Article

G-Networks with Adders

Jean-Michel Fourneau [1],[†] and Erol Gelenbe [2],[*],[†]

[1]   Laboratoire DAVID, Université de Versailles-Saint-Quentin, 78000 Versailles, France;
     jean-michel.fourneau@uvsq.fr
[2]   Intelligent Systems & Networks Group, Imperial College, London SW7 2AZ, UK
[*]   Correspondence: e.gelenbe@imperial.ac.uk; Tel.: +44-207-594-6342
[†]   These authors contributed equally to this work.

Abstract: Queueing networks are used to model the performance of the Internet, of manufacturing and job-shop systems, supply chains, and other networked systems in transportation or emergency management. Composed of service stations where customers receive service, and then move to another service station till they leave the network, queueing networks are based on probabilistic assumptions concerning service times and customer movement that represent the variability of system workloads. Subject to restrictive assumptions regarding external arrivals, Markovian movement of customers, and service time distributions, such networks can be solved efficiently with "product form solutions" that reduce the need for software simulators requiring lengthy computations. G-networks generalise these models to include the effect of "signals" that re-route customer traffic, or negative customers that reject service requests, and also have a convenient product form solution. This paper extends G-networks by including a new type of signal, that we call an "Adder", which probabilistically changes the queue length at the service center that it visits, acting as a load regulator. We show that this generalisation of G-networks has a product form solution.

Keywords: G-networks; internet; computer and network performance; queueing networks; transportation tetworks; product form solutions

1. Introduction

Queueing models with "product form" solutions and their efficient computational algorithms [1,2] are useful in many engineering fields including computer systems and networks [3,4], machine learning [5–7], transportation systems [8,9], job-shop and manufacturing systems [10], and emergency evacuation [11,12]. G-networks [13–15] are a significant extension of earlier queueing models [16]. They include "triggers" which re-route traffic [17] for load balancing among multiple servers, "negative customers", "batch removal" [18] of customers, and "string transitions" [19] which extend trigger multiple steps of customer movements.

Starting from the G-network model, this paper discusses a new type of "customer" we call an "Adder". When a service ends at some node, the departing customer can be transformed into an Adder which replaces the number of customers it finds, at the queue where it arrives, with a random number of customers which depends on two probability distributions: (a) the probability that queue to which it arrives was previously empty, and (b) an externally defined arbitrary probability distribution on the non-negative integers, which depends on where the Adder comes from and which queue it targets. We show that the resulting system obeys a system of non-linear traffic equations, and has a steady-state joint probability distribution for the network with "product form" solution.

2. New Types of Customers: The Adders

In the model we consider, N distinct servers have queues where ordinary customers can line up waiting for service, and the server handles them individually, in some order which does not depend on the individual customers' service needs. The service times of successive customers are independent and identically distributed exponential variables at each of the queues, and the service rates for the different queues are $\mu_i \geq 0$ for $i = 1, ..., N$. Denote by $(K_1(t), ..., K_N(t))$ the queue lengths where $K_i(t)$ is the number of normal customers at queue i at time t, and let $p(\vec{k}, t) = Pr[(K_1(t), ..., K_N(t)) = \vec{k}]$.

In addition to positive customers, negative customers, triggers, new types of customers that we call Adders can arrive to a queue from some node (possibly the same one). External arrivals of positive and negative arrivals occur according to mutually independent Poisson processes, with rates λ_i^+, λ_i^-, respectively, and:

- Only positive customers can remain in a queue to wait for service. Negative customers, triggers and adders disappear once they have accomplished their mission.
- Negative customers are a special case of triggers, since the customer they remove from a queue will then directly leave the network rather than joining another queue.
- Positive customers, negative customers (including batch removal) and triggers, are covered in early work [18] and lead to a " product form" solution for the joint probability of network state in equilibrium, provided external arrivals are Poisson processes, service times are independent and identically distributed, and the movement of customers is described by a Markov chain.

