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An Efficient Inertial Picard-CR Method for Solving Fixed-Point Problems with Applications

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Abstract

The aim of this paper is to introduce an inertial Picard–CR iterative method for approximating fixed points of a family of nonexpansive mappings in real Hilbert spaces. The proposed method is designed to improve the convergence performance of existing schemes while maintaining simplicity in implementation. Under suitable conditions, we establish strong convergence results for the proposed iterative process. We apply the algorithm to solve split feasibility problems, signal processing problems, polynomiography problems, fractional differential, and integral equations. Several numerical experiments demonstrating the superiority of the proposed method over existing methods are presented. The obtained results extend and improve several known results in the literature.

Keywords: fixed point; fractional differential and integral equations; split feasibility problem; convex minimization problem; weak convergence and strong convergence

1. Introduction

Let G and D be two nonempty closed convex subsets such that $\emptyset \neq G \subset H_1$, $\emptyset \neq D \subset H_2$, where H_1 and H_2 are real Hilbert spaces. The split feasibility problem (SFP) can be modeled as follows:

$$\text{find } p^+ \in G \text{ such that } Ep^+ \in D, \quad (1)$$

where the operator $E : H_1 \rightarrow H_2$ is bounded and linear. In [1], Censor and Elfving considered the SFP in finite-dimensional Hilbert spaces. The SFP has applications in several fields in engineering and applied sciences. For instance, it can be applied to signal processing and medical image deblurring. For examples, see [2–7].

Fixed point theory has become an essential tool in modern nonlinear analysis due to its wide range of applications in various branches of mathematics and applied sciences, including differential equations, integral equations, optimization theory, economics, and engineering. Many problems arising in these fields can be formulated as fixed-point problems for nonlinear operators defined on appropriate spaces. Consequently, the development of efficient iterative methods for approximating fixed points of nonlinear mappings has attracted considerable attention from researchers over the past few decades.

One of the central problems in fixed point theory is the approximation of solutions of the equation

$$Sk = k, \quad (2)$$



Academic Editor: María A. Navascués

Received: 1 June 2026

Revised: 8 July 2026

Accepted: 12 July 2026

Published: 13 July 2026

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where S is a nonlinear operator defined on a subset of a Banach or Hilbert space. The existence and approximation of such solutions have been studied extensively using various iterative schemes. Classical iterative methods such as the Picard, Mann [8], Ishikawa [9], Noor [10], S [11], and CR [12] iterations have played significant roles in the study of convergence behavior of nonlinear mappings. These methods have been further extended and modified to improve the convergence speed, stability, and applicability to broader classes of mappings [13–15].

In recent years, considerable efforts have been devoted to designing new iterative algorithms with stronger convergence properties and better computational performance. Several researchers have introduced modified and hybrid iterative schemes for approximating fixed points of nonexpansive, pseudocontractive, and other classes of nonlinear operators [16–19]. Such developments are motivated by the need to handle more complex nonlinear problems that arise in practical applications.

The SFP (Equation (1)) can be expressed as a fixed-point problem as follows:

$$P_G(I - sE^*(I - P_D)E)p^\dagger = p^\dagger, \quad (3)$$

where $s > 0$, I denotes the identity operator on H , E^* stands for the adjoint of E , and P_G and P_D are the (orthogonal) projections onto G and D , respectively. That is, $p^\dagger$ is a solution of the SFP (1) if and only if $p^\dagger$ is a solution of the fixed point equation (Equation (3)) (see [20]). The readers may consult [21–25] for more details.

The following algorithm, introduced by Byrne [26], is known as the CQ algorithm:

$$k_{t+1} = P_G(I - sE^*(I - P_D)E)k_t, \quad t \geq 0, \quad (4)$$

where I denotes the identity operator on H , $E^* : H_2 \rightarrow H_1$ is the adjoint of E , $0 < s < \frac{2}{\|E\|^2}$, and P_G and P_D are the projections onto G and D , respectively. The CQ method (Equation (4)) has been instrumental for finding the solution of the SFP because of its computational simplicity. Several modifications of the CQ algorithm have been studied in the literature. For examples, see [23,27].

In [28], Dang and Gao proposed a three-step iteration algorithm for solving the SFP. The authors obtained strong convergence theorems for their algorithm under some standard parametric control conditions. Recently, Kittiratanawasin et al. [29] introduced an efficient iterative algorithm for solving the SFP. With numerical experiments, the authors demonstrated that their method performed better than many existing methods in the literature.

For about a decade now, many authors have shown that the incorporation of the Picard iteration into existing methods enhances the speed of convergence of these methods, as observed in cases such as the Picard–Mann hybrid [30], Picard–Krasnoselskii hybrid [19], Picard–Ishikawa hybrid [18], Picard–Noor hybrid [31], Picard– S hybrid [32], and Picard–CR hybrid [33].

The inertial technique was introduced by Polyak [34] to speed up the convergence of iterative methods. It is well known as the heavy ball technique of the second-order time-dynamic system. In recent years, many iterative algorithms embedding inertial techniques have been studied. For some recent results on inertial related methods, the reader may see more details in [17,35–41].

Despite the significant progress made in this direction, there is still a need to develop more efficient iterative methods capable of providing faster convergence and improved stability under weaker conditions on the underlying operators. In particular, the study of iterative methods that combine different acceleration techniques or incorporate structural properties of nonlinear operators remains an active area of research.

The aim of this paper is to introduce an inertial Picard–CR iterative method for approximating fixed points of a family of nonexpansive mappings in real Hilbert spaces. The proposed method is designed to improve the convergence performance of existing schemes while maintaining simplicity in its implementation. Under suitable conditions, we establish strong convergence results for the proposed iterative process. The proposed algorithm is applied to solve split feasibility, signal processing, and polynomiography problems. Several numerical experiments showing the superiority of our method over several existing methods are presented. The obtained results extend and improve several known results in the literature.

The remainder of this paper is organized as follows. In Section 2, we present some preliminary definitions and results that will be used throughout the paper. Section 3 introduces the new iterative scheme and establishes its convergence properties. In Section 4, we consider the applications of the proposed method to solve real-world problems. Furthermore, we provide numerical illustrations and comparisons with some existing iterative methods to demonstrate the efficiency of the proposed approach. Finally, our conclusions are presented in Section 5.

2. Preliminaries

Let H be a real Hilbert space with an induced norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $F(S)$ stand for the set of fixed points of the self mapping $S : H \rightarrow H$ defined by $F(S) = \{k \in H : Sk = k\}$. We denote the strong and weak convergence of the sequence $\{k_t\}$ converging to k in H by $k_t \rightarrow k$ and $k_t \rightharpoonup k$, respectively. An operator S on H is said to be L -Lipschitz-continuous if for each $k, l \in H$, we have $\|Sk - Sl\| \leq L\|k - l\|$ for a constant $L > 0$. If $L = 1$, then we call S a nonexpansive map, and if $L \in [0, 1)$, then we say that S is a contraction mapping.

Let the solution set $\Gamma = \{k \in G : Ek \in D\} = G \cap E^{-1}D$ of the SFP (Equation (1)) be a closed, convex, and nonempty set.

Lemma 1. Let H be a real Hilbert space. Then, for every $k, l \in H$ and $c \in \mathbb{R}$, one gets

- (i) $\|k + l\|^2 \leq \|k\|^2 + 2\langle l, k + l \rangle$;
- (ii) $\|k + l\|^2 = \|k\|^2 + 2\langle k, l \rangle + \|l\|^2$;
- (iii) $\|ck + (1 - c)l\|^2 = c\|k\|^2 + (1 - c)\|l\|^2 - c(1 - c)\|k - l\|^2$.

Lemma 2 ([42]). If $\{a_t\}$ is a positive sequence of real numbers and $\{\delta_t\}$ is a sequence of real numbers, then

$$a_{t+1} \leq (1 - \theta_t)a_t + \theta_t\delta_t, \quad \forall t \geq 1,$$

where $\{\theta_t\} \subset (0, 1)$, satisfying $\sum_{t=0}^{\infty} \theta_t = \infty$. Suppose that $\limsup_{t \rightarrow \infty} \delta_t \leq 0$ for every subsequence $\{a_{t_j}\}$ of $\{a_t\}$. If the following inequality holds:

$$\liminf_{j \rightarrow \infty} (a_{t_{j+1}} - a_{t_j}) \geq 0,$$

then $\lim_{t \rightarrow \infty} a_t = 0$.

Definition 1 ([43]). Let $S : G \rightarrow G$ be a nonexpansive mapping and $S_t : G \rightarrow G$ be a family of nonexpansive mappings satisfying $\emptyset \neq F(S) \subset \Gamma = \bigcap_{t=1}^{\infty} F(S_t)$, where $F(S_t)$ denotes the fixed point set of S_t for every $t \in \mathbb{N}$. Then, we say that $\{S_t\}$ satisfies the NST-condition (I) with S if, for any bounded sequence $\{k_t\} \in G$, we have

$$\lim_{t \rightarrow \infty} \|k_t - S_t k_t\| = 0 \implies \lim_{t \rightarrow \infty} \|k_t - S k_t\| = 0.$$

Definition 2 ([44,45]). Let $\{k_t\} \subset H$ be a bounded sequence and $S_t : G \rightarrow G$ be a family of nonexpansive mappings with $\bigcap_{t=1}^{\infty} F(S_t) \neq \emptyset$. We say that $\{S_t\}$ satisfies condition (Z) if

$$\|k_t - S_t k_t\| \rightarrow 0$$

implies that every weak cluster point of $\{k_t\}$ belongs to $\bigcap_{t=1}^{\infty} F(S_t)$.

Following the demicloseness of $I - S$, where $S : G \rightarrow G$ is a nonexpansive mapping, we have the following remark.

Remark 1 ([46]). If the mapping S is nonexpansive, and $\{S_t\}$ is a family of nonexpansive mappings that satisfies NST-condition (I) with S , then $\{S_t\}$ fulfills condition (Z).

Definition 3. Let H be a real Hilbert space, and let G be a nonempty closed convex subset of H . The metric projection $P_G : H \rightarrow G$ is defined by

$$P_G k = \arg \min_{l \in G} \|k - l\|, \quad k \in H.$$

That is, for every $k \in H$, the point $P_G k$ is the unique element of G satisfying

$$\|k - P_G k\| = \inf_{l \in G} \|k - l\|.$$

Lemma 3 (Lemma 3.2 in [47]). Let H be a real Hilbert space, and let G be a nonempty closed convex subset of H . Let $P_G : H \rightarrow G$ denote the metric projection onto G . Then, the following properties hold:

- (1) $P_G k = k$ for every $k \in G$;
- (2) P_G is nonexpansive such that

$$\|P_G k - P_G l\| \leq \|k - l\|, \quad \forall k, l \in H;$$

- (3) P_G is firmly nonexpansive, namely

$$\|P_G k - P_G l\|^2 \leq \langle P_G k - P_G l, k - l \rangle, \quad \forall k, l \in H.$$

Lemma 4 ([48]). Suppose $S = P_G[I - sE^*(I - P_D)E]$, where $0 < s < \frac{2}{\|E\|^2}$. Then, the map S is nonexpansive.

