



Article

Beyond Classical Metrics: Fixed Point Dynamics in tvs -Valued Cone Suprametric Spaces with Fractional Differential Applications

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Abstract

The main aim of this work is to introduce the notion of a tvs -valued cone suprametric space, which unifies and generalizes cone suprametric spaces, tvs -valued cone metric spaces, and cone metric spaces, and to establish new fixed point theorems for single-valued and multivalued generalized contractive mappings within this framework. The presented results extend and unify some well-known outcomes available in the existing literature. To demonstrate the applicability and significance of the proposed theory, a number of illustrative examples are provided. In addition, the obtained fixed point results are employed to investigate the existence of solutions for a Caputo fractional differential equation as well as a Volterra–Fredholm integral equation.

Keywords: fixed point; tvs -valued cone suprametric space; Caputo fractional differential equation; Volterra–Fredholm integral equation

MSC: 34K37; 47H10; 54H25

1. Introduction

Fixed point (FP) theory is a well-established and influential branch of mathematics, widely recognized for its applications across various scientific and engineering fields. The concept of metric spaces (MSs), which serves as a cornerstone for FP theory, was first introduced by M. Fréchet in 1906, providing a rigorous framework for analyzing distances and convergence. This foundational idea has motivated extensive research into generalizations and extensions of MSs, leading to significant advancements in the study of FPs. Stefan Banach [1] is credited with laying the groundwork of FP theory through the introduction of contraction mappings and the celebrated Banach Contraction Principle (BCP). Subsequent research has focused on broadening the applicability of this principle. Notably, Kannan [2] formulated a FP theorem that replaces the classical Lipschitz-type contraction with a weaker condition involving only the distances between points and their images, thereby ensuring the existence and uniqueness of FPs for a wider class of mappings. Reich [3] later developed a theorem that generalizes both Banach's and Kannan's results. Additionally, Nadler [4] extended the BCP to multivalued mappings, further expanding its scope and utility. A significant development in FP theory within ordered structures was initiated by Ran et al. [5]. In this work, the authors established



Academic Editor: Ricardo Almeida

Received: 11 May 2026

Revised: 1 July 2026

Accepted: 4 July 2026

Published: 22 July 2026

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a BCP in partially ordered MSs by incorporating the order structure into the contractive framework. Their result assumes monotonicity of the mapping and requires the contraction condition only for comparable elements, leading to existence and uniqueness of FPs. Subsequently, Nieto et al. [6] extended and refined these ideas and developed more general FP theorems in partially ordered MSs by weakening certain assumptions, particularly regarding continuity and monotonicity. Their framework proved especially effective in treating ordinary differential equations, where existence and uniqueness of solutions were obtained by transforming the problem into an ordered FP setting.

Meanwhile, Huang et al. [7] defined cone metric spaces (cone MSs) equipped with normal cones, although similar ideas had appeared earlier in the literature. Their approach relied on a partial ordering in a real Banach space induced by a cone, and they presented key notions including convergence, completeness, and continuity, ultimately establishing the BCP in this context. Building on these results, Rezapour et al. [8] showed that the theorems remain valid even when the cone is not normal, representing a significant generalization. Later, Janković et al. [9] provided a thorough review of cone MSs, helping to consolidate the theoretical foundations of the field. Further developments by Cho et al. [10] expanded the theory by introducing a generalized Hausdorff distance and establishing a FP theorem for multivalued mappings. Subsequently, Cho et al. [11] extended these results to multivalued mappings in cone MSs that satisfy Kannan-type contractive conditions. Beg et al. [12] broadened the framework of cone MSs by putting forward topological vector space (*tv*s)-valued cone MSs. Later, Azam et al. [13] further developed this concept through the development of FP theorems for mappings subject to new contractive conditions. Subsequently, Shatanawi et al. [14] reformulated the Hausdorff distance in the setting of *tv*s-valued cone MSs with the solid cones and derived FP theorems for multivalued mappings. Azam et al. [15] advanced this line of research by obtaining additional FP results for multivalued mappings in *tv*s-valued cone MSs. Alzumi et al. [16] further strengthened the concept of *tv*s-valued cone MSs by establishing new FP theorems for multivalued mappings satisfying Kikkawa and Suzuki-type contractions. Additional results related to this topic can be found in [17–21].

The concept of a suprametric space (SMS) was introduced by Berzig [22] as a generalization of the classical MS. A suprametric space is defined by relaxing the standard triangle inequality and replacing it with a weaker inequality involving an additional multiplicative term. This modification allows the distance function to exhibit non-linear behavior while still preserving essential properties such as symmetry and the identity of indiscernibles. Berzig [22] also constructed suprametrics in partially ordered vector spaces and employed them to establish some FP theorems. For more details in this direction, we refer the readers to [23,24].

The main aim of this paper is to introduce the notion of a *tv*s-valued cone SMS, which unifies and generalizes cone SMSs, *tv*s-valued cone MSs, and cone MSs. Within this unified framework, new FP theorems are established for both single-valued and multivalued generalized contractive mappings. The obtained results extend and unify several well-known FP theorems available in the existing literature. To demonstrate the applicability and significance of the proposed theory, a number of illustrative examples are presented. Furthermore, the developed FP results are applied to investigate the existence of solutions for a Caputo fractional differential equation and a nonlinear Volterra–Fredholm integral equation.

The remainder of this paper is organized as follows. Section 2 presents the fundamental definitions, preliminary concepts, and auxiliary results that are required throughout the paper. Section 3 contains the main results and is divided into two subsections. In Section 3.1, new FP theorems are established for single-valued self-mappings in *tv*s-valued

cone SMSs, whereas Section 3.2 is devoted to FP results for multivalued mappings in the same framework. Section 4 presents some consequences of the main theorems by deriving corresponding FP results in cone SMSs, *tv*s-valued cone MSs, cone MSs, and SMSs. Section 5 is devoted to applications of the developed theory. Specifically, Section 5.1 establishes the existence of solutions for a Caputo fractional differential equation, while Section 5.2 investigates the existence of solutions for a nonlinear Volterra–Fredholm integral equation by employing the main FP results.

2. Preliminaries

In the current section, we recall the basic definitions, concepts, and results that will be used across the research article. For the sake of clarity, we denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{R}$ the set of all real numbers, and by $\mathbb{R}^+$ the set of positive real numbers.

Definition 1 ([12]). Let $(\mathbb{E}, \tau)$ always be a *tv*s with zero element θ and $P \subset \mathbb{E}$ be a subset. Then P is called a cone if the following conditions hold:

- (i) P is a nonempty, closed and $P \neq \{\theta\}$;
- (ii) for all $\varsigma, \kappa \in P$ and for every pair of nonnegative real numbers a, b , the linear combination $a\varsigma + b\kappa$ also belongs to P ;
- (iii) $P \cap (-P) = \{\theta\}$.

Definition 2 ([12]). Let $(\mathbb{E}, \tau)$ always be a *tv*s with zero element θ and let $P \subset \mathbb{E}$ be a cone. The cone P induces a partial order $\preceq$ on $\mathbb{E}$ defined by

$$\varsigma \preceq \kappa \text{ if and only if } \kappa - \varsigma \in P.$$

In this case, the pair $(\mathbb{E}, P)$ is called an ordered *tv*s. Furthermore, we write $\varsigma \prec \kappa$, whenever

$$\varsigma \preceq \kappa \text{ and } \varsigma \neq \kappa$$

and $\varsigma \ll \kappa$ whenever

$$\kappa - \varsigma \in \text{int}P,$$

where $\text{int}P$ denotes the interior of P . The cone P is called normal provided that the space $\mathbb{E}$ admits a neighborhood base at θ consisting entirely of order-convex sets. Finally, the cone P is called solid if its interior is nonempty.

Definition 3 ([12]). Let $\Phi \neq \emptyset$, and consider $(\mathbb{E}, P)$ as an ordered *tv*s. Assume that the function $d : \Phi \times \Phi \rightarrow \mathbb{E}$ satisfies the following conditions:

- (*tv*s₁) $\theta \preceq d(\varsigma, \kappa)$ and $d(\varsigma, \kappa) = 0 \iff \varsigma = \kappa$,
- (*tv*s₂) $d(\varsigma, \kappa) = d(\kappa, \varsigma)$,
- (*tv*s₃) $d(\varsigma, \kappa) \preceq d(\varsigma, z) + d(z, \kappa)$,

for all $\varsigma, \kappa, z \in \Phi$, then (Φ, d) is called a *tv*s-valued cone MS.

Remark 1 ([12]). The concept of a cone MS can be viewed as a particular instance of a *tv*s-valued cone MS when the underlying *tv*s $\mathbb{E}$ is taken to be a real Banach space.

The concept of a SMS was introduced by Berzig in [22].

Definition 4 ([22]). Let $\Phi \neq \emptyset$, $\wp \geq 0$ and $d : \Phi \times \Phi \rightarrow \mathbb{R}^+ \cup \{0\}$. If the function d fulfills the following properties:

- (*sms*₁) $0 \leq d(\varsigma, \kappa)$ and $d(\varsigma, \kappa) = 0 \iff \varsigma = \kappa$,

$$(sms_2) \quad d(\zeta, \kappa) = d(\kappa, \zeta),$$

$$(sms_3) \quad d(\zeta, \kappa) \leq d(\zeta, z) + d(z, \kappa) + \wp d(\zeta, z)d(z, \kappa)$$

$\forall \zeta, \kappa, z \in \Phi$. Then, the pair (Φ, d) is called a SMS.

In this way, we define the notion of tvs-valued cone SMS.

Definition 5. Let $\Phi \neq \emptyset$, and let $(\mathbb{E}, \tau)$ be a tvs with a solid cone $P \subset \mathbb{E}$. Assume further that $\mathbb{E}$ is equipped with a continuous bilinear multiplication

$$\cdot : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{E}$$

making it a topological algebra, and that the cone satisfies $P \cdot P \subseteq P$. Assume that the function $d : \Phi \times \Phi \rightarrow \mathbb{E}$ satisfies the following conditions:

$$(csms_1) \quad \theta \preceq d(\zeta, \kappa) \text{ and } d(\zeta, \kappa) = 0 \iff \zeta = \kappa,$$

$$(csms_2) \quad d(\zeta, \kappa) = d(\kappa, \zeta),$$

$$(csms_3) \quad d(\zeta, \kappa) \preceq d(\zeta, z) + d(z, \kappa) + \wp \cdot [d(\zeta, z) \cdot d(z, \kappa)]$$

for all $\zeta, \kappa, z \in \Phi$, where $\wp \in P$, then (Φ, d) is called a tvs-valued cone SMS.

Definition 6. Let (Φ, d) be a tvs-valued cone SMS over an ordered tvs $(\mathbb{E}, P)$. For any $\zeta \in \Phi$ and $\theta \ll c \in \mathbb{E}$, we define the open ball

$$B(\zeta, c) = \{\kappa \in \Phi : d(\zeta, \kappa) \ll c\}.$$

The family

$$\tau = \{U \subseteq \Phi : \forall \zeta \in U, \exists \theta \ll c \in \mathbb{E} \text{ such that } B(\zeta, c) \subseteq U\}$$

defines a topology on Φ , called the topology induced by the tvs-valued cone SMS.

