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# Core Stability and Core Selection in a Decentralized Labor Matching Market

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**Abstract:** We propose a dynamic model of decentralized many-to-one matching in the context of a competitive labor market. Through wage offers and wage demands, firms compete over workers and workers compete over jobs. Firms make hire-and-fire decisions dependent on the wages of their own workers and on the alternative workers available on the job market. Workers bargain for better jobs; either individually or collectively as unions, adjusting wage demands upward/downward depending on whether they are currently employed/unemployed. We show that such a process is absorbed into the core with probability one in finite time. Moreover, within the core, allocations are selected that are characterized by surplus splitting according to a bargaining solution such that $(i)$ firms and workforce share total revenue according to relative bargaining strengths, and $(ii)$ workers receive equal workforce shares above their individual outside options. These results bridge empirical evidence and provide a rich set of testable predictions.

**Keywords:** cooperative games; core; evolutionary games; matching; generalized Nash bargaining solution

**JEL:** C71, C73, C78, D83

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## 1. Motivation

In this paper, we study a dynamic labor market matching model where many firms compete over many workers repeatedly over time. The key characteristic of the market is that it is decentralized, as many labor markets are. The stage game underlying our model is a generalized many-to-one matching game with transferable utility. Each firm may employ many workers, but each worker works for at most one firm. The resulting company (the firm-plus-workers coalition) is associated with a revenue that is transferable (from the firm to the workers). The firm's profit is what is left of the revenue after paying wages. The revenue depends on the identities of the firm and of its workers. In addition to the wage component, each worker's utility also depends on the identity of the firm and of their coworkers.

The dynamics are driven by the following three adjustments that may occur at any given time:

*Hire and fire* Firms hire new workers if, in light of relevant wage payments, this promises higher profits. Firms fire workers whose wages exceed their marginal revenue contributions.

*Job search* Unemployed workers look for jobs at their wage demands, which they reduce if they find none. Employed workers look for better jobs and increase their wage demands if better(-paid) jobs are available.

*Intra-company negotiation* In existing companies, unions of workers collectively bargain with their firm counterparts, resulting either in a wage cut or a wage increase for that firm's entire workforce.

The principal motivation for this paper is to formulate a dynamic model of hire-and-fire in a decentralized labor market setting. Too little is known about the convergence properties of such markets compared to centralized markets. Our model is designed to provide a formal game-theoretic link from well-established micro-level bargaining ingredients (e.g., [1,2]) to macro-level labor market phenomena that have been empirically documented (e.g., [3,4]). Motivated by stylized ingredients of real-world labor markets, we extend the standard (static) model in several ways. First, we allow peer effects; that is, workers' utilities and firms' revenues may depend on the composition of the workforce. Second, we specify a fully dynamic model applicable to arbitrary states of the economy, allowing mismatch. Third, we let coalitions of workers form unions, thus adding a further coalitional element to the many-to-one model where the bargaining strengths of unions depend on their internal homogeneity in terms of wages. Finally, we presume that there is initial uncertainty concerning the productivity of newly employed workers.

We obtain convergence results for such markets provided the underlying game has a nonempty core. The convergence results summarize with two main messages. First, from any starting state, the process is absorbed into the core with probability one in finite time. Second, within the core, outcomes are selected that are characterized by a "weighted firm-workforce Nash bargaining solution". These results provide a rich set of testable predictions; for example, homogeneity in wages leads to higher average wages. The long-run stable outcomes of the process are such that, on the one hand, firms and workforce split total revenue according to relative bargaining strengths, and, on the other hand, all workers receive equal shares of their workforce split above outside options.

The rest of this paper is structured as follows. In the next section, we discuss related literature. Section 3 introduces the model's static and dynamic components, as well as relevant solution concepts. Section 4 contains our main results. Section 5 concludes.