3. Proposed Method

In this section, we present a modified Picard–CR iterative algorithm. The method incorporates the viscosity technique to guarantee its strong convergence. It also embeds an inertial extrapolation step to accelerate its convergence. The strong convergence result of the proposed method will be established under the following mild conditions:

- (A1) Let G be a nonempty closed convex subset of H ;
- (A2) Let the self map $f : G \rightarrow G$ be a ρ -contraction where $0 \leq \rho < 1$;
- (A3) Let $S_t : G \rightarrow G$ be a family of nonexpansive mappings that satisfy the condition (Z), where $\Omega = \bigcap_{t=1}^{\infty} F(S_t) \neq \emptyset$;

Let $\{\eta_t\}$, $\{\zeta_t\}$, $\{\gamma_t\}$ and $\{\sigma_t\}$ be positive real sequences satisfying the following:

- (B1) $0 < p_1 \leq \gamma_t \leq p_2 < 1$;
- (B2) $0 < \zeta_t, \eta_t < 1$, $\lim_{t \rightarrow \infty} \eta_t \zeta_t = 0$ and $\sum_{t=1}^{\infty} \eta_t \zeta_t = \infty$;
- (B3) $\lim_{t \rightarrow \infty} \sigma_t = 0$.

The proposed Algorithm 1 is presented as follows:

Algorithm 1 An inertial viscosity modified Picard–CR algorithm (IVMPCR).

Initialization: Choose $k_0, k_1 \in G$ arbitrarily.

Iterative steps: For $t \geq 1$, k_{t+1} is calculated as follows:

Step 1. Compute an inertial parameter

$$\psi_t = \begin{cases} \min \left\{ \frac{q_t - 1}{q_{t+1}}, \frac{\eta_t \zeta_t \sigma_t}{\|k_t - k_{t-1}\|} \right\}, & \text{if } k_t \neq k_{t-1}, \\ \frac{q_t - 1}{q_{t+1}}, & \text{otherwise,} \end{cases} \quad (5)$$

where $q_1 = 1$ and $q_{t+1} = \frac{1 + \sqrt{1 + 4q_t^2}}{2}$.

Step 2. Compute

$$\begin{aligned} z_t &= P_G(k_t + \psi_t(k_t - k_{t-1})), \\ y_t &= (1 - \eta_t)z_t + \eta_t f(z_t), \\ w_t &= (1 - \zeta_t)S_t z_t + \zeta_t S_t y_t, \\ m_t &= (1 - \gamma_t)w_t + \gamma_t S_t w_t \\ k_{t+1} &= S_t m_t. \end{aligned}$$

Remark 2. The proposed IVMPCR algorithm extends several existing iterative methods by integrating adaptive inertial extrapolation, viscosity approximation, and the modified Picard–CR iteration into a unified framework. When $G = H$, the metric projection satisfies $P_G = I$, and the proposed algorithm reduces to its original unprojected form. In particular, if $\psi_t = 0$, then the inertial extrapolation vanishes, and the algorithm reduces to a viscosity-modified Picard–CR-type iteration. If $\eta_t = 0$, then the viscosity approximation is removed, yielding an inertial Picard–CR-type iteration. Furthermore, when both $\psi_t = 0$ and $\eta_t = 0$, the proposed algorithm reduces to the underlying modified Picard–CR framework. Therefore, the IVMPCR algorithm provides a genuine generalization of Picard–CR-type methods by incorporating adaptive inertia, viscosity regularization, and projection while preserving strong convergence under the stated assumptions.

Theorem 1. A sequence $\{k_t\}$ generated by Algorithm 1 converges strongly to an element $p^\dagger \in \Omega$, where $p^\dagger = P_\Omega f(p^\dagger)$, provided that it satisfies conditions (A1–A3) and (B1–B3).

Proof. Suppose that $p^\dagger = P_\Omega f(p^\dagger)$. Then, $p^\dagger \in \bigcap_{t=1}^\infty F(S_t)$. Firstly, we prove that $\{k_t\}$ is bounded.

Since $z_t = P_G(k_t + \psi_t(k_t - k_{t-1}))$ and $p^\dagger \in \Omega \subseteq G$, we have $P_G(p^\dagger) = p^\dagger$. Under the nonexpansiveness of the metric projection P_G , it follows that

$$\begin{aligned} \|z_t - p^\dagger\| &= \|P_G(k_t + \psi_t(k_t - k_{t-1})) - P_G(p^\dagger)\| \\ &\leq \|k_t + \psi_t(k_t - k_{t-1}) - p^\dagger\| \\ &\leq \|k_t - p^\dagger\| + \psi_t \|k_t - k_{t-1}\|. \end{aligned} \quad (6)$$

From the definition of y_t , we have

$$\begin{aligned} \|y_t - p^\dagger\| &\leq \eta_t \|f(z_t) - p^\dagger\| + (1 - \eta_t) \|z_t - p^\dagger\| \\ &\leq \eta_t \|f(z_t) - f(p^\dagger)\| + \eta_t \|f(p^\dagger) - p^\dagger\| + (1 - \eta_t) \|z_t - p^\dagger\| \\ &\leq \eta_t \rho \|z_t - p^\dagger\| + \eta_t \|f(p^\dagger) - p^\dagger\| + (1 - \eta_t) \|z_t - p^\dagger\| \\ &= [1 - \eta_t(1 - \rho)] \|z_t - p^\dagger\| + \eta_t \|f(p^\dagger) - p^\dagger\|. \end{aligned} \quad (7)$$

According to Equations (6) and (7) and the definition of w_t , we obtain

$$\begin{aligned}
 \|w_t - p^\dagger\| &\leq (1 - \zeta_t)\|S_t z_t - p^\dagger\| + \zeta_t\|S_t y_t - p^\dagger\| \\
 &\leq (1 - \zeta_t)\|z_t - p^\dagger\| + \zeta_t\|y_t - p^\dagger\| \\
 &\leq (1 - \zeta_t)\|z_t - p^\dagger\| + \zeta_t[1 - \eta_t(1 - \rho)]\|z_t - p^\dagger\| + \eta_t\|f(p^\dagger) - p^\dagger\| \\
 &= \|z_t - p^\dagger\| - \zeta_t\eta_t(1 - \rho)\|z_t - p^\dagger\| + \zeta_t\eta_t\|f(p^\dagger) - p^\dagger\| \\
 &= [1 - \zeta_t\eta_t(1 - \rho)](\|k_t - p^\dagger\| + \psi_t\|k_{t-1} - k_t\|) + \zeta_t\eta_t\|f(p^\dagger) - p^\dagger\|. \quad (8)
 \end{aligned}$$

Through Equation (8) and the definition of m_t , we have

$$\begin{aligned}
 \|m_t - p^\dagger\| &\leq (1 - \gamma_t)\|w_t - p^\dagger\| + \gamma_t\|S_t w_t - p^\dagger\| \\
 &\leq (1 - \gamma_t)\|w_t - p^\dagger\| + \gamma_t\|w_t - p^\dagger\| \\
 &= \|w_t - p^\dagger\| \\
 &\leq [1 - \zeta_t\eta_t(1 - \rho)](\|k_t - p^\dagger\| + \psi_t\|k_{t-1} - k_t\|) + \zeta_t\eta_t\|f(p^\dagger) - p^\dagger\|. \quad (9)
 \end{aligned}$$

With Equation (9) and the definition of k_{t+1} , we have

$$\begin{aligned}
 \|k_{t+1} - p^\dagger\| &= \|S_t m_t - p^\dagger\| \\
 &\leq \|m_t - p^\dagger\| \\
 &\leq [1 - \zeta_t\eta_t(1 - \rho)](\|k_t - p^\dagger\| + \psi_t\|k_{t-1} - k_t\|) + \zeta_t\eta_t\|f(p^\dagger) - p^\dagger\| \\
 &\leq [1 - \zeta_t\eta_t(1 - \rho)]\|k_t - p^\dagger\| + \left(\psi_t\|k_{t-1} - k_t\| + \zeta_t\eta_t\|f(p^\dagger) - p^\dagger\|\right) \\
 &= [1 - \zeta_t\eta_t(1 - \rho)]\|k_t - p^\dagger\| + \zeta_t\eta_t\left(\frac{\psi_t}{\zeta_t\eta_t}\|k_{t-1} - k_t\| + \|f(p^\dagger) - p^\dagger\|\right).
 \end{aligned}$$

Under condition (B3), we obtain

$$\frac{\psi_t}{\zeta_t\eta_t}\|k_{t-1} - k_t\| \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (10)$$

Through Equation (10), it follows that there exists $Q > 0$ such that $\frac{\psi_t}{\zeta_t\eta_t}\|k_{t-1} - k_t\| \leq Q$ for all $t \geq 1$. Therefore, we have

$$\begin{aligned}
 \|k_{t+1} - p^\dagger\| &\leq [1 - \zeta_t\eta_t(1 - \rho)]\|k_t - p^\dagger\| + (1 - \rho)\zeta_t\eta_t\left(\frac{\|f(p^\dagger) - p^\dagger\| + Q}{(1 - \rho)}\right) \\
 &\leq \max\left\{\|k_t - p^\dagger\|, \frac{\|f(p^\dagger) - p^\dagger\| + Q}{(1 - \rho)}\right\}. \quad (11)
 \end{aligned}$$

Via mathematical induction, we get

$$\|k_t - p^\dagger\| \leq \max\left\{\|k_1 - p^\dagger\|, \frac{\|f(p^\dagger) - p^\dagger\| + Q}{(1 - \rho)}\right\},$$

for all $t \geq 1$. This means that $\{k_t\}$ is bounded, and hence, $\{y_t\}$, z_t , and $\{w_t\}$ are also bounded. Through Lemma 1 and Algorithm 1, we have

$$\begin{aligned}
\|y_t - p^\dagger\|^2 &= \|\eta_t f(z_t) + (1 - \eta_t)z_t - p^\dagger\|^2 \\
&= \|\eta_t(f(z_t) - f(p^\dagger)) + (1 - \eta_t)(z_t - p^\dagger) + \eta_t(f(p^\dagger) - p^\dagger)\|^2 \\
&\leq \|\eta_t(f(z_t) - f(p^\dagger)) + (1 - \eta_t)(z_t - p^\dagger)\|^2 + 2\eta_t\langle f(p^\dagger) - p^\dagger, y_t - p^\dagger \rangle \\
&\leq \eta_t\|f(z_t) - f(p^\dagger)\|^2 + (1 - \eta_t)\|z_t - p^\dagger\|^2 + 2\eta_t\langle f(p^\dagger) - p^\dagger, y_t - p^\dagger \rangle \\
&\leq (1 - \eta_t(1 - \rho))\|z_t - p^\dagger\|^2 + 2\eta_t\langle f(p^\dagger) - p^\dagger, y_t - p^\dagger \rangle.
\end{aligned} \tag{12}$$

