



Article

A Two-Stage Strategy for Q Full-Waveform Inversion in Viscoacoustic Media with Adam Optimization

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Abstract

Accurate characterization of subsurface attenuation hinges on the quality factor (Q), and high-fidelity Q models are indispensable for high-resolution imaging in exploration seismology. In principle, full-waveform inversion (FWI) offers the most accurate framework for parameter estimation, and decades of research have delivered substantial advances. Despite this, Q -FWI remains challenging in attenuating media, as velocity and Q exhibit strong crosstalk. Additionally, the sensitivity of data to Q is much weaker than to velocity, and absorption further diminishes signal strength. To address these issues, we introduce a staged FWI workflow that leverages the Adam optimization algorithm. To mitigate velocity– Q crosstalk in multiparameter inversion, we adopt a two-stage synchronous (joint–reset–joint) scheme: in stage one, velocity and Q are inverted jointly but the artifact-contaminated Q model is discarded; in stage two, velocity and Q are again inverted jointly using the updated velocity and the initial Q model. To avoid the additional trial forward simulations required by line-search step-length estimation, we use the Adam optimizer with parameter-specific adaptive updates. Numerical experiments on three synthetic models show that this workflow reduces velocity-related artifacts in the recovered Q models and improves the stability of dual-parameter viscoacoustic FWI.

Keywords: full-waveform inversion; Adam optimization algorithm; viscoacoustic media; multiparameter inversion; synchronous inversion strategy

1. Introduction

Field and laboratory observations consistently show that subsurface media exhibit pervasive absorptive attenuation [1,2].

The quality factor (Q) is commonly used to characterize this behavior and serves as a key parameter in viscoacoustic and viscoelastic constitutive models for describing intrinsic attenuation and velocity dispersion [3,4]. Because of causality, seismic attenuation is necessarily accompanied by dispersion, resulting in both amplitude decay and phase distortion.

The theoretical relationship between attenuation and causal dispersion was established by Futterman [5]. Based on this relationship, Hargreaves and Calvert [6] developed an inverse Q -filtering method to compensate for the amplitude loss and phase distortion of seismic waves, while Wang [7] introduced a stabilized inverse Q -filtering approach for seismic-resolution enhancement.



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Although these methods can improve attenuated seismic records, their effectiveness depends strongly on the accuracy of the available Q model. An inaccurate Q model may lead to insufficient compensation or unstable amplification of high-frequency energy. Moreover, propagation through attenuating media causes spectral downshift, resolution loss, and amplitude distortion [8], while spatial variations in Q can bias spectral attributes and subsequent inversion results [9]. Reliable Q -model building is therefore a prerequisite for stable attenuation compensation, high-resolution imaging, and quantitative seismic interpretation.

The development of Q model building traces back to the 1970s. Ref. [10] formalized the spectral-ratio method in geophysics, noting that, in a constant- Q medium, the logarithmic amplitude spectrum varies linearly with frequency. Ref. [11] applied the spectral-ratio approach to VSP, estimating interval Q from the logarithmic spectral ratio between each downhole pulse and a reference pulse. Building on constant- Q theory and nonstationary convolutional model building, ref. [12] proposed the complex spectral ratio (CSR), which leverages both amplitude and phase to reduce the bias inherent in purely amplitude-based ratios. Subsequently, ref. [13] presented a parallel evaluation of spectral ratio, spectral modeling, matching techniques, and time-domain matched filtering, offering practical guidelines for different data types and noise levels. Ref. [14] recast “spectral division” as a shaping-regularized inversion and incorporated multichannel spatial coherence, markedly improving noise robustness for reflection data. The spectral-ratio method is physically transparent and easy to implement and partially cancels source and coupling effects; however, it is highly sensitive to noise and band selection, tends to conflate nonintrinsic effects (e.g., geometric spreading and tuning) with Q , and typically yields path-averaged estimates.

In the early 1990s, cross-well experiments exploited the decay of ray amplitude with path length to reconstruct path-integrated absorption coefficients, seeding early amplitude and diffraction tomography and providing numerical and experimental testbeds for subsequent Q tomography [15]. Ref. [9] showed that, under constant- Q approximation, the centroid frequency decreases linearly with propagation distance in proportion to the path integral of attenuation, thereby enabling Q tomography to use the centroid-frequency difference as the observable. As the bandwidth and SNR of surface reflection data improved, reflection-based Q -tomography workflows emerged [16]. To mitigate errors from spectral estimation and noise sensitivity, practitioners introduced multichannel constraints, regularization, and robust objective functions, alleviating the ill-posedness associated with band selection and spectral division and making reflection-based Q tomography more practical in noisy, complex settings [17]. Building on finite-frequency theory and adjoint model building, sensitivity-kernel/adjoint-based Q tomography was developed, expanding the observables beyond centroid-frequency/spectral attributes to a broader set of measurements and explicitly addressing velocity- Q coupling and other multiparameter trade-offs [18].

Compared with single-trace spectral estimates, Q tomography incorporates information from multiple propagation paths and can reconstruct a spatially varying $Q(\mathbf{x})$ model. Nevertheless, its resolution and stability remain strongly dependent on acquisition geometry, the accuracy of the background velocity model, the selected measurements, and regularization. Errors in the velocity model may be mapped into the recovered attenuation model, while the limited bandwidth and signal-to-noise ratio of seismic data further increase the non-uniqueness of Q estimation. These limitations motivate waveform-based approaches that use the complete phase and amplitude information contained in seismic records.

All Q -model building approaches presuppose a reliable velocity model. When velocity is also unknown, the two fields must be estimated jointly. Extending full-waveform inver-

sion (FWI) from acoustic to viscoacoustic/viscoelastic physics enables Q -FWI, which seeks either simultaneous or staged updates of velocity and attenuation. The central difficulty is parameter decoupling: velocity and Q exhibit similar radiation patterns, making it nontrivial to isolate their contributions in the viscoacoustic wave equation [19]. In practice, many workflows adopt a two-stage scheme to curb cross-talk—first, resolve velocity in the target zone, then estimate Q [20]. In theory, this reduces leakage; in practice, if velocity has not converged, single-parameter Q updates driven by record residuals can be overstated [21]. An alternative multi-level design inverts velocity and Q in both stages yet often discards the stage-one Q in favor of the initial model before stage two, weakening continuity. When updating multiple parameters together, disparities in gradient scale produce unbalanced steps. A common remedy is gradient normalization—rendering velocity and Q gradients dimensionless and scaled to $[0, 1]$ as a form of preconditioning [22]. However, such scaling can bias the solution and degrade Q accuracy [23]. To address illumination and scaling imbalance without sacrificing fidelity, Yong et al. [24] approximated the diagonal of the Hessian as an adaptive preconditioner so that velocity and Q gradients were of comparable magnitude, which improved stability and accuracy in attenuating-media FWI.

In addition to tailored inversion strategies, higher-order optimization can further mitigate parameter cross-talk. First-order schemes (steepest descent, conjugate gradient) do not fundamentally resolve coupling; leveraging second-order information is essential [25], for example, via Newton-type algorithms in attenuating-media FWI. Direct Newton methods, however, demand prohibitive computation and memory and are impractical for large, complex models. To balance fidelity and cost, Keating and Innanen [26] implemented multi-parameter Q -FWI with a truncated Newton solver, reduced the number of frequency scales, and subsequently proposed a multi-resolution Q -FWI in which the model grid density varies with frequency [27]. This design applies higher-order optimization at low frequencies, while at high frequencies, it limits the risk of cross-talk to smaller, better-conditioned updates. More recently, Li [28] developed a decoupled attenuating-media FWI based on a truncated Newton solver, deriving Hessian-vector products and their Gauss–Newton approximation via the Lagrange multiplier framework to obtain accurate descent directions that decouple velocity and Q .

This study focuses on an efficient inversion strategy for dual-parameter viscoacoustic FWI rather than on the development of a new fractional wave-propagation equation. We adopt the established fixed-order fractional-Laplacian constant- Q formulation and its stabilized attenuation-compensation treatment from Chen et al. [29]. This formulation separates the dispersion- and absorption-related operators, making it possible to compensate for attenuated wavefield amplitudes during residual backpropagation. The corresponding sensitivity kernels are evaluated using the adjoint-state framework. Because reversing the absorption term may amplify unstable high-frequency components, a low-pass stabilization operator is applied to the compensated wavefield following the cited formulation.

To avoid the additional wave-equation solutions required by line-search step-length estimation, we employ the Adam optimizer following its application to seismic waveform inversion [30]. Adam constructs element-wise adaptive updates from the first and second moments of the gradients, which is useful for multiparameter inversion because velocity and Q have different physical units, gradient magnitudes, and admissible update ranges. In this study, the computational advantage of Adam refers specifically to the removal of trial wave propagations for line-search step-length estimation; it does not imply that Adam necessarily requires fewer total iterations than all conventional optimization methods.

The main contribution of this study is a two-stage synchronous inversion strategy for reducing velocity– Q cross-talk. In the first stage, velocity and Q are updated simultaneously so that the velocity model is reconstructed while accounting for attenuation effects.

Because the first-stage Q model is strongly contaminated by velocity-related cross-talk, it is not transferred to the next stage. In the second stage, simultaneous inversion is restarted using the updated velocity model and the initial Q model. This joint–reset–joint workflow differs from conventional velocity-then- Q sequential inversion because Q participates in both stages, while the contaminated first-stage Q estimate is prevented from biasing the second-stage reconstruction.

The proposed strategy is evaluated using an anomalous-body model, the BP model, and the Marmousi model. The sequential and two-stage synchronous strategies are compared in terms of reconstructed velocity and Q models, objective-function histories, extracted model profiles, and attenuation-compensated reverse-time-migration images. The numerical results indicate that the proposed workflow reduces the velocity-related artifacts in the recovered Q models for the synthetic examples considered in this study.

