

Article

Multi-Modal Dynamic Transit Assignment for Transit Networks Incorporating Bike-Sharing

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Abstract

Traditional multi-modal dynamic transit assignment (DTA) models predominantly focus on bus and rail systems, overlooking the role of bike-sharing in passenger flow distribution. To bridge this gap, a multi-modal dynamic transit assignment model incorporating bike-sharing (MMDTA-BS) is proposed. This model integrates bike-sharing, buses, rail services, and walking into a unified framework. Represented by the variational inequality (VI), the MMDTA-BS model is proven to satisfy the multi-modal dynamic transit user equilibrium conditions. To solve the VI formulation, a projection-based approach with dynamic path costing (PA-DPC) is developed. This approach dynamically updates path costs to accelerate convergence. Experiments conducted on real-world networks demonstrate that the PA-DPC approach achieves rapid convergence and outperforms all compared algorithms. The results also reveal that bike-sharing can serve as an effective means for transferring passengers to rail modes and attracting short-haul passengers. Moreover, the model can quantify bike-sharing demand imbalances and offer actionable insights for optimizing bike deployment and urban transit planning.

Keywords: dynamic transit assignment; multi-modal transit; bike-sharing; variational inequality; projection-based approach with dynamic path costing

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1. Introduction

With global urbanization accelerating, urban population growth has worsened transit congestion [1,2]. Rational transit resource allocation and congestion reduction have become key challenges for cities [3]. Traditional transit analysis algorithms, focusing on rail and buses, often fail to capture diverse travel patterns. As an emerging mode, bike-sharing fills the public transit system’s “last mile” gap [4,5], enabling passengers to choose transportation more flexibly, especially for short distances, where bicycles have become many citizens’ preferred option [6].

The emergence of dockless bike-sharing systems, in particular, has introduced a new layer of flexibility and complexity to urban transit. Unlike traditional station-based systems, dockless bikes can be parked within a designated area, which more closely mirrors real-world operations in many modern cities. The dynamic transit assignment

(DTA) assigns time-varying passenger demand to transit modes in a network via passenger travel paths. It is a critical tool for evaluating transit service levels, enabling planners and operators to optimize transit planning and scheduling [7,8]. However, existing multi-modal DTA models predominantly focus on traditional bus and rail systems while overlooking bike-sharing. This oversight leads to imbalanced transit assignment and an inability to fully reflect inter-modal interactions. For instance, without accounting for bike-sharing, models may overestimate demand for bus services on short routes and underestimate the potential for bike-and-ride combinations, leading to suboptimal resource allocation.

To achieve equilibrium transit assignment in a transit network incorporating bike-sharing, a novel multi-modal dynamic transit assignment model incorporating bike-sharing (MMDTA-BS) is developed. A projection-based approach with dynamic path costing (PA-DPC) is proposed to solve the model. Validation in real transit networks reveals the impact of bike-sharing on public transit and provides insights for scheduling and parking planning of bike-sharing. The core innovation lies in the explicit modeling of bike-sharing as a first-class transit mode, complete with its own cost structure, resource constraints, and dynamic interactions with other modes.

2. Related Works

Existing multi-modal DTA models mainly focus on bus and rail systems [9] while overlooking bike-sharing's impact on transit networks and rider paths. This section reviews both single-modal and multi-modal DTA models, along with bike-sharing's application in transit networks.

2.1. Single-Modal and Multi-Modal Dynamic Transit Assignment Models

Single-modal DTA models primarily focus on passenger path selection for a single transit mode (e.g., buses or rail), typically modeled using Dynamic User Equilibrium (DUE). DUE stipulates that at equilibrium, all passengers choose paths with identical and minimal actual generalized costs (e.g., travel time, fares, discomfort), and no one can reduce their costs by unilaterally switching paths. Studies by Hickman & Bernstein [10] and Nuzzolo et al. [11] provided the theoretical foundation for single-modal DTA. While simple and suitable for small networks, these models struggle to reflect multi-modal interactions in modern urban transit systems [12]. Their primary limitation is the assumption of a homogenous transit system, which fails to capture the competitive and complementary relationships between different modes, such as buses and bike-sharing competing for short trips or bike-sharing feeding into rail stations.

Multi-modal DTA models (MMDTA) extend single-modal models by incorporating multiple transit modes (e.g., bus and rail), enabling more comprehensive simulation of passenger choice behavior in multi-modal networks. Models proposed by Cats [13] and Zhou & Mahmassani [14] better handle urban transit flows, with some studies also integrating private cars and ride-hailing services [15,16]. However, MMDTA models are more complex and computationally intensive, particularly for large-scale networks, where computational efficiency and convergence speed remain key challenges.

2.2. Application Studies of Bike-Sharing in Transit Networks

As bike-sharing gains popularity, studies have begun to focus on how to integrate it into existing transit network models [17–19], especially with its impact on the public transit system [20]. Studies show that bike-sharing not only relieves the pressure on the public transit system [21] but also improves the accessibility of public transit [22]. However, most existing multi-modal transit assignment models fail to effectively integrate bike-sharing. Kamel et al. [23] pointed out that many models failed to consider

the dynamic characteristics of bike-sharing, especially the dynamic changes in leasing and return, which led to the models' inability to accurately predict the use of bike-sharing in the actual network. This dynamic leasing and returning process creates a feedback loop: passenger choices affect bike availability, which in turn influences subsequent passenger choices. Ignoring this interdependency can lead to significant prediction errors, such as underestimating waiting times at empty parking areas or congestion at full ones. These studies show that despite the important role of bike-sharing in the "last-mile" travel, the limitations of the existing models prevent the potential of bike-sharing from being fully utilized. Therefore, this paper proposes a new model for the multi-modal dynamic transit assignment incorporating bike-sharing (MMDTA-BS) problem to more comprehensively simulate passengers' travel choices and transit distribution.

3. The MMDTA-BS Problem

The MMDTA-BS problem is concerned with the assignment of time-dependent passenger demand along passenger paths to a multi-modal transit network incorporating bike-sharing, with the objective of satisfying user equilibrium conditions. This requires first describing the bike-sharing-integrated network, defining passenger paths for demand assignment, and then presenting the user equilibrium objective and problem constraints.

3.1. A Multi-Modal Transit Network Model Incorporating Bike-Sharing

The MMDTA-BS problem is rooted in a transit network, which can be modeled as a graph $G(N, A)$, where N denotes a set of nodes and A denotes a set of arcs that connect these nodes. Unlike conventional multi-modal transit network models, this network incorporates bike nodes and biking arcs. Specifically, only dockless shared bikes, whose adoption has grown rapidly in recent years, are taken into account. We assume that parking areas have unlimited capacity for bike storage; this setup may lead to bike accumulation, a phenomenon consistent with real-world scenarios.

The multi-modal transit network includes four node types: (1) a *bike node* denoting a parking area for bike-sharing; (2) a *rail node* denoting a station of a rail line; (3) a *bus node* denoting a stop of a bus line; and (4) a *passenger node* denoting a passenger origin/destination point. Within this network, passengers can travel via bus, rail, shared bike, or walking, leading to the definition of the four arc types shown in Figure 1.

Bus and rail services operate on fixed lines: *rail arcs* connect consecutive rail nodes on the same line (Figure 1a), and *bus arcs* link consecutive bus nodes on the same line (Figure 1b). For biking, passengers rent/return bikes at parking areas, so *biking arcs* connect any pair of bike nodes (Figure 1c). This reflects the free-floating nature of dockless bike-sharing, where a direct connection is assumed between any two parking areas, with the cost (time, fare) being a function of the distance. *Walking arcs* (Figure 1d) exist between any two nodes to enable transfers, but only if walking time is under a threshold (e.g., 15 min). The threshold is determined according to passenger travel habits within the transit network. Let A_r , A_b , A_s , and A_w denote rail, bus, biking, and walking arc sets, respectively, such that the arc set $A = A_r \cup A_b \cup A_s \cup A_w$.

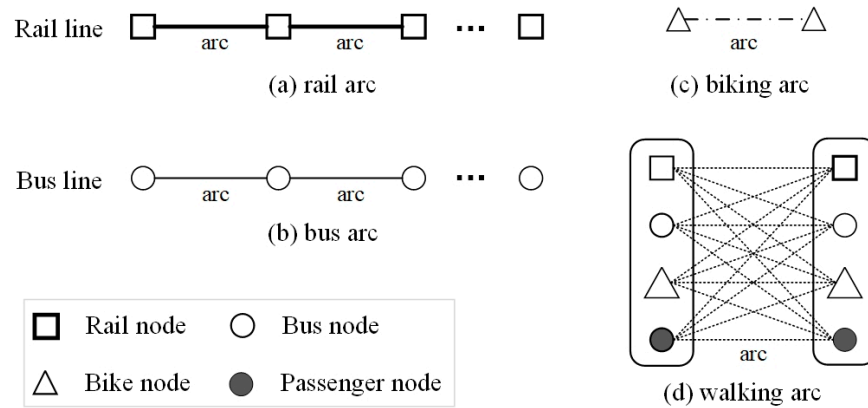


Figure 1. An illustration of the arcs.