- If $\wp = \theta$, condition $(csms_3)$ reduces to the standard triangle inequality

$$d(\zeta, \kappa) \preceq d(\zeta, z) + d(z, \kappa),$$

which is precisely the definition of a tvs-valued cone MS. Therefore, every tvs-valued cone MS is a tvs-valued cone SMS, but the converse is not true in general.

- When $\mathbb{E} = \mathbb{R}$, $P = [0, +\infty)$, and $\wp \in [0, +\infty)$ the definition reduces to the classical (real-valued) SMS introduced by Berzig [22].
- The algebraic structure of $\mathbb{E}$ is essential: without a well-defined multiplication on $\mathbb{E}$, the term $d(\zeta, z) \cdot d(z, \kappa)$ would not be meaningful. The condition $P \cdot P \subseteq P$ ensures that products of cone elements remain in the cone, preserving the order structure.

Example 1. Let $\Phi = \{a, b, c\}$, $\mathbb{E} = \mathbb{R}^2$, equipped with the usual vector addition and scalar multiplication. Consider $\mathbb{E}$ as a topological algebra with the continuous bilinear multiplication defined componentwise by

$$(\zeta_1, \kappa_1) \cdot (\zeta_2, \kappa_2) = (\zeta_1 \zeta_2, \kappa_1 \kappa_2).$$

Let

$$P = \mathbb{R}_+^2 = \{(\zeta, \kappa) : \zeta, \kappa \geq 0\},$$

which is the standard cone in $\mathbb{E}$. Clearly, $P \cdot P \subseteq P$. Choose

$$\wp = (1, 1) \in P.$$

Define $d : \Phi \times \Phi \rightarrow \mathbb{E}$ by

$$d(a, a) = d(b, b) = d(c, c) = (0, 0),$$

$$d(a, b) = d(b, a) = \left(\frac{1}{2}, \frac{1}{2}\right),$$

$$d(a, c) = d(c, a) = (1, 1),$$

$$d(b, c) = d(c, b) = (2, 2).$$

Clearly,

$$(0, 0) \preceq d(\zeta, \kappa),$$

and

$$d(\zeta, \kappa) = (0, 0) \iff \zeta = \kappa.$$

Hence (csms_1) holds. Also from the definition, we have

$$d(\zeta, \kappa) = d(\kappa, \zeta),$$

so the condition (csms_2) is satisfied directly. Moreover, for the condition (csms_3) , the only nontrivial case is

$$\zeta = b, \quad \kappa = c, \quad z = a.$$

Then

$$d(b, c) = (2, 2),$$

while

$$d(b, a) + d(a, c) + \wp d(b, a) \cdot d(a, c) = \left(\frac{1}{2}, \frac{1}{2}\right) + (1, 1) + (1, 1) \cdot \left[\left(\frac{1}{2}, \frac{1}{2}\right) \cdot (1, 1)\right].$$

Since the algebra multiplication on $\mathbb{R}^2$ is componentwise,

$$\left(\frac{1}{2}, \frac{1}{2}\right) \cdot (1, 1) = \left(\frac{1}{2}, \frac{1}{2}\right),$$

and thus

$$(1, 1) \cdot \left[\left(\frac{1}{2}, \frac{1}{2}\right) \cdot (1, 1)\right] = (1, 1) \cdot \left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Therefore

$$\begin{aligned} d(b, a) + d(a, c) + \wp \cdot [d(b, a) \cdot d(a, c)] &= \left(\frac{1}{2}, \frac{1}{2}\right) + (1, 1) + \left(\frac{1}{2}, \frac{1}{2}\right) \\ &= (2, 2). \end{aligned}$$

Hence

$$d(b, c) \preceq d(b, a) + d(a, c) + \wp \cdot [d(b, a) \cdot d(a, c)].$$

All remaining cases are immediate. Therefore, (Φ, d) is a tvs-valued cone SMS. On the other hand, in a tvs-valued cone MS, one requires

$$d(\zeta, \kappa) \preceq d(\zeta, z) + d(z, \kappa).$$

However,

$$d(b, c) = (2, 2),$$

whereas

$$\begin{aligned} d(b,a) + d(a,c) &= \left(\frac{1}{2}, \frac{1}{2}\right) + (1,1) \\ &= \left(\frac{3}{2}, \frac{3}{2}\right). \end{aligned}$$

Since

$$(2,2) \not\leq \left(\frac{3}{2}, \frac{3}{2}\right),$$

which shows that the triangle inequality of tvs-valued cone MS fails. Hence, (Φ, d) is not a tvs-valued cone MS.

Example 2. Let

$$\Phi = C([0,1]), \quad \mathbb{E} = C([0,1]),$$

where $\mathbb{E}$ is endowed with the usual operations of function addition and scalar multiplication. Consider $\mathbb{E}$ as a topological algebra with the continuous bilinear multiplication defined pointwise by

$$(f \cdot g)(t) = f(t)g(t), \quad t \in [0,1].$$

Let

$$P = \{f \in \mathbb{E} : f(t) \geq 0, \forall t \in [0,1]\}.$$

Clearly, $P \cdot P \subseteq P$. Let $\wp(t) = c \geq 0$, for all $t \in [0,1]$, so $\wp \in P$. Define $d : \Phi \times \Phi \rightarrow \mathbb{E}$ by

$$d(f,g)(t) = |f(t) - g(t)|e^t, \quad t \in [0,1].$$

It is immediate that conditions $(csms_1)$ and $(csms_2)$ hold. For any $f, g, h \in \Phi$ and $t \in [0,1]$, we obtain

$$\begin{aligned} d(f,g)(t) &= |f(t) - g(t)|e^t \\ &\leq |f(t) - h(t)|e^t + |h(t) - g(t)|e^t \\ &\quad + c|f(t) - h(t)||h(t) - g(t)|e^{2t} \\ &= d(f,h)(t) + d(h,g)(t) + \wp \cdot [d(f,h)(t) \cdot d(h,g)(t)], \end{aligned}$$

where the last term uses the pointwise multiplication in the algebra $\mathbb{E}$. Hence

$$d(f,g) \preceq d(f,h) + d(h,g) + \wp \cdot [d(f,h) \cdot d(h,g)],$$

which verifies condition $(csms_3)$. Therefore, (Φ, d) is a tvs-valued cone SMS.

Definition 7. Let (Φ, d) be a tvs-valued cone SMS, $\varsigma \in \Phi$ and $\{\varsigma_n\}_{n \geq 1}$ be a sequence in Φ . Then

- (i) $\{\varsigma_n\}_{n \geq 1}$ converges to ς whenever for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number N such that

$$d(\varsigma_n, \varsigma) \ll c$$

for all $n \geq N$. We denote this by

$$\lim_{n \rightarrow \infty} \varsigma_n = \varsigma \quad \text{or} \quad \varsigma_n \rightarrow \varsigma.$$

- (ii) $\{\varsigma_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in \mathbb{E}$ with $\theta \ll c$ there is a natural number N such that

$$d(\varsigma_n, \varsigma_m) \ll c$$

for all $n, m \geq N$.

(iii) (Φ, d) is a complete tvs-valued cone SMS if every Cauchy sequence is convergent.

Motivated by [21], we state and prove the following lemma, which is fundamental for proving the main results.

Lemma 1. Consider (Φ, d) as a tvs-valued cone SMS associated with a solid cone P within ordered tvs $\mathbb{E}$. The subsequent properties will be frequently used directly or indirectly in the sequel.

- (a) For $u, v, w \in \mathbb{E}$, if $u \ll v$ and $v \preceq w$, then it follows that $u \ll w$.
- (b) Whenever $u \preceq v$ and $v \ll w$, then $u \ll w$, for all $u, v, w \in \mathbb{E}$.
- (c) For $u, v, w \in \mathbb{E}$, the relations $u \ll v$ and $v \ll w$ imply that $u \ll w$.
- (d) Suppose that $\theta \preceq u$ and $u \ll v$ for all $v \in \text{Int}P$. Then it follows that $u = \theta$.
- (e) If $u \in P$ and $u \preceq \lambda u$ for some $\lambda \in [0, 1)$, then necessarily $u = \theta$.
- (f) If $\{u_n\} \subset \mathbb{E}$ and $u_n \rightarrow \theta$ as $n \rightarrow \infty$ and if $v \in \text{int}P$, then there exists $n_0 \in \mathbb{N}$ such that

$$u_n \ll v$$

for all $n > n_0$.

Proof. (a) Since $u \ll v$, we have

$$v - u \in \text{int}P.$$

Since $v \preceq w$, then

$$w - v \in P.$$

Now

$$w - u = (w - v) + (v - u).$$

Because $P + \text{int}P \subseteq \text{int}P$, we obtain $w - u \in \text{int}P$. Hence $u \ll w$.

(b) Since $u \preceq v$, then

$$v - u \in P.$$

Since $v \ll w$, so

$$w - v \in \text{int}P.$$

Now

$$w - u = (w - v) + (v - u).$$

Again using $P + \text{int}P \subseteq \text{int}P$, we get that $w - u \in \text{int}P$. Hence $u \ll w$.

(c) Suppose that $u \ll v$ and $v \ll w$, then

$$v - u \in \text{int}P \text{ and } w - v \in \text{int}P.$$

Now since

$$w - u = (w - v) + (v - u),$$

and $\text{int}P + \text{int}P \subseteq \text{int}P$, so we have $w - u \in \text{int}P$. Hence $u \ll w$.

(d) Suppose $u \neq \theta$. Then $u \in P \setminus \{\theta\}$. Since P is solid, $\text{int}P \neq \emptyset$, choose $w \in \text{int}P$. For sufficiently small $t > 0$, let

$$v = tw \in \text{int}P.$$

Since $u \ll v$, we have

$$v - u \in \text{int}P.$$

On the other hand, for sufficiently small $t > 0$, this is impossible unless $u = \theta$. Hence we conclude that

$$u = \theta.$$

(e) From $u \preceq \lambda u$, we get that $\lambda u - u \in P$. Hence

$$-(1 - \lambda)u = (\lambda - 1)u = \lambda u - u \in P. \quad (1)$$

Since $u \in P$ and P is a cone, we have

$$(1 - \lambda)u \in P, \quad (2)$$

for $(1 - \lambda) > 0$. By (1) and (2), we have

$$(1 - \lambda)u \in P \text{ and } -(1 - \lambda)u \in P.$$

It means

$$(1 - \lambda)u \in P \text{ and } (1 - \lambda)u \in -P.$$

Hence

$$(1 - \lambda)u \in P \cap (-P).$$

But P is a cone, so

$$P \cap (-P) = \{\theta\}.$$

Hence $(1 - \lambda)u = \theta$. Since $(1 - \lambda) > 0$, it follows that $u = \theta$.