2. Methods

2.1. Fundamentals of Full-Waveform Inversion

Full-waveform inversion estimates subsurface model parameters by minimizing an objective function that measures the misfit between synthetic and observed data. We measure the data misfit with an arbitrary differentiable and symmetric residual cost ($\psi(\cdot)$) and impose prior and smoothing constraints through $R(\mathbf{m})$. Over a multi-source set (S), the objective function is

$$\min_{\mathbf{m}, \{\mathbf{u}_s\}} F(\mathbf{m}) = \sum_{s \in S} \int_{\Gamma_r} \int_0^T \psi\left((P\mathbf{u}_s)(\mathbf{x}, t) - \mathbf{d}_s^{\text{obs}}(\mathbf{x}, t)\right) dt d\Gamma + \beta R(\mathbf{m}), \quad (1)$$

subject to the forward-modeling constraint:

$$L(\mathbf{m}) \mathbf{u}_s(\mathbf{x}, t) = \mathbf{f}_s(\mathbf{x}, t), \quad s \in S. \quad (2)$$

where L is the wave operator, P projects the field onto the receivers, and $\mathbf{m}$ denotes the parameters to be inverted. Here, S is the source index set; Ω is the computational domain; Γ_r denotes the receiver locations; T is the recording time; $\mathbf{u}_s$ and $\mathbf{f}_s$ are the wavefield and source for shot s , respectively; and $\mathbf{d}_s^{\text{obs}}$ is the corresponding observed-data vector. The P operator samples the wavefield at the receivers, $R(\mathbf{m})$ is an optional model regularizer, and β is its weight. Zero initial conditions, i.e., $\mathbf{u}_s(\mathbf{x}, 0) = \partial_t \mathbf{u}_s(\mathbf{x}, 0) = \mathbf{0}$, are assumed.

To minimize F under the constraint, we construct the following Lagrangian functional:

$$\begin{aligned} J(\mathbf{m}, \{\mathbf{u}_s\}, \{\lambda_s\}) = & \sum_{s \in S} \left[\int_{\Gamma_r} \int_0^T \psi\left((P\mathbf{u}_s) - \mathbf{d}_s^{\text{obs}}\right) dt d\Gamma \right. \\ & \left. + \langle \lambda_s, L(\mathbf{m})\mathbf{u}_s - \mathbf{f}_s \rangle_{\Omega \times [0, T]} \right] \\ & + \beta R(\mathbf{m}). \end{aligned} \quad (3)$$

Taking the first-order optimality condition with respect to the wavefield yields the adjoint equation:

$$L(\mathbf{m})^* \lambda_s(\mathbf{x}, t) = -P^* \left[\psi' \left((P\mathbf{u}_s)(\mathbf{x}, t) - \mathbf{d}_s^{\text{obs}}(\mathbf{x}, t) \right) \right], \quad \lambda_s(\mathbf{x}, T) = \mathbf{0}, \quad \partial_t \lambda_s(\mathbf{x}, T) = \mathbf{0}. \quad (4)$$

The Lagrange multiplier (λ_s) is the adjoint wavefield, L^* and P^* denote the adjoints of L and P , and $\langle \cdot, \cdot \rangle_{\Omega \times [0, T]}$ is the space–time inner product.

Accordingly, the model gradient is given by the adjoint–forward interaction term:

$$\nabla_{\mathbf{m}} F(\mathbf{m}) = \beta \nabla_{\mathbf{m}} R(\mathbf{m}) + \sum_{s \in S} \left\langle \lambda_s, \frac{\partial L(\mathbf{m})}{\partial \mathbf{m}} \mathbf{u}_s \right\rangle_{\Omega \times [0, T]}. \quad (5)$$

In practice, the gradient can be preconditioned using a diagonal pseudo-Hessian, angle-domain imaging weights, L-BFGS or Gauss–Newton approximations, followed by an update with backtracking line search.

$$\mathbf{m}^{k+1} = \Pi_{[\mathbf{m}_{\min}, \mathbf{m}_{\max}]} \left(\mathbf{m}^k - \alpha_k \mathbf{B}_k^{-1} \nabla_{\mathbf{m}} F(\mathbf{m}^k) \right), \quad (6)$$

where Π denotes the projection onto box constraints and $\mathbf{B}_k^{-1}$ is the preconditioning operator.

2.2. Viscoacoustic Full-Waveform Inversion

For the numerical implementation, the general objective function in Equation (1) is specialized by choosing the quadratic residual penalty ($\psi(\mathbf{r}) = \frac{1}{2} \|\mathbf{r}\|_2^2$) and omitting an explicit model regularizer (i.e., $\beta = 0$). Accordingly, the data-misfit function is written as

$$F(\mathbf{m}) = \frac{1}{2} \sum_{s \in S} \|\delta \mathbf{d}_s\|_2^2, \quad \delta \mathbf{d}_s = P \mathbf{u}_s - \mathbf{d}_s^{\text{obs}}. \quad (7)$$

where $\mathbf{m}$ is the model-parameter vector and $\delta \mathbf{d}_s$ is the modeled-minus-observed-data residual for source s . The L_2 objective is used consistently in all synthetic tests reported below.

The constant- Q viscoacoustic wave equation adopted from Chen et al. [29] can be expressed as follows:

$$\frac{1}{\lambda c_0^2} \frac{\partial^2 u}{\partial t^2} = \left(1 - \frac{32}{\pi Q} \right) \nabla^2 u - \frac{32}{\pi Q} \left(\frac{c_0}{\omega_d} \right)^{\frac{1}{16}} (-\nabla^2)^{\frac{33}{32}} u - \frac{1}{Q c_0} \frac{\partial}{\partial t} (-\nabla^2)^{\frac{17}{32}} u, \quad (8)$$

where $u(\mathbf{x}, t)$ is the pressure wavefield, $Q(\mathbf{x})$ is the quality factor, and $c_0(\mathbf{x})$ is the phase velocity at the reference angular frequency (ω_0). Hereafter, unless otherwise stated, the term “velocity model” refers specifically to the reference-frequency velocity ($c_0(\mathbf{x})$). Here, $\omega_d = 2\pi f_d$ and $\omega_0 = 2\pi f_0$ denote the dominant and reference angular frequencies, respectively, and

$$\lambda = \left(\frac{\omega_d}{\omega_0} \right)^{\frac{2}{\pi Q}} \quad (9)$$

is the frequency-dispersion correction coefficient used in the adopted constant- Q approximation. The fixed fractional orders (33/32 and 17/32) replace the spatially varying fractional orders with the fixed orders of the approximation in Chen et al. [29]; the spatially varying $Q(\mathbf{x})$ is retained in the operator coefficients. The fractional Laplacian is defined in the spatial Fourier domain as

$$\mathcal{F} \left[(-\nabla^2)^\gamma u \right] = |\mathbf{k}|^{2\gamma} \hat{u}, \quad (10)$$

where $\mathbf{k}$ is the wavenumber vector and $\hat{u}$ is the spatial Fourier transform of u [31]. In the numerical implementation, the second-order time derivative is advanced by finite differences, whereas the fractional Laplacians are evaluated by the Fourier pseudospectral relation in Equation (10). Equation (8) describes propagation away from the source; including the source gives $L_f \mathbf{u}_s = \mathbf{f}_s$, consistently with Equation (2).

For clarity, we distinguish the forward-propagation operator from the attenuation-compensated back-propagation operator. Moving all terms in Equation (8) to the left gives the following forward operator:

$$L_f = \frac{1}{\lambda c_0^2} \frac{\partial^2}{\partial t^2} + \left(\frac{32}{\pi Q} - 1 \right) \nabla^2 + \frac{32}{\pi Q} \left(\frac{c_0}{\omega_d} \right)^{\frac{1}{16}} (-\nabla^2)^{\frac{33}{32}} + \frac{1}{Q c_0} \frac{\partial}{\partial t} (-\nabla^2)^{\frac{17}{32}}. \quad (11)$$

For attenuation compensation during backpropagation, only the sign of the absorption term is reversed, while the dispersion terms are retained.

$$L_c = \frac{1}{\lambda c_0^2} \frac{\partial^2}{\partial t^2} + \underbrace{\left[\left(\frac{32}{\pi Q} - 1 \right) \nabla^2 + \frac{32}{\pi Q} \left(\frac{c_0}{\omega_d} \right)^{\frac{1}{16}} (-\nabla^2)^{\frac{33}{32}} \right]}_{L_d} - \underbrace{\frac{1}{Q c_0} \frac{\partial}{\partial t} (-\nabla^2)^{\frac{17}{32}}}_{L_a}. \quad (12)$$

The L_d operator contains the acoustic and dispersion terms that control wave kinematics and phase dispersion, whereas L_a denotes the absorption term controlling amplitude decay.

Equation (12) is used for attenuation-compensated backpropagation: relative to L_f , the sign of L_a is reversed, whereas the signs of the second-order time derivative and the dispersion terms remain unchanged. This sign reversal compensates for energy loss but also amplifies high-wavenumber components. Therefore, following Chen et al. [29], a time-invariant low-pass wavenumber filter is applied at every backward time step. Its cutoff is selected from the highest frequency retained in the current multiscale band and the velocity range of the model and is kept fixed within that band. A cutoff that is too low removes useful model wavenumbers and weakens deep updates, whereas a cutoff that is too high permits exponentially growing compensation noise. The filter is consequently used only to restrict the wavenumber range participating in the sign-reversed absorption compensation. Because this filtering operation modifies the exact continuous adjoint, L_c denotes the stabilized compensated backpropagation operator rather than the strict adjoint operator.