3.2. Passenger Path

Assigning the passenger demand to a given transit network involves allocating it to a set of passenger paths within the transit network. Consequently, this section begins by defining a passenger path, then gives a classification of path modes and the cost of paths.

3.2.1. Definition of a Passenger Path

On a transit network, passengers begin by walking from an origin passenger node to a bus, rail, or bike node, make transfers between modes, and end with a walk to a destination passenger node. Therefore, a *passenger path* can be defined as a sequence of arcs, starting and ending with walking arcs, as shown in Figure 2, where O and D denote origin node and destination node, respectively.

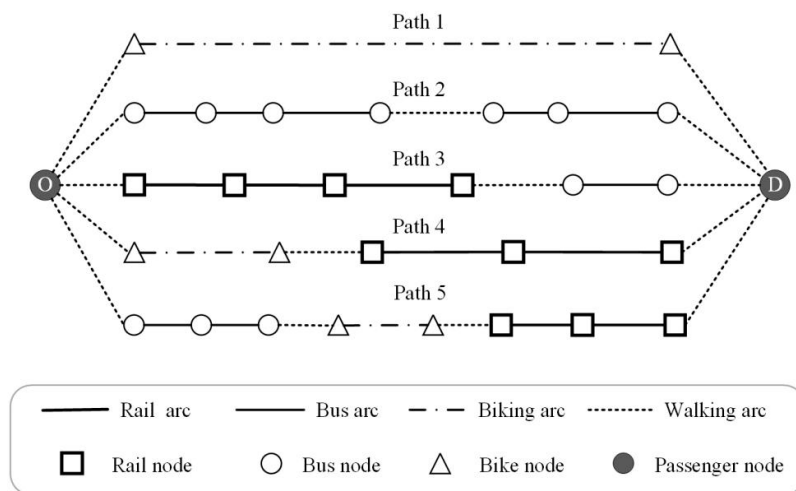


Figure 2. An illustration of a set of paths between an OD pair.

For an OD pair, one or more passenger paths may exist. Unlike conventional transit models, these paths can include bike-sharing, which are used for first/last-mile travel (e.g., Paths 1, 4) or convenient transfers between stations. Path 5, for example, demonstrates a multi-modal journey where a bike is used to connect two rail stations, potentially bypassing a slower or more congested bus transfer. This highlights the role of bike-sharing not just as an access mode but also as a competitive transfer mode within the network.

3.2.2. Classification of Path Modes

A passenger path (i.e., a passenger's journey) comprises one or more arc types. Except for walking arcs, each arc corresponds to a public transit mode (bus, rail, or shared bike). Seven types of path modes arise from combining these three transit modes, defined as follows: $M = \{bus, rail, bike-sharing, bus + rail, bus + bike-sharing, rail + bike-sharing, bus + rail + bike-sharing\}$.

All these path modes support both transfers between different public transit modes and transfers within the same public transit modes.

3.2.3. The Cost of a Passenger Path

The cost of a passenger path can be straightforwardly defined as the aggregate cost of all arcs comprising the path. For any arc a with departing time t , its cost $c_{a,t}$ is defined below, contingent upon the type of arc involved.

(a) Cost of the rail arc and bus arc

The cost of a bus arc or a rail arc is influenced by the travel time, fare, and congestion, as defined in Equation (1), where β and γ denote the coefficients.

$$c_{a,t} = H_{a,t} + \beta Q_a + \gamma U_{a,t}, \forall t \in T, a \in A_r \cup A_b \quad (1)$$

Each bus or rail arc has a corresponding travel time, which is time-varying. Hence, the travel time of any arc a with departing time t is denoted as $H_{a,t}$. The departure time of each vehicle on each arc is determined by the pre-known timetable, and the vehicles on the arcs follow the First In, First Out (FIFO) principle, with no schedule delays occurring.

In practice, bus and rail journey fares are fixed, regardless of the number of arcs. Suppose that the fare of a journey is Q , which cannot be simply imposed on each arc included in the journey. As already mentioned, a walking arc always exists before a passenger boards a bus or rail. Therefore, the fare of any bus or rail arc a can be defined as $Q_a = Q$ if the arc a is immediately following a walking arc; otherwise, $Q_a = 0$, since passengers pay the fare only once upon boarding.

Crowding is frowned upon; hence a crowding penalty $U_{a,t}$ is introduced into the cost of each rail or bus arc a with departing time t , as defined in Equation (2) [24,25], where $H_{a,t}$ denotes the travel time, $V_{a,t}$ denotes the number of passengers aboard the first vehicle (e.g., bus or train) that passes through the arc a after time t , and N_a denotes the number of seats in the vehicle.

$$U_{a,t} = \begin{cases} 0, & \text{if } V_{a,t} < N_a \\ H_{a,t}(V_{a,t} - N_a), & \text{otherwise} \end{cases}, \forall t \in T, a \in A_r \cup A_b \quad (2)$$

(b) Cost of the biking arc

The cost of a biking arc is influenced by biking time, fare, and long biking penalty, as defined in Equation (3), where β and σ represent the coefficients for normalizing different dimensions.

$$c_{a,t} = B_{a,t} + \beta K_{a,t} + \sigma D_a, \forall t \in T, a \in A_s \quad (3)$$

The biking time $B_{a,t}$ is assumed to be time-varying for generality, while the fare $K_{a,t}$ is a function of biking time, as defined in Equation (4), where u denotes the unit time of charge (e.g., 30 min), and v denotes the fare per unit time.

$$K_{a,t} = v \lceil B_{a,t} / u \rceil, \forall t \in T, a \in A_s \quad (4)$$

The riding distance of shared bikes is generally less than 6 km [26], and passengers are averse to long-distance rides. Therefore, a penalty for long-distance rides needs to be

incorporated. A penalty D_a is added to biking arc costs (Equation (5)), where L_a is arc length and L is the long-distance threshold, which is preset based on the travel habits of citizens.

$$D_a = \begin{cases} 0, & \text{if } L_a < L \\ L_a - L, & \text{otherwise} \end{cases}, \forall t \in T, a \in A_s \quad (5)$$

(c) Cost of the walking arc

The cost of a walking arc is influenced by the walking time, waiting time, and delay time, as defined in Equation (6), where φ and ϕ denote the coefficients for waiting time and delay time, respectively.

$$c_{a,t} = W_{a,t} + \varphi G_{a,t} + \phi Z_{a,t}, \forall t \in T, a \in A_w \quad (6)$$

Each walking arc has a corresponding *walking time*, assumed to be time-varying for generality and denoted as $W_{a,t}$.

For a walking arc a leading to a bus or rail node (i.e., station), a *waiting time* $G_{a,t}$ should be included in the arc's cost, as passengers must wait for vehicle arrival. If the arc a does not lead to a bus or rail node, $G_{a,t} = 0$ for all time t .

A reasonable assumption is that passengers at bus or rail nodes adhere to the FIFO principle: passengers who arrive first board the bus first. This operates with buses arriving at stops per the timetable, where the timetable defines departure times from the first station, and subsequent arrival times at other stops are derived from arc travel times. If a bus is full, waiting passengers must board the next arriving one.

Similarly, passengers at bike nodes also adhere to the FIFO principle: those who arrive first at a parking area use a shared bike first. If no bikes are available at the current parking area, passengers need to seek alternative bikes. The time spent searching for bikes translates to additional delay, so a delay time should be added to the walking arc.

For a walking arc a leading to a bus or rail node, a *delay time* $Z_{a,t}$ occurs when passengers cannot board the first arriving vehicle due to overloading, requiring waiting for the next vehicle; for a walking arc a leading to a biking node (parking area), $Z_{a,t}$ occurs when no bikes are available, representing the search time for alternative bikes.

3.3. The MMDTA-BS Model

In this section, symbol definitions are presented first, followed by objectives and constraints, to model the MMDTA-BS problem.

3.3.1. Symbol Definitions

The notations and their descriptions are shown in Table 1.

Table 1. The notations and descriptions used hereafter.