(f) Since $v \in \text{int}P$, so the set $v - \text{int}P$ is an open neighborhood of θ . Because $\{u_n\} \rightarrow \theta$ as $n \rightarrow \infty$. So there exists $n_0 \in \mathbb{N}$ such that

$$u_n \in v - \text{int}P, \text{ for all } n > n_0.$$

Thus

$$v - u_n \in \text{int}P.$$

Hence $u_n \ll v$, for all $n > n_0$. $\square$

3. Main Results

This section is devoted to establishing the main FP theorems in the framework of *tv*s-valued cone SMSs. The obtained results provide sufficient conditions for the existence and uniqueness of FPs for both single-valued and multivalued generalized contractive mappings, thereby extending and unifying some well-known results in the existing literature. Throughout this section, we assume that $\mathbb{E}$ is a topological algebra with a continuous bilinear multiplication $\cdot : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{E}$ and a multiplicative identity $1_{\mathbb{E}}$. The cone $P \subset \mathbb{E}$ is assumed to be solid and satisfies $P \cdot P \subseteq P$ and (Φ, d) denotes a complete *tv*s-valued cone SMS with a solid cone P .

3.1. Fixed Point Theorems for Self-Mappings

In this subsection, we establish some FP theorems for self-mappings in the context of *tv*s-valued cone SMSs. The contractive condition employed in this study extends and unifies classical contraction principles, including the Banach contraction and the Kannan contraction, thereby providing a broader setting for the analysis of nonlinear operators.

Theorem 1. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exist the constants $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ with $\sigma_1 + \sigma_2 + \sigma_3 < 1$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma_1 d(\zeta, \kappa) + \sigma_2 d(\zeta, \mathcal{Q}\zeta) + \sigma_3 d(\kappa, \mathcal{Q}\kappa), \quad (3)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

Proof. Let $\varsigma_0 \in \Phi$ be an arbitrary element. Define a sequence $\{\varsigma_n\}$ such that

$$\varsigma_{n+1} = \mathcal{Q}\varsigma_n$$

for all $n \in \mathbb{N}$. If there exists a natural number n^* such that $\varsigma_{n^*} = \varsigma_{n^*+1}$, then ς_{n^*} is a FP of the mapping $\mathcal{Q}$. So we assume that $\varsigma_n \neq \varsigma_{n+1}$, for all $n \in \mathbb{N}$. Now by (3), we have

$$\begin{aligned} d(\varsigma_n, \varsigma_{n+1}) &= d(\mathcal{Q}\varsigma_{n-1}, \mathcal{Q}\varsigma_n) \preceq \sigma_1 d(\varsigma_{n-1}, \varsigma_n) + \sigma_2 d(\varsigma_{n-1}, \mathcal{Q}\varsigma_{n-1}) + \sigma_3 d(\varsigma_n, \mathcal{Q}\varsigma_n) \\ &= \sigma_1 d(\varsigma_{n-1}, \varsigma_n) + \sigma_2 d(\varsigma_{n-1}, \varsigma_n) + \sigma_3 d(\varsigma_n, \varsigma_{n+1}), \end{aligned}$$

which further yields

$$d(\varsigma_n, \varsigma_{n+1}) \preceq \left(\frac{\sigma_1 + \sigma_2}{1 - \sigma_3} \right) d(\varsigma_{n-1}, \varsigma_n) = \lambda d(\varsigma_{n-1}, \varsigma_n), \quad (4)$$

for all $n \in \mathbb{N}$. Similarly by (3), we have

$$\begin{aligned} d(\varsigma_{n-1}, \varsigma_n) &= d(\mathcal{Q}\varsigma_{n-2}, \mathcal{Q}\varsigma_{n-1}) \preceq \sigma_1 d(\varsigma_{n-2}, \varsigma_{n-1}) + \sigma_2 d(\varsigma_{n-2}, \mathcal{Q}\varsigma_{n-2}) + \sigma_3 d(\varsigma_{n-1}, \mathcal{Q}\varsigma_{n-1}) \\ &= \sigma_1 d(\varsigma_{n-2}, \varsigma_{n-1}) + \sigma_2 d(\varsigma_{n-2}, \varsigma_{n-1}) + \sigma_3 d(\varsigma_{n-1}, \varsigma_n), \end{aligned}$$

which implies

$$d(\varsigma_{n-1}, \varsigma_n) \preceq \left(\frac{\sigma_1 + \sigma_2}{1 - \sigma_3} \right) d(\varsigma_{n-2}, \varsigma_{n-1}) = \lambda d(\varsigma_{n-2}, \varsigma_{n-1}), \quad (5)$$

for all $n \in \mathbb{N}$. Then by (4) and (5), we have

$$d(\varsigma_n, \varsigma_{n+1}) \preceq \lambda d(\varsigma_{n-1}, \varsigma_n) \preceq \lambda^2 d(\varsigma_{n-2}, \varsigma_{n-1}),$$

for all $n \in \mathbb{N}$. Continuing in this way, we get

$$d(\varsigma_n, \varsigma_{n+1}) \preceq \lambda d(\varsigma_{n-1}, \varsigma_n) \preceq \cdots \preceq \lambda^n d(\varsigma_0, \varsigma_1), \quad (6)$$

for all $n \in \mathbb{N}$. For $m > n$, using (csms₃) repeatedly, we obtain

$$\begin{aligned} d(\varsigma_n, \varsigma_m) &\preceq d(\varsigma_n, \varsigma_{n+1}) + d(\varsigma_{n+1}, \varsigma_m) + \wp \cdot [d(\varsigma_n, \varsigma_{n+1}) \cdot d(\varsigma_{n+1}, \varsigma_m)] \\ &= d(\varsigma_n, \varsigma_{n+1}) + (1_{\mathbb{E}} + \wp \cdot d(\varsigma_n, \varsigma_{n+1})) \cdot d(\varsigma_{n+1}, \varsigma_m). \end{aligned} \quad (7)$$

Applying the triangle inequality (csms₃) once again to $d(\varsigma_{n+1}, \varsigma_m)$, we get

$$\begin{aligned} d(\varsigma_{n+1}, \varsigma_m) &\preceq d(\varsigma_{n+1}, \varsigma_{n+2}) + d(\varsigma_{n+2}, \varsigma_m) + \wp \cdot [d(\varsigma_{n+1}, \varsigma_{n+2}) \cdot d(\varsigma_{n+2}, \varsigma_m)] \\ &= d(\varsigma_{n+1}, \varsigma_{n+2}) + (1_{\mathbb{E}} + \wp \cdot d(\varsigma_{n+1}, \varsigma_{n+2})) \cdot d(\varsigma_{n+2}, \varsigma_m). \end{aligned}$$

Proceeding similarly for the subsequent terms, we have

$$\begin{aligned} d(\varsigma_{n+2}, \varsigma_m) &\preceq d(\varsigma_{n+2}, \varsigma_{n+3}) + d(\varsigma_{n+3}, \varsigma_m) + \wp \cdot [d(\varsigma_{n+2}, \varsigma_{n+3}) \cdot d(\varsigma_{n+3}, \varsigma_m)] \\ &= d(\varsigma_{n+2}, \varsigma_{n+3}) + (1_{\mathbb{E}} + \wp \cdot d(\varsigma_{n+2}, \varsigma_{n+3})) \cdot d(\varsigma_{n+3}, \varsigma_m). \end{aligned}$$

Continuing this recursive process, we obtain

$$\begin{aligned} d(\varsigma_{m-2}, \varsigma_m) &\preceq d(\varsigma_{m-2}, \varsigma_{m-1}) + d(\varsigma_{m-1}, \varsigma_m) + \wp \cdot [d(\varsigma_{m-2}, \varsigma_{m-1}) \cdot d(\varsigma_{m-1}, \varsigma_m)] \\ &= d(\varsigma_{m-2}, \varsigma_{m-1}) + (1_{\mathbb{E}} + \wp \cdot d(\varsigma_{m-2}, \varsigma_{m-1})) \cdot d(\varsigma_{m-1}, \varsigma_m). \end{aligned}$$

By successively substituting each inequality into its predecessor in (7) and simplifying, we derive

$$d(\zeta_n, \zeta_m) \preceq \sum_{k=n}^{m-1} d(\zeta_k, \zeta_{k+1}) \cdot \prod_{j=n}^{k-1} (1_{\mathbb{E}} + \wp \cdot d(\zeta_j, \zeta_{j+1})),$$

where all products are taken in the topological algebra $\mathbb{E}$. Using inequality (6), it follows that

$$d(\zeta_n, \zeta_m) \preceq \sum_{k=n}^{m-1} \lambda^k d(\zeta_0, \zeta_1) \cdot \prod_{j=n}^{k-1} (1_{\mathbb{E}} + \wp \cdot \lambda^j d(\zeta_0, \zeta_1)). \quad (8)$$

Define

$$u_j = \wp \cdot \lambda^j d(\zeta_0, \zeta_1) \in P.$$

Since $\lambda \in [0, 1)$ and $d(\zeta_0, \zeta_1) \in P$, the sequence u_j converges to θ in $\mathbb{E}$. Moreover, the series $\sum_{j=0}^{\infty} u_j$ converges in the order-topological sense because

$$\sum_{j=0}^{\infty} \lambda^j$$

converges in $\mathbb{R}$ and $\mathbb{E}$ is a topological algebra with continuous multiplication, ensuring that products of convergent sequences converge. Therefore, the infinite product

$$\prod_{j=n}^{\infty} (1_{\mathbb{E}} + u_j)$$

converges in $\mathbb{E}$ to some element. Consequently, the partial products

$$\prod_{j=n}^{k-1} (1_{\mathbb{E}} + u_j)$$

are uniformly bounded in $\mathbb{E}$. Moreover,

$$\prod_{j=n}^{k-1} (1_{\mathbb{E}} + u_j) \rightarrow 1_{\mathbb{E}} \text{ as } n \rightarrow \infty,$$

uniformly in $k \geq n$. Hence, there exists $M \in \mathbb{E}$, with $\theta \prec M$, such that

$$\prod_{j=n}^{k-1} (1 + u_j) \preceq M, \quad \forall k \geq n. \quad (9)$$

Using (9) in (8), we obtain

$$d(\zeta_n, \zeta_m) \preceq M \cdot d(\zeta_0, \zeta_1) \cdot \sum_{k=n}^{m-1} \lambda^k.$$

Since

$$\sum_{k=n}^{m-1} \lambda^k \leq \frac{\lambda^n}{1 - \lambda},$$

we conclude

$$d(\zeta_n, \zeta_m) \preceq \frac{M}{1 - \lambda} \cdot \lambda^n d(\zeta_0, \zeta_1). \quad (10)$$

Let $c \in \mathbb{E}$ with $c \gg \theta$. Since $\lambda^n \rightarrow 0$, from (10) there exists $N \in \mathbb{N}$ such that

$$\frac{M}{1-\lambda} \cdot \lambda^n d(\zeta_0, \zeta_1) \ll c, \text{ for all } n \geq N.$$

Thus, for all $m > n \geq N$,

$$d(\zeta_n, \zeta_m) \ll c.$$

This shows that $\{\zeta_n\}$ is a Cauchy sequence in the *tvs*-valued cone SMS (Φ, d) . The completeness of Φ implies the existence of ζ^* such that $\zeta_n \rightarrow \zeta^* \in \Phi$ as $n \rightarrow \infty$.