Let $\delta \mathbf{d}_s = P \mathbf{u}_s - \mathbf{d}_s^{\text{obs}}$ denote the data residual for source s . The residual wavefield ($\mathbf{v}_s$) is obtained by solving

$$L_c \mathbf{v}_s = P^* \delta \mathbf{d}_s, \quad \mathbf{v}_s(\mathbf{x}, T) = \partial_t \mathbf{v}_s(\mathbf{x}, T) = \mathbf{0}.$$

Without stabilization filtering, $\mathbf{v}_s = -\lambda_s$ under the sign convention of Equation (4); the following expressions are then the adjoint-state gradients. With the low-pass filter included, the same expressions define the stabilized compensated gradient directions used in the inversion:

$$\frac{\partial F}{\partial c_0} = - \sum_{s \in S} \left\langle \mathbf{v}_s, \frac{\partial L_f}{\partial c_0} \mathbf{u}_s \right\rangle_{\Omega \times [0, T]}, \quad (13)$$

$$\frac{\partial F}{\partial Q} = - \sum_{s \in S} \left\langle \mathbf{v}_s, \frac{\partial L_f}{\partial Q} \mathbf{u}_s \right\rangle_{\Omega \times [0, T]}. \quad (14)$$

Here, $\mathbf{u}_s$ is the forward wavefield and $\mathbf{v}_s$ is the compensated backpropagated residual wavefield. The model derivatives entering Equations (13) and (14) are the Fréchet derivatives of the forward operator (L_f), not derivatives of the compensated backpropagation operator. They are given by

$$\frac{\partial L_f}{\partial c_0} = - \frac{2}{\lambda c_0^3} \frac{\partial^2}{\partial t^2} + \frac{2}{\pi Q} \left(\frac{1}{\omega_d} \right)^{\frac{1}{16}} c_0^{-\frac{15}{16}} (-\nabla^2)^{\frac{33}{32}} - \frac{1}{Q c_0^2} \frac{\partial}{\partial t} (-\nabla^2)^{\frac{17}{32}}. \quad (15)$$

$$\begin{aligned} \frac{\partial L_f}{\partial Q} = & \frac{2}{\pi \lambda c_0^2 Q^2} \log\left(\frac{f_d}{f_0}\right) \frac{\partial^2}{\partial t^2} - \frac{32}{\pi Q^2} \nabla^2 \\ & - \frac{32}{\pi Q^2} \left(\frac{c_0}{\omega_d}\right)^{\frac{1}{16}} (-\nabla^2)^{\frac{33}{32}} - \frac{1}{Q^2 c_0} \frac{\partial}{\partial t} (-\nabla^2)^{\frac{17}{32}}. \end{aligned} \quad (16)$$

Equations (15) and (16) are obtained by differentiating the adopted forward operator with respect to c_0 and Q while holding the other parameter fixed. In particular,

$$\frac{\partial c_0^{1/16}}{\partial c_0} = \frac{1}{16} c_0^{-15/16}, \quad \frac{\partial \lambda^{-1}}{\partial Q} = \frac{2}{\pi Q^2 \lambda} \log\left(\frac{\omega_d}{\omega_0}\right), \quad \frac{\partial}{\partial Q} \left(\frac{32}{\pi Q}\right) = -\frac{32}{\pi Q^2},$$

which gives the coefficients and signs in Equations (15) and (16). The wave-equation form is adopted from the fixed-order constant- Q formulation of Chen et al. [29]; only the operator differentiation required for the present inversion is shown here rather than repeating the complete derivation of the fractional wave equation.

Before optimization, the two gradient components are balanced using the diagonal preconditioner employed in the multiparameter implementation, i.e.,

$$\begin{bmatrix} \tilde{\mathbf{g}}_{c_0} \\ \tilde{\mathbf{g}}_Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} \mathbf{g}_{c_0} \\ \mathbf{g}_Q \end{bmatrix}, \quad 0.01 \leq \delta \leq 0.05.$$

The δ coefficient is reduced as the multiscale frequency increases. This operation balances the relative update magnitudes of the two parameter classes; it is a diagonal preconditioner and not a separate normalization of each gradient to $[0, 1]$.

2.3. Adam Optimization Algorithm

First-order schemes such as gradient descent and conjugate gradient are widely used in FWI; beyond these, the adaptive moment estimation (Adam) optimizer—validated in deep learning—also proves effective for seismic inversion, providing stable and efficient parameter updates in practical workflows [32].

The Adam optimizer [30,32] is also applicable to seismic inversion.

Unlike line-search methods, Adam constructs an element-wise adaptive update from exponential moving averages of the first and second moments of the gradient.

In the present dual-parameter inversion, independent moment estimates and learning rates are used for c_0 and Q after applying the diagonal gradient-balancing operator defined above. Adam removes the extra wave-equation solves required by line-search trials and therefore reduces the line-search overhead per iteration; however, it does not guarantee a monotonic objective-function decrease at every iteration or remove the physical cross-talk between parameter classes.

For either model parameter ($m \in \{c_0, Q\}$), Adam updates the first- and second-moment estimates independently according to

$$\mathbf{a}_k = \beta_1 \mathbf{a}_{k-1} + (1 - \beta_1) \mathbf{g}_k, \quad \mathbf{b}_k = \beta_2 \mathbf{b}_{k-1} + (1 - \beta_2) \mathbf{g}_k \odot \mathbf{g}_k, \quad (17)$$

followed by bias corrections, i.e.,

$$\hat{\mathbf{a}}_k = \frac{\mathbf{a}_k}{1 - \beta_1^k}, \quad \hat{\mathbf{b}}_k = \frac{\mathbf{b}_k}{1 - \beta_2^k}, \quad (18)$$

and the following update:

$$\mathbf{m}_{k+1} = \mathbf{m}_k - \alpha \frac{\hat{\mathbf{a}}_k}{\sqrt{\hat{\mathbf{b}}_k + \varepsilon}}. \quad (19)$$

Here, $\mathbf{g}_k$ is the balanced gradient component at iteration k , $\odot$ denotes element-wise multiplication, and α denotes either α_{c_0} or α_Q . The commonly used hyperparameters are $\beta_1 = 0.9$, $\beta_2 = 0.999$, and $\varepsilon = 1 \times 10^{-8}$ [30]. Because the two parameter classes have different physical units and gradient scales, the raw learning-rate values are not treated as universal constants. Instead, α_{c_0} and α_Q are selected independently so that the initial effective model increments satisfy

$$10 \text{ m s}^{-1} \leq \|\Delta c_{0,0}\|_{\infty} \leq 100 \text{ m s}^{-1}, \quad 1 \leq \|\Delta Q_0\|_{\infty} \leq 5. \quad (20)$$

Here, $\Delta c_{0,0}$ and ΔQ_0 are the first Adam increments for a multiscale inversion stage. Equation (20) reports the model-space increment ranges used in the implementation rather than a single dimensionless learning rate. Adam subsequently adapts the element-wise increments through Equations (17)–(19). The ranges are implementation controls and should not be interpreted as universal optimal values [30].

2.4. Multiparameter Inversion Strategies

Simultaneous inversion updates multiple parameters within a single stage using the same data residual. In practice, the objective function is expected to decrease overall as the iterations proceed, but a monotonic decrease at every Adam iteration is not guaranteed.

When the prescribed convergence or iteration criterion is met, the final parameter model is produced [33]. A multiscale inversion strategy partitions FWI into successive stages, each emphasizing different inversion objectives, thereby leveraging staged updates to improve multiparameter recovery. The choice of strategy is guided by differences in radiation patterns and the expected degree of cross-talk among parameters.

(1) *The sequential inversion strategy* prioritizes parameters and performs staged single-parameter updates. To mitigate cross-talk, practical multiparameter workflows assign each stage to a single parameter according to differences in radiation patterns. In viscoacoustic media, velocity primarily controls phase, whereas attenuation (via dispersion) affects both phase and amplitude, coupling their data sensitivities. In two-parameter viscoacoustic inversion, the cross-talk from Q into velocity is generally weaker than from velocity into Q . Accordingly, stage one performs a velocity-only inversion starting from the initial velocity and quality-factor models; stage two then fixes the updated velocity from stage one and, together with the initial quality-factor model, carries out a Q -only inversion (Figure 1).

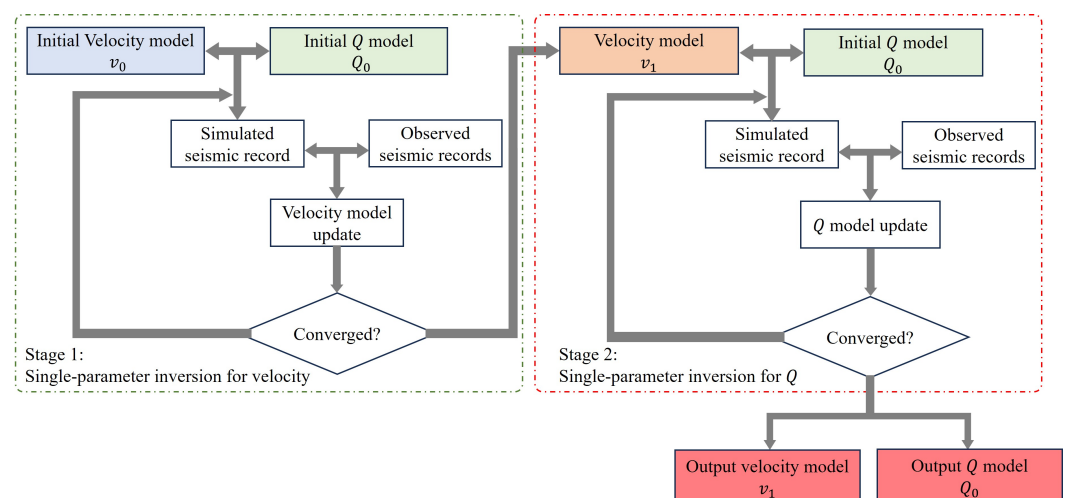


Figure 1. Workflow of the sequential inversion strategy for viscoacoustic media.

