamma] & k < 0, \end{cases}$$
- (3)

$$[u - v]^\gamma = [u]^\gamma + (-1)[v]^\gamma = [\underline{u}^\gamma - \overline{v}^\gamma, \overline{u}^\gamma - \underline{v}^\gamma].$$

Definition 2. The metric structure is determined by the Hausdorff distance, which is defined as follows:

$$d_\infty : R_F \times R_F \rightarrow \mathbb{R}_+ \cup \{0\},$$

$$d_\infty(u, v) = \sup_{\gamma \in [0, 1]} \max\{|\underline{u}^\gamma - \underline{v}^\gamma|, |\overline{u}^\gamma - \overline{v}^\gamma|\}.$$

(R_F, d_∞) is a complete metric space and the subsequent properties are widely recognized:

$$\begin{aligned} d_\infty(u + j, v + j) &= d_\infty(u, v), \quad \forall u, v, j \in R_F, \\ d_\infty(k \odot u, k \odot v) &= |k|d_\infty(u, v), \quad \forall k \in \mathbb{R}, u, v \in R_F, \\ d_\infty(u + j, v + e) &\leq d_\infty(u, v) + d_\infty(j, e), \quad \forall u, v, j, e \in R_F. \end{aligned}$$

Definition 3. Let $\mu_1, \mu_2 \in R_F$. If there exists a $\mu_3 \in R_F$ such that $\mu_1 = \mu_2 + \mu_3$ then μ_3 is called the H -difference of μ_1, μ_2 and it can be expressed by $\mu_1 \ominus \mu_2$.

Throughout this paper, we use the symbol $\ominus$ to represent the H -difference. Note that $\mu_1 \ominus \mu_2 \neq \mu_1 + (-1)\mu_2 = \mu_1 - \mu_2$, and we have

$$[\mu_1 \ominus \mu_2]^\gamma = [\underline{\mu}_1^\gamma - \underline{\mu}_2^\gamma, \overline{\mu}_1^\gamma - \overline{\mu}_2^\gamma].$$

Let $I = [0, T]$ be a compact interval on the real axis $\mathbb{R}$, and let $C^F(I)$ and $L^F(I)$ be the spaces of all continuous fuzzy-valued functions and all Lebesgue integrable fuzzy-valued functions on I , respectively.

To rigorously address the initial singularity at $t = 0$ induced by the fractional integral kernel for $\alpha \in (1/2, 1)$, we introduce the weighted continuous function space $C_{1-\alpha}(I, R_F)$. Let the closed interval be $I = [0, T]$. The space $C_{1-\alpha}(I, R_F)$ is defined as the set of all functions $y \in C((0, T], R_F)$ such that the weighted function $t^{1-\alpha}y(t)$ can be continuously extended to the entire closed interval I . This space is a Banach space when equipped with the weighted supremum norm:

$$\|y\|_{C_{1-\alpha}} = \sup_{t \in [0, T]} \|t^{1-\alpha}y(t)\|.$$

This ensures that the solutions are well-defined on the closed interval $[0, T]$ despite the fractional singularity at the origin.

Definition 4 ([33]). Let $f \in C^F(I) \cap L^F(I)$. The fuzzy Riemann–Liouville integral of fuzzy-valued function f is defined as follows:

$$(I_{0+}^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f(s)ds, \quad t > 0, \quad \frac{1}{2} < \alpha < 1,$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt$ is the gamma function.

Definition 5 ([33]). Let $f \in C^F(I) \cap L^F(I)$, $t_0 > 0$ and $F(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(s)}{(t-s)^{\alpha}}ds$. We say f is Riemann–Liouville H -differentiable of order $\frac{1}{2} < \alpha < 1$ at t_0 , if there exists an element $D_{0+}^{\alpha}f(t_0) \in R_F$, such that

(c1) for all $h > 0$ sufficiently close to 0, the H -differences $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ exist and the limits

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = D_{0+}^{\alpha}f(t_0),$$

or

(c2) for all $h > 0$ sufficiently close to 0, the H -differences $F(t_0) \ominus F(t_0 + h)$, $F(t_0 - h) \ominus F(t_0)$ exist and the limits

$$\lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = D_{0+}^{\alpha}f(t_0).$$

Definition 6. Let $f \in C^F(I) \cap L^F(I)$. We say ${}^{RL}[(1) - \alpha]$ is differentiable on I , if f is differentiable in the sense (c1). Similarly we can define ${}^{RL}[(2) - \alpha]$ as differentiable on I .

Remark 1. To clarify the distinction and application of these derivatives, the choice between ${}^{RL}[(1) - \alpha]$ -differentiable and ${}^{RL}[(2) - \alpha]$ -differentiable fundamentally depends on the evolution of the system's uncertainty (the diameter of the fuzzy level sets) over time.

- ${}^{RL}[(1) - \alpha]$ -differentiable: This definition applies to scenarios where the fuzziness (uncertainty) increases as time progresses.
- ${}^{RL}[(2) - \alpha]$ -differentiable: This definition is applied to scenarios where the fuzziness decreases or resolves over time.

Crucially, the evolution of uncertainty (Type 1 or Type 2) and the system's internal drift (determined by the sign of λ) are independent factors. As explicitly demonstrated in Lemma 1, their combinations yield four distinct mathematical solution structures (involving $E_{\alpha, \alpha}$, Ch_{α} , and Sh_{α}) to capture all possible physical dynamics.

Theorem 1. Let $f \in C^F(I) \cap L^F(I)$, $t_0 > 0$ and $\frac{1}{2} < \alpha < 1$, put $[f]^{\gamma} = [\underline{f}^{\gamma}, \bar{f}^{\gamma}]$ for each $\gamma \in [0, 1]$.

- (1) If f is ${}^{\text{RL}}[(1) - \alpha]$ -differentiable then $\underline{f}$ and $\bar{f}$ are differentiable functions and $[D_{0+}^{\alpha} f]^{\gamma} = [D_{0+}^{\alpha} \underline{f}^{\gamma}, D_{0+}^{\alpha} \bar{f}^{\gamma}]$.
- (2) If f is ${}^{\text{RL}}[(2) - \alpha]$ -differentiable then $\underline{f}$ and $\bar{f}$ are differentiable functions and $[D_{0+}^{\alpha} f]^{\gamma} = [D_{0+}^{\alpha} \bar{f}^{\gamma}, D_{0+}^{\alpha} \underline{f}^{\gamma}]$.

Let $\langle \cdot \rangle : \mathbb{R} \rightarrow R_F$ be an embedding of $\mathbb{R}$ into R_F ; i.e., for $r \in \mathbb{R}$, one has

$$\langle r \rangle(a) = \begin{cases} 1, & \text{if } a = r, \\ 0, & \text{if } a \neq r. \end{cases}$$

Remark 2. (Clarification of RL differentiability and embedding). It is crucial to distinguish how crisp real numbers interact with fuzzy spaces in our framework.

1. The two conditions (c1) and (c2) correspond to two possible Hukuhara differentiability types of a fuzzy-valued function.

If the lower and upper endpoint functions preserve their natural order after fractional differentiation, we use the ${}^{\text{RL}}[(1) - \alpha]$ -differentiability. In this case,

$$[D_{0+}^{\alpha} y(t)]^{\gamma} = [D_{0+}^{\alpha} \underline{y}^{\gamma}(t), D_{0+}^{\alpha} \bar{y}^{\gamma}(t)].$$

If the fractional differentiation reverses the order of the endpoint functions, we use the ${}^{\text{RL}}[(2) - \alpha]$ -differentiability. In this case,

$$[D_{0+}^{\alpha} y(t)]^{\gamma} = [D_{0+}^{\alpha} \bar{y}^{\gamma}(t), D_{0+}^{\alpha} \underline{y}^{\gamma}(t)].$$

Therefore, the choice between the ${}^{\text{RL}}[(1) - \alpha]$ -solution and the ${}^{\text{RL}}[(2) - \alpha]$ -solution depends on the orientation induced by the Hukuhara difference in the corresponding solution representation.