From the definitions of k_{t+1} , together with Lemma 1 and Equation (12), we have

$$\begin{aligned}
\|k_{t+1} - p^\dagger\|^2 &= \|S_t m_t - p^\dagger\|^2 \\
&\leq \|m_t - p^\dagger\|^2 \\
&= \|\gamma_t w_t + (1 - \gamma_t)S_t w_t - p^\dagger\|^2 \\
&= \gamma_t\|w_t - p^\dagger\|^2 + (1 - \gamma_t)\|S_t w_t - p^\dagger\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&\leq \|w_t - p^\dagger\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&= \|(1 - \zeta_t)S_t z_t + \zeta_t S_t y_t - p^\dagger\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&= (1 - \zeta_t)\|S_t z_t - p^\dagger\|^2 + \zeta_t\|S_t y_t - p^\dagger\|^2 \\
&\quad - \zeta_t(1 - \zeta_t)\|S_t z_t - S_t y_t\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&\leq (1 - \zeta_t)\|z_t - p^\dagger\|^2 + \zeta_t\|y_t - p^\dagger\|^2 \\
&\quad - \zeta_t(1 - \zeta_t)\|S_t z_t - S_t y_t\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&\leq (1 - \zeta_t)\|z_t - p^\dagger\|^2 + \zeta_t\|y_t - p^\dagger\|^2 \\
&\quad + \zeta_t[(1 - \eta_t(1 - \rho))\|z_t - p^\dagger\|^2 + 2\eta_t\langle f(p^\dagger) - p^\dagger, y_t - p^\dagger \rangle] \\
&\quad - \zeta_t(1 - \zeta_t)\|S_t z_t - S_t y_t\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&\leq [1 - \zeta_t\eta_t(1 - \rho)](\|k_t - p^\dagger\| + \psi_t\|k_{t-1} - k_t\|)^2 \\
&\quad + 2\zeta_t\eta_t\|f(p^\dagger) - p^\dagger\|\|y_t - w_t\| + 2\zeta_t\eta_t\langle f(p^\dagger) - p^\dagger, w_t - p^\dagger \rangle \\
&\quad - \zeta_t(1 - \zeta_t)\|S_t z_t - S_t y_t\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&\leq [1 - \zeta_t\eta_t(1 - \rho)]\|k_t - p^\dagger\|^2 + 2\psi_t\|k_t - p^\dagger\|\|k_{t-1} - k_t\| \\
&\quad + \psi_t^2\|k_{t-1} - k_t\|^2 + 2\zeta_t\eta_t\|f(p^\dagger) - p^\dagger\|\|y_t - w_t\| + 2\zeta_t\eta_t\langle f(p^\dagger) - p^\dagger, w_t - p^\dagger \rangle \\
&\quad - \zeta_t(1 - \zeta_t)\|S_t z_t - S_t y_t\|^2 - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&= [1 - \zeta_t\eta_t(1 - \rho)]\|k_t - p^\dagger\|^2 - \zeta_t(1 - \zeta_t)\|S_t z_t - S_t y_t\|^2 \\
&\quad - \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 + (1 - \rho)\zeta_t\eta_t c_t,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
c_t &= \frac{1}{1 - \rho} \left\{ 2\|k_t - p^\dagger\| \left(\frac{\psi_t}{\zeta_t\eta_t} \|k_{t-1} - k_t\| \right) + \psi_t\|k_{t-1} - k_t\| \left(\frac{\psi_t}{\zeta_t\eta_t} \|k_{t-1} - k_t\| \right) \right. \\
&\quad \left. + 2\zeta_t\eta_t\|f(p^\dagger) - p^\dagger\|\|y_t - w_t\| + 2\zeta_t\eta_t\langle f(p^\dagger) - p^\dagger, w_t - p^\dagger \rangle \right\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\zeta_t(1 - \zeta_t)\|S_t z_t - S_t y_t\|^2 + \gamma_t(1 - \gamma_t)\|S_t w_t - w_t\|^2 \\
&\leq \|k_t - p^\dagger\|^2 - \|k_{t+1} - p^\dagger\|^2 + \zeta_t\eta_t(1 - \rho)P',
\end{aligned} \tag{14}$$

where $P' = \sup\{c_t : t \in \mathbb{N}\}$.

Next, we prove that $\{k_t\}$ strongly converges to $p^\dagger$. To use Lemma 2, we let $a_t = \|k_t - p^\dagger\|^2$ and $\theta_t = \zeta_t \eta_t (1 - \rho)$. Using (13), the following inequality is obtained:

$$a_{t+1} \leq (1 - \theta_t)a_t + \theta_t \delta_t.$$

If $\{a_{t_j}\}$ is a subsequence of $\{a_t\}$ with $\liminf_{j \rightarrow \infty} (a_{t_{j+1}} - a_{t_j}) \geq 0$, then from Equation (14) and the assumption $\lim_{t \rightarrow \infty} \zeta_t \eta_t = 0$, we have

$$\begin{aligned} \limsup_{j \rightarrow \infty} \zeta_{t_j} (1 - \zeta_{t_j}) \|S_{t_j} z_{t_j} - S_{t_j} y_{t_j}\|^2 + \limsup_{j \rightarrow \infty} \gamma_{t_j} (1 - \gamma_{t_j}) \|S_{t_j} w_{t_j} - w_{t_j}\|^2 \\ \leq \limsup_{j \rightarrow \infty} (a_{t_j} - a_{t_{j+1}} + \zeta_{t_j} \eta_{t_j} (1 - \rho) P') \\ \leq \limsup_{j \rightarrow \infty} (a_{t_j} - a_{t_{j+1}}) + (1 - \rho) P' \limsup_{j \rightarrow \infty} \gamma_{t_j} \\ = -\liminf_{j \rightarrow \infty} (a_{t_{j+1}} - a_{t_j}) \\ \leq 0. \end{aligned}$$

It follows that

$$\lim_{j \rightarrow \infty} \|S_{t_j} z_{t_j} - S_{t_j} y_{t_j}\| = 0 \quad (15)$$

$$\lim_{j \rightarrow \infty} \|S_{t_j} w_{t_j} - w_{t_j}\| = 0. \quad (16)$$

Now, we show that $\limsup_{j \rightarrow \infty} \delta_{t_j} \leq 0$. Clearly, it suffices to illustrate that

$$\limsup_{j \rightarrow \infty} \langle f(p^\dagger) - p^\dagger, w_{t_j} - p^\dagger \rangle \leq 0.$$

Now, let $w_{t_{j_i}}$ be a subsequence of w_{t_j} such that

$$\lim_{i \rightarrow \infty} \langle f(p^\dagger) - p^\dagger, w_{t_{j_i}} - p^\dagger \rangle = \limsup_{j \rightarrow \infty} \langle f(p^\dagger) - p^\dagger, w_{t_j} - p^\dagger \rangle.$$

Due to the boundedness of $\{w_{t_{j_i}}\}$, there exists a subsequence $w_{t_{j_{i_h}}}$ of $\{w_{t_{j_i}}\}$ and $z \in H$ such that $w_{t_{j_{i_h}}} \rightharpoonup z$. Since $\{w_t\}$ is bounded, and $\|S_t w_t - w_t\| \rightarrow 0$ under Equation (15), condition (Z) implies that every weak cluster point of $\{w_t\}$ belongs to Ω . Hence, if $w_{t_{j_{i_h}}} \rightharpoonup z$, then $z \in \Omega$. Therefore, the equation $p^\dagger = P_\Omega f(p^\dagger)$ yields

$$\limsup_{j \rightarrow \infty} \langle f(p^\dagger) - p^\dagger, w_{t_j} - p^\dagger \rangle = \lim_{h \rightarrow \infty} \langle f(p^\dagger) - p^\dagger, w_{t_{j_{i_h}}} - p^\dagger \rangle = \langle f(p^\dagger) - p^\dagger, z - p^\dagger \rangle \leq 0.$$

Through Lemma 2, one can conclude that $k_t \rightarrow p^\dagger$ as $t \rightarrow \infty$. This completes the proof. $\square$

Remark 3. In Theorem 1, if $S_t = \mathcal{S}$ for all $t \in \mathbb{N}$, then following the demiclosedness principle of $\mathcal{S}$, we have that $\{S_t\}$ satisfies condition (Z).

From the remark above, we have weak and strong convergence results for approximating fixed points of nonexpansive mapping.

Corollary 1. Let $S : H \rightarrow H$ be a nonexpansive operator with a nonempty fixed point set. Let $f : H \rightarrow H$ be a contraction. Meanwhile, $\{k_t\}$ is a sequence iteratively generated by $k_0, k_1 \in G$, and

$$\begin{cases} z_t = P_G(k_t + \psi_t(k_t - k_{t-1})), \\ y_t = (1 - \eta_t)z_t + \eta_t f(z_t), \\ w_t = (1 - \zeta_t)S_t z_t + \zeta_t S_t y_t, \\ m_t = (1 - \gamma_t)w_t + \gamma_t S_t w_t \\ k_{t+1} = S_t m_t, \end{cases} \quad (17)$$

where the inertial factor ψ_t is defined by Equation (5). Suppose conditions (B1–B3) hold. Then, $\{k_t\}$ converges strongly to $p^+ \in F(S)$, where $p^+ = P_{F(S)}f(p^+)$.

4. Applications

In this section, we show the applications of the new method to fractional differential and integral equations, split feasibility, signal recovery, and polynomiography problems.

4.1. Split Feasibility Problem

Let $\chi = \{k \in G : Ek \in D\}$ denote the solution set of Equation (1). Let $g : H_1 \rightarrow \mathbb{R}$ be differentiable, weakly lower semicontinuous, and convex on H_1 , and let the function g be defined by $g(k) = \frac{1}{2}\|Ek - P_D Ek\|^2$ for all $k \in H_1$. Thus, we get $\nabla g(k) = E^*(I - P_D)Ek$ for all $k \in H_1$.

It is known that ∇g is $\frac{1}{\|E\|^2}$ -inverse and strongly monotone [2]. Hence, Equation (4) can be expressed as follows:

$$k_{t+1} = P_G(k_t - s_t \nabla g(k_t)),$$

where $0 < s_t < \frac{2}{\|E\|^2}$. Now, we have the following result.

Corollary 2. Let G and D be closed, convex, and nonempty subsets of the real Hilbert spaces H_1 and H_2 , respectively, and let $E : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that $\chi \neq \emptyset$. Let $\{k_t\}$ be a sequence in H defined by

$$\begin{cases} z_t = P_G(k_t + \psi_t(k_t - k_{t-1})), \\ y_t = (1 - \eta_t)z_t + \eta_t f(z_t), \\ w_t = (1 - \zeta_t)P_G(z_t - s_t \nabla g(z_t)) + \zeta_t P_G(y_t - s_t \nabla g(y_t)), \\ m_t = (1 - \gamma_t)w_t + \gamma_t P_G(w_t - s_t \nabla g(w_t)) \\ k_{t+1} = P_G(m_t - s_t \nabla g(m_t)), \end{cases} \quad (18)$$

where the inertial factor ψ_t is defined by Equation (5). Suppose conditions (B1–B3) hold. Then, $\{k_t\}$ converges strongly to $p^+ \in \chi$, where $p^+ = P_\chi f(p^+)$.