(2) *The two-stage synchronous (joint–reset–joint) strategy* performs multiparameter updates at every stage, then selectively retains results before moving on. In viscoacoustic FWI, stage one carries out simultaneous inversion for velocity and the quality factor (Q). Because the recovered Q is strongly contaminated by cross-talk, it is discarded at the end of this stage. In stage two, we again perform simultaneous inversion, now using the velocity obtained from stage one, together with the initial Q model. During the first stage, continuous updates of Q accompany a dynamically refined velocity, which improves the accuracy of the velocity model; a more accurate velocity then limits spurious Q updates in stage two. This two-stage synchronous design yields a more reliable joint recovery of velocity and Q (Figure 2).

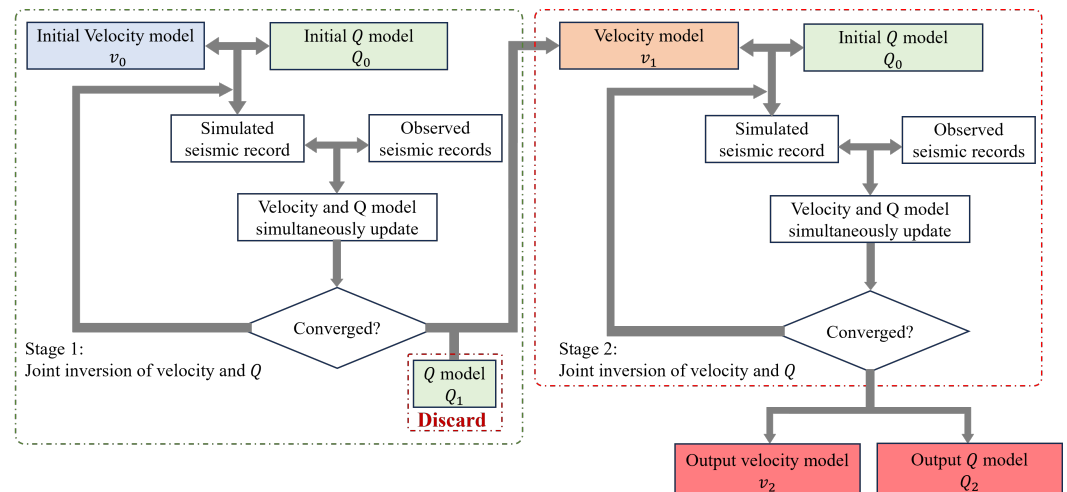


Figure 2. Workflow of the two-stage synchronous (joint–reset–joint) strategy for viscoacoustic media.

The advantage of the synchronous strategy is that it accounts for the mutual influence of velocity and the quality factor throughout the inversion. In viscoacoustic media, wave propagation depends on both velocity and attenuation; therefore, simultaneous updates provide a more faithful description of the medium. In stage one, joint inversion delivers a reasonably accurate velocity model, which, in turn, supplies a reliable basis for the Q update in stage two. In stage two, we invert again using the velocity from stage one, together with the initial Q model, thereby improving the accuracy of the recovered Q while avoiding the excessive Q updates that arise from velocity errors. The strategy is intended to reduce the transfer of the contaminated first-stage Q estimate into the second stage; it does not constitute a complete physical decoupling of velocity and attenuation. The diagonal gradient balancing and parameter-specific Adam updates limit excessive Q increments during the two joint inversions.

3. Results

For each convergence plot, the normalized objective value is defined as F_k/F_0 , where F_0 is the objective value at the beginning of the corresponding inversion stage and frequency band. Because each stage is normalized by its own initial value, these curves quantify the within-stage reduction of the data misfit and are not used to compare the absolute objective values of different stages.

3.1. Anomalous-Body Model

To investigate cross-talk in dual-parameter inversion, we perform numerical simulations on a diffractor model (Figure 3). The velocity model contains an anomalous low-velocity body on the left, and the quality-factor (Q) model contains an anomalous low- Q body on the right. The model size is 2000 m × 1000 m, with both horizontal and

vertical grid spacing of 10 m. Twenty-one shots are evenly deployed at the surface, with a shot spacing of 100 m, and a full-receiver array is used. A Ricker wavelet with a dominant frequency of 20 Hz serves as the source; the total record length is 4 s, with a sampling interval of 0.5 ms. Multiscale FWI is conducted with frequency bands in ascending order: 3.5, 7, 10.5, 14, 17.5, and 22 Hz. FWI with attenuation compensation is performed using both the sequential inversion approach and the synchronous inversion approach.

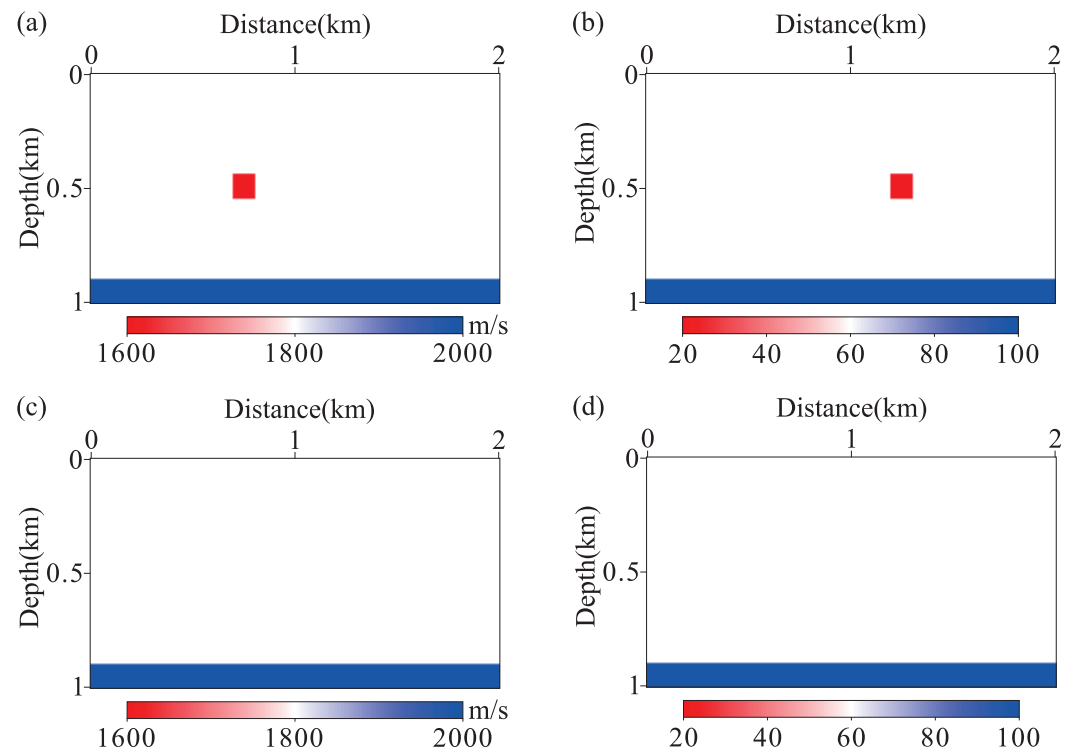


Figure 3. Anomalous-body model: (a) real velocity model; (b) real Q model; (c) initial velocity model; (d) initial Q model.

In the first stage of sequential inversion, the initial velocity and quality-factor models are used to invert for velocity alone. When the initial Q is inaccurate, severe cross-talk occurs in the velocity model, producing shadow artifacts of the anomalous Q body on the right side of the model (Figure 4). In the second stage, the velocity model obtained from the first stage is employed for single-parameter inversion of Q . The results show that when the velocity model exhibits significant errors, the inverted Q model is also unreliable, with cross-talk from the anomalous velocity body (Figure 5). Therefore, the sequential inversion strategy of recovering velocity first, then Q is not suitable for anomalous-body models.

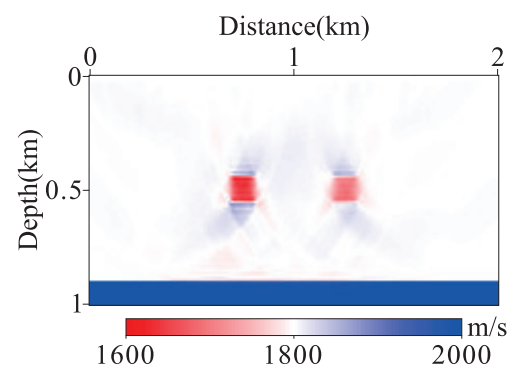


Figure 4. Velocity inversion result of the first stage of sequential inversion for the anomalous-body model.

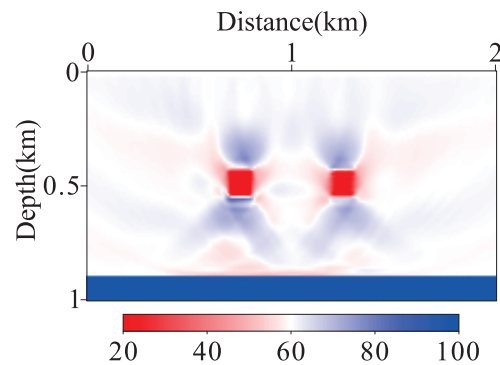


Figure 5. Q inversion result of the second stage of sequential inversion for the anomalous-body model.

To mitigate cross-talk, a synchronous inversion strategy is applied, in which Q is updated, together with velocity, during the first stage. The results indicate that the inverted velocity model is more reliable than the quality-factor model (Figure 6a), while the Q model suffers from cross-talk with velocity, producing shadow artifacts of velocity diffractions on the left side of the model (Figure 6b). Due to its low accuracy, the Q model is unsuitable for subsequent migration and seismic interpretation. To eliminate this effect, the low-accuracy Q model is discarded in the second stage, and the initial Q model is employed for two-parameter simultaneous inversion.