## 2. Related Literature

This paper is inspired by the Crawford–Knoer [5] mechanism for core implementations in many-to-one matching markets (also see [6,7]), an ingenious generalization of the deferred acceptance algorithm for one-to-one matching with non-transferable utility [8]. This mechanism proceeds as follows. Starting with wages set at workers' outside options ("minimum wages"), firms make offers to their preferred sets of workers at current wage levels. Receiving the offers, each worker holds on to his most preferred offer, while the wages of all workers who rejected an offer are incrementally increased. Given the new set of wages, firms make offers to their preferred sets of workers, *etc.* The auction terminates when no new offer is made, in which case the last offer is implemented and wages are paid accordingly.[1] The outcome thus reached is a core outcome.

The same chain of adjustments into the core may also be reinterpreted as a process of dynamic recontracting where workers actually work for firms whose offers they do not reject until finding a better(-paid) job. This reinterpretation has been attributed to John Conlisk (see [5]). Indeed, this interpretation is precisely in the spirit of Walrasian tâtonnement equilibration, because the wage evolution follows a stepwise adjustment process driven by over-demand of workers, and leads, over time, to core-stable wage contracts. The aim of our paper is to reformulate Conlisk's interpretation of the Crawford–Knoer mechanism as a formal, fully dynamic model of decentralized recontracting.

The dynamic and decentralized approach to studying matching markets contrasts with typical matching theory. Matching theory mainly develops blueprints for matching programs, such as for university admission, organ transplantation, or hospital residency (see [9] for the classic textbook).

1 Alternatively, the auction may begin with maximum wages, and with reductions being made by workers without offers.

These are used in markets that are regulated by central authorities with well-defined social welfare targets, regularly employing suitable clearing algorithms to enforce the "best" matchings. Often, utility is non-transferable.

Compared to non-transferable utility matching theory, relatively little is known about decentralized matching with transferable utility. [2] However, such markets are also of high practical relevance. Most labor markets, for example, are decentralized. In this paper, we obtain convergence results for a class of matching markets that mirror key features of a decentralized and dynamic labor market. These results rely on nonemptiness of the core, which can be guaranteed by appropriate taxation [11]. Generally, however, the core of a many-to-one matching game can be empty. In the absence of peer effects, [7,12,13] show that *gross substitutability* guarantees nonemptiness of the core.[3] [14,15] generalize gross substitutability. In the presence of peer effects, [16,17] provide algorithms that lead to core outcomes when the core is nonempty.

Non-emptiness of the core is ensured in the context of one-to-one matching. For assignment games, [18,20,21] introduce dynamics that lead to the core and select within. These results motivate the present paper. Their main feature is that, without a central market authority, the core of the economy is reached by local adjustments; matched agents occasionally trying to ratchet up their demands in the hope of finding a better match, and unmatched agents reducing their demands in the hope of becoming matched. Similar arguments had previously been made in the context of Gale–Shapley [22–24]. Core implementation (but not selection) for the assignment game and several generalizations is also shown by [18,19,25–30], and for non-transferable utility (NTU) matching by [31], and particularly by [32,33] in many-to-one NTU matching markets.

Our model is based on Nax and Pradelski [20], from which we differ in two respects. First, as a function of the generalization to many-to-one, we allow peer effects in the workforce, and productivities of firms to depend on the constitution of their workforce. Second, we allow for coalition formation amongst workers that lead to coalitional bargaining. While Nax and Pradelski [20] provide general convergence results for assignment games that hold for almost any "realistic" kind of dynamic, this present paper builds on these results in formulating a model that bridges from the experimental literature on bargaining (e.g., [1,2]) to the empirical literature on (decentralized) labor markets (e.g., [3,4]).

More broadly, our model fits into coalitional bargaining models that lead to core-stable outcomes based on generalizations of Young's [34] evolutionary bargaining model (see [35–37]). An important difference between this strand of models and ours is the assumption regarding information that is available to agents. More specifically, in contrast to these models, our results link up with a growing body of literature studying the evolution of group behavior based on individual adjustments that do not require knowledge about other players' payoffs. Such *uncoupled* dynamics do not lead to Nash equilibrium in noncooperative games in general [38], but several natural dynamics lead to Nash equilibrium in games satisfying genericity conditions [39–42].