Next, we present a numerical example in $\mathbb{R}^2$ with a primary focus on the study of the SFP as given in Corollary 2. We compare the performance of our IVMPGR algorithm with the Noor-Type Split Projected Gradient Algorithm (NTSPGA), Inertial Viscosity Modified SP Algorithm (IVMSPA) in [46], and Multi-Inertial Triple Projection Algorithm (MITPA) in [49].

Example 1. Consider the optimization problem $\min_{k \in G} g(k)$, where $k = (l_1, l_2)^T$, $g(l_1, l_2) = l_1^3 + 2l_2^2 + 1$, and $G = \{k \in \mathbb{R}^2 : \|k - (2, 3)\| \leq 5\}$.

The metric projection onto G is given by

$$P_G(k) = \begin{cases} (2,3) + \frac{5}{\|k - (2,3)\|} (k - (2,3)), & \|k - (2,3)\| > 5, \\ k, & \|k - (2,3)\| \leq 5. \end{cases}$$

We consider four cases of initial points as follows:

$$\text{Case 1: } k_0 = (8, -8)^T, \quad k_1 = (-6, 10)^T,$$

$$\text{Case 2: } k_0 = (-2.8, -7)^T, \quad k_1 = (0.5, 7)^T,$$

$$\text{Case 3: } k_0 = (3, -2)^T, \quad k_1 = (4, 5)^T,$$

$$\text{Case 4: } k_0 = (-2, 9)^T, \quad k_1 = (-1, -9)^T.$$

The stopping criterion is defined by $D_t = \frac{1}{2} \|k_t - P_G(k_t - s_t \nabla g(k_t))\| < 10^{-5}$. Since $\nabla g(l_1, l_2) = \begin{pmatrix} 3l_1^2 \\ 4l_2 \end{pmatrix}$, we estimate the Lipschitz constant with $L = \max\{36M^2, 16\}$, $M = \max\{|l_1|, |l_2|\}$ and choose the variable step size $s_t = \frac{1.99t}{(t+1)L}$. Let $f(k) = 0.5k$.

The control parameters for the various methods are presented in Table 1.

Table 1. Control parameters used for the numerical experiments.

Method	Control Parameters
MITPA	$\gamma_t = 0.70$, $\zeta_t = \frac{1}{\sqrt{t+1}}$, $\eta_t = \frac{1}{\sqrt{t+1}}$, $\delta_{1,t} = 0.30$, $s_t = \frac{1.99t}{(t+1)L}$, where $L = \max\{36M^2, 16\}$, $M = \max\{ l_1 , l_2 \}$.
IVMPCR	$\gamma_t = 0.70$, $\zeta_t = \frac{1}{\sqrt{t+1}}$, $\eta_t = \frac{1}{\sqrt{t+1}}$, $\sigma_t = \frac{1}{4}(t+1)\ k_t - k_{t-1}\ $, $\psi_t = \begin{cases} \min\left\{\frac{q_t - 1}{q_{t+1}}, \frac{\eta_t \zeta_t \sigma_t}{\ k_t - k_{t-1}\ }\right\}, & k_t \neq k_{t-1}, \\ \frac{q_t - 1}{q_{t+1}}, & \text{otherwise,} \end{cases}$ $q_1 = 1$, $q_{t+1} = \frac{1 + \sqrt{1 + 4q_t^2}}{2}$, $s_t = \frac{1.99t}{(t+1)L}$.
NTSPGA	$\gamma_t = \frac{1}{(3t^2 + 1)^4}$, $\zeta_t = \frac{1}{(4t^2 + 1)^3}$, $\eta_t = \frac{1}{t^{0.2} + 1}$, $s_t = \frac{1.99t}{(t+1)L}$.
IVMSPA	$\gamma_t = \frac{1}{(2t^2 + 1)^4}$, $\zeta_t = \frac{1}{(5t^2 + 1)^3}$, $\eta_t = 0.45$, $\theta_t = \frac{1}{t^{0.2} + 1}$, $s_t = \frac{1.99t}{(t+1)L}$.

The numerical results are shown in Table 2 and Figure 1.

Table 2. Comparison of the numerical performance of the MITPA, IVMPCR, NTSPGA and IVMSPA for Example 1 under different initial points.

Case	MITPA		IVMPCR		NTSPGA		IVMSPA	
	Iterations	CPU Time (s)	Iterations	CPU Time (s)	Iterations	CPU Time (s)	Iterations	CPU Time (s)
1	90	0.008695	54	0.004232	500	0.004901	388	0.007314
2	178	0.000638	56	0.000529	500	0.002755	500	0.002066
3	162	0.000717	59	0.000414	500	0.002109	500	0.001731
4	140	0.000685	51	0.000201	500	0.001337	500	0.000890

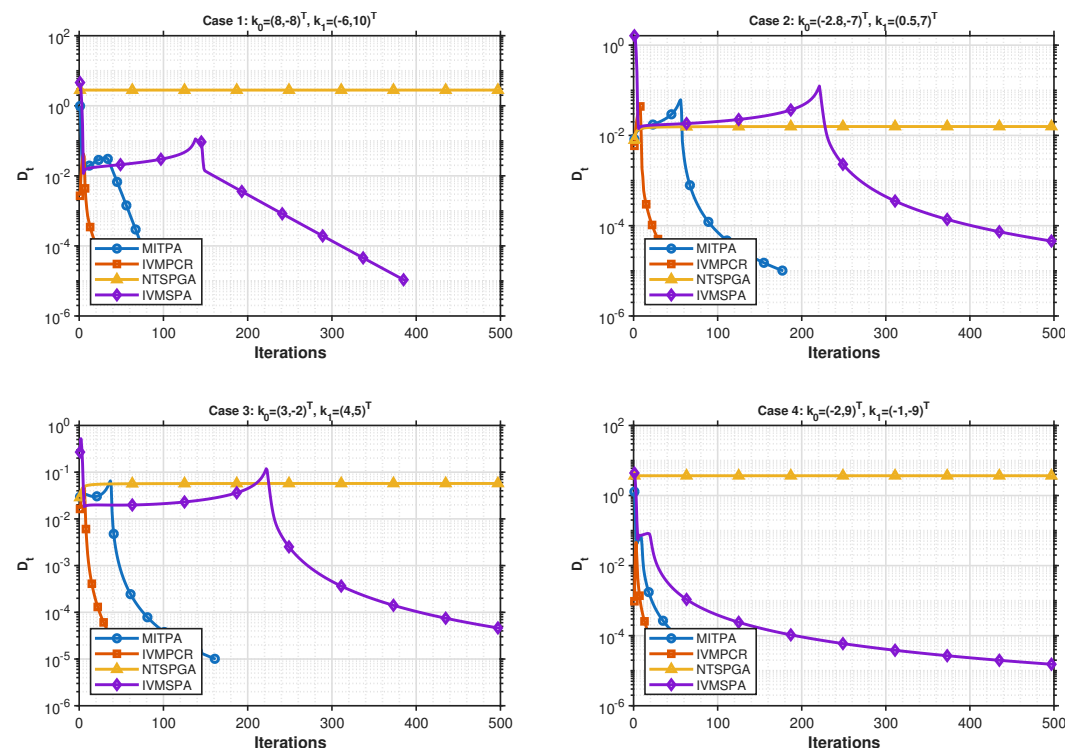


Figure 1. Log-scale convergence history of the stopping criterion D_t for the MITPA, IVMPQR, NTSPGA and IVMSPA applied to Example 1 under four different initial points: Case 1, Case 2, Case 3, and Case 4. The vertical axis represents the stopping criterion D_t on a logarithmic scale, while the horizontal axis denotes the number of iterations.

From the results presented in Table 2 and Figure 1, it is evident that our method required fewer iterations and less computation time. Hence, our proposed IVMPQR algorithm outperformed the competing methods in terms of iteration count and computational time.

4.2. Application to Compressed Sensing

In this section, we present numerical experiments arising from compressed sensing. We compare the performance of the proposed IVMPQR algorithm with the NTSPGA, IVMSPA, and MITPA.

In signal processing, the compressed sensing model can be formulated as follows:

$$l = Fk + v, \quad (19)$$

where $F \in \mathbb{R}^{M \times N}$ is a sensing matrix with $M < N$, $k \in \mathbb{R}^N$ is the sparse signal to be recovered, $v \in \mathbb{R}^M$ denotes noise, and $l \in \mathbb{R}^M$ is the observed measurement vector.

The recovery problem associated with Equation (19) can be formulated as the constrained least squares problem:

$$\min_{k \in \mathbb{R}^N} \frac{1}{2} \|l - Fk\|_2^2 \quad \text{subject to} \quad \|k\|_1 \leq s, \quad (20)$$

where $s > 0$ is a prescribed regularization bound. Let $G = \{k \in \mathbb{R}^N : \|k\|_1 \leq s\}$. Then, Equation (20) can be viewed as a constrained convex optimization problem over the closed convex set G . Moreover, by introducing the set $D = \{l\} \subset \mathbb{R}^M$, the recovery task can also be interpreted within the split feasibility framework, where one seeks a signal $k \in G$ such that Fk is close to the measured data l . Hence, projection-type and CQ-type algorithms are applicable to Equation (20).

For the numerical experiments, the sparse signal $k^\dagger \in \mathbb{R}^N$ was generated with m nonzero components whose values were uniformly distributed in $[-2, 2]$. The sensing matrix F was generated from the standard normal distribution with a mean of zero and variance of one. The observed data were generated according to

$$l = Fk^\dagger + \nu,$$

where ν is white Gaussian noise corresponding to $\text{SNR} = 40$. The algorithms were initialized with $k_1 = \mathbf{1}_N$, and we set $s = m$. The same control parameters as those in Example 1 were used for all the algorithms. The stopping criterion used in the implementation is

$$D_t = \|k_{t+1} - k_t\|_2 < 10^{-5}. \quad (21)$$

This criterion was used only to terminate the iterative process.

To evaluate the quality of the recovered signal, we used the mean squared error (MSE), defined by

$$\text{MSE} = \frac{1}{N} \|k^\dagger - k_t\|_2^2, \quad (22)$$

where $k^\dagger$ denotes the exact sparse signal and k_t denotes the recovered signal at iteration t . Smaller values for the MSE indicate better reconstruction accuracy.

We also used the signal-to-noise ratio (SNR) as a performance metric, defined by

$$\text{SNR} = 20 \log_{10} \left(\frac{\|k^\dagger\|_2}{\|k^\dagger - k_t\|_2} \right). \quad (23)$$

A larger SNR indicates better quality of signal reconstruction.

Note that the stopping criterion (Equation (21)) and the performance measures (Equations (22) and (23)) serve different purposes. The stopping criterion determines when the iterative process terminates, while the MSE and SNR are used only to assess the accuracy and quality of the recovered signal.