Notation	Description
Parameters:	
P_m^r	the set of paths in mode m of OD pair r ;
q_t^r	the total passenger demand of OD pair r departing at time t ;
$c_{m,p,t}^r$	the cost of path p in mode m of OD pair r departing at time t ;
$d_{m,p,t}^r$	the delay cost of path p in mode m of OD pair r departing at time t ;
$\mu_{m,t}^r$	the cost of the shortest path in mode m of OD pair r departing at time t ;
$x_{m,p,e}^r$	binary parameter, equal to 1 if the path p in mode m contains a biking arc that originates from parking area e and 0 otherwise;
$y_{m,p,e}^r$	binary parameter, equal to 1 if the path p in mode m contains a biking arc that points to parking area e and 0 otherwise;

$z_{m,p,a}^r$	binary parameter, equal to 1 if the path p in mode m contains the arc a and 0 otherwise;
$V_{e,t}$	the number of bikes in parking area e at time t ;
V_a	denotes the vehicle passenger capacity on the line of the arc a ;
Variables:	
$q_{m,t}^r$	the total passenger flow in mode m of OD pair r departing at time t ;
$q_{m,p,t}^r$	the passenger flow on path p in mode m of OD pair r departing at time t ;

3.3.2. Objectives Represented by User Equilibrium Conditions

The MMDTA-BS problem aims to determine a passenger flow distribution that satisfies user equilibrium conditions. Within transit networks, passengers make two key choices: first, selecting an optimal path mode, and second, choosing a specific path within that mode. User equilibrium conditions thus include the following: (1) inter-modal equilibrium, which describes passengers' mode choice behavior, and (2) intra-modal equilibrium, which describes passengers' path choice behavior within a given path mode. Inter-modal equilibrium conditions are based on the minimum path cost of each mode, while within-mode route choice is treated as a deterministic shortest path choice [27,28].

For a passenger OD pair r , let the passenger choose path mode m and path p departing at time $t \in T$. The intra- and inter-modal equilibrium conditions are as follows.

(1) Inter-modal equilibrium condition

The inter-modal equilibrium condition is typically modeled using the logit model shown in Equation (7), which indicates that the passenger flow in the path under each mode is inversely proportional to the cost of the shortest path in the mode. The shortest path is the path with the minimum path cost. The θ is a parameter in the logit model.

$$\frac{q_{m,t}^r}{q_t^r} = \frac{\exp(-\theta \mu_{m,t}^r)}{\sum_{m' \in M} \exp(-\theta \mu_{m',t}^r)}, \forall r \in R, m \in M, t \in T \quad (7)$$

(2) Intra-modal equilibrium condition

The intra-modal equilibrium condition is typically modeled as Equation (8), which indicates that if there is passenger flow on path p in mode m , then the path p must be the shortest path in mode m . Multiple shortest paths in mode m with the same cost may all have passenger flows on them.

$$q_{m,p,t}^r (c_{m,p,t}^r - \mu_{m,t}^r) = 0, \forall r \in R, m \in M, t \in T, p \in P_m^r \quad (8)$$

3.3.3. Constraints

The MMDTA-BS problem encompasses three primary constraints: flow balance constraint, dynamic bike resource constraint, and vehicle capacity constraint.

Passenger travel necessitates the selection of a passenger path, whereby the total passenger demand between any two passenger nodes equals the sum of passengers traversing all paths connecting them. This is referred to as the *flow balance constraint*, and it is expressed in Equation (9).

$$\sum_{m \in M} \sum_{p \in P_m^r} q_{m,p,t}^r = q_t^r, \forall r \in R, t \in T \quad (9)$$

Each parking area has a limited number of bikes, with rental quantities capped by this limit. Bike counts dynamically change as passengers rent/return bikes, defined by the *dynamic bike resource constraint* in Equation (10), where $t' = t - \lambda_{p,e,t}$ and $\lambda_{p,e,t}$ denotes the travel time from origin node to parking area e at time t via path p .

$$\sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} x_{m,p,e}^r q_{m,p,t'}^r \leq V_{e,t}, \forall e \in E, t \in T \quad (10)$$

$V_{e,t}$ will increase as passengers return bikes to the parking area e and decrease as passengers rent bikes from the parking area e . The calculation of $V_{e,t}$ is in Equation (11). Bike rebalancing is not taken into account; the bike count at parking area $V_{e,t}$ changes only in response to passengers' bike rental and return activities.

$$V_{e,t+1} = V_{e,t} + \sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} (x_{m,p,e}^r q_{m,p,t'}^r - y_{m,p,e}^r q_{m,p,t'}^r), \forall e \in E, t \in T \quad (11)$$

Each bus/train has a passenger capacity, typically uniform for the same line. For each time t , total passengers departing on a bus/rail arc must not exceed the line's capacity, defined by the *vehicle capacity constraint* in Equation (12), where $\bar{t} = t - \xi_{p,a,t}$, and $\xi_{p,a,t}$ is the travel time from the origin node to board the vehicle on arc a at time t via path p .

$$\sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} z_{m,p,a}^r q_{m,p,\bar{t}}^r \leq V_a, \forall a \in A_r \cup A_s, t \in T \quad (12)$$

The number of passengers on any path is nonnegative, leading to the *nonnegativity constraint* in (13).

$$q_{m,p,t}^r \geq 0, \forall r \in R, m \in M, p \in P_m^r, t \in T \quad (13)$$

4. Multi-Modal Dynamic User Equilibrium Model for Transit Networks Incorporating Bike-Sharing

The MMDTA-BS model is recast as a multi-modal dynamic user equilibrium model, which is represented by a variational inequality (VI) formulation. Subsequently, proof of consistency is provided, establishing the equivalence between the VI formulation and the user equilibrium conditions.

4.1. Variational Inequality Formulation

A VI formulation is established in this section to capture the multi-modal dynamic user equilibrium in a transit network incorporating bike-sharing. Let $\mathbf{q}$ be the vector of decision variable $q_{m,p,t}^r$. The VI formulation is to identify a path flow vector $\mathbf{q}^* \in \Omega$ that satisfies Equation (14). The decision space Ω of the VI formulation is contingent upon the constraints (9)–(13).

$$F(\mathbf{q}^*)(\mathbf{q} - \mathbf{q}^*) \geq 0, \forall \mathbf{q} \in \Omega \quad (14)$$

The function F is specifically constructed to guarantee that the solution $\mathbf{q}^*$ of VI Formulation (14) satisfies the user equilibrium conditions (7)–(8). The function F is depicted in Equation (15). Given the path flow $\mathbf{q}^*$, $c_{m,p,t}^r(\mathbf{q}^*)$ represents the path cost, and $d_{m,p,t}^r(\mathbf{q}^*)$ represents the cost of delay time, associated with path p departing at time t in mode m of OD pair r . Additionally, $q_{m,t}^r(\mathbf{q}^*)$ denotes the total flow of OD pair r in mode m departing at time t .

$$F(\mathbf{q}^*) = c_{m,p,t}^r(\mathbf{q}^*) - d_{m,p,t}^r(\mathbf{q}^*) + \frac{1}{\theta} \ln(q_{m,t}^r(\mathbf{q}^*)) \quad (15)$$

Although it is assumed that passengers are served on a FIFO basis, it remains difficult to strictly prove other properties of the model. The VI model is extremely hard to prove rigorously, due to the complicated structure of the simulation model, the path cost, and

the uncertainty in bike travel. However, the explanations for the existence and continuity of the VI model for DTA can be found in Pi et al. (2019) [27].

4.2. Proof of VI Formulation Equivalent to Multi-Modal Dynamic Transit User Equilibrium Conditions

This section is to prove that the proposed VI formulation is equivalent to the inter-modal and intra-modal user equilibrium conditions, respectively.

4.2.1. Proof of VI Formulation Equivalent to Inter-Modal Equilibrium Condition

Proving that the proposed VI formulation is equivalent to the inter-modal equilibrium condition is tantamount to demonstrating that the solution $\mathbf{q}^*$ of the VI formulation (14) satisfies the inter-modal equilibrium condition (7).

Since the satisfaction of the KKT (Karush–Kuhn–Tucker) conditions is necessary for a point to be a solution to a VI formulation, the solution $\mathbf{q}^*$ of the VI formulation (14) must satisfy these conditions. Thus, proving that $\mathbf{q}^*$ meets the inter-modal equilibrium condition is equivalent to demonstrating that the VI's KKT conditions align with (7).