$$\lim_{n \rightarrow \infty} \zeta_n = \zeta^*.$$

Thus,

$$\begin{aligned} d(\zeta^*, Q_{\zeta^*}) &\preceq d(\zeta^*, \zeta_{n+1}) + d(\zeta_{n+1}, Q_{\zeta^*}) + \wp \cdot [d(\zeta^*, \zeta_{n+1}) \cdot d(\zeta_{n+1}, Q_{\zeta^*})] \\ &= d(\zeta^*, \zeta_{n+1}) + d(Q_{\zeta_n}, Q_{\zeta^*}) + \wp \cdot [d(\zeta^*, \zeta_{n+1}) \cdot d(Q_{\zeta_n}, Q_{\zeta^*})] \\ &\preceq d(\zeta^*, \zeta_{n+1}) + \sigma_1 d(\zeta_n, \zeta^*) + \sigma_2 d(\zeta_n, Q_{\zeta_n}) + \sigma_3 d(\zeta^*, Q_{\zeta^*}) \\ &\quad + \wp \cdot [d(\zeta^*, \zeta_{n+1}) \cdot \{\sigma_1 d(\zeta_n, \zeta^*) + \sigma_2 d(\zeta_n, Q_{\zeta_n}) + \sigma_3 d(\zeta^*, Q_{\zeta^*})\}] \\ &= d(\zeta^*, \zeta_{n+1}) + \sigma_1 d(\zeta_n, \zeta^*) + \sigma_2 d(\zeta_n, \zeta_{n+1}) + \sigma_3 d(\zeta^*, Q_{\zeta^*}) \\ &\quad + \wp \cdot [d(\zeta^*, \zeta_{n+1}) \cdot \{\sigma_1 d(\zeta_n, \zeta^*) + \sigma_2 d(\zeta_n, \zeta_{n+1}) + \sigma_3 d(\zeta^*, Q_{\zeta^*})\}]. \end{aligned}$$

Since $\zeta_n \rightarrow \zeta^*$ in (Φ, d) , it follows that

$$d(\zeta^*, \zeta_{n+1}) \rightarrow \theta, \quad d(\zeta_n, \zeta^*) \rightarrow \theta, \quad d(\zeta_n, \zeta_{n+1}) \rightarrow \theta$$

as $n \rightarrow \infty$. Let $c \in \mathbb{E}$ with $c \gg \theta$. Since P is a solid cone, convergence is characterized in the order-topological sense. Hence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(\zeta^*, \zeta_{n+1}) \ll c, \quad d(\zeta_n, \zeta^*) \ll c, \quad d(\zeta_n, \zeta_{n+1}) \ll c.$$

Using the continuity of vector addition and multiplication in the ordered *tvs* $\mathbb{E}$, and passing to the limit as $n \rightarrow \infty$ in the above inequality, we obtain

$$d(\zeta^*, Q_{\zeta^*}) \preceq \sigma_3 d(\zeta^*, Q_{\zeta^*}).$$

Since $0 \leq \sigma_3 < 1$, we have

$$(1_{\mathbb{E}} - \sigma_3 1_{\mathbb{E}}) \cdot d(\zeta^*, Q_{\zeta^*}) \preceq \theta.$$

Since $(1_{\mathbb{E}} - \sigma_3 1_{\mathbb{E}}) \in \text{int } P$ and $d(\zeta^*, Q_{\zeta^*}) \in P$, their product belongs to P . The cone property

$$P \cap (-P) = \{\theta\}$$

implies

$$(1_{\mathbb{E}} - \sigma_3 1_{\mathbb{E}}) \cdot d(\zeta^*, Q_{\zeta^*}) = \theta.$$

Since $(1_{\mathbb{E}} - \sigma_3 1_{\mathbb{E}})$ is invertible, it follows that

$$d(\zeta^*, Q_{\zeta^*}) = \theta.$$

By condition (csm_{s_1}) , we conclude that

$$Q_{\zeta^*} = \zeta^*.$$

Therefore, ζ^* is a FP of the mapping $\mathcal{Q}$. Assume that there exists another FP ζ' of $\mathcal{Q}$, that is, $\mathcal{Q}\zeta' = \zeta'$ but $\zeta^* \neq \zeta'$. Then by (3), we have

$$\begin{aligned} d(\zeta^*, \zeta') &= d(\mathcal{Q}\zeta^*, \mathcal{Q}\zeta') \preceq \sigma_1 d(\zeta^*, \zeta') + \sigma_2 d(\zeta^*, \mathcal{Q}\zeta^*) + \sigma_3 d(\zeta', \mathcal{Q}\zeta') \\ &= \sigma_1 d(\zeta^*, \zeta'). \end{aligned}$$

Thus

$$(1_{\mathbb{E}} - \sigma_1 1_{\mathbb{E}}) \cdot d(\zeta^*, \zeta') \preceq \theta.$$

Since $(1_{\mathbb{E}} - \sigma_1 1_{\mathbb{E}}) \in \text{int } P$ and $d(\zeta^*, \zeta') \in P$, their product belongs to P . The cone property

$$P \cap (-P) = \{\theta\}$$

implies

$$(1_{\mathbb{E}} - \sigma_1 1_{\mathbb{E}}) \cdot d(\zeta^*, \zeta') = \theta.$$

Since $(1_{\mathbb{E}} - \sigma_1 1_{\mathbb{E}})$ is invertible, it follows that

$$d(\zeta^*, \zeta') = \theta.$$

Thus $\zeta^* = \zeta'$. $\square$

Corollary 1. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma d(\zeta, \kappa)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

Proof. Take $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 0$ in Theorem 1. $\square$

Corollary 2. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, \frac{1}{2})$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma(d(\zeta, \mathcal{Q}\zeta) + d(\kappa, \mathcal{Q}\kappa))$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

Proof. Take $\sigma_1 = 0$ and $\sigma_2 = \sigma_3 = \sigma$ in Theorem 1. $\square$

Example 3. Let Φ be the upper half of the unit circle in the complex plane, that is,

$$\Phi = \{e^{i\theta} : 0 \leq \theta \leq \pi\},$$

where,

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Let

$$\mathbb{E} = C[0, \pi],$$

the space of continuous real-valued functions on $[0, \pi]$, with the supremum norm

$$\|f\| = \sup_{t \in [0, \pi]} |f(t)|$$

So this is a Banach space. Define the cone

$$P = \{f \in \mathbb{E} : f(t) \geq 0, \text{ for all } t \in [0, \pi]\},$$

Then P is a solid cone with nonempty interior

$$\text{int}(P) = \{f \in \mathbb{E} : f(t) > 0, \text{ for all } t \in [0, \pi]\}.$$

Consider $\mathbb{E}$ as a topological algebra with the continuous bilinear multiplication defined pointwise by

$$(f \cdot g)(t) = f(t)g(t), \quad t \in [0, \pi].$$

Clearly, $P \cdot P \subseteq P$. Let $\wp(t) = e^{i(-t)}$, for all $t \in [0, \pi]$, then $\wp \in P$ and $\wp \neq \theta$. Define $d : \Phi \times \Phi \rightarrow \mathbb{E}$ by

$$\begin{aligned} \left(d\left(e^{i\theta_1}, e^{i\theta_2}\right)\right)(t) &= \sqrt{(\cos(\theta_1) - \cos(\theta_2))^2 + (\sin(\theta_1) - \sin(\theta_2))^2} \cdot e^t \\ &= \left|e^{i\theta_1} - e^{i\theta_2}\right| e^t \end{aligned}$$

where $0 \leq t \leq \pi$. Then (Φ, d) is a complete tvs-valued cone SMS. Define $\mathcal{Q} : \Phi \rightarrow \Phi$ by

$$\mathcal{Q}\left(e^{i\theta}\right) = e^{i\frac{\theta}{2}}$$

for $\theta \in [0, \pi]$. For any $\varsigma = e^{i\theta_1}, \kappa = e^{i\theta_2} \in \Phi$, we have

$$d(\mathcal{Q}\varsigma, \mathcal{Q}\kappa)(t) = \left|e^{i\frac{\theta_1}{2}} - e^{i\frac{\theta_2}{2}}\right| e^t.$$

Using the inequality

$$\left|e^{i\alpha} - e^{i\beta}\right| \leq |\alpha - \beta|,$$

we obtain

$$\left|e^{i\frac{\theta_1}{2}} - e^{i\frac{\theta_2}{2}}\right| \leq \frac{1}{2}|\theta_1 - \theta_2|.$$

Thus

$$\begin{aligned} d(\mathcal{Q}\varsigma, \mathcal{Q}\kappa)(t) &= \left|e^{i\frac{\theta_1}{2}} - e^{i\frac{\theta_2}{2}}\right| e^t \\ &\leq \frac{1}{2}|\theta_1 - \theta_2| e^t. \end{aligned} \tag{11}$$

Also, from

$$\left|e^{i\theta_1} - e^{i\theta_2}\right| = 2 \sin\left(\frac{|\theta_1 - \theta_2|}{2}\right)$$

and

$$\sin u \geq \frac{2u}{\pi}$$

for $u \in [0, \frac{\pi}{2}]$. So

$$\left|e^{i\theta_1} - e^{i\theta_2}\right| \geq \frac{2}{\pi}|\theta_1 - \theta_2|.$$

Hence

$$|\theta_1 - \theta_2| \leq \frac{\pi}{2} \left|e^{i\theta_1} - e^{i\theta_2}\right|.$$

Substituting this fact in inequality (11), we have

$$\begin{aligned}
d(Q_\zeta, Q_\kappa)(t) &\leq \frac{1}{2}|\theta_1 - \theta_2|e^t \\
&\leq \frac{1}{2} \cdot \frac{\pi}{2} |e^{i\theta_1} - e^{i\theta_2}|e^t \\
&= \frac{\pi}{4}d(\zeta, \kappa)(t).
\end{aligned}$$

Thus in the cone order,

$$d(Q_\zeta, Q_\kappa) \preceq \frac{\pi}{4}d(\zeta, \kappa).$$

Taking $\sigma_1 = \frac{\pi}{4}, \sigma_2 = \sigma_3 = 0$ gives $\sigma_1 + \sigma_2 + \sigma_3 = \frac{\pi}{4} < 1$. Thus all the conditions of Theorem 1 are satisfied and $\zeta^* = e^{i0} = 1$ is a unique FP of Q .

3.2. Fixed Point Results for Multivalued Mappings

In this subsection, we establish FP results for multivalued mappings in the framework of tv s-valued cone SMSs. To attain this target, we define a generalized Hausdorff distance function in tv s-valued cone SMSs and establish new FP theorems. These results broaden the scope of FP theory in tv s-valued environments and provide a foundation for further applications to nonlinear operator equations. In what follows, (Φ, d) represents a complete tv s-valued cone SMS equipped with a solid cone P in ordered locally convex space $\mathbb{E}$.