After a positive customer ends its service at time $t \geq 0$ at queue i, we will have $K_i(t^+) = K_i(t) - 1$, and then the customer can then leave the network with probability d_i, or it joins some queue j as:

- A positive customer with probability P_{ij}^+; hence $K_j(t^+) = K_j(t) + 1$ and we can have $P_{ii} \geq 0$,
- A negative customer with probability P_{ij}^-, but we require that $P_{ii}^- = 0$, so that $K_j(t^+) = max[0, K_j(t) - 1]$,
- A trigger that moves one positive customer from some other node j to node m, if there is at least one such customer, with probability Q_{ijm}; we require that $Q_{ijm} = 0$ if $i = j$ or $i = m$ or $j = m$. As a result $K_j(t^+) = max[0, K_j(t) - 1].1[K_i(t) > 0]$, and $K_m(t^+) = K_m(t) + 1[K_i(t) > 0, K_j(t) > 0]$,
- Finally as the new customer type, the Adder with probability P_{ij}^A, and we have $d_i + \sum_{j=1}^N [P_{ij}^A + P_{ij}^+ + P_{ij}^- + \sum_{m=1}^N Q_{ijm}] = 1$ for each queue i.

Let $w_{ij}(k), k \in \{1, 2, ...\}$ be a (possibly non-honest) probability distribution with $\sum_{k=1}^\infty w_{ij}(k) \leq 1$, and let its generating function be: $W_{ij}(y) = \sum_{k=1}^\infty w_{ij}(k)y^k, |y| \leq 1$.

The Adder will replace the current number of customers at the queue where it arrives, by a random number of customers which depends on the probability $R_j(t)$ that the queue was busy just before the Adder arrived, where $R_j(t) \equiv Pr[K_j(t) > 0]$. The queue length $K_j(t^+)$, just after the Adder arrives at j, is a random variable whose probability distribution is:

$$Pr[K_j(t^+) = x] \equiv \gamma_{ij}(x, t) = [1 - R_j(t)]W_{ij}(R_j(t))(R_j(t))^x, \ x > 0,$$
$$= 1 - R_j(t)W_{ij}(R_j(t)) \ if \ x = 0,$$

so that:

$$Pr[K_j(t^+) > 0] = R_j(t)W_{ij}(R_j(t)). \tag{1}$$

We immediately see the potential application of Adders, as a way to reduce queue length and make the system more stable, since we have $Pr[K_j(t^+) > 0] \leq Pr[K_j(t) > 0]$ since $W_{ij}(R_j(t)) \leq 1$.

Chapman-Kolmogorov Equations

Let $\vec{e}_i$ be the *N*-vector which is 0 everywhere, except in position *i* where it has the value 1: $\vec{e}_i = (0, .., 1, .., 0)$. The Chapman-Kolmogorov equations, or "master equations" [20,21] for the system we have defined are:

$$\frac{dp(\vec{k},t)}{dt} = -\sum_{i=1}^{N} \left[\lambda_i^+ + (\mu_i + \lambda_i^-)1_{\{k_i>0\}} \right] p(\vec{k},t)$$

$$+ \sum_{i=1}^{N} \lambda_i^+ p(\vec{k} - \vec{e}_i, t)1_{\{k_i>0\}} + \sum_{i=1}^{N} (\mu_i d_i + \lambda_i^-) p(\vec{k} + \vec{e}_i, t)$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i p(\vec{k} + \vec{e}_i - \vec{e}_j, t)P_{ij}^+ 1_{\{k_j>0\}}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} [\mu_i p(\vec{k} + \vec{e}_i + \vec{e}_j, t)P_{ij}^- + \mu_i p(\vec{k} + \vec{e}_i, t)P_{ij}^- 1_{\{k_j=0\}}]$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{m=1}^{N} \mu_i \, p(\vec{k} + \vec{e}_i + \vec{e}_m - \vec{e}_j, t)Q_{imj}1_{\{k_j>0\}}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{m=1}^{N} \mu_i \, p(\vec{k} + \vec{e}_i, t)Q_{ijm}1_{\{k_j=0\}} + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{y=0}^{\infty} \mu_i \, P_{ij}^A \, \gamma_{ij}(k_j, t) p(\vec{k} + \vec{e}_i + \vec{e}_j(y - k_j), t).$$