Proof. According to the KKT conditions of the VI formulation [24], the solution $\mathbf{q}^*$ satisfies Equation (16).

$$q_{m,p,t}^{r*} [c_{m,p,t}^{r*} - d_{m,p,t}^{r*} + \frac{1}{\theta} \ln(q_{m,t}^{r*}) - \gamma_t^r - \sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{e \in E} (x_{m,p,e}^r - y_{m,p,e}^r) \omega_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t}] = 0, \forall r \in R, m \in M, p \in P_m^r, t \in T \quad (16)$$

The $\gamma_t^r, \delta_{e,t}, \omega_{e,t}, \pi_{a,t}$ are the dual variables of the constraint (9)–(12), respectively. The $c_{m,p,t}^{r*}$ denotes the path cost under equilibrium flow $\mathbf{q}^*$. The $d_{m,p,t}^{r*}$ denotes the cost of delay time under equilibrium flow $\mathbf{q}^*$. According to Equation (16), for any $q_{m,p,t}^{r*} > 0$, we have that

$$c_{m,p,t}^{r*} - d_{m,p,t}^{r*} + \frac{1}{\theta} \ln(q_{m,t}^{r*}) - \gamma_t^r - \sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{e \in E} (x_{m,p,e}^r - y_{m,p,e}^r) \omega_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t} = 0, \forall r \in R, m \in M, p \in P_m^r, t \in T \quad (17)$$

The biking arc must point from one bike node to another. Therefore, for each path $\sum_{e \in E} (x_{m,p,e}^r - y_{m,p,e}^r) = 0$, we have

$$c_{m,p,t}^{r*} - d_{m,p,t}^{r*} + \frac{1}{\theta} \ln(q_{m,t}^{r*}) - \gamma_t^r - \sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t} = 0 \quad (18)$$

Moving $q_{m,t}^{r*}$ to the left-hand side, we obtain

$$q_{m,t}^{r*} = \exp \left(-\theta (c_{m,p,t}^{r*} - d_{m,p,t}^{r*} - \gamma_t^r - \sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t}) \right) \quad (19)$$

Since $q_t^r = \sum_{m \in M} q_{m,t}^{r*}$, dividing both sides of Equation (19) by q_t^r , we get

$$\frac{q_{m,t}^{r*}}{q_t^r} = \frac{\exp\left(-\theta(c_{m,p,t}^{r*} - d_{m,p,t}^{r*} - \gamma_t^r - \sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t})\right)}{\sum_{m' \in M} \exp\left(-\theta(c_{m',p,t}^{r*} - d_{m',p,t}^{r*} - \gamma_t^r - \sum_{e \in E} x_{m',p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m',p,a}^r \pi_{a,t})\right)} \quad (20)$$

Simplifying Equation (20) by factoring out common terms, we get

$$\frac{q_{m,t}^{r*}}{q_t^r} = \frac{\exp\left(-\theta(c_{m,p,t}^{r*} - d_{m,p,t}^{r*} - \sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t})\right)}{\sum_{m' \in M} \exp\left(-\theta(c_{m',p,t}^{r*} - d_{m',p,t}^{r*} - \sum_{e \in E} x_{m',p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m',p,a}^r \pi_{a,t})\right)} \quad (21)$$

According to Lam et al. [29], $-\sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t}$ in Equation (21) is equal to the cost of delay time as shown in Equation (22).

$$-\sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t} = d_{m,p,t}^r \quad (22)$$

Therefore, substituting Equation (22) into Equation (21) yields

$$\frac{q_{m,t}^{r*}}{q_t^r} = \frac{\exp(-\theta c_{m,p,t}^{r*})}{\sum_{m' \in M} \exp(-\theta c_{m',p,t}^{r*})} \quad (23)$$

If $c_{m,p,t}^{r*}$ equals the shortest path cost $\mu_{m,t}^r$, then Equation (23) is equivalent to Equation (7), which means that the KKT conditions of the VI formulation are equivalent to inter-modal equilibrium condition (7).

4.2.2. Proof of VI Formulation Equivalent to Intra-Modal Equilibrium Condition

Proving that the proposed VI formulation is equivalent to the intra-modal equilibrium condition is equivalent to proving that the solution $\mathbf{q}^*$ of the VI formulation (14) satisfies the intra-modal equilibrium condition (8). This requires **Proposition 1**.

Proposition 1 [30]. Let $F: \mathbf{R}^n \mapsto \mathbf{R}^n$ and $\mathbf{R}_+^n$ denote the nonnegative orthant in $\mathbf{R}^n$. The following complementarity problem: Find $\mathbf{x}^* \geq 0$ such that $F(\mathbf{x}^*) \geq 0$ and $F(\mathbf{x}^*)^T \mathbf{x}^* = 0$ have precisely the same solutions as VI($F, \mathbf{R}_+^n$), if any.

Let $\hat{\mathbf{F}} = (c_{m,p,t}^r - \mu_{m,t}^r \mid \forall r \in R, m \in M, p \in P_m^r, t \in T)$; the intra-modal equilibrium condition (8) can be written as

$$\hat{\mathbf{F}}^T \mathbf{q}^* = 0 \quad (24)$$

Since the feasible domain Ω of the VI formulation (14) is nonnegative, according to **Proposition 1**, Equation (24) and VI($\hat{\mathbf{F}}, \Omega$) have precisely the same solutions. The VI($\hat{\mathbf{F}}, \Omega$) is to find a $\mathbf{q}^*$ that satisfies $\hat{\mathbf{F}}^T (\mathbf{q} - \mathbf{q}^*) \geq 0, \forall \mathbf{q} \in \Omega$. Therefore, proving that $\hat{\mathbf{F}}^T (\mathbf{q} - \mathbf{q}^*)$ is equal to $\mathbf{F}^T (\mathbf{q} - \mathbf{q}^*)$ means that the intra-modal equilibrium condition (8) and the VI formulation (14) have precisely the same solutions.

Proof. Moving $\mu_{m,t}^r$ to the left side of Equation (1), we get

$$\mu_{m,t}^r = -\frac{1}{\theta} \ln(q_{m,t}^r) - \frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) \quad (25)$$

Subtracting $c_{m,p,t}^r$ from both sides of Equation (25), we obtain

$$c_{m,p,t}^r - \mu_{m,t}^r = c_{m,p,t}^r + \frac{1}{\theta} \ln(q_{m,t}^r) + \frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) \quad (26)$$

Adding and subtracting $d_{m,p,t}^r$ on the right-hand side of Equation (26) yields

$$c_{m,p,t}^r - \mu_{m,t}^r = c_{m,p,t}^r - d_{m,p,t}^r + \frac{1}{\theta} \ln(q_{m,t}^r) + d_{m,p,t}^r + \frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) \quad (27)$$

We split the right-hand side of Equation (27) into three parts as follows:

$$\mathbf{F} = \left(c_{m,p,t}^r - d_{m,p,t}^r + \frac{1}{\theta} \ln(q_{m,t}^r) \mid \forall r \in R, m \in M, p \in P_m^r, t \in T \right) \quad (28)$$

$$\mathbf{L} = \left(\frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) \mid \forall r \in R, m \in M, p \in P_m^r, t \in T \right) \quad (29)$$

$$\mathbf{D} = (d_{m,p,t}^r \mid \forall r \in R, m \in M, p \in P_m^r, t \in T) \quad (30)$$

Therefore, Equation (26) transforms into Equation (31).

$$\hat{\mathbf{F}} = \mathbf{F} + \mathbf{L} + \mathbf{D} \quad (31)$$

By multiplying both sides of Equation (31) by $(\mathbf{q} - \mathbf{q}^*)$, we obtain

$$\hat{\mathbf{F}}^T (\mathbf{q} - \mathbf{q}^*) = \mathbf{F}^T (\mathbf{q} - \mathbf{q}^*) + \mathbf{L}^T (\mathbf{q} - \mathbf{q}^*) + \mathbf{D}^T (\mathbf{q} - \mathbf{q}^*) \quad (32)$$

To prove $\hat{\mathbf{F}}^T (\mathbf{q} - \mathbf{q}^*) = \mathbf{F}^T (\mathbf{q} - \mathbf{q}^*)$, it is sufficient to prove that $\mathbf{D}^T (\mathbf{q} - \mathbf{q}^*)$ and $\mathbf{L}^T (\mathbf{q} - \mathbf{q}^*)$ are equal to zero.

Firstly, to prove $\mathbf{L}^T (\mathbf{q} - \mathbf{q}^*) = 0$,

we have

$$\mathbf{L}^T (\mathbf{q} - \mathbf{q}^*) = \sum_{t \in T} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} (q_{m,p,t}^r - q_{m,p,t}^{r*}) \frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) \quad (33)$$

where

$$\begin{aligned} & \sum_{p \in P_m^r} (q_{m,p,t}^r - q_{m,p,t}^{r*}) \frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) \\ &= \frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) \sum_{p \in P_m^r} (q_{m,p,t}^r - q_{m,p,t}^{r*}) \\ &= \frac{1}{\theta} \ln \left(\frac{1}{q_t^r} \sum_{m' \in M} \exp(-\theta \mu_{m',t}^r) \right) (q_{m,t}^r - q_{m,t}^{r*}) = 0 \end{aligned} \quad (34)$$

Therefore, $\mathbf{L}^T (\mathbf{q} - \mathbf{q}^*) = 0$.