2. It is crucial to distinguish how crisp real numbers interact with fuzzy spaces in our framework: when a crisp real function (such as $4 \cos^2(t)$ or the Mittag–Leffler function) scales a fuzzy variable, we consistently treat it as a crisp scalar and use the scalar multiplication operator $\odot$, which does not change the crisp nature of the scalar, while the embedding operator $\langle \cdot \rangle$ is strictly and exclusively applied to transform a real-valued random variable (specifically, the result of the crisp stochastic Itô integral $\int g dB$) into a degenerate fuzzy random variable within the fuzzy space R_F , and we never mix these two distinct mathematical operations.

Definition 7. By fuzzy stochastic Itô integral we mean the fuzzy random variable $\left\langle \int_0^t v(s) dB(s) \right\rangle$.

For every $t \in I$, consider the fuzzy stochastic Itô integral $\left\langle \int_0^t v(s) dB(s) \right\rangle$, which may be interpreted as follows

$$\left\langle \int_0^t v(s) dB(s) \right\rangle = \left\langle \int_0^T \chi_{[0,t]}(s) v(s) dB(s) \right\rangle.$$

Proposition 1. If $u, v \in \mathcal{L}^2(I \times \Omega, R_F; \mathbb{R})$, then $\forall t \in I$ we have

$$d_{\infty}^2 \left(\left\langle \int_0^t u(s) dB(s) \right\rangle, \left\langle \int_0^t v(s) dB(s) \right\rangle \right) = \int_0^t d_{\infty}^2(\langle u(s) \rangle, \langle v(s) \rangle) ds.$$

Proposition 2 ([34]). For $u, v \in \mathcal{L}^p(I \times \Omega, R_F; \mathbb{R})$ and $p \geq 1$, we have

$$E \sup_{a \in [0,t]} d_{\infty}^p \left(\int_0^a u(s) ds, \int_0^a v(s) ds \right) \leq t^{p-1} \int_0^t E d_{\infty}^p(u(s), v(s)) ds.$$

3. Main Results

Consider the following FFSDs:

$$\begin{cases} d(D_{0+}^{\alpha-1}y(t)) = [\lambda \odot y(t) + f(t, y(t))]dt + d\left\langle \int_0^t g(s, y(s))dB(s) \right\rangle, & t > 0, \frac{1}{2} < \alpha < 1, \\ D_{0+}^{\alpha-1}y(0) = y_0 \in R_F, \end{cases} \quad (1)$$

where $f : I \times R_F \rightarrow R_F$ is a fuzzy-valued function, and $g : I \times R_F \rightarrow \mathbb{R}$ is a real-valued function, and $(B(t))_{t \geq 0}$ is a standard Brownian motion on a probability space $(\Omega, F_T, \mathbf{P})$ for $T > 0$, with filtration $F_T = \{\mathfrak{F}_t\}_{t \in [0, T]}$ satisfying usual conditions.

Here, the notation $\langle g(t, y(t))dB(t) \rangle$ is understood in the sense of the Malinowski–Michta integral framework. Specifically, $g(t, y(t))$ implicitly represents the real-valued mapping (such as the distance to the fuzzy zero element or the selection of level-set components) derived from the fuzzy stochastic process $y(t)$. This ensures that $g(t, y(t))$ acts as the real-valued representative suitable for the standard Itô integral, which is then embedded into the fuzzy space R_F via the operator $\langle \cdot \rangle$.

Definition 8. Let $y : I \rightarrow R_F$ be a fuzzy-valued function. If it is ${}^{RL}[(1) - \alpha]$ -differentiable and y and $d(D_{0+}^{\alpha-1}y(t))$ satisfy Equation (1), we say y is ${}^{RL}[(1) - \alpha]$ -solution of Equation (1). Similarly, if it is ${}^{RL}[(2) - \alpha]$ -differentiable and y and $d(D_{0+}^{\alpha-1}y(t))$ satisfy Equation (1), we say y is ${}^{RL}[(2) - \alpha]$ -solution of Equation (1).

Lemma 1. A function $y(\cdot) \in C_{1-\alpha}([0, T], R_F)$ is a solution to the Equation (1) on the interval $[0, T]$ if and only if it is a solution to the integral equation.

- for case $\lambda > 0$, ${}^{RL}[(1) - \alpha]$ -solution

$$\begin{aligned} y_1(t) = & t^{\alpha-1}E_{\alpha, \alpha}(\lambda t^\alpha) \odot y_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot f(s, y_1)ds \\ & + \left\langle \int_0^t (t-s)^{\alpha-1}E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot g(s, y_1)dB(s) \right\rangle, \end{aligned}$$

- for case $\lambda < 0$, ${}^{RL}[(2) - \alpha]$ -solution

$$\begin{aligned} y_2(t) = & t^{\alpha-1}E_{\alpha, \alpha}(\lambda t^\alpha) \odot y_0 \ominus (-1) \int_0^t (t-s)^{\alpha-1}E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot f(s, y_2)ds \\ & \ominus (-1) \left\langle \int_0^t (t-s)^{\alpha-1}E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot g(s, y_2)dB(s) \right\rangle, \end{aligned}$$

- for case $\lambda < 0$, ${}^{RL}[(1) - \alpha]$ -solution

$$\begin{aligned} y_3(t) = & t^{\alpha-1}(Ch_\alpha(\lambda t^\alpha) \odot y_0 + Sh_\alpha(\lambda t^\alpha) \odot y_0) \\ & + \int_0^t (t-s)^{\alpha-1}(Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_3) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_3))ds \\ & + \left\langle \int_0^t (t-s)^{\alpha-1}(Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_3) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_3))dB(s) \right\rangle, \end{aligned}$$

- for case $\lambda > 0$, ${}^{RL}[(2) - \alpha]$ -solution

$$\begin{aligned} y_4(t) = & t^{\alpha-1}(Ch_\alpha(\lambda t^\alpha) \odot y_0 \ominus (-1)Sh_\alpha(\lambda t^\alpha) \odot y_0) \\ & \ominus (-1) \int_0^t (t-s)^{\alpha-1}(Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_4) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_4))ds \\ & \ominus (-1) \left\langle \int_0^t (t-s)^{\alpha-1}(Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_4) \right. \\ & \left. + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_4))dB(s) \right\rangle. \end{aligned}$$

It should be noted that due to the singular factor $t^{\alpha-1}$ for $0 < \alpha < 1$, the solution is continuous on $(0, T]$. However, the weighted function $t^{1-\alpha}y(t)$ remains continuous on the closed interval $[0, T]$.

We deal with the existence and uniqueness of solutions for the Equation (1). The following assumptions are given:

(A1) There exists a constant $C_1 > 0$ such that $\forall t \in I$ and $\forall y \in R_F$ we have $d_\infty^2(f(t, y), \hat{0}) \leq C_1^2(1 + d_\infty^2(y, \hat{0}))$ and $\|g(t, y)\|^2 = d_\infty^2(\langle g(t, y) \rangle, \hat{0}) \leq C_1^2(1 + d_\infty^2(y, \hat{0}))$, where $\hat{0}$ denotes the fuzzy zero.

(A2) There exists a constant $C_2 > 0$ such that $\forall t \in I$ and $\forall x, y \in R_F$ we have $d_\infty^2(f(t, x), f(t, y)) \leq C_2 d_\infty^2(x, y)$ and $\|g(t, x) - g(t, y)\|^2 = d_\infty^2(\langle g(t, x) \rangle, \langle g(t, y) \rangle) \leq C_2 d_\infty^2(x, y)$.