Let 2^Φ and $CB(\Phi)$ represent the collection of all non-empty subsets (respectively, closed and bounded subsets) of a subset of a tv s-valued cone SMSs. Let (Φ, d) be a tv s-valued cone SMS with a solid cone $P \subset \mathbb{E}$, where $\mathbb{E}$ is a topological algebra. Here, a subset $\Omega \subseteq \Phi$ is considered closed if, for each sequence $\{\zeta_n\} \subseteq \Omega, \forall n \in \mathbb{N}$ such that $\zeta_n \rightarrow \zeta$, the limit ζ also belongs to Ω . A subset $\Omega \subseteq \Phi$ is called bounded if there exists an element $\omega \in \mathbb{P}$ such that

$$d(\zeta, \kappa) \preceq \omega,$$

for all $\zeta, \kappa \in \Omega$. In other words, all distances between points in Ω are bounded above by a fixed element of the cone $\mathbb{P}$. Now, let us define a function $s : \mathbb{E} \rightarrow \mathbb{E}$ by

$$s(p) = \{q \in \mathbb{E} : p \preceq q\}$$

for $p \in \mathbb{E}$, and

$$s(\zeta, \Omega) = \bigcup_{\kappa \in \Omega} s(d(\zeta, \kappa)) = \bigcup_{\kappa \in \Omega} \{z \in \mathbb{E} : d(\zeta, \kappa) \preceq z\}$$

for $\zeta \in \Phi$ and $\Omega \in 2^\Phi$.

For $\Omega_1, \Omega_2 \in CB(\Phi)$, we define the generalized Hausdorff distance in this way

$$s(\Omega_1, \Omega_2) = \left(\bigcap_{\zeta \in \Omega_1} s(\zeta, \Omega_2) \right) \cap \left(\bigcap_{\kappa \in \Omega_2} s(\kappa, \Omega_1) \right).$$

Remark 2. If $\mathbb{E} = \mathbb{R}$ and $P = [0, +\infty)$, then (Φ, d) is a SMS. Moreover, for $A, B \in CB(\Phi)$, we have

$$H(A, B) = \inf s(A, B)$$

is the Hausdorff distance induced by d . Also,

$$s(\{\zeta\}, \{\kappa\}) = s(d(\zeta, \kappa))$$

for all $\zeta, \kappa \in \Phi$.

Theorem 2. Let $Q, W : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(Q_\zeta, W_\kappa) \tag{12}$$

for all $\varsigma, \kappa \in \Phi$. Then there exists $\varsigma^* \in \Phi$ such that $\varsigma^* \in \mathcal{Q}_{\varsigma^*} \cap \mathcal{W}_{\varsigma^*}$.

Proof. Let $\varsigma_0 \in \Phi$ be an arbitrary element. Since $\mathcal{Q}_{\varsigma_0} \in CB(\Phi)$, $\mathcal{Q}_{\varsigma_0}$ is non-empty, so that there exists some $\varsigma_1 \in \mathcal{Q}_{\varsigma_0}$. By (12), we have $\square$

$$\sigma d(\varsigma_0, \varsigma_1) \in s(\mathcal{Q}_{\varsigma_0}, \mathcal{W}_{\varsigma_1}).$$

By the definition of $s(\mathcal{Q}_{\varsigma_0}, \mathcal{W}_{\varsigma_1})$, we have

$$\sigma d(\varsigma_0, \varsigma_1) \in \left(\bigcap_{\varsigma \in \mathcal{Q}_{\varsigma_0}} s(\varsigma, \mathcal{W}_{\varsigma_1}) \right) \cap \left(\bigcap_{\varsigma \in \mathcal{W}_{\varsigma_1}} s(\varsigma, \mathcal{Q}_{\varsigma_0}) \right),$$

which implies that

$$\sigma d(\varsigma_0, \varsigma_1) \in \left(\bigcap_{\varsigma \in \mathcal{Q}_{\varsigma_0}} s(\varsigma, \mathcal{W}_{\varsigma_1}) \right)$$

which further yields

$$\sigma d(\varsigma_0, \varsigma_1) \in s(\varsigma, \mathcal{W}_{\varsigma_1}), \text{ for all } \varsigma \in \mathcal{Q}_{\varsigma_0}.$$

Since $\varsigma_1 \in \mathcal{Q}_{\varsigma_0}$, so we have

$$\sigma d(\varsigma_0, \varsigma_1) \in s(\varsigma_1, \mathcal{W}_{\varsigma_1}).$$

By the definition of $s(\varsigma, \mathcal{W}_{\varsigma_1})$, we have

$$\sigma d(\varsigma_0, \varsigma_1) \in s(\varsigma_1, \mathcal{W}_{\varsigma_1}) = \bigcup_{\varsigma \in \mathcal{W}_{\varsigma_1}} s(d(\varsigma_1, \varsigma)).$$

So there exists some $\varsigma_2 \in \mathcal{W}_{\varsigma_1}$ such that

$$\sigma d(\varsigma_0, \varsigma_1) \in s(d(\varsigma_1, \varsigma_2)).$$

Thus by the definition, we have

$$d(\varsigma_1, \varsigma_2) \preceq \sigma d(\varsigma_0, \varsigma_1). \quad (13)$$

Then by (12), we have

$$\sigma d(\varsigma_1, \varsigma_2) = \sigma d(\varsigma_2, \varsigma_1) \in s(\mathcal{Q}_{\varsigma_2}, \mathcal{W}_{\varsigma_1}) = s(\mathcal{W}_{\varsigma_1}, \mathcal{Q}_{\varsigma_2}).$$

By the definition of $s(\mathcal{W}_{\varsigma_1}, \mathcal{Q}_{\varsigma_2})$, we have

$$\sigma d(\varsigma_1, \varsigma_2) \in \left(\bigcap_{\varsigma \in \mathcal{W}_{\varsigma_1}} s(\varsigma, \mathcal{Q}_{\varsigma_2}) \right) \cap \left(\bigcap_{\varsigma \in \mathcal{Q}_{\varsigma_2}} s(\varsigma, \mathcal{W}_{\varsigma_1}) \right),$$

which implies that

$$\sigma d(\varsigma_1, \varsigma_2) \in \left(\bigcap_{\varsigma \in \mathcal{W}_{\varsigma_1}} s(\varsigma, \mathcal{Q}_{\varsigma_2}) \right),$$

which further yields

$$\sigma d(\varsigma_1, \varsigma_2) \in s(\varsigma, \mathcal{Q}_{\varsigma_2}), \text{ for all } \varsigma \in \mathcal{W}_{\varsigma_1}.$$

Since $\varsigma_2 \in \mathcal{W}_{\varsigma_1}$, so we have

$$\sigma d(\varsigma_1, \varsigma_2) \in s(\varsigma_2, \mathcal{Q}_{\varsigma_2}).$$

By the definition, we have

$$\sigma d(\zeta_1, \zeta_2) \in s(\zeta_2, \mathcal{Q}_{\zeta_2}) = \bigcup_{\zeta \in \mathcal{Q}_{\zeta_2}} s(d(\zeta_2, \zeta)).$$

so there exists some $\zeta_3 \in \mathcal{Q}_{\zeta_2}$ such that

$$\sigma d(\zeta_1, \zeta_2) \in s(d(\zeta_2, \zeta_3)).$$

Thus by the definition, we have

$$d(\zeta_2, \zeta_3) \preceq \sigma d(\zeta_1, \zeta_2). \quad (14)$$

Using (14) in (13), we have

$$d(\zeta_2, \zeta_3) \preceq \sigma d(\zeta_1, \zeta_2) \preceq \sigma^2 d(\zeta_0, \zeta_1).$$

Continuing this procedure we can construct a sequence $\{\zeta_n\}$ in Φ such that

$$\zeta_{2n+1} \in \mathcal{Q}_{\zeta_{2n}}, \zeta_{2n+2} \in \mathcal{W}_{\zeta_{2n+1}}$$

and

$$d(\zeta_{2n}, \zeta_{2n+1}) \preceq \sigma d(\zeta_{2n-1}, \zeta_{2n}) \quad (15)$$

$$d(\zeta_{2n+1}, \zeta_{2n+2}) \preceq \sigma d(\zeta_{2n}, \zeta_{2n+1}). \quad (16)$$

From (15) and (16), we can conclude that

$$d(\zeta_n, \zeta_{n+1}) \preceq \sigma d(\zeta_{n-1}, \zeta_n), \quad (17)$$

which further implies

$$d(\zeta_n, \zeta_{n+1}) \preceq \sigma d(\zeta_{n-1}, \zeta_n) \preceq \sigma^2 d(\zeta_{n-2}, \zeta_{n-1}) \preceq \dots \preceq \sigma^n d(\zeta_0, \zeta_1)$$

for all $n \in \mathbb{N}$. By employing arguments analogous to those used in the proof of Theorem 1 (with the topological algebra structure and the continuity of multiplication), we conclude that $\{\zeta_n\}$ is a Cauchy sequence in Φ . Since Φ is complete, there exists $\zeta^* \in \Phi$ such that $\zeta_n \rightarrow \zeta^*$. Let $c \gg \theta$. Since P is solid, choose $c_1 \gg \theta$ and $N(c_1) \in \mathbb{N}$ such that

$$d(\zeta^*, \zeta_{2n+1}) \ll c_1, \quad d(\zeta_{2n}, \zeta^*) \ll c_1 \text{ and } d(\zeta^*, \zeta_{2n-1}) \ll c_1, \quad (18)$$

and

$$c_1 + \sigma c_1 + \wp \cdot (\sigma c_1 \cdot c_1) \ll c,$$

for $n \geq N(c_1)$. From (12), we have

$$\sigma d(\zeta_{2n}, \zeta^*) \in s(\mathcal{Q}_{\zeta_{2n}}, \mathcal{W}_{\zeta^*}).$$

By the definition of $s(\mathcal{Q}_{\zeta_{2n}}, \mathcal{W}_{\zeta^*})$, we obtain

$$\sigma d(\zeta_{2n}, \zeta^*) \in \left(\bigcap_{\zeta \in \mathcal{Q}_{\zeta_{2n}}} s(\zeta, \mathcal{W}_{\zeta^*}) \right) \cap \left(\bigcap_{\zeta \in \mathcal{W}_{\zeta^*}} s(\zeta, \mathcal{Q}_{\zeta_{2n}}) \right),$$

which implies that

$$\sigma d(\varsigma_{2n}, \varsigma^*) \in \left(\bigcap_{\varsigma \in \mathcal{Q}_{\varsigma_{2n}}} s(\varsigma, \mathcal{W}_{\varsigma^*}) \right),$$

which further yields

$$\sigma d(\varsigma_{2n}, \varsigma^*) \in s(\varsigma, \mathcal{W}_{\varsigma^*}), \text{ for all } \varsigma \in \mathcal{Q}_{\varsigma_{2n}}.$$

Since $\varsigma_{2n+1} \in \mathcal{Q}_{\varsigma_{2n}}$, so we have

$$\sigma d(\varsigma_{2n}, \varsigma^*) \in s(\varsigma_{2n+1}, \mathcal{W}_{\varsigma^*}).$$

By the definition, we have

$$\sigma d(\varsigma_{2n}, \varsigma^*) \in s(\varsigma_{2n+1}, \mathcal{W}_{\varsigma^*}) = \bigcup_{\varsigma \in \mathcal{W}_{\varsigma^*}} s(d(\varsigma_{2n+1}, \varsigma)).$$

so there exists some $u_n \in \mathcal{W}_{\varsigma^*}$ such that

$$\sigma d(\varsigma_{2n}, \varsigma^*) \in s(d(\varsigma_{2n+1}, u_n)).$$

and thus

$$d(\varsigma_{2n+1}, u_n) \preceq \sigma d(\varsigma_{2n}, \varsigma^*).$$

Now, using (csms₃) with the algebra multiplication,

$$\begin{aligned} d(\varsigma^*, u_n) &\preceq d(\varsigma^*, \varsigma_{2n+1}) + d(\varsigma_{2n+1}, u_n) + \wp \cdot [d(\varsigma^*, \varsigma_{2n+1}) \cdot d(\varsigma_{2n+1}, u_n)] \\ &\preceq d(\varsigma^*, \varsigma_{2n+1}) + \sigma d(\varsigma_{2n}, \varsigma^*) + \wp \cdot [\sigma d(\varsigma^*, \varsigma_{2n+1}) \cdot d(\varsigma_{2n}, \varsigma^*)] \\ &\ll c_1 + \sigma c_1 + \wp \cdot (\sigma c_1 \cdot c_1) \ll c. \end{aligned}$$