Secondly, to prove $\mathbf{D}^T(\mathbf{q} - \mathbf{q}^*) = 0$,
we have

$$\mathbf{D}^T(\mathbf{q} - \mathbf{q}^*) = \sum_{t \in T} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} (q_{m,p,t}^r - q_{m,p,t}^{r*}) d_{m,p,t}^r \quad (35)$$

According to Equation (22), $-\sum_{e \in E} x_{m,p,e}^r \delta_{e,t} - \sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t} = d_{m,p,t}^r$; therefore,

Equation (35) becomes

$$\mathbf{D}^T(\mathbf{q} - \mathbf{q}^*) = \mathbf{D}_1^T(\mathbf{q} - \mathbf{q}^*) + \mathbf{D}_2^T(\mathbf{q} - \mathbf{q}^*) \quad (36)$$

where $\mathbf{D}_1^T = -\sum_{e \in E} x_{m,p,e}^r \delta_{e,t}$, $\mathbf{D}_2^T = -\sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t}$.

Firstly, to prove $\mathbf{D}_1^T(\mathbf{q} - \mathbf{q}^*) = 0$, we split $\mathbf{D}_1^T(\mathbf{q} - \mathbf{q}^*)$ by parking area and arc according to whether the dynamic bike resource constraint (10) is valid or not, and we obtain

$$\mathbf{D}_1^T(\mathbf{q} - \mathbf{q}^*) = -\sum_{t \in T} \left(\sum_{\substack{e \in E, \\ v_{e,t} < V_{e,t}}} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} (q_{m,p,t}^r - q_{m,p,t}^{r*}) x_{m,p,e}^r \delta_{e,t} \right) + \sum_{\substack{e \in E, \\ v_{e,t} = V_{e,t}}} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} (q_{m,p,t}^r - q_{m,p,t}^{r*}) x_{m,p,e}^r \delta_{e,t} \quad (37)$$

where $v_{e,t} = \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} x_{m,p,e}^r q_{m,p,t}^r$ denotes the number of bikes rented from the parking area e at time t . For parking areas with $v_{e,t} < V_{e,t}$, the cost of delay time $\sum_{e \in E} x_{m,p,e}^r \delta_{e,t} = 0$, and Equation (37) becomes

$$\begin{aligned} \mathbf{D}_1^T(\mathbf{q} - \mathbf{q}^*) &= -\sum_{t \in T} \sum_{\substack{e \in E, \\ v_{e,t} = V_{e,t}}} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} (q_{m,p,t}^r - q_{m,p,t}^{r*}) x_{m,p,e}^r \delta_{e,t} \\ &= \sum_{t \in T} \sum_{\substack{e \in E, \\ v_{e,t} = V_{e,t}}} (V_{e,t} - v_{e,t}) \delta_{e,t} = 0 \end{aligned} \quad (38)$$

Then, to prove $\mathbf{D}_2^T(\mathbf{q} - \mathbf{q}^*) = 0$, we split $\mathbf{D}_2^T(\mathbf{q} - \mathbf{q}^*)$ by arc according to whether the vehicle capacity constraint (12) is valid or not, and we obtain

$$\mathbf{D}_2^T(\mathbf{q} - \mathbf{q}^*) = -\sum_{t \in T} \left(\sum_{\substack{a \in A_r \cup A_s, \\ v_{a,t} < V_{a,t}}} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} (q_{m,p,t}^r - q_{m,p,t}^{r*}) z_{m,p,a}^r \pi_{a,t} \right) + \sum_{\substack{a \in A_r \cup A_s, \\ v_{a,t} = V_{a,t}}} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} (q_{m,p,t}^r - q_{m,p,t}^{r*}) z_{m,p,a}^r \pi_{a,t} \quad (39)$$

where $v_{a,t} = \sum_{r \in R} \sum_{m \in M} \sum_{p \in P'_m} z_{m,p,a}^r q_{m,p,t}^r$ denotes the total number of passengers who depart

on arc a at time t . For arcs with $v_{a,t} < V_{a,t}$, the cost of delay time $\sum_{a \in A_r \cup A_s} z_{m,p,a}^r \pi_{a,t} = 0$,

Equation (39) becomes

$$\begin{aligned} \mathbf{D}_1^T(\mathbf{q}-\mathbf{q}^*) &= -\sum_{t \in T} \sum_{\substack{a \in A_r \cup A_s, \\ V_{a,t} < V'_{a,t}}} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} (q_{m,p,t}^r - q_{m,p,t}^{r*}) z_{m,p,a}^r \pi_{a,t} \\ &= \sum_{t \in T} \sum_{\substack{a \in A_r \cup A_s, \\ V_{a,t} < V'_{a,t}}} (V_{a,t} - V'_{a,t}) \pi_{a,t} = 0 \end{aligned} \quad (40)$$

Therefore, $\hat{\mathbf{F}}^T(\mathbf{q}-\mathbf{q}^*)$ is equal to $\mathbf{F}^T(\mathbf{q}-\mathbf{q}^*)$.

5. The Projection-Based Approach with Dynamic Path Costing

A projection-based approach incorporating dynamic path costing (PA-DPC) is proposed to solve the VI formulation. The basic idea is projecting the gradient of the objective function at the current solution onto the feasible region to derive an iterative direction, which is then used to update the solution iteratively toward optimality.

To determine the iterative direction for the current solution, a projection operator, which is tailored specifically for the VI formulation, is proposed. To accurately compute the operator's parameters, a dynamic path costing method is devised. Subsequently, an initial solution generation method is designed to generate an initial solution that complies with the dynamic bicycle resource constraint. Lastly, the framework of the PA-DPC is presented.

5.1. Projection Operator

In the projection-based approach, given the current transit assignment solution $\mathbf{q}$ and step length coefficient τ , the iterative direction for the VI formulation (14) is the projection of $\mathbf{q} - \tau \mathbf{F}(\mathbf{q})$ onto the feasible region Ω . A commonly employed form of projection operator, denoted as $\Pi_{\Omega}(\mathbf{q} - \tau \mathbf{F}(\mathbf{q})) = \operatorname{argmin}_{\mathbf{g} \in \Omega} \theta_{\tau}(\mathbf{q})$, is utilized in this paper, where $\theta_{\tau}(\mathbf{q})$ represents the regularized merit function (RMF), defined as $\theta_{\tau}(\mathbf{q}) = \min_{\mathbf{g} \in \Omega} \mathbf{F}(\mathbf{q})^T(\mathbf{g} - \mathbf{q}) + \frac{1}{2\tau} \|\mathbf{g} - \mathbf{q}\|_2^2$. Therefore, the projection operator $\Pi_{\Omega}(\mathbf{q} - \tau \mathbf{F}(\mathbf{q}))$ can be expressed as the solution of the following quadratic programming model.

$$\min \sum_{\substack{g_{m,p,t}^r \\ r \in R}} \sum_{m \in M} \sum_{p \in P_m^r} \sum_{t \in T} \left(2\tau(g_{m,p,t}^r - q_{m,p,t}^r) \left(c_{m,p,t}^r - d_{m,p,t}^r + \frac{1}{\theta} \ln(q_{m,t}^r) \right) + \frac{1}{2\tau} (g_{m,p,t}^r - q_{m,p,t}^r)^2 \right) \quad (41)$$

subject to (9)–(13).

In the above model, given a set of OD pairs R , path modes M , times T , and the set of paths P_m^r for mode m of OD pair r , $g_{m,p,t}^r$ is the decision variable, representing the projected path flow. The parameters $q_{m,p,t}^r$, $q_{m,t}^r$, $c_{m,p,t}^r$, and $d_{m,p,t}^r$ are influenced by the passenger flow distribution and are updated during each iteration of the model.

The projection operator is a strict descent direction for $\theta_{\tau}(\mathbf{q})$ when $\mathbf{F}(\mathbf{q})$ is Lipschitz continuous and monotone [31]. According to formula (15) of $\mathbf{F}(\mathbf{q})$, when the path cost is Lipschitz continuous and monotone, $\mathbf{F}(\mathbf{q})$ also satisfies the properties of being Lipschitz continuous and monotone. However, even under the FIFO condition assumption, it remains difficult to provide strict mathematical proof for the properties of the path cost. Nevertheless, the projection method is still a common approach for solving DTA problems [32,33], and in this case, the selection of the step length coefficient τ directly affects the algorithm results.

The objective function (41) can be further streamlined by incorporating a constant term and subsequently combining terms through the application of the binomial theorem. The new objective function is presented in formula (42).

$$\min_{g_{m,p,t}^r} \sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} \sum_{t \in T} \left(g_{m,p,t}^r - q_{m,p,t}^r + \tau \left(c_{m,p,t}^r - d_{m,p,t}^r + \frac{1}{\theta} \ln(q_{m,t}^r) \right) \right)^2 \quad (42)$$

To reduce the model scale, the quadratic programming model is discretized by time t and solved sequentially during the solution process, using a time interval shorter than the shortest travel time of any arc.

5.2. Dynamic Path Costing Method

The parameters $c_{m,p,t}^r$ and $d_{m,p,t}^r$ of the projection operator at time t are typically calculated based on the passenger flow distribution from the previous iteration. However, they will change with the variation of path flow prior to time t . To compute them more accurately at time t , a dynamic path costing method is proposed: update path flows from time 1 to $t-1$ using projection operator results to form a new flow distribution, then simulate passenger travel in the transit network with dynamic network loading for this new distribution; the cost of a path departing at t equals its simulated travel cost.