(A3) For any $x \in R_F$, there exist averaged coefficients $\tilde{f}(x)$ and $\tilde{g}(x)$ and positive functions $\beta_1(T), \beta_2(T)$ with $\lim_{T \rightarrow \infty} \beta_i(T) = 0$ ($i = 1, 2$) such that

$$\frac{1}{T} \int_0^T d_\infty^2(f(s, x), \tilde{f}(x)) ds \leq \beta_1(T)(1 + d_\infty^2(x, \hat{0})),$$

and

$$\frac{1}{T} \int_0^T \|g(s, x) - \tilde{g}(x)\|^2 ds \leq \beta_2(T)(1 + d_\infty^2(x, \hat{0})).$$

For any $x_i, y_i \in R_F, i = 1, 2, \dots, n$, the following inequality is true

$$d_\infty^2(x_1 + x_2 + \dots + x_n, y_1 + y_2 + \dots + y_n) \leq n \sum_{i=1}^n d_\infty^2(x_i, y_i).$$

Let's consider the standard forms of four integral equations:

- for case $\lambda > 0, {}^{RL}[(1) - \alpha]$ -solution

$$y_{1\varepsilon}(t) = t^{\alpha-1} E_{\alpha, \alpha}(\lambda t^\alpha) \odot y_0 + \varepsilon \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot f(s, y_{1\varepsilon}) ds \\ + \sqrt{\varepsilon} \left\langle \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot g(s, y_{1\varepsilon}) dB(s) \right\rangle,$$

- for case $\lambda < 0, {}^{RL}[(2) - \alpha]$ -solution

$$y_{2\varepsilon}(t) = t^{\alpha-1} E_{\alpha, \alpha}(\lambda t^\alpha) \odot y_0 \ominus (-1)\varepsilon \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot f(s, y_{2\varepsilon}) ds \\ \ominus (-1)\sqrt{\varepsilon} \left\langle \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot g(s, y_{2\varepsilon}) dB(s) \right\rangle,$$

- for case $\lambda < 0, {}^{RL}[(1) - \alpha]$ -solution

$$y_{3\varepsilon}(t) = t^{\alpha-1} (Ch_\alpha(\lambda t^\alpha) \odot y_0 + Sh_\alpha(\lambda t^\alpha) \odot y_0) \\ + \varepsilon \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_{3\varepsilon}) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_{3\varepsilon})) ds \\ + \sqrt{\varepsilon} \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_{3\varepsilon}) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_{3\varepsilon})) dB(s) \right\rangle,$$

- for case $\lambda > 0$, ${}^{RL}[(2) - \alpha]$ -solution

$$\begin{aligned} y_{4\varepsilon}(t) = & t^{\alpha-1}(Ch_{\alpha}(\lambda t^{\alpha}) \odot y_0 \ominus (-1)Sh_{\alpha}(\lambda t^{\alpha}) \odot y_0) \\ & \ominus (-1)\varepsilon \int_0^t (t-s)^{\alpha-1}(Ch_{\alpha}(\lambda(t-s)^{\alpha}) \odot f(s, y_{4\varepsilon}) \\ & + Sh_{\alpha}(\lambda(t-s)^{\alpha}) \odot f(s, y_{4\varepsilon}))ds \\ & \ominus (-1)\sqrt{\varepsilon} \langle \int_0^t (t-s)^{\alpha-1}(Ch_{\alpha}(\lambda(t-s)^{\alpha}) \odot g(s, y_{4\varepsilon}) \\ & + Sh_{\alpha}(\lambda(t-s)^{\alpha}) \odot g(s, y_{4\varepsilon}))dB(s) \rangle. \end{aligned}$$

The next step is to demonstrate the original solution y_{ε} converges, as ε goes to zero, to the solution Y_{ε} of the averaged system:

- for case $\lambda > 0$, ${}^{RL}[(1) - \alpha]$ -solution

$$\begin{aligned} Y_{1\varepsilon}(t) = & t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) \odot y_0 + \varepsilon \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{f}(s, Y_{1\varepsilon})ds \\ & + \sqrt{\varepsilon} \langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{g}(s, Y_{1\varepsilon})dB(s) \rangle, \end{aligned}$$

- for case $\lambda < 0$, ${}^{RL}[(2) - \alpha]$ -solution

$$\begin{aligned} Y_{2\varepsilon}(t) = & t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) \odot y_0 \ominus (-1)\varepsilon \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{f}(s, Y_{2\varepsilon})ds \\ & \ominus (-1)\sqrt{\varepsilon} \langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{g}(s, Y_{2\varepsilon})dB(s) \rangle, \end{aligned}$$

- for case $\lambda < 0$, ${}^{RL}[(1) - \alpha]$ -solution

$$\begin{aligned} Y_{3\varepsilon}(t) = & t^{\alpha-1}(Ch_{\alpha}(\lambda t^{\alpha}) \odot y_0 + Sh_{\alpha}(\lambda t^{\alpha}) \odot y_0) \\ & + \varepsilon \int_0^t (t-s)^{\alpha-1}(Ch_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{f}(s, Y_{3\varepsilon}) + Sh_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{f}(s, Y_{3\varepsilon}))ds \\ & + \sqrt{\varepsilon} \langle \int_0^t (t-s)^{\alpha-1}(Ch_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{g}(s, Y_{3\varepsilon}) + \\ & Sh_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{g}(s, Y_{3\varepsilon}))dB(s) \rangle, \end{aligned}$$

- for case $\lambda > 0$, ${}^{RL}[(2) - \alpha]$ -solution

$$\begin{aligned} Y_{4\varepsilon}(t) = & t^{\alpha-1}(Ch_{\alpha}(\lambda t^{\alpha}) \odot y_0 \ominus (-1)Sh_{\alpha}(\lambda t^{\alpha}) \odot y_0) \\ & \ominus (-1)\varepsilon \int_0^t (t-s)^{\alpha-1}(Ch_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{f}(s, Y_{4\varepsilon}) \\ & + Sh_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{f}(s, Y_{4\varepsilon}))ds \\ & \ominus (-1)\sqrt{\varepsilon} \langle \int_0^t (t-s)^{\alpha-1}(Ch_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{g}(s, Y_{4\varepsilon}) \\ & + Sh_{\alpha}(\lambda(t-s)^{\alpha}) \odot \tilde{g}(s, Y_{4\varepsilon}))dB(s) \rangle. \end{aligned}$$

Lemma 2. Let $T > 0$ and $u(t), a(t)$ be non-negative measurable functions on $[0, T]$. If $C > 0$ and $\alpha > 1/2$ satisfy the integral inequality

$$u(t) \leq a(t) + C \int_0^t (t-s)^{2\alpha-2}u(s)ds, \quad \forall t \in [0, T],$$

then there exists a constant depending only on α and C such that

$$u(t) \leq a(t)E_{2\alpha-1}\left(C\Gamma(2\alpha-1)t^{2\alpha-1}\right),$$

where $E_\gamma(\cdot)$ is the one-parameter Mittag-Leffler function and $\Gamma(\cdot)$ is the Gamma function.

Theorem 2. Suppose that Assumptions (A1)–(A3) are satisfied and the fractional order $\alpha \in (\frac{1}{2}, 1)$. For any arbitrarily small number $\delta_1 > 0$, any given constant $L > 0$, and any parameter $\beta \in (0, 1)$, there exists a sufficiently small constant $\varepsilon_1 > 0$ (depending on δ_1 , L , and β) such that for all $\varepsilon \in (0, \varepsilon_1]$, the following inequality holds:

$$E \sup_{t \in [0, \tau_\varepsilon]} d_\infty^2(y_\varepsilon(t), Y_\varepsilon(t)) \leq \delta_1,$$

where $\tau_\varepsilon = \min\{L\varepsilon^{-\beta}, T\}$.

Proof. Case 1. $\lambda > 0$, ${}^{RL}[(1) - \alpha]$ -solution.