3. The Product Form

Consider the following system of non-linear equations:

$$\Lambda_i^+ = \lambda_i^+ + \sum_{j=1}^{N} [\mu_j \rho_j P_{ji}^+ + \mu_j \rho_j P_{ji}^A \rho_i W_{ji}(\rho_i) + \sum_{m=1}^{N} \mu_j \rho_j \rho_m Q_{jmi}], \tag{2}$$

$$\Lambda_i^- = \lambda_i^- + \sum_{j=1}^{N} [\mu_j \rho_j P_{ji}^- + \sum_{m=1}^{N} \mu_j \rho_j Q_{jim} + \mu_j \rho_j P_{ji}^A \frac{1 - \rho_i W_{ji}(\rho_i)}{1 - \rho_i}], \tag{3}$$

$$\rho_i = \frac{\Lambda_i^+}{\mu_i + \Lambda_i^-}. \tag{4}$$

Theorem 1. *If the system of 3N Equations (2)–(4) have a solution with $0 \le \rho_i < 1$, for $i = 1, ..., N$, then the equilibrium probability for the G-network with Adders is:*

$$p(\vec{k}) \equiv lim_{t\to\infty} Pr[K_1(t) = k_1, ..., K_N(t) = k_N] = \prod_{i=1}^{N} p_i(k_i), \ k_i \ge 0, \tag{5}$$

where the marginal queue length probabilities are $p_i(k_i) = lim_{t\to\infty} Pr[K_i(t) = k_i] = (\rho_i)^{k_i}(1 - \rho_i), \ k_i \ge 0.$

A Simple Example

In the introduction we had mentioned that the G-network with Adders has a "stabilising" or load reducing property, and we would like to illustrate this with a simple example before we develop the proof of the Theorem. Consider a system with just a single queue, and assume that the arrival rate of positive customers to this queue is identical to the service rate, so that $\lambda = \mu$. Assume that there are no negative customers arriving to the queue, nor any feedback of customers back to the queue after

service, so that $\lambda^- = 0$ and $P_{11}^+ = P_{11}^- = 0$. Also assume that there are no "true" departures so that each customer after a service turns into an Adder, and we have $P_{11}^A = 1$.

Then (2)–(4) give us the utilisation factor for the queue that is expressed as:

$$\rho = \frac{\lambda^+ + \mu\rho^2 W_{11}(\rho)}{\mu + \mu\rho\frac{1-\rho W_{11}(\rho)}{1-\rho}} = \frac{\frac{\lambda^+}{\mu}}{1 + \rho\frac{1-W_{11}(\rho)}{1-\rho}} = \frac{\lambda^+}{\mu}\frac{1-\rho}{1-\rho W_{11}(\rho)} \leq \frac{\lambda^+}{\mu}, \tag{6}$$

while if their were no Adders, and if each customer departed after service, the utilisation factor would be larger at the value $\frac{\lambda^+}{\mu}$. Clearly, the average queue length with Adders will also be smaller than the average queue length without Adders but where every customer departs the queue after a service.

Thus quite obviously and surprisingly, the queue without Adders and with ordinary departures is less stable than a queue without any customer departures but with Adders.

4. Proof of the Theorem

Proof. We now write the Chapman-Kolmogorov equations in steady-state, with the term $\gamma_{ij}(x) \equiv lim_{t\to\infty}\gamma_{ij}(x,t)$, so that:

$$
\begin{aligned}
\sum_{i=1}^{N} &\left[\lambda_i^+ + (\mu_i + \lambda_i^-)1_{\{k_i>0\}}\right] p(\vec{k}) \\
&= \sum_{i=1}^{N}[\lambda_i^+ p(\vec{k}-\vec{e}_i)1_{\{k_i>0\}} + (\mu_i d_i + \lambda_i^-)p(\vec{k}+\vec{e}_i)] + \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i p(\vec{k}+\vec{e}_i-\vec{e}_j)P_{ij}^+ 1_{\{k_j>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}[\mu_i p(\vec{k}+\vec{e}_i+\vec{e}_j)P_{ij}^- + \mu_i p(\vec{k}+\vec{e}_i)P_{ij}^- 1_{\{k_j=0\}}] + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\mu_i\, p(\vec{k}+\vec{e}_i+\vec{e}_m-\vec{e}_j)Q_{imj}1_{\{k_j>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\mu_i\, p(\vec{k}+\vec{e}_i)\, Q_{ijm}1_{\{k_j=0\}} + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{y=0}^{\infty}\mu_i\, P_{ij}^A\, \gamma_{ij}(k_j)p(\vec{k}+\vec{e}_i+\vec{e}_j(y-k_j)),
\end{aligned}
\tag{7}
$$

into which we substitute (5):

$$
\begin{aligned}
\sum_{i=1}^{N}&\left[\lambda_i^+ + (\mu_i + \lambda_i^-)1_{\{k_i>0\}}\right] \\
&= \sum_{i=1}^{N}\frac{\lambda_i^+ 1_{\{k_i>0\}}}{\rho_i} + \sum_{i=1}^{N}(\mu_i d_i + \lambda_i^-)\rho_i + \sum_{i=1}^{N}\mu_i\frac{\rho_i}{\rho_j}P_{ij}^+ 1_{\{k_j>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i\rho_j P_{ij}^- + \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i P_{ij}^- 1_{\{k_j=0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\mu_i\frac{\rho_i\rho_m}{\rho_j}Q_{imj}1_{\{k_j>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\mu_i\rho_i Q_{ijm}1_{\{k_j=0\}} + \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i 1_{\{k_j=0\}}P_{ij}^A\frac{1-\rho_j W_{ij}(\rho_j)}{1-\rho_j} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\frac{\rho_i\rho_j}{\rho_j}1_{\{k_j>0\}}P_{ij}^A W_{ij}(\rho_j).
\end{aligned}
\tag{8}
$$

Using $1_{\{k_j=0\}} = 1 - 1_{\{k_j>0\}}$, we regroup terms on the RHS to the LHS (left-hand-side) changing sign, exchange some j's and i's, and use ρ_i in (4), and (3), yielding:

$$
\begin{aligned}
\sum_{i=1}^{N}&\left[\lambda_i^+ + \frac{\Lambda_i^+}{\rho_i}1_{\{k_i>0\}}\right] \\
&= \sum_{i=1}^{N}\frac{\lambda_i^+ 1_{\{k_i>0\}}}{\rho_i} + \sum_{i=1}^{N}(\mu_i d_i + \lambda_i^-)\rho_i + \sum_{i=1}^{N}\mu_i\frac{\rho_i}{\rho_j}P_{ij}^+ 1_{\{k_j>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i\rho_j P_{ij}^- + \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i P_{ij}^- + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\mu_i\frac{\rho_i\rho_m}{\rho_j}Q_{imj}1_{\{k_j>0\}} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\mu_i\rho_i Q_{ijm} \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\rho_i P_{ij}^A\frac{1-\rho_j W_{ij}(\rho_j)}{1-\rho_j} + \sum_{i=1}^{N}\sum_{j=1}^{N}\mu_i\frac{\rho_i\rho_j}{\rho_j}1_{\{k_j>0\}}P_{ij}^A W_{ij}(\rho_j).
\end{aligned}
\tag{9}
$$

With (2) we get:

$$\sum_{i=1}^{N} \lambda_i^+ = \sum_{i=1}^{N} \rho_i [\lambda_i^- + \mu_i [d_i + \sum_{j=1}^{N} (\rho_j P_{ij}^- + P_{ij}^- + P_{ij}^A \frac{1 - \rho_j W_{ij}(\rho_j)}{1 - \rho_j} + \sum_{m=1}^{N} \mu_i Q_{ijm})]]. \tag{10}$$