The dynamic path costing method can assess the impact of transit assignment changes on path cost, enabling accurate calculations and accelerating approach convergence.

5.3. Criterion for Convergence

The criterion for convergence is used to determine whether the passenger flow distribution meets the equilibrium conditions. Generally, the equilibrium relative gap (Rgap) is used to measure the difference between the current distribution and the equilibrium distribution [34]. Rgap refers to the relative deviation between the total travel cost of all passengers and the travel cost when all passengers choose the shortest path, and its calculation is shown in Equation (43).

$$\text{Rgap} = \frac{\sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} \sum_{t \in T} c_{m,p,t}^r q_{m,p,t}^r - \sum_{r \in R} \sum_{m \in M} \sum_{t \in T} \mu_{m,t}^r \delta_{m,t}^r}{\sum_{r \in R} \sum_{m \in M} \sum_{p \in P_m^r} \sum_{t \in T} c_{m,p,t}^r q_{m,p,t}^r} \quad (43)$$

The $\delta_{m,t}^r$ represents the total passenger demand of OD pair r in mode m departing at time t when achieving equilibrium. To ensure that Rgap can accurately reflect the equilibrium error, $\delta_{m,t}^r$ is calculated based on the inter-mode equilibrium condition (7), as shown in Equation (44).

$$\delta_{m,t}^r = \frac{\exp(-\theta \mu_{m,t}^r)}{\sum_{m' \in M} \exp(-\theta \mu_{m',t}^r)} q_t^r \quad (44)$$

The approach converges when the Rgap is smaller than a given acceptable value.

5.4. Initial Solution Generation Method

The existing method for generating initial solutions does not consider bike-sharing and fails to satisfy the dynamic bike resource constraints. Therefore, a new initial solution generation method is designed.

Firstly, initial path sets are generated for each OD pair of all path modes using Dijkstra's method. Then, all passenger demands at each time are assigned to the shortest path from the path set, forming a passenger flow distribution. Such a passenger flow distribution may violate the dynamic bike resource constraint. Next, the travel of passengers is simulated in the transit network using the passenger flow distribution.

Finally, the passengers who fail to rent bikes at each parking area are funded to reassign to another path with shorter cost.

5.5. Framework of the Projection-Based Approach Based on Path Costing

Given a transit network with passenger demand q_t^r for each OD pair r at any time t , an initial number of bikes $V_{e,1}$ at each parking area e , a maximum number of iterations K , an iteration count k , an acceptable Rgap ε , and a current Rgap g , the framework of the PA-DPC is given as follows.

Step 1: Set $k = 1$ and $g = 999$, generate a path set P^r for each OD pair r using the Dijkstra's method, and generate initial solution $\mathbf{q}^k$ associated with the P^r using the initial solution generation method;

Step 2: Set $t = 1$, and initialize the dynamic network loading simulation environment;

Step 3: Calculate the cost $c_{m,p,t}^r$ of each path p at time t using the dynamic path costing method;

Step 4: Calculate the path flow $\mathbf{q}_t^k$ at time t using the projection operator;

Step 5: $t++$, if $t \leq T$, go to Step 3; otherwise, go to Step 6;

Step 6: Construct the solution $\mathbf{q}^k = \{\mathbf{q}_t^k | t \leq T\}$ for the k -th iteration and calculate the Rgap g of $\mathbf{q}^k$, $k++$; if $k < K$ and $g > \varepsilon$, generate shortest paths p^m for each mode using the Dijkstra's method, add p^m to P^r if $p^m \notin P^r$, and go to Step 2; otherwise, terminate.

6. Experimental Results

Two regions, Boston and Jiaxing, are selected for experiments. First, the network and parameter settings are given. Then, the effectiveness of the approach is verified. The effect of bike-sharing on passenger flow distributions is further investigated through experiments. Finally, the bike demand at the parking area is analyzed.

6.1. Case Study and Parameter Settings

Two regions, Boston and Jiaxing, are selected, establishing a total of four public transit networks categorized into two types: those with shared bikes and those without. The public transit network in Boston without shared bikes (BO network for short) is relatively smaller, containing 34 nodes and 114 arcs; the Boston network with bike-sharing (BO-BS network for short) comprises 45 nodes and 133 arcs. In contrast, the public transit network in Jiaxing without shared bikes (JX network for short) is larger, including 65 nodes and 384 arcs; the Jiaxing network with bike-sharing (JX-BS network for short) consists of 84 nodes and 726 arcs. In the four networks, the layout and timetable of transit routes and OD demands are given.

Two categories of parameters (parameters of path cost and parameters of approach) are presented in Table 2. All experiments were conducted on a computer equipped with an Intel i7 7700K processor with 16 GB of RAM.

Table 2. Parameter settings.

Type	Parameter	Value	Parameter	Value
Parameters of path cost	β	0.7	σ	0.00067
	γ	0.4	u	15
	ν	1.5	φ	1
	L_a	5000	ϕ	1
Parameters of approach	τ	0.3	ε	0.01
	θ	0.5	K	100

In the experiments, the OD demand is assigned to the paths through the MMDTA algorithms, with the expectation of achieving DUE. The R_{gap} is applied to represent the deviation between the current passenger flow distribution and DUE, where an R_{gap} of 0 indicates that DUE has been achieved.

6.2. Experimental Results and Parameter Sensitivity Analysis

PA-DPC calculates the path flow of all OD pairs across four networks. One Boston OD pair was selected as an example to demonstrate the impact of shared bikes, and its path passenger flows in the BO and BO-BS networks are plotted in Figure 3. Additionally, the path flows for an OD pair in JX are plotted in Figure 4.

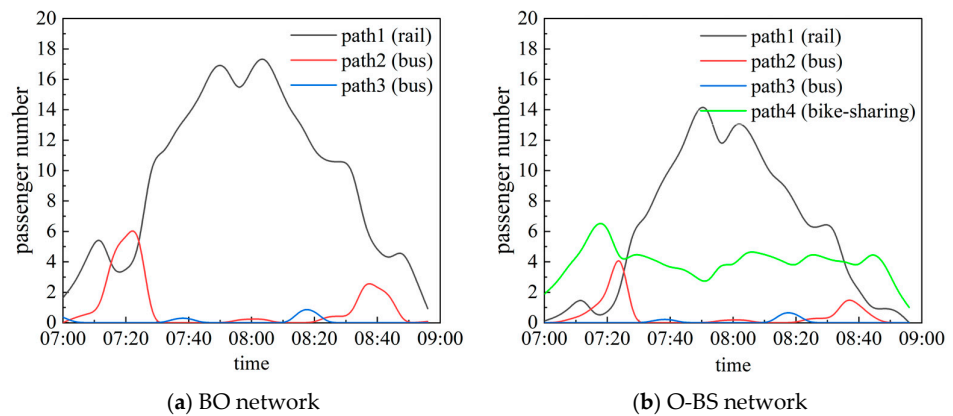


Figure 3. Passenger path flow for an OD pair in Boston.

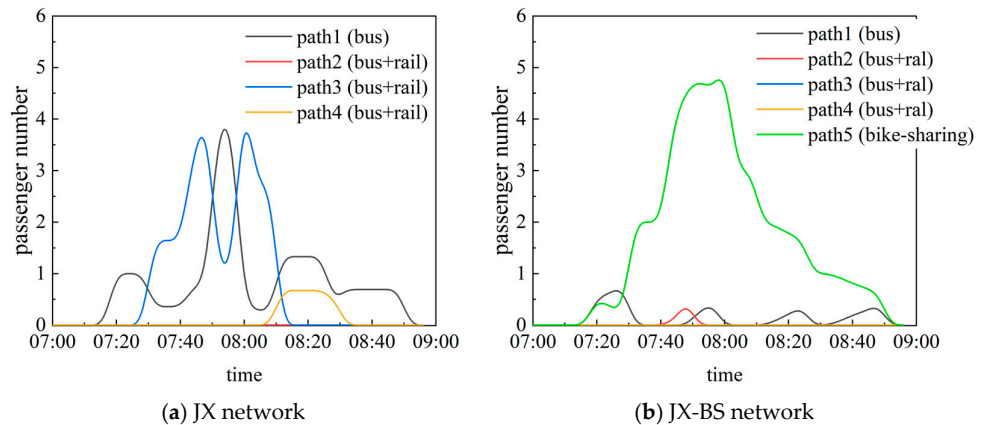


Figure 4. Passenger path flow for an OD pair in Jiaxing.

As shown in Figures 3 and 4, the introduction of bike-sharing has an obvious impact on passenger flow, leading to a significant decrease in passenger flow for bus and rail modes.

To explore the effects of step size τ on the PA-DPC, a sensitivity analysis of step size τ is conducted, as shown in Figure 5.