Let $\tau_\varepsilon \equiv \min\{L\varepsilon^{-\beta}, T\}$. For any $w \in [0, \tau_\varepsilon] \subset [0, T]$, all integral estimates established below still hold since $w \leq T$ stays inside our global definition interval $I = [0, T]$. For any $t \in [0, w]$, we have

$$\begin{aligned} & E \sup_{t \in [0, w]} d_\infty^2(y_\varepsilon(t), Y_\varepsilon(t)) \\ &= E \sup_{t \in [0, w]} d_\infty^2(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) \odot y_0 + \varepsilon \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot f(s, y_\varepsilon) ds \\ &\quad + \sqrt{\varepsilon} \langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot g(s, y_\varepsilon) dB(s) \rangle, \\ &\quad t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) \odot y_0 + \varepsilon \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\varepsilon(s)) ds \\ &\quad + \sqrt{\varepsilon} \langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\varepsilon(s)) dB(s) \rangle) \\ &\leq 4\varepsilon^2 E \sup_{t \in [0, w]} d_\infty^2(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot f(s, y_\varepsilon) ds, \\ &\quad \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot f(s, Y_\varepsilon) ds) \\ &\quad + 4\varepsilon^2 E \sup_{t \in [0, w]} d_\infty^2(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot f(s, Y_\varepsilon) ds, \\ &\quad \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\varepsilon(s)) ds) \\ &\quad + 4\varepsilon E \sup_{t \in [0, w]} d_\infty^2(\langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot g(s, y_\varepsilon) dB(s) \rangle \\ &\quad \langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot g(s, Y_\varepsilon) dB(s) \rangle) \\ &\quad + 4\varepsilon E \sup_{t \in [0, w]} d_\infty^2(\langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot g(s, Y_\varepsilon) dB(s) \rangle, \\ &\quad \langle \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\varepsilon(s)) dB(s) \rangle). \end{aligned}$$

We set

$$\begin{aligned}
 J_{11} &= 4\epsilon^2 E \sup_{t \in [0, w]} d_\infty^2 \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon) ds, \right. \\
 &\quad \left. \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon) ds \right) \\
 &\leq 4\epsilon^2 w \int_0^w (t-s)^{2\alpha-2} [E_{\alpha, \alpha}(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(f(s, y_\epsilon), f(s, Y_\epsilon))] ds \\
 &\leq 4w C_2 \epsilon^2 [E_{\alpha, \alpha}(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0, s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds.
 \end{aligned}$$

$$\begin{aligned}
 J_{12} &= 4\epsilon^2 E \sup_{t \in [0, w]} d_\infty^2 \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon) ds, \right. \\
 &\quad \left. \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s)) ds \right) \\
 &\leq 4\epsilon^2 w \int_0^w (t-s)^{2\alpha-2} [E_{\alpha, \alpha}(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(f(s, Y_\epsilon), \tilde{f}(Y_\epsilon(s)))] ds \\
 &\leq 4\epsilon^2 w^{2\alpha} [E_{\alpha, \alpha}(\lambda w^\alpha)]^2 \beta_1(w) [1 + \sup_{t \in [0, w]} E d_\infty^2(Y_\epsilon(t), \hat{0})].
 \end{aligned}$$

$$\begin{aligned}
 J_{21} &= 4\epsilon E \sup_{t \in [0, w]} d_\infty^2 \left(\left\langle \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon) dB(s) \right\rangle, \right. \\
 &\quad \left. \left\langle \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon) dB(s) \right\rangle \right) \\
 &\leq 4\epsilon \int_0^w (t-s)^{2\alpha-2} [E_{\alpha, \alpha}(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(\langle g(s, y_\epsilon) \rangle, \langle g(s, Y_\epsilon) \rangle)] ds \\
 &\leq 4\epsilon C_2 [E_{\alpha, \alpha}(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0, s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds.
 \end{aligned}$$

$$\begin{aligned}
 J_{22} &= 4\epsilon E \sup_{t \in [0, w]} d_\infty^2 \left(\left\langle \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon) dB(s) \right\rangle, \right. \\
 &\quad \left. \left\langle \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s)) dB(s) \right\rangle \right) \\
 &\leq 4\epsilon \int_0^w (t-s)^{2\alpha-2} [E_{\alpha, \alpha}(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(\langle g(s, Y_\epsilon) \rangle, \langle \tilde{g}(Y_\epsilon(s)) \rangle)] ds \\
 &\leq 4\epsilon w^{2\alpha-1} [E_{\alpha, \alpha}(\lambda w^\alpha)]^2 \beta_2(w) [1 + \sup_{t \in [0, w]} E d_\infty^2(Y_\epsilon(t), \hat{0})].
 \end{aligned}$$

Then, we can get

$$\begin{aligned}
 J &\leq J_{11} + J_{12} + J_{21} + J_{22} \\
 &\leq 4wC_2\epsilon^2[E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds \\
 &\quad + 4\epsilon^2 w^{2\alpha} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beta_1(w) [1 + \sup_{t \in [0,w]} E d_\infty^2(Y_\epsilon(t), \hat{0})] \\
 &\quad + 4\epsilon C_2 [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds \\
 &\quad + 4\epsilon w^{2\alpha-1} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beta_2(w) [1 + \sup_{t \in [0,w]} E d_\infty^2(Y_\epsilon(t), \hat{0})] \\
 &= 4\epsilon^2 w^{2\alpha} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beth_1 + 4\epsilon w^{2\alpha-1} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beth_2 \\
 &\quad + 4wC_2\epsilon^2 [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds \\
 &\quad + 4\epsilon C_2 [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds \\
 &\leq 4\epsilon^2 w^{2\alpha} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beth_1 + 4\epsilon w^{2\alpha-1} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beth_2 \\
 &\quad + \left(4wC_2\epsilon^2 [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 + 4\epsilon C_2 [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \right) \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds,
 \end{aligned}$$

where

$$\beth_1 = \beta_1(w) [1 + \sup_{t \in [0,w]} E d_\infty^2(Y_\epsilon(t), \hat{0})]$$

and

$$\beth_2 = \beta_2(w) [1 + \sup_{t \in [0,w]} E d_\infty^2(Y_\epsilon(t), \hat{0})].$$

Hence, by using the generalized Gronwall inequality (Lemma 2), we get

$$\begin{aligned}
 E \sup_{t \in [0,w]} d_\infty^2(y_\epsilon(t), Y_\epsilon(t)) &\leq \left[4\epsilon^2 w^{2\alpha} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beth_1 + 4\epsilon w^{2\alpha-1} [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 \beth_2 \right] \\
 &\quad E_{2\alpha-1} \left((4wC_2\epsilon^2 [E_{\alpha,\alpha}(\lambda w^\alpha)]^2 + 4\epsilon C_2 [E_{\alpha,\alpha}(\lambda w^\alpha)]^2) \Gamma(2\alpha-1) w^{2\alpha-1} \right).
 \end{aligned}$$

Choose $\beta \in (0, 1)$ and $L > 0$. By setting $w = \tau_\epsilon = \min\{L\epsilon^{-\beta}, T\}$, we consider two cases:

- If $L\epsilon^{-\beta} \leq T$, then $\tau_\epsilon = L\epsilon^{-\beta}$. Substituting $w = L\epsilon^{-\beta}$ into the above inequality, we obtain

$$E \sup_{t \in [0, \tau_\epsilon]} d_\infty^2(y_\epsilon(t), Y_\epsilon(t)) \leq k_1 L^{2\alpha-1} \epsilon^{\beta-2\beta\alpha+1},$$

where

$$\begin{aligned}
 k_1 &= \left[4L\epsilon^{1-\beta} [E_{\alpha,\alpha}(\lambda L^\alpha \epsilon^{-\beta\alpha})]^2 \beth_1 + 4 [E_{\alpha,\alpha}(\lambda L^\alpha \epsilon^{-\beta\alpha})]^2 \beth_2 \right] \\
 &\quad E_{2\alpha-1} \left((4L\epsilon^{2-\beta} C_2 [E_{\alpha,\alpha}(\lambda L^\alpha \epsilon^{-\beta\alpha})]^2 + 4\epsilon C_2 [E_{\alpha,\alpha}(\lambda L^\alpha \epsilon^{-\beta\alpha})]^2) \Gamma(2\alpha-1) L^{2\alpha-1} \epsilon^{\beta-2\beta\alpha} \right).
 \end{aligned}$$

It is crucial to note that under this specific case condition ($L\epsilon^{-\beta} \leq T$), all terms involving $\epsilon^{-\beta}$ inside k_1 are strictly bounded by the global constant T . Consequently, the Mittag-Leffler function satisfies $E_{\alpha,\alpha}(\lambda L^\alpha \epsilon^{-\beta\alpha}) \leq E_{\alpha,\alpha}(\lambda T^\alpha)$, and the exponential terms are similarly bounded. This guarantees that k_1 is uniformly bounded by a finite constant and will not blow up as $\epsilon \rightarrow 0$.