In the following experiments, we considered two compressed sensing settings:

$$\text{Case 1: } N = 1050, \quad M = 525, \quad m = 15,$$

and

$$\text{Case 2: } N = 512, \quad M = 256, \quad m = 40.$$

For each case, we report the objective function values, MSE, SNR, number of iterations, and CPU time for the IVMPCR algorithm as well as the NTSPGA, IVMSPA, and MITPA. The numerical findings are presented as follows.

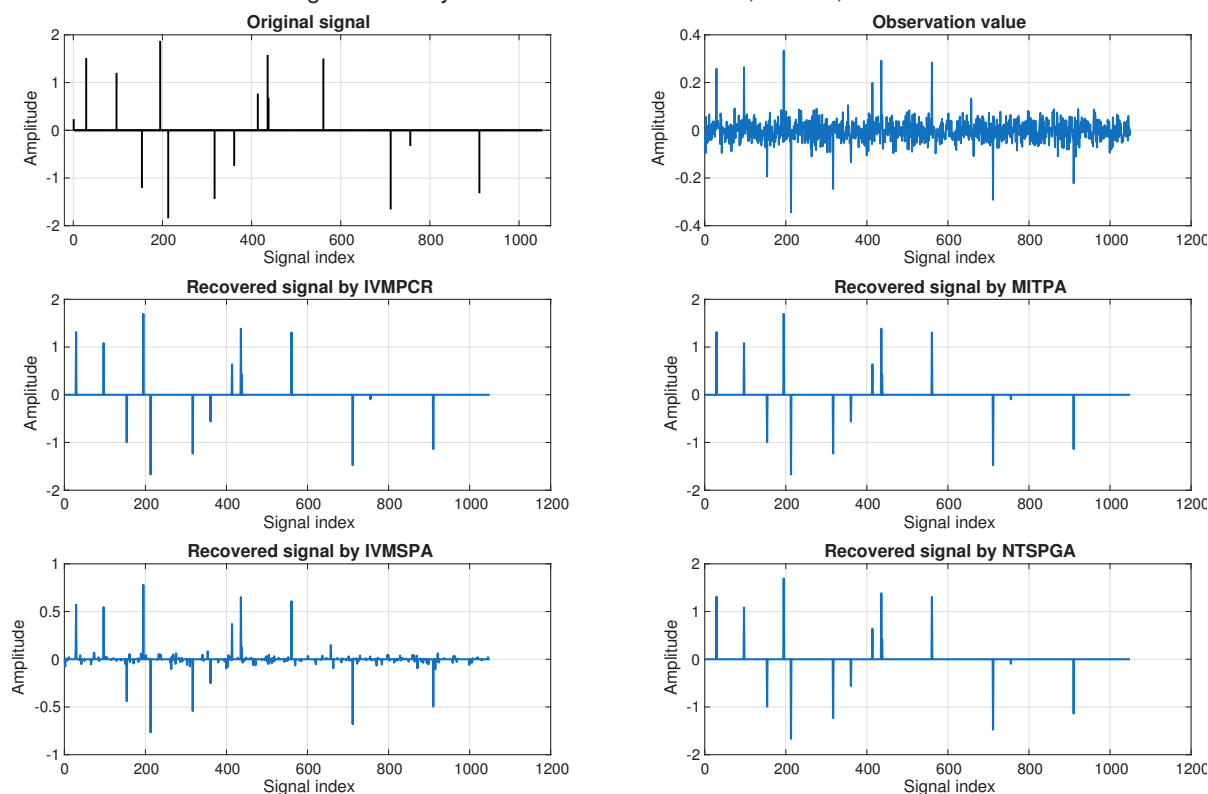
Table 3 shows the performance of IVMPCR and other methods. The analyses of the results are presented in terms of computation time and the number of iterations. The table shows signals of sizes $N = 1050$, $M = 525$ and $N = 512$, $M = 256$ when the number of signal bars was set to $m = 15$ and $m = 40$, respectively. The produced results show that our method required the fewest iterations and less computation time than the other methods in both considered cases.

The plots in Figure 2 indicates the original signal, observed data, and signal recovered via various algorithms when $N = 1050$, $M = 525$, $m = 15$. The recovery time and number of iterations for each algorithm are shown. As can be clearly seen in the figures for both cases, the recovery by the IVMPCR algorithm yielded the closest match to the original signal and required less time, unlike the other algorithms, which required more iterations and computation time to recover the signal.

Table 3. Comparison of the numerical performance of the IVMPCR algorithm and MITPA, IVMSPA, and NTSPGA for the compressed sensing problem under two test cases.

Case	Method	Iterations	CPU Time (s)
Case 1	IVMPCR	15	0.009178
	MITPA	29	0.013099
	IVMSPA	500	0.157839
	NTSPGA	315	0.164066
Case 2	IVMPCR	43	0.006617
	MITPA	85	0.009731
	IVMSPA	500	0.024612
	NTSPGA	500	0.040637

Figure 3 displays the relationship between the SNR and the number of iterations, as well as the MSE value and the number of iterations, when $N = 1050$, $M = 525$, and $m = 15$. The plots in Figure 4 indicate the original signal, observed data, and signal recovered via various algorithms when $N = 512$, $M = 256$, $m = 40$. Meanwhile, Figure 5 shows the relationship between the SNR and the number of iterations, as well as the MSE value and the number of iterations, when $N = 512$, $M = 256$, and $m = 40$.

Signal recovery results for Case 1: $N=1050$, $M=525$, $m=15$ **Figure 2.** Signal recovery results for Case 1 ($N = 1050$, $M = 525$, $m = 15$). The six subfigures display the original sparse signal, the observation signal, and the reconstructed signals obtained by the IVMPCR algorithm, MITPA, IVMSPA, and NTSPGA. It can be observed that the proposed IVMPCR method reconstructed the sparse signal more accurately than the competing methods.

For all the numerical experiments conducted, the obtained results indicate that IMVPCR outperformed the other algorithms in terms of iterations and CPU time.

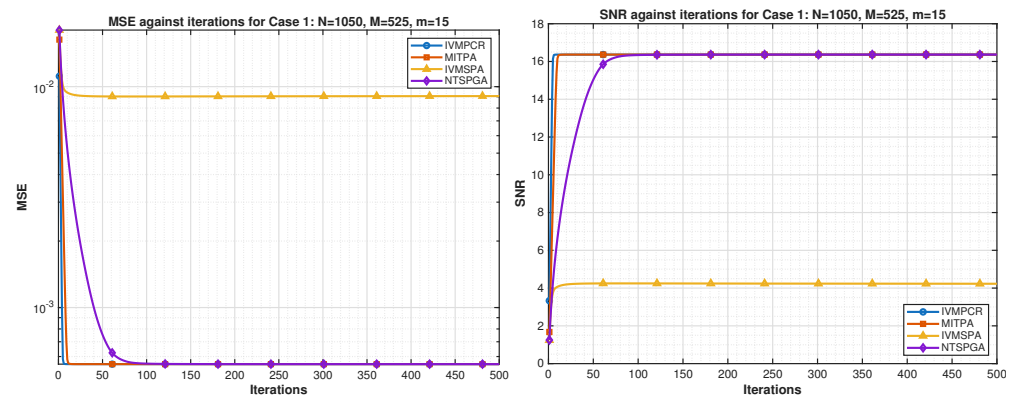


Figure 3. (left) SNR vs. number of iterations and (right) MSE vs. number of iterations when $N = 1050$, $M = 525$, and $m = 15$.

Signal recovery results for Case 2: $N = 512$, $M = 256$, $m = 40$

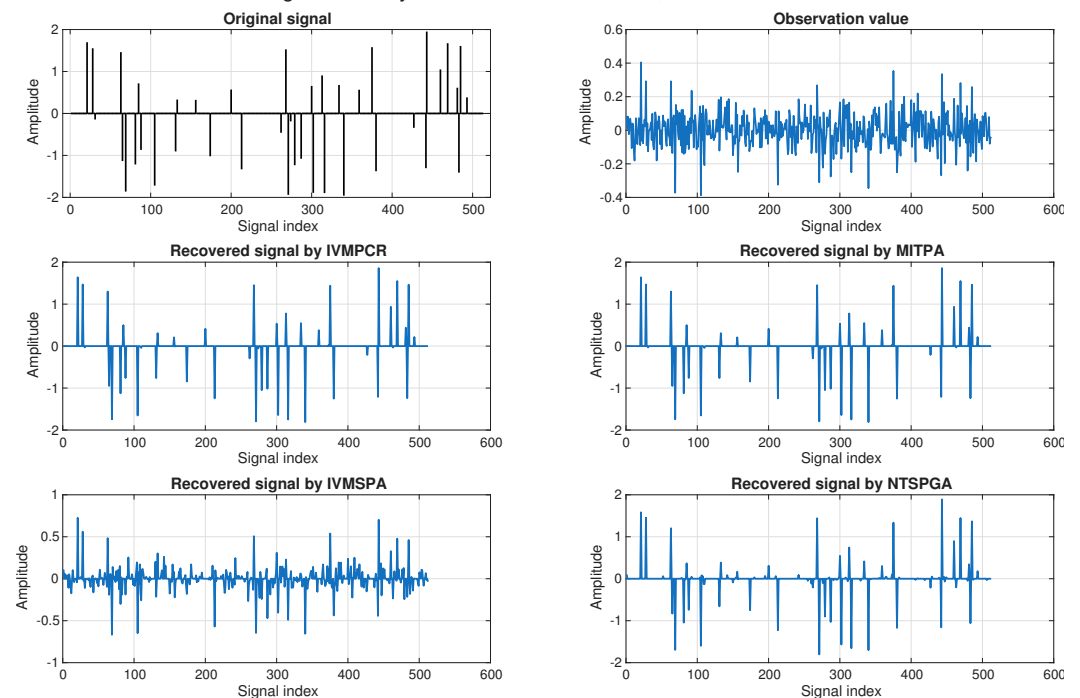


Figure 4. Signal recovery results for Case 2 ($N = 512$, $M = 256$, $m = 40$). The six subfigures show the original sparse signal, the observation signal, and the reconstructed signals produced by the IVMPCR algorithm, MITPA, IVMSPA, and NTSPGA. The proposed IVMPCR method provided the closest approximation to the original sparse signal, demonstrating its superior recovery capability.

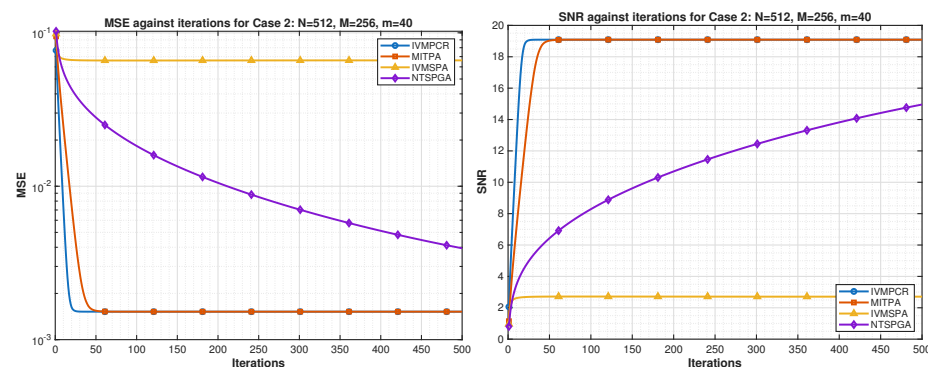


Figure 5. (left) SNR vs. number of iterations and (right) MSE vs. number of iterations when $N = 512$, $M = 256$, and $m = 40$.