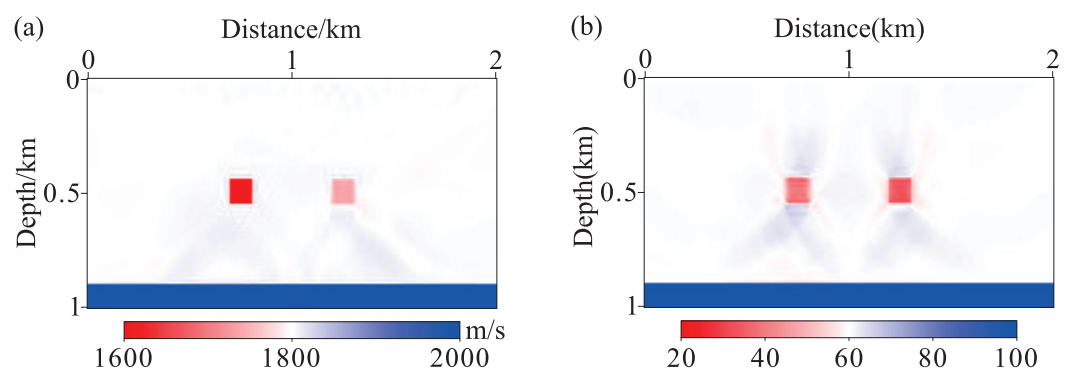


Figure 6. First-stage synchronous inversion results for the anomalous-body model: (a) velocity inversion result; (b) Q inversion result.

Compared with the first-stage results, the second-stage inversion (Figure 7) of the quality factor (Q) exhibits a clear reduction in cross-talk, and the diffractor is recovered at the correct location; profiles extracted (Figure 8) at the diffractor position indicate that the inverted velocity and Q values are close to the true values.

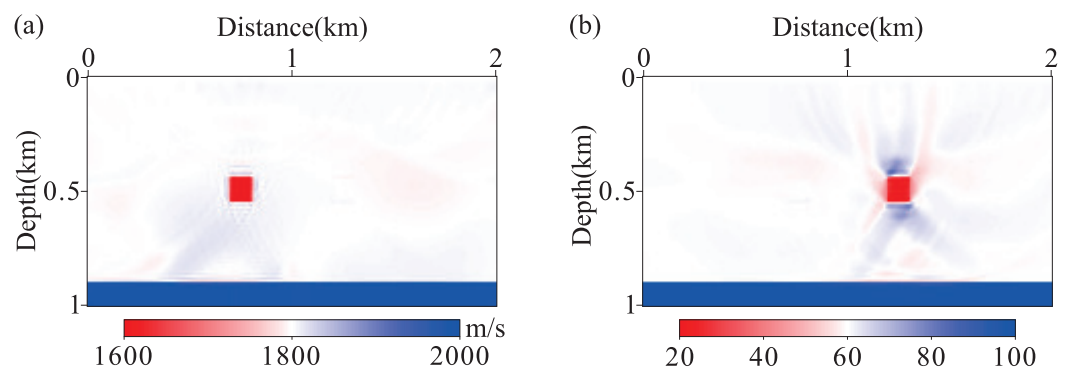


Figure 7. Second-stage synchronous inversion results for the anomalous-body model: (a) velocity inversion result; (b) Q inversion result.

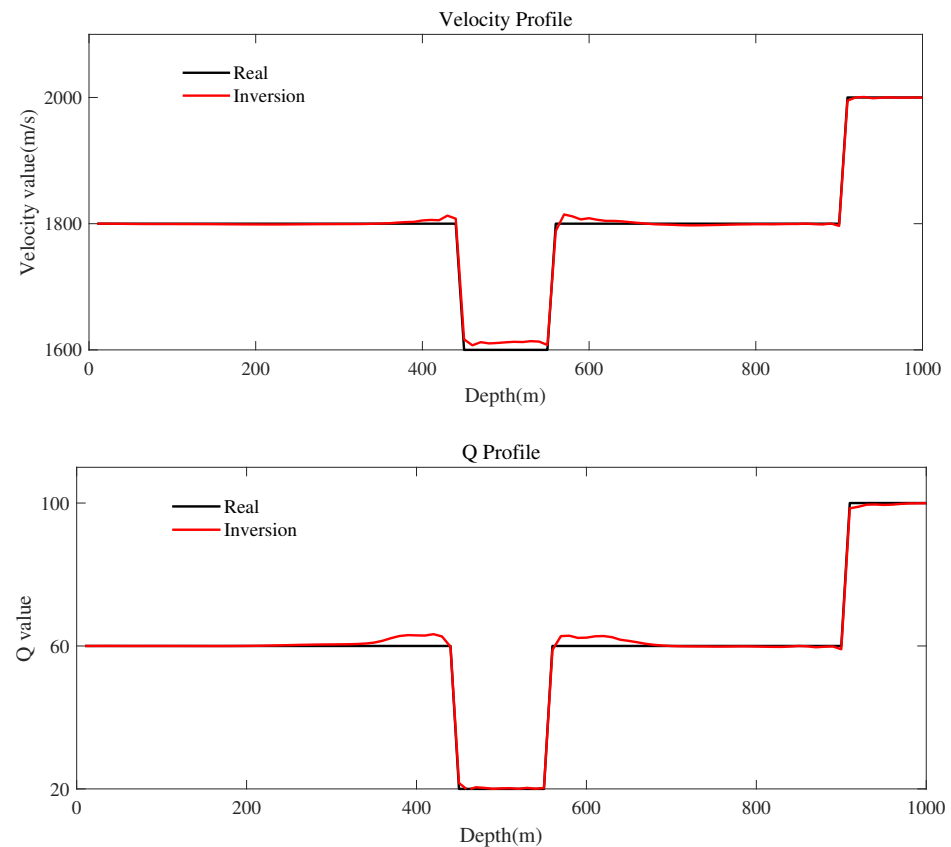


Figure 8. Profiles extracted from the inversion results for the anomalous-body model: **(above)** velocity profile at $x = 750$ m; **(below)** Q profile at $x = 1250$ m.

At the highest scale in both inversion stages, the convergence curves (Figure 9) show that the first-stage simultaneous inversion and the second-stage two-parameter simultaneous inversion both reduce their respective normalized objective values as the iteration count increases. Together with the extracted profiles, these results support the expected behavior of the staged inversion strategy for Q -compensated dual-parameter FWI on the simple model, laying the foundation for the next section's inversion of a more complex model.

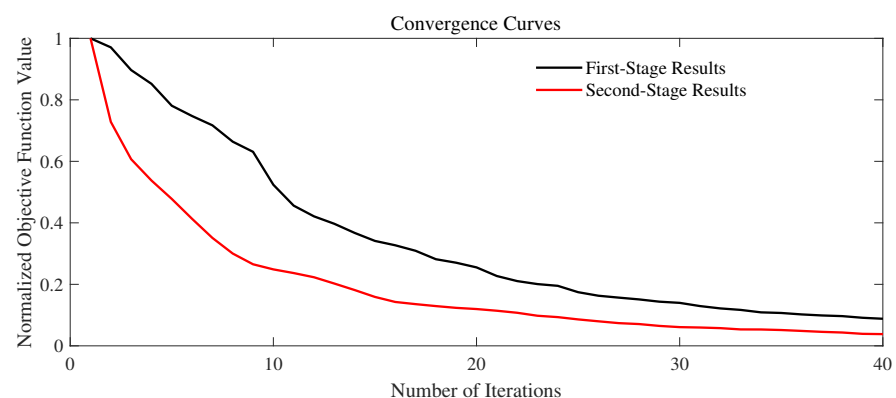


Figure 9. Convergence curves of the different inversion stages for the anomalous-body model.

3.2. BP Model

In practical data processing and parameter inversion, velocity and the quality factor (Q) are generally unavailable; therefore, the two parameters must be jointly inverted from the seismic data residuals. We test on the BP model (Figure 10), using a highly smoothed version of the true model as the initial model. The model size is $3000 \text{ m} \times 1370 \text{ m}$, with both horizontal and vertical grid spacing of 10 m. Thirty-one shots are evenly deployed at the

surface, with a shot spacing of 100 m. A full-receiver spread is used. A Ricker wavelet with a dominant frequency of 20 Hz serves as the source, the total record length is 6 s, and the sampling interval is 0.5 ms.