As shown in Figure 5, as the step size τ increases, R_{gap} increases progressively. When τ is less than 0.4, R_{gap} remains small and varies slightly. Therefore, after comprehensive consideration, τ for PA-DPC is set to 0.3.

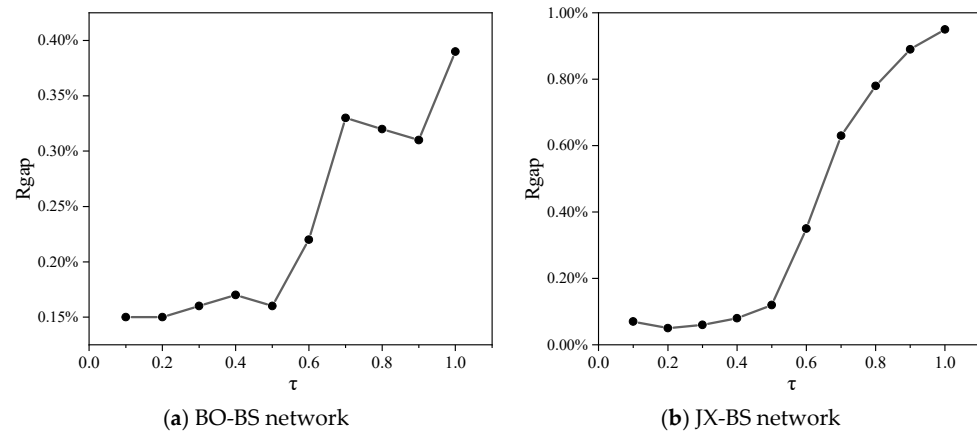


Figure 5. Effect of step size τ on the PA-DPC algorithm.

6.3. Comparison of the Solution Efficiency Between PA-DPC and Other Algorithms

To validate the proposed algorithm and model, PA-DPC is compared with the projection-based algorithm (PA) and the classical method of successive averages (MSA). Results are in Tables 3 and 4: NOI (1%) is iterations until $R_{gap} < 1\%$; Time (1%) refers to the runtime (in seconds) when the R_{gap} first drops below 1%; and Time (PO), Time (DNL), and Time (PU) refer to the runtime (in seconds) of the projection operator, dynamic network loading simulation, and path set update, respectively. R_{gap} is the R_{gap} value after 100 iterations.

Table 3. Comparative results of algorithms based on the BO and BO-BS networks.

		PA-DPC	PA	MSA
BO network	NOI (1%)	13	14	48
	Time (1%)	8.04 s	8.67 s	5.20 s
	Time (PO)	5.64 s	5.88 s	-
	Time (DNL)	0.02 s	0.02 s	0.02 s
	Time (PU)	0.88 s	0.92 s	4.36 s
	Rgap	0.04%	0.04%	0.05%
BO-BS network	NOI (1%)	13	14	Not Available (>1%)
	Time (1%)	8.04 s	8.67 s	
	Time (PO)	5.64 s	5.74 s	
	Time (DNL)	0.02 s	0.02 s	
	Time (PU)	0.86 s	0.98 s	
	Rgap	0.06%	0.06%	7.69%

In Table 3, “Not Available (>1%)” means that the algorithm’s R_{gap} remains above 1% after 100 iterations. Table 3 shows that PA-DPC and PA need similar iterations and time to reach a 1% R_{gap} . The time consumption of both PA and PADPC algorithms is mainly spent on the projection operator. After 100 iterations, both PA-DPC and PA achieve much lower R_{gap} values than others, particularly in the BO-BS network.

As shown in Table 4, R_{gap} rises consistently with network scale. In the JX-BS network, PA-DPC achieves a post-iteration R_{gap} of 0.16%, significantly outperforming MSA. However, the time consumption of both PA and PADPC algorithms is mainly spent on the path set update, which is different from the network in Boston. This is because as the network scale increases, the time consumed in finding the shortest path becomes longer and longer, and improvements to the shortest path method can be considered in the future.

Table 4. Comparative results of algorithms based on the JX and JX-BS networks.

		PA-DPC	PA	MSA
JX network	NOI (1%)	13	14	
	Time (1%)	22.50 s	23.20 s	
	Time (PO)	8.88 s	9.41 s	Not Available (>1%)
	Time (DNL)	0.04 s	0.05 s	
	Time (PU)	12.22 s	13.00 s	
	Rgap	0.05%	0.05%	2.75%
JX-BS network	NOI (1%)	11	12	
	Time (1%)	19.98 s	21.15 s	
	Time (PO)	8.90 s	9.71 s	Not Available (>1%)
	Time (DNL)	0.05 s	0.05 s	
	Time (PU)	10.23 s	10.35 s	
	Rgap	0.16%	0.15%	23.44%

PA and PA-DPC leverage gradient information from the MMDTA-BS model to reach equilibrium in bike-sharing networks, driving solutions to lower Rgap. This confirms that the MMDTA-BS model effectively captures dynamic transit flow equilibrium with bike-sharing. Convergence curves for all five algorithms are plotted in Figure 6.

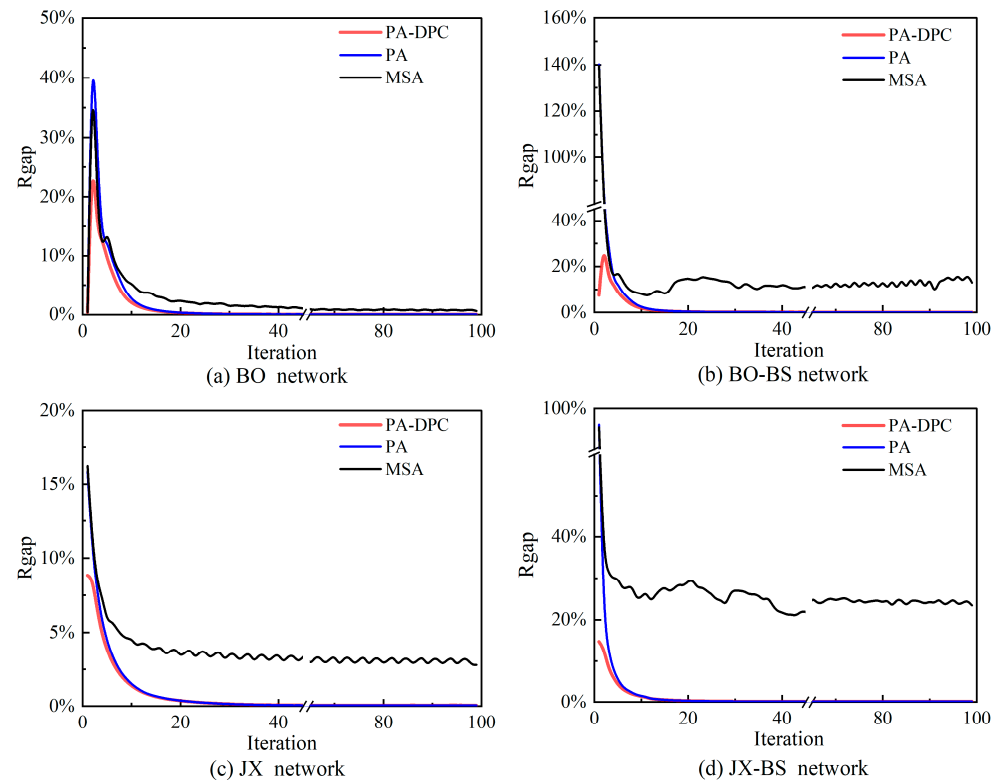
**Figure 6.** Convergence Curves of Algorithms.

Figure 6 shows that MSA exhibits late-stage zigzagging, while PA and PA-DPC reduce oscillations via gradient-guided steps. PA-DPC converges faster than PA by accurately evaluating path costs.

The experimental results reveal that introducing bike-sharing increases the final Rgap, complicating equilibrium attainment due to shifts in passenger flow distribution.

Consequently, further analysis comparing passenger flow distributions with/without bike-sharing is needed.

6.4. Insights into the Effects of Bike-Sharing on Passenger Flow Distributions

Figure 7 shows the passenger flow distribution by path mode from PA-DPC's results to analyze bike-sharing impacts on flow patterns.