4.3. Application to Polynomiography

Polynomials play an important role in several areas of mathematics and applied sciences. The problem of locating the roots of polynomials has received renewed attention through the work of Kalantari [50], who introduced the concept of polynomiography. Polynomiography is the art and science of visualizing the approximation of polynomial roots by means of iterative functions [50,51]. The images obtained through this process are called polynomiographs. These images are different from fractals, whose shapes are often determined entirely by the input data of the generating function [52]. In contrast, polynomiographs can be controlled more predictably through the choice of the polynomial and the iterative method.

Let $p \in \mathbb{C}[z]$ be a polynomial. The classical Newton method for solving

$$p(k) = 0$$

is given by

$$k_{t+1} = k_t - \frac{p(k_t)}{p'(k_t)}, \quad (24)$$

where $p'(k_t) \neq 0$. We define the Newton fixed point operator $T : \mathbb{C} \rightarrow \mathbb{C}$ by

$$T(k) = k - \frac{p(k)}{p'(k)}. \quad (25)$$

Then, we have

$$p(k^*) = 0 \iff T(k^*) = k^*.$$

Thus, every root of p is a fixed point of the operator T . Therefore, the polynomial root-finding problem can be treated as a fixed-point problem.

In this setting, we apply the proposed IVMPQR algorithm by taking the fixed-point operator in the algorithm to be the Newton operator T defined in Equation (25). Hence, the IVMPQR scheme becomes

$$\begin{cases} z_t = k_t + \psi_t(k_t - k_{t-1}), \\ y_t = (1 - \eta_t)z_t + \eta_t T(z_t), \\ w_t = (1 - \zeta_t)T(z_t) + \zeta_t T(y_t), \\ m_t = (1 - \gamma_t)w_t + \gamma_t T(w_t), \\ k_{t+1} = T(m_t). \end{cases} \quad (26)$$

Consequently, the convergence of the sequence generated by Equation (26) corresponds to the approximation of a root of the polynomial p , provided that the assumptions required for convergence are satisfied locally around the root.

The control parameters employed in the numerical experiments are summarized in Table 4. Unless otherwise stated, these parameters are kept fixed throughout the computations for all polynomial examples to ensure a fair and consistent comparison between the proposed IVMPQR algorithm and the competing methods.

To generate the polynomiographs, we considered a rectangular region $A \subset \mathbb{C}$, a maximum number of iterations $M \in \mathbb{N}$, and a tolerance $\varepsilon > 0$. For each initial point $k_0 \in A$, the iteration was stopped whenever

$$\|k_{t+1} - k_t\| < \varepsilon. \quad (27)$$

The color assigned to each point was determined by the number of iterations needed to satisfy Equation (27), following the standard iteration count coloring technique [53].

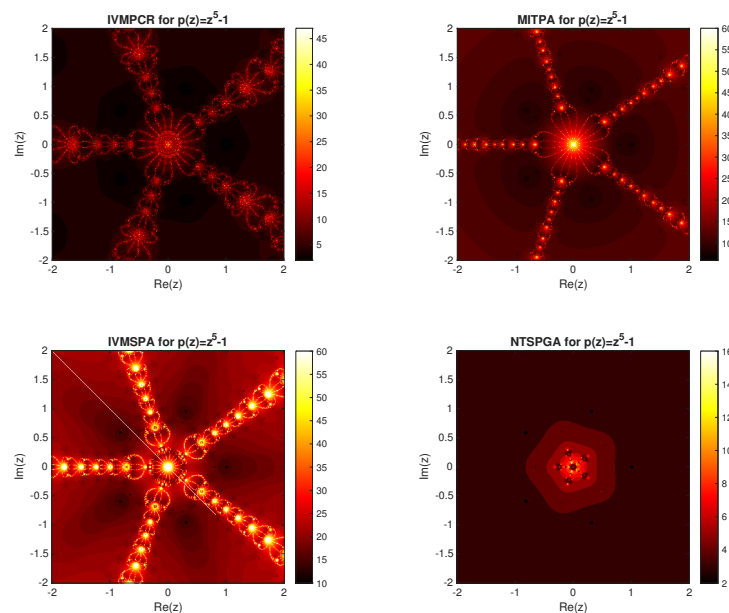
Table 4. Control parameters used in the polynomiography experiments.

Method	Control Parameters
IVMPCR	$\eta_t = \frac{1}{t^{0.2} + 1}, \zeta_t = \frac{1}{t^{0.8} + 1}, \gamma_t = 0.99, \psi_t = 0.25$
MITPA	$\alpha_t = 0.70, \beta_t = 0.70, \gamma_t = 0.45, \delta_t = 0.30$
IVMSPA	$\alpha_t = \frac{1}{(2t^2 + 1)^4}, \beta_t = \frac{1}{(5t^2 + 1)^3}, \gamma_t = 0.45, \theta_t = \frac{1}{t^{0.2} + 1}$
NTSPGA	$\alpha_t = \frac{1}{(3t^2 + 1)^4}, \beta_t = \frac{1}{(4t^2 + 1)^3}, \gamma_t = \frac{1}{t^{0.2} + 1}$
Common parameters: Maximum iterations $M = 60$, stopping tolerance $\varepsilon = 10^{-6}$.	

Example 2. We compared the polynomiographs generated by the IVMPCR algorithm, MITPA, IVMSPA, and NTSPGA for the following polynomials:

$$(a) \quad p(z) = z^5 - 1, \quad (b) \quad p(z) = z^6 - z^3 + 1.$$

Figures 6 and 7 compare the polynomiographs generated by the proposed IVMPCR algorithm with the MITPA, IVMSPA, and NTSPGA for polynomials $(p(z) = z^5 - 1)$ and $(p(z) = z^6 - z^3 + 1)$, respectively. Each colored region represents the basin of attraction associated with one of the polynomial roots, while the color intensity indicates the number of iterations required for convergence.

**Figure 6.** Polynomiographs generated by IVMPCR, MITPA, IVMSPA, and NTSPGA for $p(z) = z^5 - 1$.

For the polynomial $(p(z) = z^5 - 1)$, all four algorithms successfully identified the five roots. However, the proposed IVMPCR algorithm produced noticeably larger and smoother convergence basins than the competing methods. The transition boundaries between neighboring basins were also less fragmented, indicating improved numerical stability and a wider region of convergence. In contrast, the MITPA exhibited relatively smaller attraction regions, whereas the IVMSPA and NTSPGA generated more intricate basin boundaries, reflecting slower convergence for the initial points near the basin interfaces.

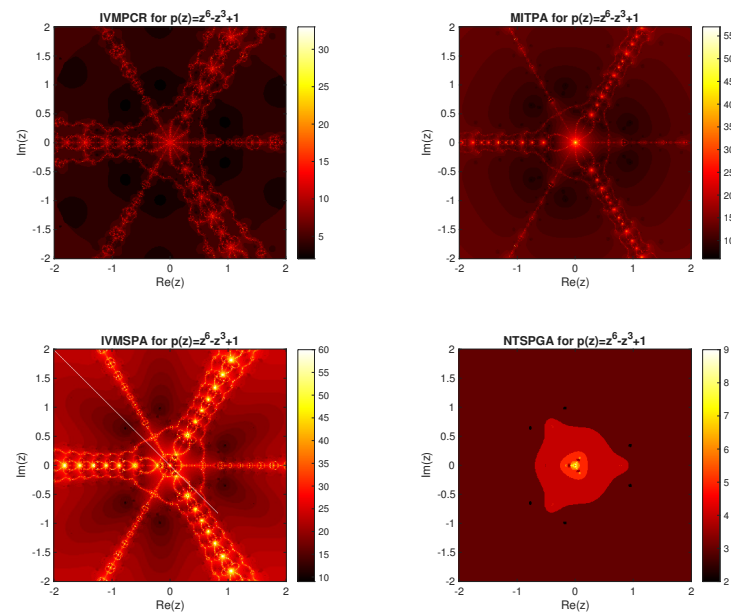


Figure 7. Polynomiographs generated by IVMPCR, MITPA, IVMSPA, and NTSPGA for $p(z) = z^6 - z^3 + 1$.

A similar behavior was observed for the polynomial ($p(z) = z^6 - z^3 + 1$). Although each method converged to the six roots, IVMPCR again displayed the largest bright convergence regions and the least irregular basin boundaries. This demonstrates that the proposed algorithm is less sensitive to the choice of initial approximations and possesses stronger global convergence characteristics. Moreover, the smoother color distribution indicates that fewer iterations were required to reach the desired stopping tolerance over a substantial portion of the computational domain.

Overall, the numerical experiments demonstrate that IVMPCR consistently outperformed the MITPA, IVMSPA, and NTSPGA in terms of convergence speed, stability, and basin coverage. The larger attraction regions and reduced number of iterations observed in the generated polynomiographs are consistent with the theoretical convergence properties established in the preceding sections, thereby confirming the effectiveness of the proposed algorithm for polynomial root finding and polynomiographic visualization.

4.4. Application to Fractional Differential and Integral Equations

Fractional differential and integral equations arise naturally in many areas of science and engineering, including viscoelasticity, signal processing, control theory, fluid mechanics, anomalous diffusion, and mathematical biology. Many classes of fractional models can be reformulated as fixed-point problems involving nonlinear integral operators. Consequently, the strong convergence results established in Section 3 can be employed to approximate their solutions.

Throughout this section, let

$$H = L^2(0, T)$$

be the real Hilbert space of square-integrable functions on $[0, T]$ equipped with the inner product

$$\langle k, m \rangle = \int_0^T k(z)m(z) dz$$

and the induced norm

$$\|k\| = \left(\int_0^T |k(z)|^2 dz \right)^{\frac{1}{2}}.$$

Furthermore, let

$$\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds, \quad \alpha > 0,$$

denote the Gamma function.

The Caputo fractional derivative of an order $\alpha \in (0, 1)$ is defined by

$${}^C D^\alpha k(z) = \frac{1}{\Gamma(1-\alpha)} \int_0^z (z-s)^{-\alpha} k'(s) ds.$$

4.4.1. Nonlinear Fractional Volterra Integral Equations

Consider the nonlinear fractional Volterra integral equation

$$k(z) = g(z) + \frac{1}{\Gamma(\alpha)} \int_0^t (z-s)^{\alpha-1} F(s, k(s)) ds, \quad z \in [0, T], \quad (28)$$

where

- $0 < \alpha < 1$;
- $k \in H$ is the unknown function;
- $g \in H$ is a given function;
- $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Define the operator $S : H \rightarrow H$ by

$$(Sk)(z) = g(z) + \frac{1}{\Gamma(\alpha)} \int_0^t (z-s)^{\alpha-1} F(s, k(s)) ds. \quad (29)$$

Observe that the solutions to Equation (28) coincide with fixed points of S .