Thus $d(\varsigma^*, u_n) \rightarrow \theta$, and since $\mathcal{W}_{\varsigma^*}$ is closed, so we conclude

$$\varsigma^* \in \mathcal{W}_{\varsigma^*}.$$

From the contractive condition (12), we have

$$\sigma d(\varsigma^*, \varsigma_{2n-1}) \in s(\mathcal{Q}_{\varsigma^*}, \mathcal{W}_{\varsigma_{2n-1}}).$$

By the definition of $s(\mathcal{Q}_{\varsigma^*}, \mathcal{W}_{\varsigma_{2n-1}})$, we obtain

$$\sigma d(\varsigma^*, \varsigma_{2n-1}) \in \left(\bigcap_{\varsigma \in \mathcal{Q}_{\varsigma^*}} s(\varsigma, \mathcal{W}_{\varsigma_{2n-1}}) \right) \cap \left(\bigcap_{\varsigma \in \mathcal{W}_{\varsigma_{2n-1}}} s(\varsigma, \mathcal{Q}_{\varsigma^*}) \right),$$

which further yields

$$\sigma d(\varsigma^*, \varsigma_{2n-1}) \in \left(\bigcap_{\varsigma \in \mathcal{W}_{\varsigma_{2n-1}}} s(\varsigma, \mathcal{Q}_{\varsigma^*}) \right)$$

that is,

$$\sigma d(\varsigma^*, \varsigma_{2n-1}) \in s(\varsigma, \mathcal{Q}_{\varsigma^*}), \text{ for all } \varsigma \in \mathcal{W}_{\varsigma_{2n-1}}.$$

Since $\varsigma_{2n} \in \mathcal{W}_{\varsigma_{2n-1}}$, so we have

$$\sigma d(\varsigma^*, \varsigma_{2n-1}) \in s(\varsigma_{2n}, \mathcal{Q}_{\varsigma^*}).$$

By the definition of $s(\zeta_{2n}, \mathcal{Q}_{\zeta^*})$, we have

$$\sigma d(\zeta^*, \zeta_{2n-1}) \in s(\zeta_{2n}, \mathcal{Q}_{\zeta^*}) = \bigcup_{\zeta \in \mathcal{Q}_{\zeta^*}} s(d(\zeta_{2n}, \zeta)).$$

so there exists some $v_n \in \mathcal{Q}_{\zeta^*}$ such that

$$\sigma d(\zeta^*, \zeta_{2n-1}) \in s(d(\zeta_{2n}, v_n)).$$

and thus

$$d(\zeta_{2n}, v_n) \preceq \sigma d(\zeta^*, \zeta_{2n-1}).$$

Now, using (csms₃) with the algebra multiplication,

$$\begin{aligned} d(\zeta^*, v_n) &\preceq d(\zeta^*, \zeta_{2n}) + d(\zeta_{2n}, v_n) + \wp \cdot [d(\zeta^*, \zeta_{2n}) \cdot d(\zeta_{2n}, v_n)] \\ &\preceq d(\zeta^*, \zeta_{2n}) + \sigma d(\zeta^*, \zeta_{2n-1}) + \wp \cdot [\sigma d(\zeta^*, \zeta_{2n}) \cdot d(\zeta^*, \zeta_{2n-1})] \\ &\ll c_1 + \sigma c_1 + \wp \cdot (\sigma c_1 \cdot c_1) \ll c. \end{aligned}$$

Thus $d(\zeta^*, v_n) \rightarrow \theta$, and since $\mathcal{Q}_{\zeta^*}$ is closed, so we conclude

$$\zeta^* \in \mathcal{Q}_{\zeta^*}.$$

Hence ζ^* is a CFP of $\mathcal{Q}$ and $\mathcal{W}$.

Example 4. Consider the interval $\Phi = [0, 1]$, and let $\mathbb{E} = C([0, 1])$, the space of all real-valued continuous functions on Φ , equipped with the supremum norm

$$\|f\|_{\infty} = \sup_{t \in [0, 1]} |f(t)|$$

So this is a Banach space. Consider $\mathbb{E}$ as a topological algebra with the continuous bilinear multiplication defined pointwise by

$$(f \cdot g)(t) = f(t)g(t), \quad t \in [0, 1].$$

Define the cone

$$P = \{f \in \mathbb{E} : f(t) \geq 0, \text{ for all } t \in [0, 1]\},$$

Then P is a solid cone with nonempty interior

$$\text{int}(P) = \{f \in \mathbb{E} : f(t) > 0, \text{ for all } t \in [0, 1]\}.$$

Clearly, $P \cdot P \subseteq P$. Let $\wp(t) = e^t$, for all $t \in [0, 1]$, so $\wp = e^t \in P$. Equipped with the usual operations of function addition and scalar multiplication, $\mathbb{E}$ becomes a vs over the field of $\mathbb{R}$. In particular, for $\zeta, \kappa \in \Phi$ and $\alpha \in \mathbb{R}$, the operations are defined by

$$(\zeta + \kappa)(t) = \zeta(t) + \kappa(t)$$

and

$$(\alpha \zeta)(t) = \alpha \zeta(t).$$

Let τ denote the finest locally convex topology on $\mathbb{E}$. Endowed with this topology, the pair (Φ, τ) becomes a tvs that is neither normable nor metrizable. Define a mapping $d : \Phi \times \Phi \rightarrow \mathbb{E}$ by

$$(d(\zeta, \kappa))(t) = |\zeta - \kappa|e^t.$$

Consequently, (Φ, d) constitutes a tvs-valued cone SMS. Now define the multivalued mappings $\mathcal{Q} : \Phi \rightarrow CB(\Phi)$ by

$$\mathcal{Q}_\zeta = \left[0, \frac{\zeta}{12}\right], \mathcal{W}_\kappa = \left[0, \frac{\kappa}{12}\right].$$

Clearly, for every $\zeta \in \Phi$, both $\mathcal{Q}_\zeta$ and $\mathcal{W}_\kappa$ are nonempty, closed and bounded subsets of Φ . Let $\zeta, \kappa \in \Phi$. Without loss of generality, assume that $\zeta \geq \kappa$. For any $u \in \mathcal{Q}_\zeta$, define

$$v = \min\left\{u, \frac{\kappa}{12}\right\} \in \mathcal{W}_\kappa.$$

Then two cases arise.

- If $u \leq \frac{\kappa}{12}$, then $v = u$ and therefore

$$|u - v| = \theta.$$

- If $u > \frac{\kappa}{12}$, then $v = \frac{\kappa}{12}$, and hence

$$|u - v| = u - \frac{\kappa}{12} \leq \frac{\zeta}{12} - \frac{\kappa}{12} = \frac{|\zeta - \kappa|}{12}.$$

Therefore, in both cases,

$$|u - v| \leq \frac{|\zeta - \kappa|}{12}.$$

Consequently,

$$d(u, v)(t) = |u - v|e^t \preceq \frac{|\zeta - \kappa|}{12}e^t = \frac{1}{12}d(\zeta, \kappa)(t).$$

Since

$$\frac{1}{12}d(\zeta, \kappa)(t) \preceq \frac{1}{2}d(\zeta, \kappa)(t),$$

it follows that

$$\frac{1}{2}d(\zeta, \kappa) \in s(\mathcal{Q}_\zeta, \mathcal{W}_\kappa).$$

The same conclusion is obtained when $\kappa \geq \zeta$. Hence,

$$\frac{1}{2}d(\zeta, \kappa) \in s(\mathcal{Q}_\zeta, \mathcal{W}_\kappa).$$

Thus the condition

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}_\zeta, \mathcal{W}_\kappa)$$

is satisfied for all $\zeta, \kappa \in \Phi$. Finally,

$$\mathcal{Q}0 \cap \mathcal{W}0 = \{0\},$$

which implies that $0 \in \mathcal{Q}0 \cap \mathcal{W}0$. Hence $\zeta^* = 0 \in \Phi$ is a common FP of the multivalued mappings $\mathcal{Q}$ and $\mathcal{W}$.

Corollary 3. Let $\mathcal{Q} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}_\zeta, \mathcal{Q}_\kappa)$$

for all $\zeta, \kappa \in \Phi$. Then there exists $\zeta^* \in \Phi$ such that $\zeta^* \in \mathcal{Q}_{\zeta^*}$.

4. Consequences

In this section, we derive several important consequences of our main theorems by specializing the underlying space $\mathbb{E}$ and the suprametric parameter $\wp$. These consequences

recover and unify various known FP results from the literature, including classical Banach contractions, Kannan contractions, and their multivalued analogues in the settings of cone MSs, SMSs, and *tv*s-valued cone MSs.

4.1. Fixed Point Analysis in Cone Suprametric Spaces

When the topological algebra $\mathbb{E}$ is taken to be a Banach algebra (such as $\mathbb{E} = \mathbb{R}$ or $\mathbb{E} = C[0, 1]$) with a solid cone P , the *tv*s-valued cone SMS reduces to a cone SMS. Throughout this subsection, (Φ, d) denotes a complete cone SMS with a solid cone P in a Banach algebra $\mathbb{E}$, and $\wp \in P$ is a fixed element. The following result is a direct consequence of Theorem 1.

Corollary 4. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exist $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ such that

$$\sigma_1 + \sigma_2 + \sigma_3 < 1$$

and

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma_1 d(\zeta, \kappa) + \sigma_2 d(\zeta, \mathcal{Q}\zeta) + \sigma_3 d(\kappa, \mathcal{Q}\kappa)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ admits a unique FP.

Corollary 5. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exist $\sigma \in [0, 1)$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma d(\zeta, \kappa)$$

for all $\zeta, \kappa \in \Phi$. Then there exists $\zeta^* \in \Phi$ such that $\zeta^* = \mathcal{Q}\zeta^*$.

Proof. Set $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 0$ in Corollary 4. $\square$

Corollary 6. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exist $\sigma \in [0, \frac{1}{2})$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma(d(\zeta, \mathcal{Q}\zeta) + d(\kappa, \mathcal{Q}\kappa))$$

for all $\zeta, \kappa \in \Phi$. Then there exists $\zeta^* \in \Phi$ such that $\zeta^* = \mathcal{Q}\zeta^*$.

Proof. Take $\sigma_1 = 0$ and $\sigma_2 = \sigma_3 = \sigma$ in Corollary 4. $\square$

The following multivalued result follows immediately from Theorem 2.

Corollary 7. Let $\mathcal{Q}, \mathcal{W} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}\zeta, \mathcal{W}\kappa)$$

for all $\zeta, \kappa \in \Phi$. Then there exists $\zeta^* \in \Phi$ such that $\zeta^* \in \mathcal{Q}\zeta^* \cap \mathcal{W}\zeta^*$.