## 3. Model

### 3.1. Generalized Many-to-One Matching

Firms, $\{f_1, f_2, ..., f_m\} \in F$, and workers, $\{w_1, w_2, ..., w_n\} \in W$, interact over time $t = 0, 1, 2, ...$ (days). Each period $t$, they may form *companies*, $(i, C)$, including exactly one firm $i \in F$, and a group of workers, $C \subseteq W$, such that every worker $j \in W$ can only be part of at most one company. We shall refer to $C$ as the *workforce* of firm $i$ (or simply as *workforce* where unambiguous).

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2 Refs [9,10] are examples of studies on decentralized matching markets with non-transferable utility.
3 Two workers are gross substitutes for one another if the demand of any firm for one does not go down if the wage of the other goes up.

*Matching* A possible *matching*, $\mathbf{M}^t = (M^t_{iC})_{m\times|\mathcal{P}(W)|}$, is given by a zero-one matrix of dimension $m \times |\mathcal{P}(W)|$ where $\mathcal{P}$ is the power-set excluding the empty set; $\sum_{i\in F}\sum_{C_j\subseteq W} M_{iC_j} \leq 1$ for all $j \in W$ (writing $C_j$ as shorthand for $C \subseteq W$ with $j \in C$), and $\sum_{C\subseteq W} M_{iC} \leq 1$ for all $i \in F$. Let $M_{iC'} = 0$ if there exists $C \supset C'$ such that $(i, C)$ are a company. Let $m^t(i) \in \{0, 1\}$ indicate whether a firm/worker is currently inactive/unemployed (0) or active/employed (1).

*Company value* Each company, $(i, C)$, is able to generate a monetary *company value* (being the revenue net of other non-labor costs that are not part of the model) $\tilde{R}_{iC} \in \mathbf{R}^+_0$.

It is this company value that needs to be split between the firm and its workers. We shall assume that the resulting utilities are linearly separable in these monetary payments that result.

*Minimum wage demand* Each worker $j \in W$ has a time-independent monetary *minimum wage demand*, $d^-_{iC_j} \in \mathbf{R}^+_0$, in a company $(i, C_j)$ which they are a part of.

This minimum wage demand corresponds to the payment at and above which the worker is willing to accept being matched into the company if no better alternative is available; all payments above are *individually rational*. If no company pays the worker's minimum wage demand, no offer is individually rational, and the worker prefers to remain unemployed.

*Full surplus* Every company $(i, C)$ produces a monetary *full surplus* of

$$\bar{\alpha}_{iC} = \left(\tilde{R}_{iC} - \sum_{j\in C} d^-_{iC_j}\right)_+ \tag{1}$$

such that $\bar{\boldsymbol{\alpha}} = (\bar{\alpha}_{iC})_{m\times|\mathcal{P}(W)|}$ describes the surplus matrix of the dimension of $\mathbf{M}$.

Consequence of Linearly Separable Utilities

Minimum wage demands can depend on the identity of their co-workers in the above formulation; "coworkers matter". We shall now show that, under linearly separable utilities, there exists an equivalence between the model where coworkers matter and the model where coworkers do not matter. This will allow us to obtain our main results in the simpler coworkers-do-not-matter environment, but our results, once obtained, can be reformulated for the more complex coworkers-matter case. The crucial consequence of linear separability of utilities is that worker's utility differentials from working for one firm rather than for another, and from working with one set of coworkers rather than with another, must both have separable utility effects that can be compensated with money. One must simply account for these compensations when moving between the two models.