Theorem 2. Suppose that

1. There exists $L > 0$ such that

$$|F(z, u) - F(z, v)| \leq L|u - v|, \quad z \in [0, T];$$

- 2.

$$\frac{LT^\alpha}{\Gamma(\alpha+1)} \leq 1;$$

3. $\text{Fix}(S) \neq \emptyset$.

Then, $S : H \rightarrow H$ is nonexpansive. Consequently, the sequence generated by Algorithm 1 converges strongly to a solution $k^\dagger \in \text{Fix}(S)$ for Equation (28).

Proof. For every $k, m \in H$, we have

$$\|Sk - Sm\| \leq \frac{LT^\alpha}{\Gamma(\alpha+1)} \|k - m\| \leq \|k - m\|.$$

Hence, S is nonexpansive. The conclusion follows directly from Corollary 1. $\square$

4.4.2. Caputo Fractional Differential Equations

Consider the Caputo fractional differential equation

$${}^C D^\alpha k(z) = F(z, k(z)), \quad k(0) = k_0, \quad 0 < \alpha < 1, \quad (30)$$

where the following definitions apply:

- $k \in H$;
- $k_0 \in \mathbb{R}$;
- $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Equation (30) is equivalent to

$$k(z) = k_0 + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} F(s, k(s)) ds. \quad (31)$$

Define $S : H \rightarrow H$ by

$$(Sk)(z) = k_0 + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} F(s, k(s)) ds.$$

Theorem 3. Assume the following:

1. F is continuous;
2. There exists $L > 0$ such that

$$|F(z, u) - F(z, v)| \leq L|u - v|;$$

3.

$$\frac{LT^\alpha}{\Gamma(\alpha+1)} \leq 1;$$

4. $\text{Fix}(S) \neq \emptyset$.

Then, the sequence generated by Algorithm 1 converges strongly to a solution to Equation (30).

Proof. The associated integral operator is nonexpansive on H . Therefore, the result follows immediately from Theorem 1. $\square$

4.4.3. Fractional Integro-Differential Equations

Consider

$${}^C D^\alpha k(z) = F\left(z, k(z), \int_0^z K(z, s, k(s)) ds\right), \quad k(0) = k_0, \quad (32)$$

where the following definitions apply:

- $0 < \alpha < 1$;
- $k \in H$;
- $k_0 \in \mathbb{R}$;
- $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous;
- $K : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous kernel.

The term

$$\int_0^z K(z, s, k(s)) ds$$

represents the memory effect of the system.

The equivalent integral equation is

$$k(z) = k_0 + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} F\left(s, k(s), \int_0^s K(s, \tau, k(\tau)) d\tau\right) ds.$$

Define $S : H \rightarrow H$ by

$$(Sk)(z) = k_0 + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} F\left(s, k(s), \int_0^s K(s, \tau, k(\tau)) d\tau\right) ds.$$

Theorem 4. Suppose there exist constants $L_1, L_2, M > 0$ such that

$$|F(z, u, v) - F(z, \bar{u}, \bar{v})| \leq L_1 |u - \bar{u}| + L_2 |v - \bar{v}|,$$

and

$$|K(z, s, u) - K(z, s, v)| \leq M |u - v|.$$

If

$$\frac{T^\alpha}{\Gamma(\alpha+1)} (L_1 + L_2 MT) \leq 1$$

and $\text{Fix}(S) \neq \emptyset$, then Algorithm 1 converges strongly to a solution for Equation (32).

Proof. The associated integral operator is nonexpansive on H . Hence, the result follows from Corollary 1. $\square$

4.4.4. Fractional Delay Differential Equations

Let

$$H_\tau = L^2(-\tau, T)$$

be the Hilbert space equipped with

$$\langle k, m \rangle = \int_{-\tau}^T k(z) m(z) dz$$

and norm

$$\|k\| = \left(\int_{-\tau}^T |k(z)|^2 dz \right)^{\frac{1}{2}}.$$

Consider the fractional delay differential equation

$${}^C D^\alpha k(z) = F(z, k(z), k(z-\tau)), \quad z \in [0, T], \quad (33)$$

subject to

$$k(z) = \phi(z), \quad z \in [-\tau, 0],$$

where the following definitions apply:

- $\tau > 0$ is the delay parameter;
- $\phi : [-\tau, 0] \rightarrow \mathbb{R}$ is the history function;
- $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

The equivalent integral equation is

$$k(z) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} F(s, k(s), k(s-\tau)) ds.$$

Define $S : H_\tau \rightarrow H_\tau$ by

$$(Sk)(z) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} F(s, k(s), k(s-\tau)) ds.$$

Theorem 5. Assume that

$$|F(z, u, v) - F(z, \bar{u}, \bar{v})| \leq L(|u - \bar{u}| + |v - \bar{v}|)$$

and

$$\frac{2LT^\alpha}{\Gamma(\alpha + 1)} \leq 1.$$

If $\text{Fix}(S) \neq \emptyset$, then the sequence generated by Algorithm 1 converges strongly to a solution to Equation (33).

Proof. The associated integral operator is nonexpansive on H_τ . Therefore, the conclusion follows from Theorem 1. $\square$

Theorem 6. Let H denote the corresponding Hilbert space associated with any of the fractional models considered above, and let $S : H \rightarrow H$ denote the induced integral operator. Suppose the following:

1. S is nonexpansive;
2. $\text{Fix}(S) \neq \emptyset$;
3. The control sequences satisfy the assumptions of Theorem 1.

Then, the sequence generated by Algorithm 1 converges strongly to the unique viscosity-selected solution

$$k^\dagger = P_{\text{Fix}(S)} f(k^\dagger).$$

Consequently, Algorithm 1 approximates solutions of all the fractional equations considered in this section.

Proof. The result follows directly from Theorem 1 and Corollary 1. $\square$

Remark 4. The above results demonstrate that the proposed inertial viscosity modified Picard–CR algorithm is applicable to a broad class of fractional models, including fractional Volterra integral equations, Caputo fractional differential equations, fractional integro-differential equations, and fractional delay differential equations. The convergence of the generated sequence follows directly from the strong convergence theory established in Section 3.

Example 3. To illustrate the applicability of the proposed IVMPCR algorithm to fractional calculus, we considered the nonlinear Caputo fractional differential equation

$${}^C D_z^\alpha k(z) = -k(z) + \arctan(k(z)), \quad 0 < z \leq 1, \quad (34)$$

with the initial condition

$$k(0) = k_0, \quad (35)$$

where $0 < \alpha < 1$. In the numerical experiment, we took $\alpha = 0.95$.

Using the standard equivalence between Caputo fractional differential equations and Volterra integral equations, the problem in Equations (34) and (35) can be written as follows:

$$k(z) = k_0 + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} [-k(s) + \arctan(k(s))] ds, \quad z \in [0, 1]. \quad (36)$$

Define the operator

$$T : C[0, 1] \rightarrow C[0, 1]$$

by

$$(Tk)(z) = k_0 + \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} [-k(s) + \arctan(k(s))] ds. \quad (37)$$

Then, solving Equation (36) is equivalent to finding a fixed point of T ; that is, $Tk = k$. Let $h(r) = -r + \arctan(r)$, $r \in \mathbb{R}$. Then, we have $h'(r) = -1 + \frac{1}{1+r^2} = -\frac{r^2}{1+r^2}$, and hence $|h'(r)| \leq 1$, $r \in \mathbb{R}$. Therefore, according to the mean value theorem, we have

$$|h(r) - h(q)| \leq |r - q|, \quad r, q \in \mathbb{R}.$$

Thus, for any $k, l \in C[0, 1]$ and $z \in [0, 1]$, we obtain

$$\begin{aligned} |(Tk)(z) - (Tl)(z)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} |h(k(s)) - h(l(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \|k - l\|_{\infty} \int_0^z (z-s)^{\alpha-1} ds \\ &= \frac{z^{\alpha}}{\Gamma(\alpha+1)} \|k - l\|_{\infty} \\ &\leq \frac{1}{\Gamma(\alpha+1)} \|k - l\|_{\infty}. \end{aligned}$$

For $\alpha = 0.95$, we have

$$\frac{1}{\Gamma(\alpha+1)} < 1.03.$$

Hence, on a sufficiently small interval $[0, T]$ satisfying

$$\frac{T^{\alpha}}{\Gamma(\alpha+1)} \leq 1,$$

the operator T is nonexpansive. In particular, since $T = 1$ gives $\frac{1}{\Gamma(1.95)} \approx 1.027$, one may either work on a slightly shorter interval or introduce a damping factor in the integral operator to obtain a strict nonexpansive mapping on $[0, 1]$.

To guarantee nonexpansiveness on $[0, 1]$, we considered the damped operator

$$(Tk)(z) = k_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} [-k(s) + \arctan(k(s))] ds, \quad (38)$$

where $0 < \lambda < \Gamma(\alpha+1)$. For the numerical implementation, we chose

$$\lambda = 0.90.$$

Then, we had

$$\begin{aligned} |(Tk)(z) - (Tl)(z)| &\leq \frac{\lambda}{\Gamma(\alpha)} \|k - l\|_{\infty} \int_0^z (z-s)^{\alpha-1} ds \\ &= \frac{\lambda z^{\alpha}}{\Gamma(\alpha+1)} \|k - l\|_{\infty} \\ &\leq \frac{\lambda}{\Gamma(\alpha+1)} \|k - l\|_{\infty}. \end{aligned}$$

Since

$$0 < \frac{\lambda}{\Gamma(\alpha+1)} < 1,$$

it follows that

$$\|Tk - Tl\|_{\infty} \leq \frac{\lambda}{\Gamma(\alpha+1)} \|k - l\|_{\infty}.$$

Therefore, T is a contraction, and hence it is nonexpansive on $C[0, 1]$.

Consequently, the fractional differential equation (Equation (34)) can be solved by applying the proposed IVMPCR algorithm to the fixed-point problem:

$$\text{find } k^* \in C[0, 1] \quad \text{such that} \quad Tk^* = k^*.$$

Since T is nonexpansive, and the viscosity mapping used in the algorithm is a contraction, the assumptions of Theorem 1 are satisfied.

Next, we compared the performance of the proposed IVMPCR algorithm with MITPA, IVMSPA, and NTSPGA for solving the nonlinear fractional Volterra integral equation. The comparison was based on the number of iterations required for convergence, CPU time, approximation error, and fixed-point residual. In addition, the convergence behavior of the four algorithms is illustrated by plotting the approximation error against the iteration number. These numerical experiments were carried out to demonstrate the computational efficiency, convergence rate, and accuracy of the proposed IVMPCR algorithm relative to the competing methods.

For the numerical implementation, the interval $[0, 1]$ was uniformly partitioned into $N = 300$ subintervals. The chosen fractional order was $\alpha = 0.95$, the damping parameter was $\lambda = 0.90$, and the initial condition was $k(0) = 1$. The initial approximations we selected were $k_0(z) = 1$, $k_{-1}(z) = 0.8$, $z \in [0, 1]$. The stopping criterion was defined by $\|k_{t+1} - k_t\|_\infty < 10^{-5}$.