- If $L\epsilon^{-\beta} > T$, then $\tau_\epsilon \equiv T$. By substituting $w = T$ directly into the Gronwall inequality, the error bound depends only on the fixed constant T and scales with ϵ . Since all

functions evaluated at T are finite constants, this bound naturally converges to zero as $\epsilon \rightarrow 0$. Therefore, the error estimate flawlessly holds on the entire global interval $[0, T]$.

Therefore, for any given arbitrarily small number $\delta_1 > 0$, we can choose $\epsilon_1 \in (0, \epsilon_0]$ sufficiently small such that for each $\epsilon \in (0, \epsilon_1]$, we have:

$$E \sup_{t \in [0, \tau_\epsilon]} d_\infty^2(y_\epsilon(t), Y_\epsilon(t)) \leq \delta_1.$$

Case 2. $\lambda < 0$, ${}^{RL}[(1) - \alpha]$ -solution.

Let $\tau_\epsilon \equiv \min\{L\epsilon^{-\beta}, T\}$. For any $w \in [0, \tau_\epsilon] \subset [0, T]$, all integral estimates established below still hold since $w \leq T$ stays inside our global definition interval $I = [0, T]$. For any $t \in [0, w]$, we have

$$\begin{aligned} & E \sup_{t \in [0, w]} d_\infty^2(y_\epsilon(t), Y_\epsilon(t)) \\ &= E \sup_{t \in [0, w]} d_\infty^2 \left(t^{\alpha-1} (Ch_\alpha(\lambda t^\alpha) \odot y_0 + Sh_\alpha(\lambda t^\alpha) \odot y_0) \right. \\ &\quad + \epsilon \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon)) ds \\ &\quad + \sqrt{\epsilon} \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon)) dB(s) \right\rangle, \\ &\quad t^{\alpha-1} (Ch_\alpha(\lambda t^\alpha) \odot y_0 + Sh_\alpha(\lambda t^\alpha) \odot y_0) \\ &\quad + \epsilon \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s))) ds \\ &\quad \left. + \sqrt{\epsilon} \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s))) dB(s) \right\rangle \right) \\ &\leq 2\epsilon^2 E \sup_{t \in [0, w]} d_\infty^2 \left(\int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon)) ds, \right. \\ &\quad \left. \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s))) ds \right) \\ &\quad + 2\epsilon E \sup_{t \in [0, w]} d_\infty^2 \left(\left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon)) dB(s) \right\rangle, \right. \\ &\quad \left. \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s))) dB(s) \right\rangle \right) \\ &\leq 4\epsilon^2 E \sup_{t \in [0, w]} d_\infty^2 \left(\int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon)) ds, \right. \\ &\quad \left. \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s))) ds \right) \\ &\quad + 4\epsilon^2 E \sup_{t \in [0, w]} d_\infty^2 \left(\int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s))) ds, \right. \\ &\quad \left. \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s))) ds \right) \\ &\quad + 4\epsilon E \sup_{t \in [0, w]} d_\infty^2 \left(\left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon)) dB(s) \right\rangle, \right. \\ &\quad \left. \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s))) dB(s) \right\rangle \right) \\ &\quad + 4\epsilon E \sup_{t \in [0, w]} d_\infty^2 \left(\left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s))) dB(s) \right\rangle, \right. \\ &\quad \left. \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s))) dB(s) \right\rangle \right). \end{aligned}$$

We set

$$\begin{aligned}
H_{11} &= 4\epsilon^2 E \sup_{t \in [0, w]} d_\infty^2 \left(\int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, y_\epsilon)) ds, \right. \\
&\quad \left. \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s))) ds \right) \\
&\leq 4\epsilon^2 w \int_0^w (t-s)^{2\alpha-2} [Ch_\alpha(\lambda(t-s)^\alpha) + Sh_\alpha(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(f(s, y_\epsilon), f(s, Y_\epsilon(s)))] ds \\
&\leq 4C_2 w \epsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0, s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds.
\end{aligned}$$

$$\begin{aligned}
H_{12} &= 4\epsilon^2 E \sup_{t \in [0, w]} d_\infty^2 \left(\int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot f(s, Y_\epsilon(s))) ds, \right. \\
&\quad \left. \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{f}(Y_\epsilon(s))) ds \right) \\
&\leq 4\epsilon^2 w \int_0^w (t-s)^{2\alpha-2} [Ch_\alpha(\lambda(t-s)^\alpha) + Sh_\alpha(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(f(s, Y_\epsilon(s)), \tilde{f}(Y_\epsilon(s)))] ds \\
&\leq 4w^{2\alpha} \epsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \beta_1(w) [1 + \sup_{t \in [0, w]} E d_\infty^2(Y_\epsilon(t), \hat{0})].
\end{aligned}$$

$$\begin{aligned}
H_{21} &= 4\epsilon E \sup_{t \in [0, w]} d_\infty^2 \left(\left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, y_\epsilon)) dB(s) \right\rangle, \right. \\
&\quad \left. \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s))) dB(s) \right\rangle \right) \\
&\leq 4\epsilon \int_0^w (t-s)^{2\alpha-2} [Ch_\alpha(\lambda(t-s)^\alpha) + Sh_\alpha(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(\langle g(s, y_\epsilon) \rangle, \langle g(s, Y_\epsilon(s)) \rangle)] ds \\
&\leq 4C_2 \epsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0, s]} d_\infty^2(y_\epsilon(\xi), Y_\epsilon(\xi)) ds.
\end{aligned}$$

$$\begin{aligned}
H_{22} &= 4\epsilon E \sup_{t \in [0, w]} d_\infty^2 \left(\left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot g(s, Y_\epsilon(s))) dB(s) \right\rangle, \right. \\
&\quad \left. \left\langle \int_0^t (t-s)^{\alpha-1} (Ch_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s)) + Sh_\alpha(\lambda(t-s)^\alpha) \odot \tilde{g}(Y_\epsilon(s))) dB(s) \right\rangle \right) \\
&\leq 4\epsilon \int_0^w (t-s)^{2\alpha-2} [Ch_\alpha(\lambda(t-s)^\alpha) + Sh_\alpha(\lambda(t-s)^\alpha)]^2 E[d_\infty^2(\langle g(s, Y_\epsilon(s)) \rangle, \langle \tilde{g}(Y_\epsilon(s)) \rangle)] ds \\
&\leq 4w^{2\alpha-1} \epsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \beta_2(w) [1 + \sup_{t \in [0, w]} E d_\infty^2(Y_\epsilon(t), \hat{0})].
\end{aligned}$$

Then, we can get

$$\begin{aligned}
 H &= H_{11} + H_{12} + H_{21} + H_{22} \\
 &\leq 4C_2 w \varepsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\varepsilon(\xi), Y_\varepsilon(\xi)) ds \\
 &\quad + 4w^{2\alpha} \varepsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \beta_1(w) [1 + \sup_{t \in [0,w]} Ed_\infty^2(Y_\varepsilon(t), \hat{0})] \\
 &\quad + 4C_2 \varepsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \int_0^t (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\varepsilon(\xi), Y_\varepsilon(\xi)) \\
 &\quad + 4w^{2\alpha-1} \varepsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \beta_2(w) [1 + \sup_{t \in [0,w]} Ed_\infty^2(Y_\varepsilon(t), \hat{0})] \\
 &\leq 4w^{2\alpha} \varepsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \mathfrak{T}_1 + 4w^{2\alpha-1} \varepsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \mathfrak{T}_2 \\
 &\quad + 4C_2 w \varepsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \int_0^w (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\varepsilon(\xi), Y_\varepsilon(\xi)) ds \\
 &\quad + 4C_2 \varepsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \int_0^t (t-s)^{2\alpha-2} E \sup_{\xi \in [0,s]} d_\infty^2(y_\varepsilon(\xi), Y_\varepsilon(\xi)) ds,
 \end{aligned}$$

where

$$\mathfrak{T}_1 = \beta_1(w) [1 + \sup_{t \in [0,w]} Ed_\infty^2(Y_\varepsilon(t), \hat{0})]$$

and

$$\mathfrak{T}_2 = \beta_2(w) [1 + \sup_{t \in [0,w]} Ed_\infty^2(Y_\varepsilon(t), \hat{0})].$$