To show that (10) is the same as (4) we write ρ_i as: $\Lambda_i^+ = \rho_i (\mu_i + \Lambda_i^-)$, in the LHS we use (2) and the RHS (3), so that summing over i the result is:

$$\sum_{i=1}^{N} [\lambda_i^+ + \sum_{j=1}^{N} [\mu_j \rho_j P_{ji}^+ + \mu_j \rho_j P_{ji}^A \rho_i W_{ji}(\rho_i) + \sum_{m=1}^{N} \mu_j \rho_j \rho_m Q_{jmi}]]$$

$$= \sum_{i=1}^{N} [\rho_i \mu_i + \lambda_i^- + \sum_{j=1}^{N} [\mu_j \rho_j P_{ji}^- + \sum_{m=1}^{N} \mu_j \rho_j Q_{jim} + \mu_j \rho_j P_{ji}^A \frac{1 - \rho_i W_{ji}(\rho_i)}{1 - \rho_i}]]. \tag{11}$$

Since $1 = d_i + \sum_{j=1}^{N} [P_{ij}^A + P_{ij}^+ + P_{ij}^- + \sum_{m=1}^{N} Q_{ijm}]$ we have:

$$\sum_{i=1}^{N} [\lambda_i^+ + \sum_{j=1}^{N} [\mu_j \rho_j P_{ji}^+ + \mu_j \rho_j P_{ji}^A \rho_i W_{ji}(\rho_i)] + \sum_{j=1}^{N} \sum_{m=1}^{N} \mu_j \rho_j \rho_m Q_{jmi}]$$

$$= \sum_{i=1}^{N} [\rho_i \mu_i [d_i + \sum_{j=1}^{N} [P_{ij}^A + P_{ij}^+ + P_{ij}^- + \sum_{m=1}^{N} Q_{ijm}]]$$

$$+ \rho_i \lambda_i^- + \sum_{j=1}^{N} \rho_i [\mu_j \rho_j P_{ji}^- + \sum_{m=1}^{N} \mu_j \rho_j Q_{jim} + \mu_j \rho_j P_{ji}^A \frac{1 - \rho_i W_{ji}(\rho_i)}{1 - \rho_i}]]. \tag{12}$$

Cancelling the term $\sum_{i=1}^{N} \sum_{j=1}^{N} \mu_j \rho_j [P_{ji}^+ + \sum_{m=1}^{N} \rho_i Q_{jim}]$ on both sides, and transferring $\sum_{i=1}^{N} \sum_{j=1}^{N} \mu_j \rho_j P_{ji}^A \rho_i W_{ji}(\rho_i)$ with a negative sign to the RHS one obtains:

$$\sum_{i=1}^{N} \lambda_i^+ = \sum_{i=1}^{N} [\rho_i \mu_i d_i + \rho_i \lambda_i^- + \sum_{j=1}^{N} \rho_i \mu_i [P_{ij}^- + \sum_{m=1}^{N} Q_{ijm}]$$

$$+ \sum_{j=1}^{N} \rho_i [\mu_j \rho_j P_{ji}^- + \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_j \rho_j [P_{ji}^A + P_{ji}^A \frac{1 - \rho_i W_{ji}(\rho_i)}{1 - \rho_i} - P_{ji}^A \rho_i W_{ji}(\rho_i)]]]. \tag{13}$$

However

$$-1 + \frac{1 - \rho_i W_{ji}(\rho_i)}{1 - \rho_i} + \rho_i W_{ji}(\rho_i) = \rho_i \frac{1 - \rho_i W_{ji}(\rho_i)}{1 - \rho_i}.$$

Therefore (13), derived from (2)–(4), is identical to (10). □

5. Conclusions

G-networks [22,23] have had numerous applications to Gene Regulatory Networks [24], Neural Networks [25] as tools to develop complex Pattern Analysis algorithms [6] and to control routing in packet networks [26,27]. Their transient behaviour [28] has been recently examined and other applications are discussed in [7].

In this paper we introduce new types of customers, the Adders, in G-networks with negative and positive customers, and triggers, and show that this generalised model has product form solution. We indicate that Adders stabilise the network through a probabilistic modification of the queue length, and illustrate this effect on a single server model. We expect that the introduction of Adders will lead to significant new applications and developments.

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