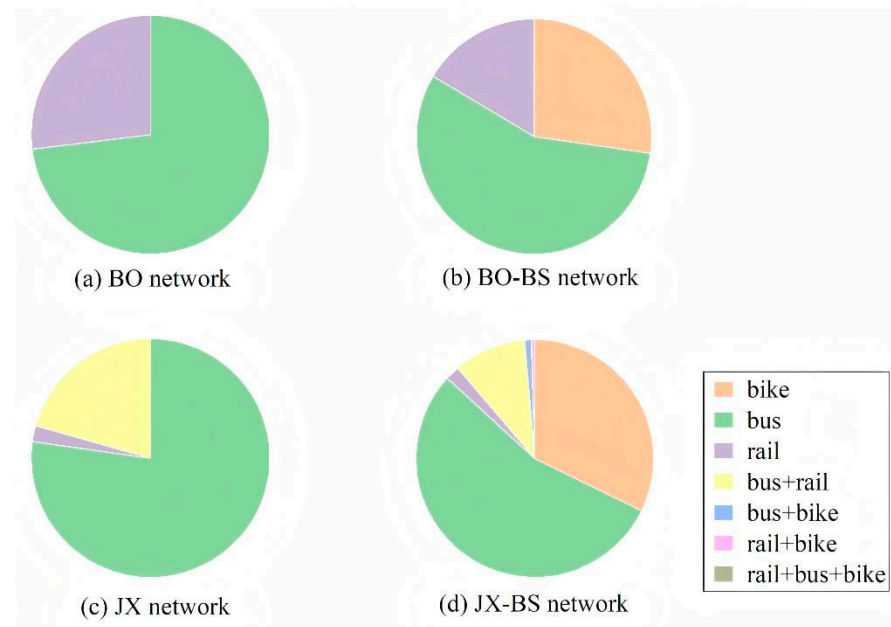


Figure 7. Passenger flow distribution obtained by the PA-DPC algorithm.

Comparing Figure 7a,c and Figure 7b,d, the number of passengers choosing buses is significantly reduced in the transit networks with bike-sharing compared to those without bike-sharing. Many passengers choose path modes that incorporate bike-sharing.

When the distance of the trip is different, the influence of bike-sharing on the passenger flow is also different. Figure 8 counts the number of passengers in various transit modes in short and long-distance situations.

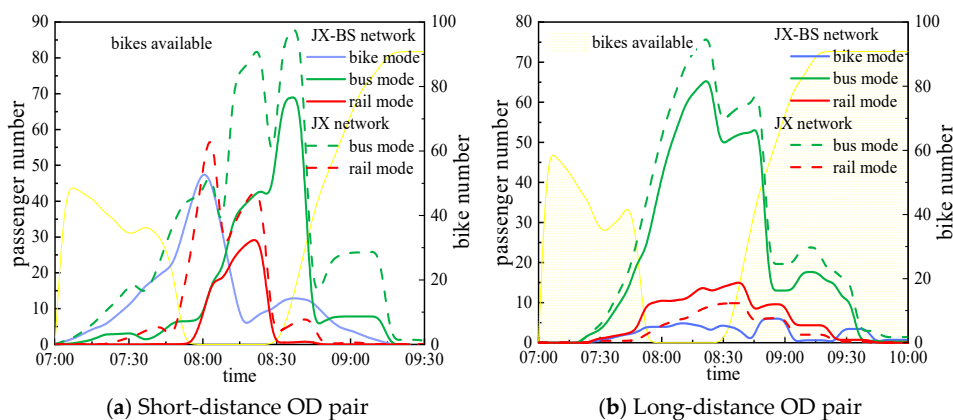


Figure 8. Passenger flows under different modes of transit.

Figure 8 shows bus ridership drops in both regions after introducing bike-sharing: short-distance travelers prefer bikes to avoid transit delays, while long-distance riders shift from bus + rail to bike + rail. Mid-peak bike-sharing use declines, as shown in Figure

8 (a), likely due to parking area shortages. Further experiments analyze real bike-sharing demand.

6.5. Demand Analysis for Shared Bikes by the MMDTA-BS Model

The number of shared bikes in parking areas changes over time. Figure 9 presents the number of shared bikes in parking areas with significant changes from the results obtained by the PA-DPC, for both the BO-BS network and the JX-BS network.

Figure 9 shows that bike counts drop to zero in some parking areas where rentals exceed returns, indicating unmet demand that can be addressed by increasing bike deployment or implementing bike dispatch.

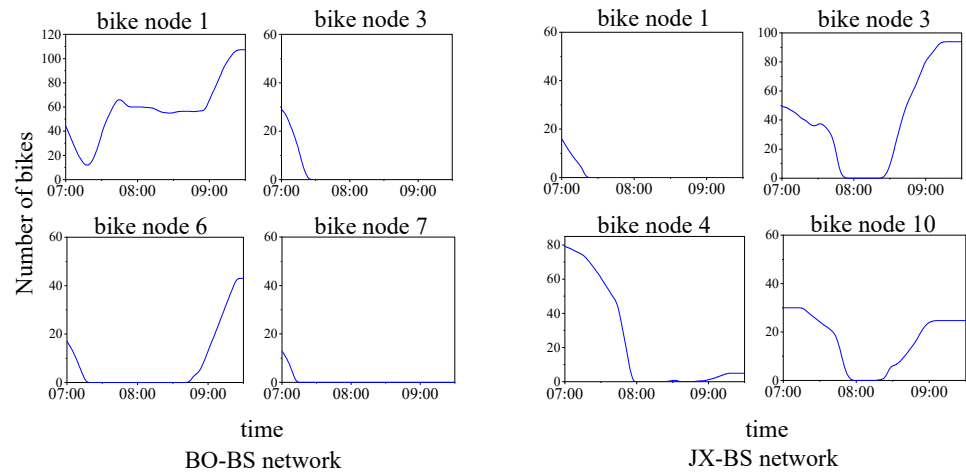


Figure 9. Variation in the number of shared bikes by parking areas.

PA-DPC experiments assuming sufficient bike availability analyze time-varying demand at each parking area using rental-return differences (rentals minus returns). Figure 10 highlights areas with significant differences: blue curves denote positive differences, while red curves show negative ones.

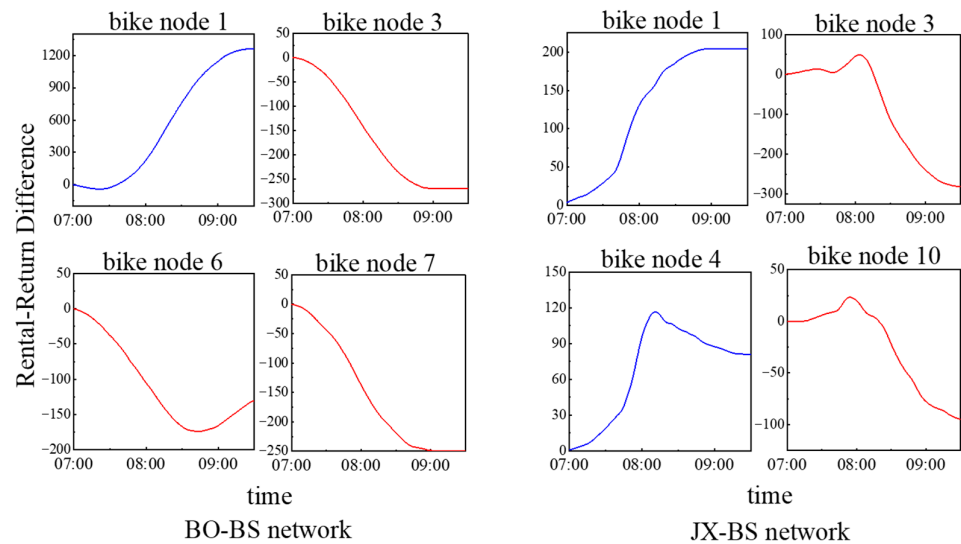


Figure 10. The rental and return differences of shared bikes in various parking.

As shown in Figure 10, rent-return differences vary over time. Smaller differences signal higher bike demand. Operators can use these to relocate bikes to high-demand areas, such as from node 1 to node 3 in the JX-BS network. The maximum difference also helps determine the optimal number of bikes to deploy per node.

The experimental results show that the MMDTA-BS model and the PA-DPC approach can estimate time-specific bike demand per parking area, making them useful for guiding bike deployment and dispatch strategies.

7. Conclusions

This paper has investigated the multi-modal dynamic transit assignment for transit networks incorporating bike-sharing, which holds significant implications for enhancing the optimization of multi-modal transit networks that include bike-sharing services. A novel model for the MMDTA-BS problem is developed and transformed into a multi-modal dynamic user equilibrium model, represented by a variational inequality (VI) formulation. The solution to the VI formulation is proven to satisfy user equilibrium conditions. Furthermore, a projection-based approach with dynamic path costing (PA-DPC) is proposed to solve the VI formulation. Additionally, a dynamic path costing method is designed to accurately calculate the path cost parameter for the projection operator.

Experiments on real-world networks demonstrate PA-DPC's effectiveness in accelerating convergence and the MMDTA-BS model's ability to reflect passenger flow distributions. Results also indicate that bike-sharing effectively transfers passengers to the rail and competes for short-distance trips. The model furthermore offers valuable insights for bike deployment and dispatching strategies.

Future research will focus on optimizing public transit networks and developing adaptive bike-sharing scheduling algorithms within the MMDTA-BS framework.

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