Hence, by using the generalized Gronwall inequality (Lemma 2), we get

$$\begin{aligned}
 E \sup_{t \in [0,w]} d_\infty^2(y_\varepsilon(t), Y_\varepsilon(t)) &\leq \left[4w^{2\alpha} \varepsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \mathfrak{T}_1 + 4w^{2\alpha-1} \varepsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 \mathfrak{T}_2 \right] \\
 &\quad E_{2\alpha-1} \left((4C_2 w \varepsilon^2 [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2 + 4C_2 \varepsilon [Ch_\alpha(\lambda w^\alpha) - Sh_\alpha(\lambda w^\alpha)]^2) \Gamma(2\alpha-1) w^{2\alpha-1} \right).
 \end{aligned}$$

Choose $\beta \in (0, 1)$ and $L > 0$. By setting $w = \tau_\varepsilon = \min\{L\varepsilon^{-\beta}, T\}$, we consider two cases:

- If $L\varepsilon^{-\beta} \leq T$, then $\tau_\varepsilon = L\varepsilon^{-\beta}$. Substituting $w = L\varepsilon^{-\beta}$ into the above inequality, we obtain

$$E \sup_{t \in [0, \tau_\varepsilon]} d_\infty^2(y_\varepsilon(t), Y_\varepsilon(t)) \leq k_2 L^{2\alpha-1} \varepsilon^{\beta-2\beta\alpha+1},$$

where

$$\begin{aligned}
 k_2 &= \left[4L\varepsilon^{1-\beta} [Ch_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha}) - Sh_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha})]^2 \mathfrak{T}_1 \right. \\
 &\quad \left. + 4[Ch_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha}) - Sh_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha})]^2 \mathfrak{T}_2 \right] \\
 &\quad E_{2\alpha-1} \left((4L\varepsilon^{2-\beta} C_2 [Ch_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha}) - Sh_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha})]^2 \right. \\
 &\quad \left. + 4\varepsilon C_2 [Ch_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha}) - Sh_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha})]^2) \Gamma(2\alpha-1) L^{2\alpha-1} \varepsilon^{\beta-2\beta\alpha} \right).
 \end{aligned}$$

It is crucial to note that under this specific case condition ($L\varepsilon^{-\beta} \leq T$), all terms involving $\varepsilon^{-\beta}$ inside k_2 are strictly bounded by the global constant T . Consequently, the fractional hyperbolic functions satisfy $Ch_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha}) \leq Ch_\alpha(\lambda T^\alpha)$ and $Sh_\alpha(\lambda L^\alpha \varepsilon^{-\beta\alpha}) \leq Sh_\alpha(\lambda T^\alpha)$, and the exponential terms are similarly bounded. This

guarantees that k_2 is uniformly bounded by a finite constant and will not blow up as $\epsilon \rightarrow 0$.

- If $L\epsilon^{-\beta} > T$, then $\tau_\epsilon = T$. By substituting $w = T$ directly into the Gronwall inequality, the error bound depends only on the fixed constant T and scales with ϵ . Since all hyperbolic and exponential functions evaluated at T are finite constants, this bound naturally converges to zero as $\epsilon \rightarrow 0$. Therefore, the error estimate flawlessly holds on the entire global interval $[0, T]$.

Therefore, for any given arbitrarily small number $\delta_1 > 0$, we can choose $\epsilon_1 \in (0, \epsilon_0]$ sufficiently small such that for each $\epsilon \in (0, \epsilon_1]$, we have

$$E \sup_{t \in [0, \tau_\epsilon]} d_\infty^2(y_\epsilon(t), Y_\epsilon(t)) \leq \delta_1.$$

The remaining two cases (namely, the ${}^{RL}[(2) - \alpha]$ -solution with $\lambda < 0$ and the ${}^{RL}[(2) - \alpha]$ -solution with $\lambda > 0$) are omitted here to avoid unnecessary redundancy. By virtue of the properties of the Hausdorff metric d_∞ (specifically $d_\infty(k \odot u, k \odot v) = |k|d_\infty(u, v)$) and the fundamental inequality for H-differences $d_\infty(u \ominus j, v \ominus e) \leq d_\infty(u, v) + d_\infty(e, j)$, the negative scalars and the inverted orientations of the H-differences are effectively factored out or bounded as absolute values. Consequently, the integral error equations reduce exactly to the same estimation framework and bounding procedures established in the previous cases. Since the alternative continuous fractional kernels ($E_{\alpha, \lambda}$ or combinations of Ch_α and Sh_α) remain uniformly bounded on the global compact interval $[0, T]$, the identical Gronwall-type analysis guarantees the same convergence results with constants independent of ϵ . $\square$

4. Examples

In this section, we give an example to illustrate our main result of this paper.

Example 1. Consider the following FFSDs:

$$\begin{cases} d(D_{0+}^{-0.2}y(t)) = [-1 \odot y(t) + (2 + \frac{1}{1+t}) \odot y(t)]dt + d\left\langle \int_0^t y(s) dB(s) \right\rangle, t > 0, \\ D_{0+}^{-0.2}y(0) = \iota \in R_F, \end{cases}$$

where $[\iota]^\gamma = [\gamma, 2 - \gamma]$.

Hence,

$$\tilde{f}(y_\epsilon) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(2 + \frac{1}{1+t}\right) \odot y_\epsilon dt = 2y_\epsilon \quad \text{and} \quad \tilde{g}(y_\epsilon) = \frac{1}{\pi} \int_0^\pi y_\epsilon dt = y_\epsilon.$$

Both ${}^{RL}[(1) - \alpha]$ and ${}^{RL}[(2) - \alpha]$ solution forms are presented since they correspond to the two Hukuhara differentiability directions defined in the preliminary section. The averaged system is derived by substituting the time-varying coefficients with their one-period time averages.

The standard form of the ${}^{RL}[(1) - 0.8]$ -solution for the above FFSDs is given as

$$\begin{aligned} y_\epsilon(t) = & t^{-0.2}(Ch_{0.8}(-t^{0.8}) \odot \iota + Sh_{0.8}(-t^{0.8}) \odot \iota) \\ & + \epsilon \int_0^t (t-s)^{-0.2}(Ch_{0.8}(-(t-s)^{0.8}) \odot (2 + \frac{1}{1+s}) \odot y_\epsilon \\ & + Sh_{0.8}(-(t-s)^{0.8}) \odot (2 + \frac{1}{1+s}) \odot y_\epsilon) ds \\ & + \sqrt{\epsilon} \left\langle \int_0^t (t-s)^{-0.2}(Ch_{0.8}(-(t-s)^{0.8}) \odot y_\epsilon + Sh_{0.8}(-(t-s)^{0.8}) \odot y_\epsilon) dB(s) \right\rangle. \end{aligned}$$

Therefore, the averaged system of the $^{RL}[(1) - 0.8]$ -solution for the above FFSDs is given as

$$\begin{aligned} Y_\epsilon(t) = & t^{-0.2}(Ch_{0.8}(-t^{0.8}) \odot \iota + Sh_{0.8}(-t^{0.8}) \odot \iota) \\ & + \epsilon \int_0^t (t-s)^{-0.2}(Ch_{0.8}(-(t-s)^{0.8}) \odot 2Y_\epsilon + Sh_{0.8}(-(t-s)^{0.8}) \odot 2Y_\epsilon) ds \\ & + \sqrt{\epsilon} \langle \int_0^t (t-s)^{-0.2}(Ch_{0.8}(-(t-s)^{0.8}) \odot Y_\epsilon + Sh_{0.8}(-(t-s)^{0.8}) \odot Y_\epsilon) dB(s) \rangle. \end{aligned}$$