Corollary 8. Let $\mathcal{Q} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}\zeta, \mathcal{Q}\kappa),$$

for all $\zeta, \kappa \in \Phi$. Then there exists $\zeta^* \in \Phi$ such that $\zeta^* \in \mathcal{Q}\zeta^*$.

Proof. Take $\mathcal{W} = \mathcal{Q}$ in the above Corollary. $\square$

4.2. Fixed Point Results in *tv*s-Valued Cone Metric Spaces

If we set $\wp = \theta$ (the zero element of $\mathbb{E}$) in Definition 5, the suprametric term vanishes, and the *tv*s-valued cone SMS framework simplifies to a *tv*s-valued cone MS. Assume

throughout that (Φ, d) denotes a complete *tv*s-valued cone MS associated with a solid cone P in an ordered topological algebra $\mathbb{E}$.

The following results are direct consequences of Theorem 1 with $\wp = \theta$.

Corollary 9. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exist $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma_1 d(\zeta, \kappa) + \sigma_2 d(\zeta, \mathcal{Q}\zeta) + \sigma_3 d(\kappa, \mathcal{Q}\kappa)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ admits a unique FP.

Corollary 10. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma d(\zeta, \kappa)$$

for all $\zeta, \kappa \in \Phi$. Then there exists $\zeta^* \in \Phi$ such that $\zeta^* \in \mathcal{Q}\zeta^*$.

Proof. Set $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 0$ in Corollary 9. $\square$

Corollary 11. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, \frac{1}{2})$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma(d(\zeta, \mathcal{Q}\zeta) + d(\kappa, \mathcal{Q}\kappa))$$

for all $\zeta, \kappa \in \Phi$. Then there exists $\zeta^* \in \Phi$ such that $\zeta^* = \mathcal{Q}\zeta^*$.

Proof. Set $\sigma_1 = 0$ and $\sigma_2 = \sigma_3 = \sigma$ in Corollary 9. $\square$

Restricting the notion of a *tv*s-valued cone SMS to a *tv*s-valued cone MS, we obtain the following result as a direct consequence of Theorem 2.

Corollary 12. Let $\mathcal{Q}, \mathcal{W} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}\zeta, \mathcal{W}\kappa)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ and $\mathcal{W}$ admits a CFP.

Corollary 13. Let $\mathcal{Q} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}\zeta, \mathcal{Q}\kappa),$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ admits a FP.

Proof. Take $\mathcal{W} = \mathcal{Q}$ in above corollary. $\square$

4.3. Existence of Fixed Points in Cone Metric Spaces

When the underlying *tv*s is a Banach space (not necessarily a Banach algebra) and $\wp = 0$, the *tv*s-valued cone SMS framework coincides with a cone MS introduced by Huang et al. [7]. Unless otherwise stated, (Φ, d) denotes a complete cone MS over a solid cone P in a Banach space $\mathbb{E}$. The next result follows straightforwardly from Theorem 1 with $\wp = 0$ and $\mathbb{E}$ a Banach space.

Corollary 14. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exist $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma_1 d(\zeta, \kappa) + \sigma_2 d(\zeta, \mathcal{Q}\zeta) + \sigma_3 d(\kappa, \mathcal{Q}\kappa)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

In this way, we derive the leading theorem (Theorem 2.3) of Rezapour et al. [8] from the above corollary.

Corollary 15. ([7]) Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma d(\zeta, \kappa)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

Proof. By setting $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 0$ in Corollary 14, we obtain the desired conclusion. $\square$

From Corollary 14, we immediately obtain the Kannan-type FP theorem (Theorem 2.6) of Rezapour et al. [8].

Corollary 16. ([7]) Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, \frac{1}{2})$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \preceq \sigma(d(\zeta, \mathcal{Q}\zeta) + d(\kappa, \mathcal{Q}\kappa))$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

Proof. The result follows by selecting $\sigma_1 = 0$ and setting $\sigma_2 = \sigma_3 = \sigma$ in Corollary 14. $\square$

By reducing the concept of a *tv*s-valued cone SMS to that of a cone MS, the following result is obtained as an immediate consequence of Theorem 2.

Corollary 17. Let $\mathcal{Q}, \mathcal{W} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}\zeta, \mathcal{W}\kappa)$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ and $\mathcal{W}$ admit a CFP.

The next result corresponds to Corollary 2.3 in Cho et al. [10].

Corollary 18. ([10]) Let $\mathcal{Q} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$\sigma d(\zeta, \kappa) \in s(\mathcal{Q}\zeta, \mathcal{Q}\kappa),$$

for all $\zeta, \kappa \in \Phi$. Then $\mathcal{Q}$ has a FP.

Proof. The result is obtained from the foregoing corollary upon identifying $\mathcal{Q}$ and $\mathcal{W}$. $\square$

4.4. Fixed Point Results in Suprametric Spaces

When $\mathbb{E} = \mathbb{R}$, $P = [0, \infty)$, and $\wp \in [0, \infty)$, the structure of *tv*s-valued cone SMS reduces to a SMS. In this case, the order $\preceq$ becomes the usual order $\leq$ on $\mathbb{R}$, and the cone P is the nonnegative real numbers. Within the subsection, (Φ, d) is regarded as a complete SMS.

The following results follow from Theorem 1 with $\mathbb{E} = \mathbb{R}$.

Corollary 19. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exist $\sigma_1, \sigma_2, \sigma_3 \in [0, 1)$ such that

$$d(\mathcal{Q}\zeta, \mathcal{Q}\kappa) \leq \sigma_1 d(\zeta, \kappa) + \sigma_2 d(\zeta, \mathcal{Q}\zeta) + \sigma_3 d(\kappa, \mathcal{Q}\kappa)$$

for all $\varsigma, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

We now derive Berzig's principal result [22] directly from the above corollary.

Corollary 20. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$d(\mathcal{Q}\varsigma, \mathcal{Q}\kappa) \leq \sigma d(\varsigma, \kappa)$$

for all $\varsigma, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

Corollary 21. Let $\mathcal{Q} : \Phi \rightarrow \Phi$. Suppose that there exists $\sigma \in [0, \frac{1}{2})$ such that

$$d(\mathcal{Q}\varsigma, \mathcal{Q}\kappa) \leq \sigma(d(\varsigma, \mathcal{Q}\varsigma) + d(\kappa, \mathcal{Q}\kappa))$$

for all $\varsigma, \kappa \in \Phi$. Then $\mathcal{Q}$ has a unique FP.

From Remark 2 and Theorem 2, the following result immediately follows.

Corollary 22. Let $\mathcal{Q}, \mathcal{W} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$H(\mathcal{Q}\varsigma, \mathcal{W}\kappa) \leq \sigma d(\varsigma, \kappa)$$

for all $\varsigma, \kappa \in \Phi$. Then there exists $\varsigma^* \in \Phi$ such that $\varsigma^* \in \mathcal{Q}\varsigma^* \cap \mathcal{W}\varsigma^*$.

Corollary 23. Let $\mathcal{Q} : \Phi \rightarrow CB(\Phi)$. Suppose that there exists $\sigma \in [0, 1)$ such that

$$H(\mathcal{Q}\varsigma, \mathcal{Q}\kappa) \leq \sigma d(\varsigma, \kappa),$$

for all $\varsigma, \kappa \in \Phi$. Then there exists $\varsigma^* \in \Phi$ such that $\varsigma^* \in \mathcal{Q}\varsigma^*$.

5. Applications

This section illustrates the applicability of the developed FP framework in *tv*s-valued cone SMSs to concrete mathematical models arising in applied analysis. By reformulating differential and integral equations as operator equations in suitable function spaces, the abstract results obtained earlier are employed to guarantee the existence and uniqueness of solutions.

5.1. Solvability of Caputo Fractional Differential Equations

Fractional differential equations have attracted considerable attention in recent years due to their ability to model memory and hereditary effects arising in various branches of science and engineering, including viscoelasticity, control theory, fluid dynamics, and biological systems. Among the several definitions of fractional derivatives, the Caputo fractional derivative is particularly significant because it allows the incorporation of classical initial conditions, making it more suitable for practical applications.

The existence and uniqueness of solutions to Caputo fractional differential equations constitute a fundamental problem in fractional calculus. FP theory provides an effective and powerful framework for addressing this problem, especially when classical analytical techniques are not applicable. By reformulating a Caputo fractional differential equation into an equivalent integral equation, one can study the associated operator in an appropriate function space and apply suitable FP theorems to guarantee the solvability of the problem.

In this subsection, we employ the established FP results in the setting of a *tv*s-valued cone SMS to investigate the existence of solutions for a Caputo fractional differential equation. The proposed approach not only ensures the existence of solutions under generalized

contractive conditions but also extends several earlier results obtained in standard metric and cone metric frameworks.

Consider the Caputo fractional differential equation of order $\alpha \in (0, 1)$

$${}^C D_a^\alpha \zeta(t) = f(t, \zeta(t)), \quad t \in [a, b] \quad (19)$$

subject to the initial condition

$$\zeta(a) = \zeta_0,$$

where $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function. It is well known that this problem is equivalent to the fractional integral equation

$$\zeta(t) = \zeta_0 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, \zeta(s)) ds, \quad t \in [a, b].$$

Let $\Phi = C([a, b], \mathbb{R}^n)$ be the space of continuous functions from $[a, b]$ to $\mathbb{R}^n$. Let $\mathbb{E} = C[a, b]$ with the supremum norm, which is a Banach algebra under pointwise multiplication

$$(f \cdot g)(t) = f(t)g(t), \quad t \in [0, 1].$$

Then $\mathbb{E}$ is a vector space over $\mathbb{R}$ under the following operations:

$$(\zeta + \kappa)(t) = \zeta(t) + \kappa(t)$$

and

$$(\alpha \zeta)(t) = \alpha \zeta(t),$$

for all $\zeta, \kappa \in \mathbb{E}$, $\alpha \in \mathbb{R}$. Let τ be the strongest vector (locally convex) topology on $\mathbb{E}$. Then (Φ, τ) is a *tv*s which is not normable and is not even metrizable. Define the cone

$$P = \{f \in \mathbb{E} : f(t) \geq 0, \text{ for all } t \in [0, 1]\},$$

which is solid with

$$\text{int}(P) = \{f \in \mathbb{E} : f(t) > 0, \text{ for all } t \in [0, 1]\},$$

and $d : \Phi \times \Phi \rightarrow \mathbb{E}$ by

$$d(\zeta, \kappa)(t) = (\|\zeta - \kappa\|_\infty) e^t.$$

Then (Φ, d) is a complete *tv*s-valued cone SMS. Define the operator $\mathcal{Q} : \Phi \rightarrow \Phi$ by

$$(\mathcal{Q}\zeta)(t) = \zeta_0 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, \zeta(s)) ds, \quad t \in [a, b].$$

Assume the following conditions:

(i) There exists $L > 0$ such that for all

$$\|f(s, \zeta) - f(s, \kappa)\| \leq L \|\zeta - \kappa\|.$$

(ii)

$$\frac{L(b-a)^\alpha}{\Gamma(\alpha+1)} < 1.$$

Theorem 3. Under the above assumptions, the Caputo fractional differential equation admits a unique solution in $C([a, b], \mathbb{R}^n)$.