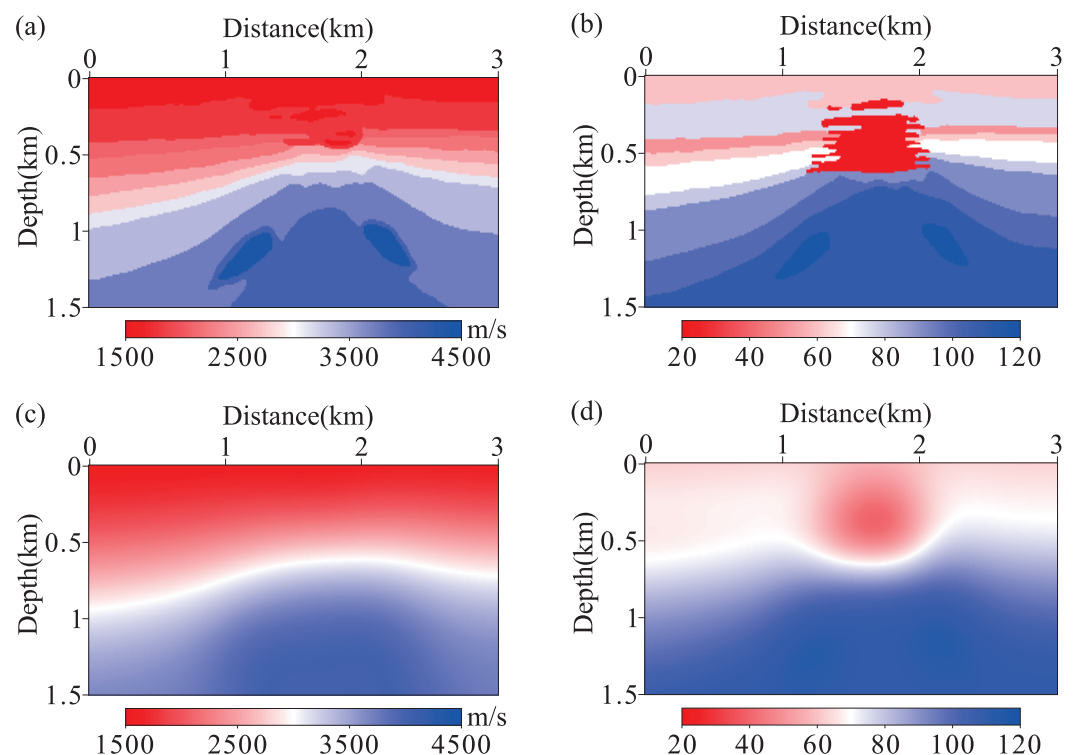


Figure 10. BP model: (a) real velocity model; (b) real Q model; (c) initial velocity model; (d) initial Q model.

To mitigate the cross-talk between velocity and the quality factor (Q), a sequential inversion strategy is applied for model testing. In the first stage, velocity is inverted as a single parameter using the coupled seismic residuals, while in the second stage, single-parameter inversion is performed for Q . The results indicate that when the Q model is inaccurate in the first stage, the inverted velocity deviates significantly from the true model (Figure 11). In the second stage, an inaccurate velocity model causes excessive updating of Q , leading to failure in both velocity and Q inversion (Figure 12). This confirms that the radiation patterns of velocity and Q are similar and that sequential inversion cannot reliably recover the parameter models.

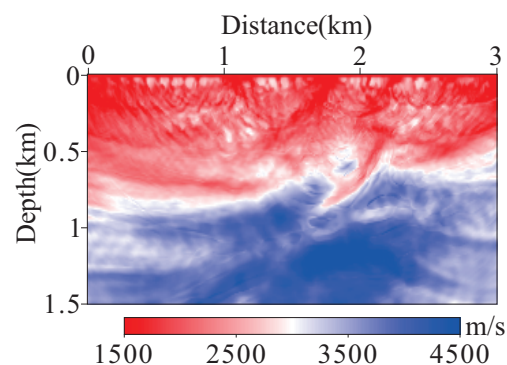


Figure 11. Velocity inversion result of the first stage of sequential inversion for the BP model.

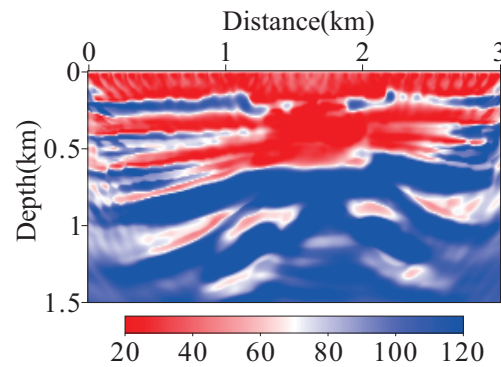


Figure 12. Q inversion result of the second stage of sequential inversion for the BP model.

To further investigate this issue, the two-stage synchronous strategy with Q compensation is applied to the same two-parameter problem. The results from the first stage show that the quality factor is updated too rapidly and fails to converge to an accurate model (Figure 13). Shadow artifacts of the velocity model appear in both the upper and middle–lower parts of the model due to parameter cross-talk, leading to erroneous results. The inverted velocity model is more reliable than the Q update, which is more vulnerable to velocity errors and parameter cross-talk, even though the seismic data are generally less sensitive to Q than to velocity. To avoid excessive updating of the Q model during the first stage, the Q gradient is downweighted by the diagonal balancing operator, and its initial effective increment is kept smaller than the velocity increment, as described in Equations (19) and (20). Subsequently, the velocity model obtained from the first stage of simultaneous inversion is retained, while the inverted quality-factor model (Q) is discarded and replaced by the initial Q model for the second-stage two-parameter simultaneous inversion. The visual comparison shows fewer velocity-related artifacts in the Q model obtained with the synchronous strategy than in the result of the sequential strategy (Figure 14). Since the velocity model is already close to the true model, only minor updates occur. Compared with the Q model obtained from the first-stage simultaneous inversion and that from the sequential inversion, the second-stage simultaneous inversion yields a more accurate Q model, particularly in the gas-cloud zone in the upper-to-middle part of the model, as well as in two small structural blocks beneath the strong-attenuation area. Therefore, the synchronous inversion strategy reduces but does not completely eliminate the velocity– Q cross-talk observed in this synthetic example.

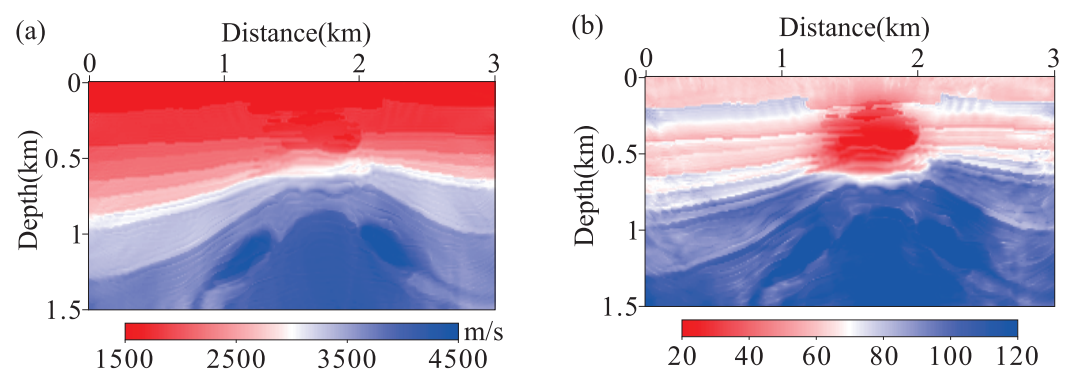


Figure 13. First-stage synchronous inversion results for the BP model: (a) velocity inversion result; (b) Q inversion result.

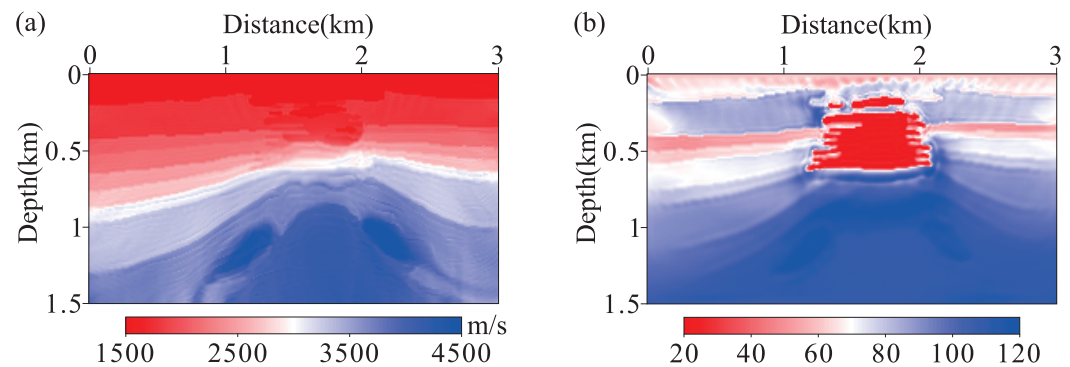


Figure 14. Second-stage synchronous inversion results for the BP model: (a) velocity inversion result; (b) Q inversion result.

The convergence curves (Figure 15) also indicate that neither the first nor the second stage of the sequential inversion strategy produces a comparable reduction in its normalized objective value, implying that when Q is inaccurate in the first stage, the velocity undergoes erroneous iterative updates. In contrast, both stages of the synchronous inversion show a sustained reduction in their normalized objective values, supporting the use of this staged strategy for viscoacoustic two-parameter inversion.

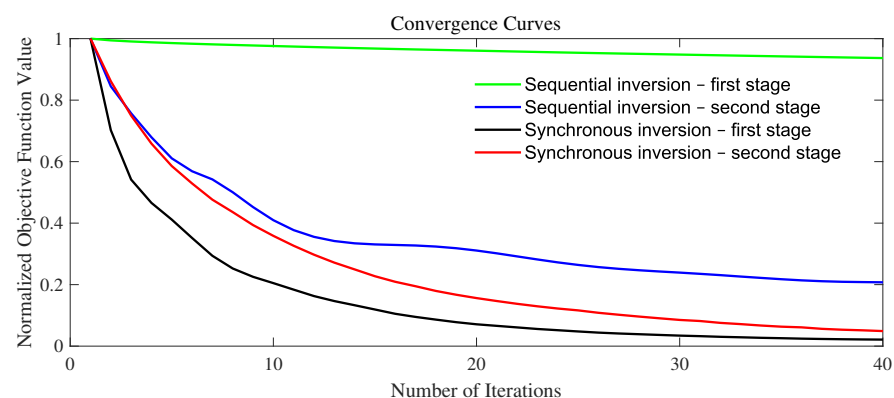


Figure 15. Convergence curves of the different inversion stages for the BP model.

To further verify the accuracy of the inversion results, Q -RTM [31] is applied to test whether the inverted model parameters can restore the subsurface reflectors to their true positions. A comparison of the three migrated images (Figure 16) shows that the migration result of the initial model exhibits inaccurate reflector positions and misaligned layers. After dual-parameter FWI, the velocity and Q models approach the true model, and the migrated image using the inversion results displays continuous structures, with reflectors restored to their correct positions and dips. Thus, Q -compensated FWI provides a solid foundation for subsequent migration processing and seismic data interpretation.