The standard form of the $^{RL}[(2) - 0.8]$ -solution for the above FFSDs is given as

$$\begin{aligned} y_\epsilon(t) = & t^{-0.2}E_{0.8,0.8}(-t^{0.8}) \odot \iota \ominus (-1)\epsilon \int_0^t (t-s)^{-0.2}E_{0.8,0.8}(-(t-s)^{0.8}) \odot \left(2 + \frac{1}{1+s}\right) \odot y_\epsilon ds \\ & \ominus (-1)\sqrt{\epsilon} \langle \int_0^t (t-s)^{-0.2}E_{0.8,0.8}(-(t-s)^{0.8}) \odot y_\epsilon dB(s) \rangle. \end{aligned}$$

Therefore, the averaged system of the $^{RL}[(2) - 0.8]$ -solution for the above FFSDs is given as

$$\begin{aligned} Y_\epsilon(t) = & t^{-0.2}E_{0.8,0.8}(-t^{0.8}) \odot \iota \ominus (-1)\epsilon \int_0^t (t-s)^{-0.2}E_{0.8,0.8}(-(t-s)^{0.8}) \odot 2Y_\epsilon ds \\ & \ominus (-1)\sqrt{\epsilon} \langle \int_0^t (t-s)^{-0.2}E_{0.8,0.8}(-(t-s)^{0.8}) \odot Y_\epsilon dB(s) \rangle. \end{aligned}$$

Then,

$$d_\infty^2(f(t, y), \hat{0}) = \left| 2 + \frac{1}{1+t} \right|^2 d_\infty^2(y, \hat{0}) \leq 9d_\infty^2(y, \hat{0}) \leq 9(1 + d_\infty^2(y, \hat{0})),$$

and

$$\|g(t, y)\|^2 = d_\infty^2(\langle g(t, y) \rangle, \hat{0}) = 1 \cdot d_\infty^2(y, \hat{0}) \leq 1 \cdot (1 + d_\infty^2(y, \hat{0})),$$

where $\hat{0}$ denotes the fuzzy zero. To satisfy Assumption (A1), we take the constant such that $C_1^2 = 9$, namely $C_1 = 3$.

Then,

$$d_\infty^2(f(t, x), f(t, y)) = \left| 2 + \frac{1}{1+t} \right|^2 d_\infty^2(x, y) \leq 9d_\infty^2(x, y),$$

and

$$\|g(t, x) - g(t, y)\|^2 = d_\infty^2(\langle g(t, x) \rangle, \langle g(t, y) \rangle) = 1 \cdot d_\infty^2(x, y).$$

To satisfy Assumption (A2), we take the constant such that $C_2 = 9$.

Furthermore, to verify Assumption (A3), we compute the time-averaged limits for f and g . For the diffusion term, since $g(s, x) = \tilde{g}(x)$, we have:

$$\frac{1}{T'} \int_0^{T'} \|g(s, x) - \tilde{g}(x)\|^2 ds = 0 \leq \beta_2(T')(1 + d_\infty^2(x, \hat{0})),$$

where we can choose $\beta_2(T') = 0$, which clearly satisfies $\lim_{T' \rightarrow \infty} \beta_2(T') = 0$.

For the drift term, we have:

$$\begin{aligned}
 \frac{1}{T'} \int_0^{T'} d_\infty^2(f(s, x), \tilde{f}(x)) ds &= \frac{1}{T'} \int_0^{T'} d_\infty^2\left(\left(2 + \frac{1}{1+s}\right) \odot x, 2 \odot x\right) ds \\
 &= \frac{1}{T'} \int_0^{T'} \frac{1}{(1+s)^2} d_\infty^2(x, \hat{0}) ds \\
 &= \frac{1}{T'} \left[1 - \frac{1}{1+T'}\right] d_\infty^2(x, \hat{0}) \\
 &= \frac{1}{1+T'} d_\infty^2(x, \hat{0}) \\
 &\leq \frac{1}{1+T'} (1 + d_\infty^2(x, \hat{0})).
 \end{aligned}$$

Thus, we can define $\beta_1(T') = \frac{1}{1+T'}$, which inherently satisfies $\lim_{T' \rightarrow \infty} \beta_1(T') = \lim_{T' \rightarrow \infty} \frac{1}{1+T'} = 0$.

The coefficients f and g satisfy the assumptions (A1)–(A2), and the coefficients f and $\tilde{f}$ satisfy the assumption (A3). Therefore, by Theorem 2, as $\epsilon \rightarrow 0$, the solutions y_ϵ and Y_ϵ are equivalent in the sense of mean square.

The numerical simulation for Example 1 is illustrated in Figure 1, which verifies the theoretical result of Theorem 2. In this simulation at the fuzzy core ($\gamma = 1$), the mean square error is evaluated as $[E_r]^2 = E[\|y_\epsilon(t) - Y_\epsilon(t)\|^2]$.

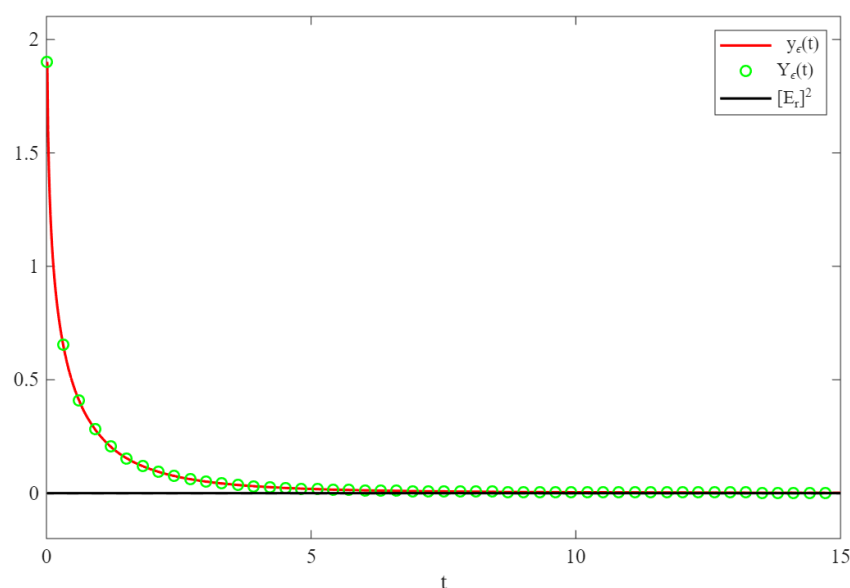


Figure 1. Numerical simulation for Example 1: Comparison of the original solution $y_\epsilon(t)$ with the averaged solution $Y_\epsilon(t)$ and their mean square error ($\epsilon = 0.00001$, $\gamma = 1$).

Example 2. Consider the following FFSDs with $\lambda = 2 > 0$:

$$\begin{cases} d(D_{0+}^{-0.3}y(t)) = [2 \odot y(t) + (3 + e^{-t}) \odot y(t)]dt + d\left\langle \int_0^t \sqrt{2}y(s) dB(s) \right\rangle, t > 0, \\ D_{0+}^{-0.3}y(0) = \iota \in R_F, \end{cases}$$

where $[\iota]^\gamma = [\gamma - 1, 1 - \gamma]$.

Hence,

$$\tilde{f}(y_\epsilon) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (3 + e^{-t}) \odot y_\epsilon dt = 3y_\epsilon \quad \text{and} \quad \tilde{g}(y_\epsilon) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sqrt{2}y_\epsilon dt = \sqrt{2}y_\epsilon.$$

Both ${}^{RL}[(1)-\alpha]$ and ${}^{RL}[(2)-\alpha]$ solution forms are presented since they correspond to the two Hukuhara differentiability directions defined in the preliminary section. The averaged system is derived by substituting the time-varying coefficients with their one-period time averages.