For the proposed IVMPCR algorithm, the following control parameters were chosen:

$$\eta_t = \frac{1}{t^{0.2} + 1}, \quad \zeta_t = \frac{1}{t^{0.8} + 1}, \quad \gamma_t = 0.99, \quad \psi_t = 0.25.$$

For the MITPA algorithm, the selected parameters were

$$\alpha_t = 0.70, \quad \beta_t = 0.70, \quad \gamma_t = 0.45, \quad \delta_t = 0.30.$$

For the IVMSPA algorithm, the chosen parameters were

$$\alpha_t = \frac{1}{(2t^2 + 1)^4}, \quad \beta_t = \frac{1}{(5t^2 + 1)^3}, \quad \gamma_t = 0.45, \quad \theta_t = \frac{1}{t^{0.2} + 1}.$$

For the NTSPGA algorithm, the chosen control parameters were

$$\alpha_t = 0.50, \quad \beta_t = 0.50, \quad \gamma_t = \frac{1}{t^{0.2} + 1}.$$

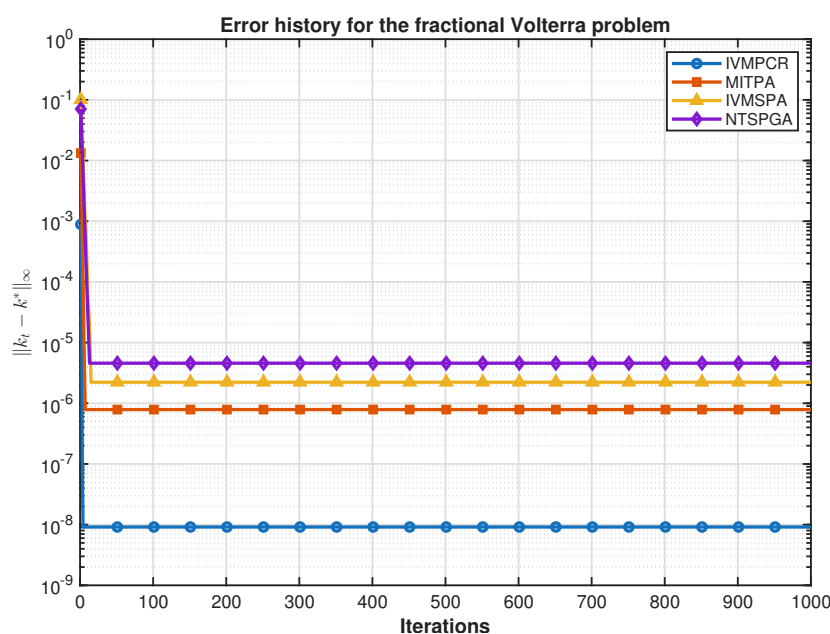
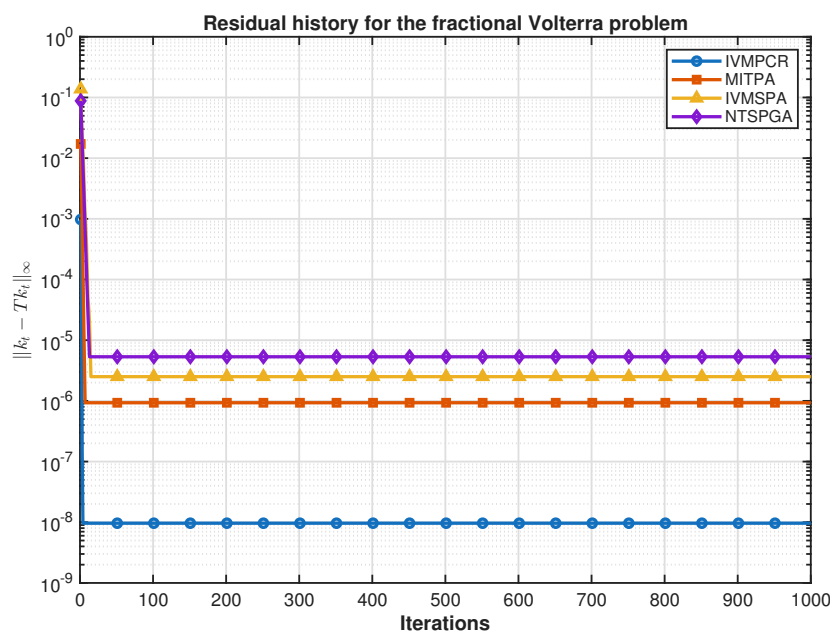
To assess the numerical performance of the algorithms, we recorded the number of iterations, CPU time, approximation error $\|k_t - k^*\|_\infty$, where k^* denotes a highly accurate reference solution computed by the Picard iteration, and the fixed point residual $\|k_t - T(k_t)\|_\infty$. Furthermore, the convergence behavior of the four algorithms was illustrated by plotting the approximation error, the fixed point residual, and the stopping criterion $\|k_{t+1} - k_t\|_\infty$ against the number of iterations.

Table 5 and Figures 8–10 present the numerical performance of the proposed IVMPCR algorithm compared with the MITPA, IVMSPA, and NTSPGA for solving the nonlinear fractional Volterra integral equation. The comparison was based on the number of iterations, CPU time, approximation error, fixed-point residual, and convergence histories of the competing methods.

Based on Table 5, the proposed IVMPCR algorithm exhibited the best overall numerical performance. It converged in only four iterations, whereas the MITPA, NTSPGA, and IVMSPA required 7, 13, and 15 iterations, respectively. Consequently, IVMPCR recorded the shortest CPU time among all the methods. Furthermore, it attained the lowest approximation error of (9.1461×10^{-9}) and the smallest fixed-point residual of (1.4939×10^{-7}) , demonstrating a more accurate approximation of the solution and a higher degree of convergence than the competing algorithms.

Table 5. Comparison of numerical performance of IVMPCR, MITPA, IVMSPA, and NTSPGA for nonlinear fractional Volterra integral equation.

Method	Iterations	CPU Time (s)	Error	Residual
IVMPCR	4	0.073203	9.1461×10^{-9}	1.4939×10^{-7}
MITPA	7	0.109318	7.8781×10^{-7}	3.2366×10^{-6}
IVMSPA	15	0.216887	2.2257×10^{-6}	5.3051×10^{-6}
NTSPGA	13	0.165939	4.5771×10^{-6}	5.3223×10^{-6}

**Figure 8.** Comparison of the approximation error against the number of iterations for the IVMPCR algorithm, MITPA, IVMSPA, and NTSPGA applied to the nonlinear fractional Volterra integral equation.**Figure 9.** Comparison of the fixed-point residual against the number of iterations for the IVMPCR algorithm, MITPA, IVMSPA, and NTSPGA applied to the nonlinear fractional Volterra integral equation.

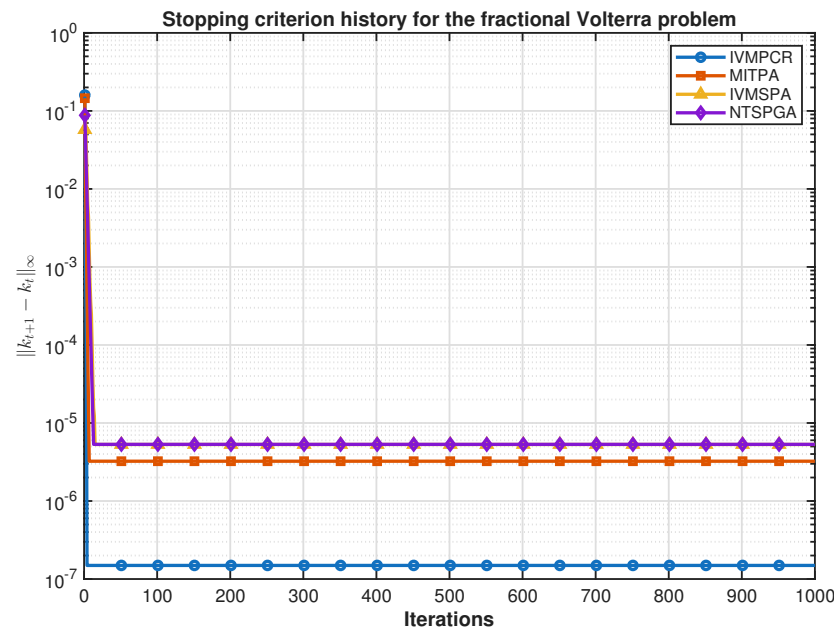


Figure 10. Comparison of the stopping criterion $\|k_{m+1} - k_m\|_{\infty}$ against the number of iterations for the IVMPCR algorithm, MITPA, IVMSPA, and NTSPGA applied to the nonlinear fractional Volterra integral equation.

Figure 8 illustrates the evolution of the approximation error with respect to the iteration number. The error associated with IVMPCR decreased much more rapidly than those of the MITPA, IVMSPA, and NTSPGA, reaching the prescribed tolerance within only a few iterations. In contrast, the remaining methods exhibited slower decay rates and required more iterations before approaching the same level of accuracy. This behavior confirms the faster convergence rate of the proposed algorithm.

Figure 9 compares the fixed-point residual histories of the four methods. The residual generated by IVMPCR decreased sharply and reached values close to machine precision much earlier than the other methods. The larger residuals observed for the MITPA, IVMSPA, and NTSPGA over the same iteration range indicate slower convergence toward the fixed point of the associated fractional integral operator.

Figure 10 presents the stopping criterion ($\|k_{t+1} - k_t\|_{\infty}$) versus the iteration number. The stopping criterion for IVMPCR decreased monotonically and satisfied the prescribed tolerance in significantly fewer iterations than the competing methods. The comparatively slower decrease exhibited by the MITPA, IVMSPA, and NTSPGA further confirms that these methods require additional iterations before convergence.

Overall, the numerical results demonstrate that the proposed IVMPCR algorithm consistently outperformed the MITPA, IVMSPA, and NTSPGA in terms of convergence speed, computational efficiency, approximation accuracy, and numerical stability when applied to the nonlinear fractional Volterra integral equation. These observations are in agreement with the theoretical convergence analysis established in the preceding sections and confirm the effectiveness of the proposed algorithm for solving fractional integral equations.

5. Conclusions

In this study, we introduced a modified Picard–CR algorithm which incorporates the inertial technique and the viscosity technique to accelerate its convergence and guarantee its strong convergence, respectively. We obtained its strong convergence using mild conditions on the control parameters. Applications of the proposed method to solve split feasibility problems, signal processing problems, and polynomiography problems, and fractional differential and integral equations are also considered. With several numerical experiments,

we demonstrate that our method outperformed many existing methods. The obtained results extend and improve several known results in the literature.

Author Contributions: Conceptualization, A.E.O.; methodology, A.E.O.; writing—original draft preparation, A.E.O.; writing—review and editing, S.P.M. and M.K.; visualization, S.P.M. and M.K.; supervision, S.P.M. and M.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sharing is not applicable for this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflicts of interest.

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