3.3. Marmousi Model

To further assess the applicability of the simultaneous inversion strategy to the model, the Marmousi model (Figure 17) is used for two-parameter FWI with Q compensation. The initial model is the result of highly smoothed models. The model size is $4530 \text{ m} \times 2700 \text{ m}$, with both lateral and vertical grid spacing of 10 m. Forty-six shots are evenly distributed on the surface, with a shot interval of 100 m, and full-aperture receivers are employed. A Ricker wavelet with a dominant frequency of 20 Hz is used as the source function, and the total seismic recording time is 6 s to ensure that deep signals are fully captured. The time

interval is set to 0.5 ms. The frequency bands are listed in increasing order as follows: 3.5 Hz, 7 Hz, 14 Hz, and 22 Hz.

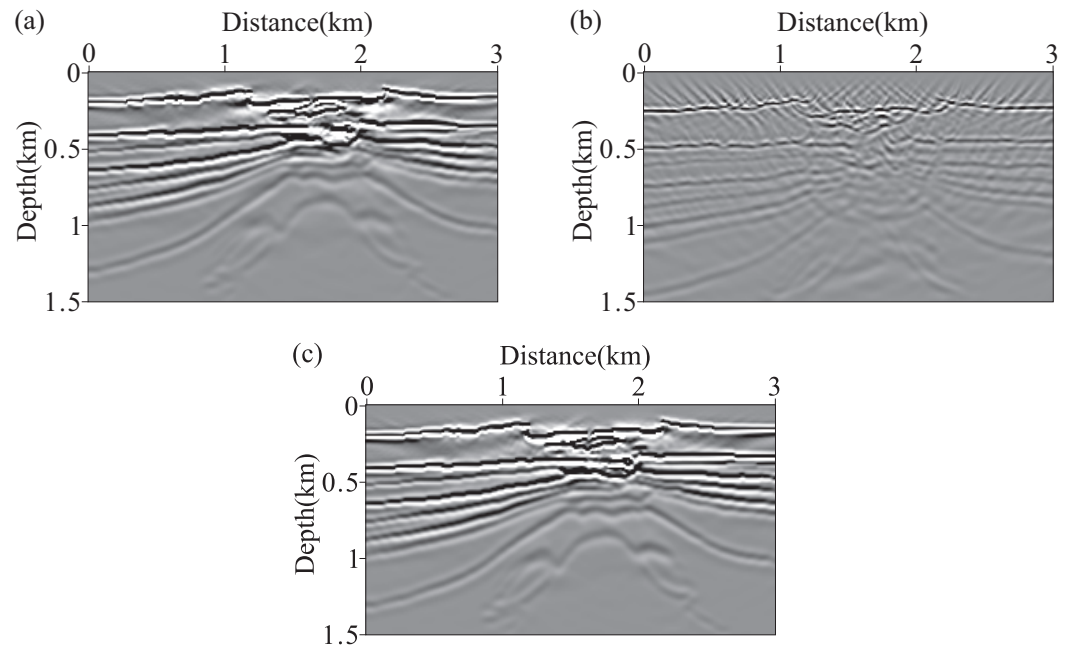


Figure 16. Q-RTM images for the BP model: (a) real-model Q-RTM image; (b) initial-model Q-RTM image; (c) inversion-model Q-RTM image.

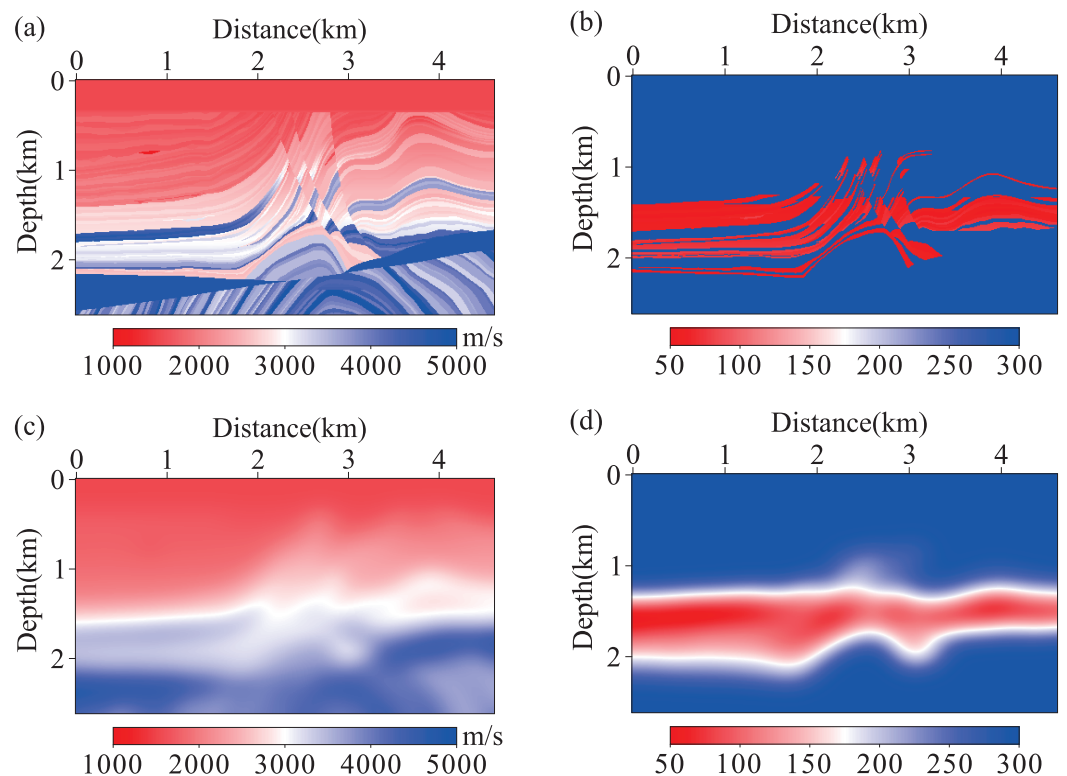


Figure 17. Marmousi model: (a) real velocity model; (b) real Q model; (c) initial velocity model; (d) initial Q model.

Since the radiation patterns of velocity and the quality factor (Q) are similar, the sequential inversion strategy cannot achieve accurate recovery. Therefore, a simultaneous inversion strategy is adopted for two-parameter inversion with Q compensation. The results show that the inverted velocity model is superior to the Q model (Figure 18): the

velocity is less affected by cross-talk from Q , allowing for accurate recovery of all layers, whereas the Q model exhibits shadow artifacts from the background velocity and fails to be updated accurately due to cross-talk. Using the velocity model obtained from the first stage, the inverted quality-factor Q model is discarded and replaced with the initial Q model for the second-stage two-parameter simultaneous inversion. The results show that, since the velocity model is already close to the true model, it undergoes only minor updates in the second stage, whereas the Q model can be accurately recovered (Figure 19). In particular, the low- Q anomaly in the middle part of the model is clearly reconstructed. Therefore, the simultaneous inversion strategy effectively alleviates the cross-talk problem associated with two-parameter inversion in viscoacoustic media. By comparing the convergence curves at the highest scale of the two stages in the simultaneous inversion strategy, it can be observed that both optimization processes reduce their respective normalized objective values, indicating stable iterative behavior for this synthetic test.

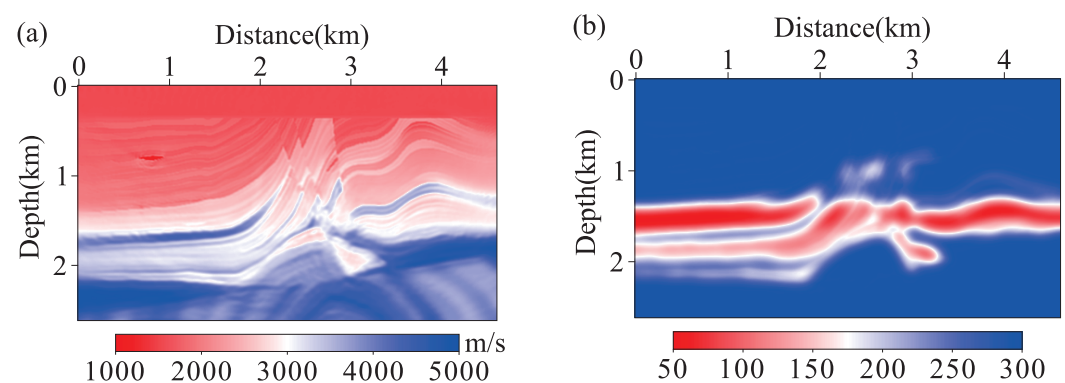


Figure 18. First-stage synchronous inversion results for the Marmousi model: (a) velocity inversion result; (b) Q inversion result.

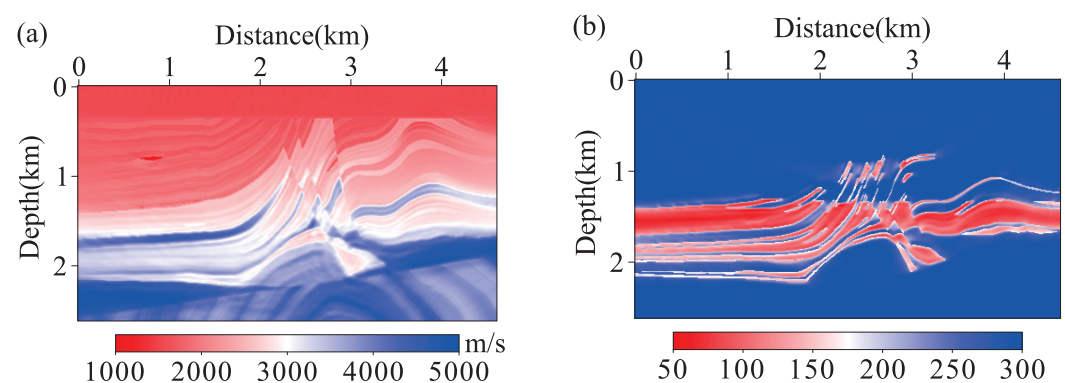


Figure 19. Second-stage synchronous inversion results for the Marmousi model: (a) velocity inversion result; (b) Q inversion result.