The standard form of the ${}^{RL}[(1) - 0.7]$ -solution for the above FFSDs is given as

$$y_\epsilon(t) = t^{-0.3}E_{0.7,0.7}(2t^{0.7}) \odot \iota + \epsilon \int_0^t (t-s)^{-0.3}E_{0.7,0.7}(2(t-s)^{0.7}) \odot (3 + e^{-s}) \odot y_\epsilon ds \\ + \sqrt{\epsilon} \langle \int_0^t (t-s)^{-0.3}E_{0.7,0.7}(2(t-s)^{0.7}) \odot \sqrt{2}y_\epsilon dB(s) \rangle.$$

Therefore, the averaged system of the ${}^{RL}[(1) - 0.7]$ -solution for the above FFSDs is given as

$$Y_\epsilon(t) = t^{-0.3}E_{0.7,0.7}(2t^{0.7}) \odot \iota + \epsilon \int_0^t (t-s)^{-0.3}E_{0.7,0.7}(2(t-s)^{0.7}) \odot 3Y_\epsilon ds \\ + \sqrt{\epsilon} \langle \int_0^t (t-s)^{-0.3}E_{0.7,0.7}(2(t-s)^{0.7}) \odot \sqrt{2}Y_\epsilon dB(s) \rangle.$$

The standard form of the ${}^{RL}[(2) - 0.7]$ -solution for the above FFSDs is given as

$$y_\epsilon(t) = t^{-0.3}(Ch_{0.7}(2t^{0.7}) \odot \iota \ominus (-1)Sh_{0.7}(2t^{0.7}) \odot \iota) \\ \ominus (-1)\epsilon \int_0^t (t-s)^{-0.3}(Ch_{0.7}(2(t-s)^{0.7}) \odot (3 + e^{-s}) \odot y_\epsilon \\ + Sh_{0.7}(2(t-s)^{0.7}) \odot (3 + e^{-s}) \odot y_\epsilon) ds \\ \ominus (-1)\sqrt{\epsilon} \langle \int_0^t (t-s)^{-0.3}(Ch_{0.7}(2(t-s)^{0.7}) \odot \sqrt{2}y_\epsilon + Sh_{0.7}(2(t-s)^{0.7}) \odot \sqrt{2}y_\epsilon) dB(s) \rangle.$$

Therefore, the averaged system of the ${}^{RL}[(2) - 0.7]$ -solution for the above FFSDs is given as

$$Y_\epsilon(t) = t^{-0.3}(Ch_{0.7}(2t^{0.7}) \odot \iota \ominus (-1)Sh_{0.7}(2t^{0.7}) \odot \iota) \\ \ominus (-1)\epsilon \int_0^t (t-s)^{-0.3}(Ch_{0.7}(2(t-s)^{0.7}) \odot 3Y_\epsilon + Sh_{0.7}(2(t-s)^{0.7}) \odot 3Y_\epsilon) ds \\ \ominus (-1)\sqrt{\epsilon} \langle \int_0^t (t-s)^{-0.3}(Ch_{0.7}(2(t-s)^{0.7}) \odot \sqrt{2}Y_\epsilon + Sh_{0.7}(2(t-s)^{0.7}) \odot \sqrt{2}Y_\epsilon) dB(s) \rangle.$$

Now, we verify the Lipschitz and linear growth conditions. For any $t > 0$, the maximum of $(3 + e^{-t})$ is 4. Then,

$$d_\infty^2(f(t, y), \hat{0}) = |3 + e^{-t}|^2 d_\infty^2(y, \hat{0}) \leq 16d_\infty^2(y, \hat{0}) \leq 16(1 + d_\infty^2(y, \hat{0})),$$

and

$$\|g(t, y)\|^2 = d_\infty^2(\langle g(t, y) \rangle, \hat{0}) = (\sqrt{2})^2 d_\infty^2(y, \hat{0}) = 2d_\infty^2(y, \hat{0}) \leq 16(1 + d_\infty^2(y, \hat{0})),$$

where $\hat{0}$ denotes the fuzzy zero. To satisfy Assumption (A1), we set the constant $C_1^2 = 16$, meaning $C_1 = 4$.

Similarly,

$$d_\infty^2(f(t, x), f(t, y)) = |3 + e^{-t}|^2 d_\infty^2(x, y) \leq 16d_\infty^2(x, y),$$

and

$$\|g(t, x) - g(t, y)\|^2 = (\sqrt{2})^2 d_\infty^2(x, y) = 2d_\infty^2(x, y) \leq 16d_\infty^2(x, y).$$

To satisfy Assumption (A2), we set $C_2 = 16$.

Furthermore, to verify Assumption (A3), we compute the time-averaged limits for f and g . Since $g(s, x) = \tilde{g}(x) = \sqrt{2}x$, we have:

$$\frac{1}{T'} \int_0^{T'} \|g(s, x) - \tilde{g}(x)\|^2 ds = 0 \leq \beta_2(T')(1 + d_\infty^2(x, \hat{0})),$$

thus we can define $\beta_2(T') = 0$, ensuring $\lim_{T' \rightarrow \infty} \beta_2(T') = 0$.

For the drift term f , we have:

$$\begin{aligned} \frac{1}{T'} \int_0^{T'} d_\infty^2(f(s, x), \tilde{f}(x)) ds &= \frac{1}{T'} \int_0^{T'} d_\infty^2((3 + e^{-s}) \odot x, 3 \odot x) ds \\ &= \frac{1}{T'} \int_0^{T'} (e^{-s})^2 d_\infty^2(x, \hat{0}) ds \\ &= \frac{1}{T'} \left[-\frac{1}{2} e^{-2s} \right]_0^{T'} d_\infty^2(x, \hat{0}) \\ &= \frac{1 - e^{-2T'}}{2T'} d_\infty^2(x, \hat{0}) \\ &\leq \frac{1 - e^{-2T'}}{2T'} (1 + d_\infty^2(x, \hat{0})). \end{aligned}$$

Thus, we can define $\beta_1(T') = \frac{1 - e^{-2T'}}{2T'}$, which inherently satisfies $\lim_{T' \rightarrow \infty} \beta_1(T') = \lim_{T' \rightarrow \infty} \frac{1 - e^{-2T'}}{2T'} = 0$.

The coefficients f and g satisfy the assumptions (A1)–(A2), and the coefficients f and $\tilde{f}$ satisfy the assumption (A3). Therefore, by Theorem 2, as $\varepsilon \rightarrow 0$, the solutions y_ε and Y_ε are equivalent in the sense of mean square.

5. Conclusions

This paper focuses on extending the averaging principle to a class of Riemann–Liouville FFSDs, addressing the research gap in the application of the averaging principle for fuzzy stochastic systems in fractional scenarios with memory. By means of Laplace transform, inverse Laplace transform, and the Mittag–Leffler function, we derived the integral form of the solution to the original FFSDs, laying a crucial technical foundation for subsequent convergence proofs. Specifically, under the rigorous assumptions of the main theorem—especially the restriction of the fractional order to $1/2 < \alpha < 1$ —we have successfully established the averaging principle on fixed finite-time intervals. We proved that the solution to the averaged FFSDs converges to that of the standard FFSDs in the mean-square fuzzy metric as ε tends to zero.

The research findings of this paper enrich the theoretical system of the averaging principle for fuzzy stochastic systems, providing a simplified analytical tool for the study of practical systems involving both “randomness” and “fuzziness” dual uncertainties. For future research directions, we intend to further explore the averaging principle for fuzzy fractional stochastic systems involving Hilfer-type or Caputo-type fractional derivatives. Furthermore, extending our current framework to variable-order or delay-type fuzzy fractional stochastic differential equations represents an interesting scope for future investigations, aiming to broaden the application scenarios of the averaging principle and enhance its adaptability to more complex systems.

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