Proof. For $\varsigma, \kappa \in \Phi$ and $t \in [a, b]$, we have

$$\begin{aligned}\|\mathcal{Q}\varsigma(t) - \mathcal{Q}\kappa(t)\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_a^t (t-s)^{\alpha-1} [f(s, \varsigma(s)) - f(s, \kappa(s))] ds \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \|f(s, \varsigma(s)) - f(s, \kappa(s))\| ds.\end{aligned}$$

Using assumption (i), we obtain

$$\begin{aligned}\|\mathcal{Q}\varsigma(t) - \mathcal{Q}\kappa(t)\| &\leq \frac{L}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \|\varsigma(s) - \kappa(s)\| ds \\ &\leq \frac{L}{\Gamma(\alpha)} \|\varsigma - \kappa\|_{\infty} \int_a^t (t-s)^{\alpha-1} ds.\end{aligned}$$

Since

$$\int_a^t (t-s)^{\alpha-1} ds = \frac{(t-a)^{\alpha}}{\alpha},$$

we get

$$\|\mathcal{Q}\varsigma(t) - \mathcal{Q}\kappa(t)\| \leq \frac{L(t-a)^{\alpha}}{\alpha\Gamma(\alpha)} \|\varsigma - \kappa\|_{\infty} \leq \frac{L(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|\varsigma - \kappa\|_{\infty}.$$

Taking supremum over $t \in [a, b]$, we have

$$\|\mathcal{Q}\varsigma - \mathcal{Q}\kappa\|_{\infty} \leq \sigma \|\varsigma - \kappa\|_{\infty},$$

where $\sigma = \frac{L(b-a)^{\alpha}}{\Gamma(\alpha+1)}$. By assumption (ii), we get $\sigma \in [0, 1)$. Hence

$$\|\mathcal{Q}\varsigma - \mathcal{Q}\kappa\|_{\infty} \leq \sigma \|\varsigma - \kappa\|_{\infty}.$$

Therefore, for all $t \in [a, b]$, we have

$$d(\mathcal{Q}\varsigma, \mathcal{Q}\kappa)(t) = \|\mathcal{Q}\varsigma - \mathcal{Q}\kappa\|_{\infty} e^t \leq \sigma \|\varsigma - \kappa\|_{\infty} e^t = \sigma d(\varsigma, \kappa)(t).$$

Thus, in the cone order,

$$d(\mathcal{Q}\varsigma, \mathcal{Q}\kappa) \preceq \sigma d(\varsigma, \kappa).$$

All the conditions of Theorem (1) are satisfied with $\sigma = \sigma_1$ and $\sigma_2 = \sigma_3 = 0$. Therefore, there exists a unique point $\varsigma^* \in \Phi$ such that $\varsigma^* = \mathcal{Q}\varsigma^*$. Hence, ς^* is the unique solution of the Caputo fractional differential equation. $\square$

5.2. Existence Theory for Nonlinear Volterra–Fredholm Integral Equations

Integral equations of Volterra–Fredholm type play a significant role in the mathematical modeling of processes where present states depend simultaneously on past history and cumulative spatial effects. Such equations frequently appear in diverse areas such as epidemiology, economics, signal processing, and nonlinear mechanics. The presence of both Volterra and Fredholm components introduces analytical complexity, which often prevents the direct use of classical solution techniques.

One effective strategy for overcoming these difficulties is to employ FP methods, which allow the problem to be studied through the properties of an associated nonlinear operator. By embedding the equation into an appropriate functional framework, FP theorems provide sufficient conditions for the existence of solutions without requiring explicit formulas.

In this part of the work, the Volterra–Fredholm integral equation is examined within the setting of a tv s-valued cone SMS. Using the FP results developed earlier, we establish the existence of solutions under generalized contractive conditions. This approach not only broadens the applicability of FP theory but also offers a unified method that encompasses and extends several existing results in the literature.

This section is devoted to establishing an existence result for solutions of a Volterra–Fredholm integral equation. Consider the nonlinear Volterra–Fredholm integral equation

$$\varsigma(t) = g(t) + \int_a^t K_1(t, s, \varsigma(s)) + \int_a^b K_2(t, s, \varsigma(s)) ds, \quad t \in [a, b], \quad (20)$$

where $\varsigma, g \in \Phi = C([a, b], \mathbb{R}^n)$, $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous functions. Let $\mathbb{E} = C[a, b]$ with the supremum norm and pointwise multiplication, and let

$$P = \{f \in \mathbb{E} : f(t) \geq 0, \text{ for all } t \in [0, 1]\}.$$

The space (Φ, d) is defined as in the previous subsection.

Define the operator

$$(\mathcal{Q}\varsigma)(t) = g(t) + \int_a^t K_1(t, s, \varsigma(s)) + \int_a^b K_2(t, s, \varsigma(s)) ds, \quad t \in [a, b].$$

Assume the following conditions hold:

(H₁) There exist the constants $L_1, L_2 > 0$ such that for all $t, s \in [a, b]$ and all $\varsigma, \kappa \in \mathbb{R}^n$,

$$\|K_i(t, s, \varsigma) - K_i(t, s, \kappa)\| \leq L_i \|\varsigma - \kappa\|, \quad i = 1, 2.$$

(H₂)

$$L_1(b - a) + L_2(b - a) < 1.$$

Theorem 4. Under assumptions (H₁)–(H₂), the Volterra–Fredholm integral Equation (20) admits a unique solution in $C([a, b], \mathbb{R}^n)$.

Proof. For any $\varsigma, \kappa \in \Phi$ and for all $t \in [a, b]$, we have

$$\begin{aligned} \|\mathcal{Q}\varsigma(t) - \mathcal{Q}\kappa(t)\| &= \left\| \int_a^t [K_1(t, s, \varsigma(s)) - K_1(t, s, \kappa(s))] ds \right\| \\ &\quad + \left\| \int_a^b [K_2(t, s, \varsigma(s)) - K_2(t, s, \kappa(s))] ds \right\| \\ &\leq \int_a^t \|K_1(t, s, \varsigma(s)) - K_1(t, s, \kappa(s))\| ds \\ &\quad + \int_a^b \|K_2(t, s, \varsigma(s)) - K_2(t, s, \kappa(s))\| ds. \end{aligned}$$

Using assumption (H_1) , we obtain

$$\begin{aligned}\|Q\zeta(t) - Q\kappa(t)\| &\leq \int_a^t L_1 \|\zeta(s) - \kappa(s)\| ds + \int_a^b L_2 \|\zeta(s) - \kappa(s)\| ds \\ &\leq (L_1(t-a) + L_2(b-a)) \|\zeta - \kappa\|_\infty \\ &\leq (L_1(b-a) + L_2(b-a)) \|\zeta - \kappa\|_\infty.\end{aligned}$$

Taking supremum over $t \in [a, b]$, we get

$$\|Q\zeta - Q\kappa\|_\infty \leq (L_1(b-a) + L_2(b-a)) \|\zeta - \kappa\|_\infty.$$

Let $\sigma = (L_1(b-a) + L_2(b-a))$. By assumption (H_2) , we have $\sigma \in [0, 1)$. Hence

$$\|Q\zeta - Q\kappa\|_\infty \leq \sigma \|\zeta - \kappa\|_\infty.$$

Therefore,

$$d(Q\zeta, Q\kappa)(t) = \|Q\zeta - Q\kappa\|_\infty e^t \leq \sigma \|\zeta - \kappa\|_\infty e^t = \sigma d(\zeta, \kappa)(t)$$

which means

$$d(Q\zeta, Q\kappa) \preceq \sigma d(\zeta, \kappa).$$

Thus all the hypotheses of the FP Theorem (1) are satisfied with $\sigma_1 = \sigma$ and $\sigma_2 = \sigma_3 = 0$. Therefore, there exists a unique point $\zeta^* \in \Phi$ such that $\zeta^* = Q\zeta^*$. Hence, ζ^* is the unique solution of the Volterra–Fredholm integral Equation (20). $\square$

6. Conclusions

This paper introduced the notion of a *tv*s-valued cone SMS, which provides a unified framework that encompasses cone SMSs, *tv*s-valued cone MSs, and cone MSs as particular cases. Within this generalized setting, new FP theorems were established for both single-valued and multivalued generalized contractive mappings. The proposed results not only extend but also unify several well-known FP theorems available in the existing literature, thereby broadening the scope of FP theory in generalized topological structures.

To demonstrate the effectiveness and applicability of the developed theory, some illustrative examples were presented, highlighting the novelty and usefulness of the established results. Furthermore, the theoretical findings were successfully applied to investigate the existence of solutions for a Caputo fractional differential equation and a Volterra–Fredholm integral equation, illustrating that the proposed framework is not only of theoretical interest but also serves as an effective analytical tool for studying nonlinear mathematical models arising in applied sciences.

Overall, the results obtained in this work contribute to the advancement of generalized metric FP theory by providing a more flexible and comprehensive framework that may facilitate further developments in both abstract FP theory and its applications to differential equations, integral equations, and other nonlinear problems.

Future Work

The framework introduced in this paper provides several promising directions for future research, including but not limited to the following:

- The concept of *tv*s-valued cone SMS may be extended to more general settings, such as *tv*s-valued *G*-cone SMSs, *tv*s-valued perturbed cone MSs, controlled cone SMSs,

and other generalized topological structures. Such extensions may lead to broader classes of FP theorems and enrich the existing theory of generalized MSs.

- Future studies may focus on establishing FP results for various nonlinear contractive conditions, including polynomial-type contractions, integral-type contractions, interpolative contractions, rational contractions, simulation function contractions, and other generalized contractive mappings. These investigations may provide a unified framework encompassing many existing contraction principles as particular cases.
- Another interesting direction is the study of FP, coincidence point, and common FP results for multivalued mappings, hybrid pairs of mappings, cyclic mappings, and mappings satisfying graph-based or order-theoretic conditions within the proposed framework.
- The developed FP theory can be employed to investigate the existence, uniqueness, stability, and qualitative behavior of solutions to various nonlinear mathematical models, including fractional differential equations, fractional boundary value problems, coupled fractional systems, integral equations, integro-differential equations, evolution equations, and dynamical systems. Such applications would further demonstrate the effectiveness of the proposed framework in modeling real-world phenomena arising in engineering, physics, biology, and applied sciences (see, for example, [25–27]).
- Future research may also address the development of iterative algorithms and numerical approximation techniques based on the established FP results. These methods could be used to approximate solutions of nonlinear equations arising in practical applications and to analyze their convergence properties.

Author Contributions: Conceptualization, H.A. and J.A.; methodology, H.A. and J.A.; formal analysis, H.A.; investigation, J.A.; validation, H.A. and J.A.; resources, H.A.; data curation, J.A.; writing—original draft preparation, H.A.; writing—review and editing, H.A. and J.A.; visualization, J.A.; supervision, H.A.; project administration, J.A.; funding acquisition, H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-25-DR-2236).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-25-DR-2236). Therefore, the authors thank the University of Jeddah for its technical and financial support.

Conflicts of Interest: The authors declare no conflicts of interest.

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