The convergence curves shown in Figure 20 indicate that the two inversion stages converge steadily. To further verify the accuracy of the inversion results, Q -RTM is applied to examine whether the model parameters obtained from simultaneous inversion can restore the subsurface structures to their true positions. Comparison of the migrated images in Figures 21 and 22 shows that the migration result from the initial model exhibits inaccurate horizons and misaligned layers. After viscoacoustic two-parameter FWI, both the velocity and quality-factor Q models approach the true models. The migrated images derived from the inverted results display continuous structures, with horizons restored to the correct positions and dips, providing a complementary qualitative indication of improved structural positioning.

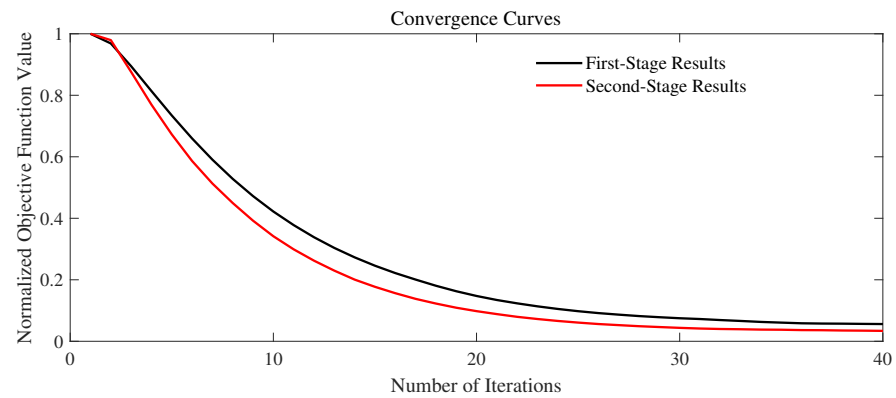


Figure 20. Convergence curves of the different inversion stages for the Marmousi model.

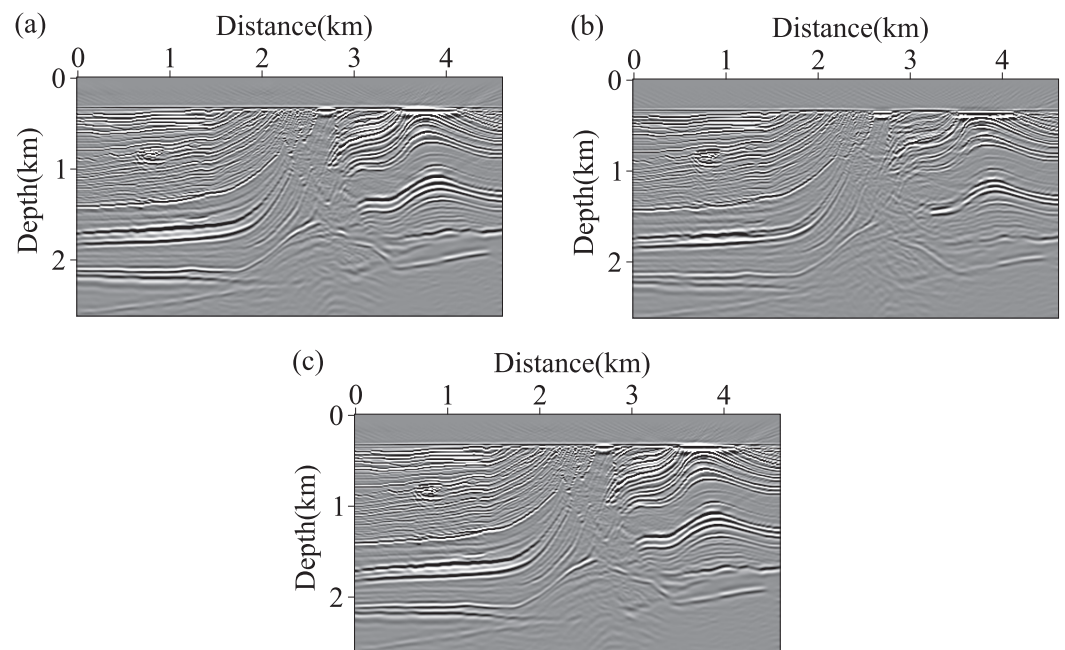


Figure 21. Q-RTM images for the Marmousi model: (a) real-model Q-RTM image; (b) initial-model Q-RTM image; (c) inversion-model Q-RTM image.

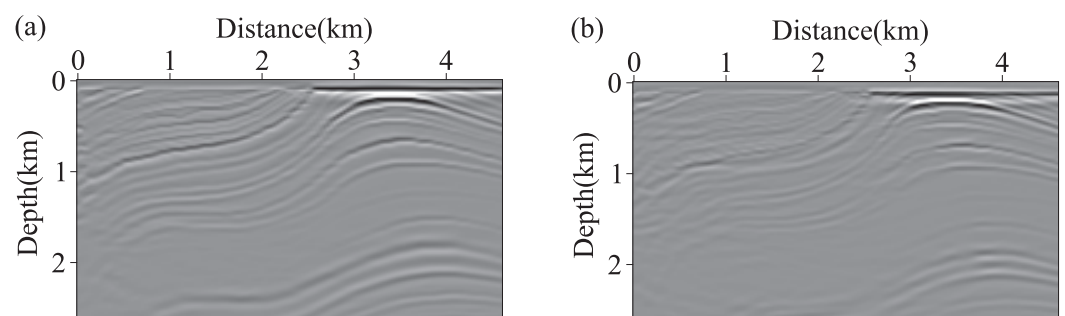


Figure 22. Cont.

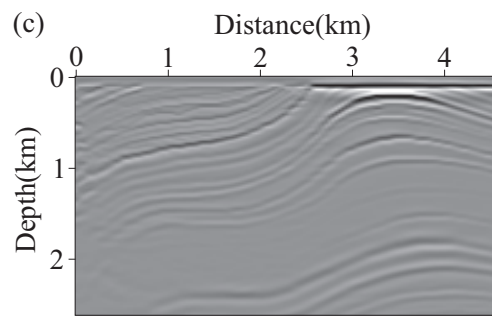


Figure 22. Local Q-RTM images for the Marmousi model: (a) real-model Q-RTM image; (b) initial-model Q-RTM image; (c) inversion-model Q-RTM image.

4. Discussion

The numerical examples illustrate why the proposed strategy improves the recovered Q model without claiming complete parameter decoupling. Velocity and attenuation have overlapping radiation patterns and different data sensitivities, so velocity errors can be mapped into the Q update [19,23]. In the first joint stage, attenuation is included while the velocity model is refined; resetting only the contaminated Q estimate then prevents that estimate from being inherited by the second joint stage. The diagonal balancing coefficient limits excessive Q updates, but the remaining trade-off is intrinsic to multiparameter viscoacoustic FWI.

Adam is used here as a low-memory first-order optimizer rather than as a physical decoupling mechanism. Its computational advantage in this workflow is specifically the avoidance of trial forward simulations for line-search step-length estimation [30]. Higher-order or Hessian-based approaches can provide stronger parameter decoupling but at greater computational and memory cost [22]. The present study therefore focuses on a practical, staged first-order strategy and does not claim that Adam is universally faster or more accurate than all conventional optimizers.

The scope of the numerical assessment is limited to two-dimensional, noise-free synthetic data and the adopted fixed-order constant- Q approximation. Field-data effects such as incomplete acquisition, source uncertainty, ambient noise, static errors, and modeling errors are not represented. In addition, the stabilized compensated backpropagation depends on the selected low-pass cutoff, and the method was not assessed here for three-dimensional propagation or more general rheological models. These aspects, together with quantitative uncertainty analysis, remain subjects for future work.

5. Conclusions

This paper presents a Q -FWI workflow based on the Adam optimizer for dual-parameter viscoacoustic inversion. By replacing conventional line-search step-length estimation with Adam, the workflow avoids the additional trial forward simulations associated with line-search step-length estimation. We adopt a two-stage simultaneous-inversion strategy: in stage one, velocity and Q are inverted jointly; in stage two, the Q model obtained in stage one is discarded, and the joint inversion is repeated, which mitigates velocity- Q cross-talk. The approach is examined on the Anomalous-Body, BP, and Marmousi models. Results show that, during stage one, Q updates rapidly, but inaccuracies in velocity can cause over-updates in the Q model. In stage two, using the more accurate velocity from stage one improves inversion accuracy, suppresses parameter cross-talk, and yields a more reliable recovery of subsurface structures. Tests on both simple and complex models show that the proposed strategy reduces velocity-related artifacts in the recovered Q models for the considered synthetic cases.

Although the method shows promising results, challenges remain, including the strong dependence of FWI on initial models, the limited ability of low-order optimization algorithms to handle cross-talk, and the restriction of this study to two-dimensional models. Future work will focus on constructing more accurate initial models with tomography or deep learning, developing more efficient optimization algorithms, and extending the methodology to three-dimensional applications.

In summary, this study provides a practical workflow for dual-parameter viscoacoustic full-waveform inversion, avoids line-search trial simulations and reduces the transfer of first-stage velocity-related artifacts into the final Q model.

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