**Proposition 1.** *Let utilities be linearly separable in money. Given any $j$ and $C_j$, $d^-_{iC_j}$ can be equivalently represented by $d^-_{ij}$, depending on the employer only.*

**Proof.** Given a worker $j$ and a firm $i$. Let, for all coalitions $C_j \subseteq W$,

$$d^-_{iC_j} = d^-_{ij} + \tilde{d}^-_{iC_j} \tag{2}$$

such that $d^-_{ij} = \min_{C_j} d^-_{iC_j}$. Therefore $\tilde{d}^-_{iC_j} \geq 0$. That is, $d^-_{ij}$ is $j$'s willingness to accept $i$ given their most preferred coalition $C$. $\tilde{d}^-_{iC_j}$ is the additional amount $j$ is asking for in order to accept $i$ when matched in coalition $C_j$. Now fix a coalition $C$:

$$\bar{\alpha}_{iC} = \left(\tilde{R}_{iC} - \sum_{j\in C} d^-_{iC}\right)_+ = \left(\underbrace{\left(\tilde{R}_{iC} - \sum_{j\in C} \tilde{d}^-_{iC}\right)}_{=:R_{iC}} - \sum_{j\in C} d^-_{ij}\right)_+ = \left(R_{iC} - \sum_{j\in C} d^-_{ij}\right)_+ \tag{3}$$

Note that $R_{iC} \leq \tilde{R}_{iC}$ now incorporates each worker's additional minimal wage at which they are willing to accept, dependent on the given coalition $C$. Further, $\bar{\alpha}_{iC}$ did not change, hence $\bar{\boldsymbol{\alpha}}$ did not change. It follows that all demands during the play of the game can be linearly transformed to coalition-independent demands without changing the game. This transformation is due to the following logic: Given a worker prefers being employed by firm $i$ under coalition $C$ over $C'$, then the firm needs to cover the worker's additional wage request when employed under $C'$ (independent of the specific period or demand). □

*3.2. Re-Formulation*

Using the equivalence, we can work in the simpler coworkers-do-not-matter model, bearing in mind that we must compensate for the separable utility effect when re-interpreting our results obtained using the simplified model formulation. Note that the issue of core existence is unaffected by this transformation. In the model we shall now outline, a firm's utility simply corresponds to its profit, and a worker's utility corresponds to their wage above their minimum demand.

Hence, we can rewrite Equation (1) by

$$\bar{\alpha}_{iC} = \left( R_{iC} - \sum_{j \in C} d^{-}_{ij} \right)_{+} \tag{4}$$

with $\bar{\boldsymbol{\alpha}} = (\bar{\alpha}_{iC})_{m \times |\mathcal{P}(W)|}$ being the full surplus matrix. Note that we write $R_{iC}$ instead of $\tilde{R}_{iC}$ for the company value, and $d^{-}_{ij}$ for the minimum wage demand instead of $d^{-}_{iC_j}$, which depended on $k \neq j$ in $C$.

The full surplus (Equation (4)) is typically only generated once the company works to full productivity. A newly-formed work arrangement, however, may be less than fully productive. We therefore define the actual "initial surplus" in the following way:

*Initial surplus* Every company $(i, C)$, in period $t$, creates a *surplus* of

$$\alpha^{t}_{iC} = \begin{cases} \bar{\alpha}_{iC} & \text{if } (i, C) \text{ was a company in period } t-1 \\ I^{t}_{iC} \cdot \bar{\alpha}_{iC} & \text{if } (i, C) \text{ was not a company in period } t-1, \end{cases} \tag{5}$$

where $I^{t}_{iC}$ are independent random variables taking values in $[0, 1]$ with 0 and 1 having positive probability bound away from zero by a constant. Let $\boldsymbol{\alpha}^{t} = (\alpha_{iC})_{m \times |\mathcal{P}(W)|}$ be the period-$t$ surplus matrix of the dimension of $\mathbf{M}$.

Note that we restrict the possible drop in productivity to one period when new workers join or leave a firm. This simplification is made for technical convenience but could readily be extended to include finitely many periods. Concerning the expected value of $I^{t}_{iC}$, it is natural to consider the possible productivity reductions to be greater the more uncertainty is present in a new company—that is, the larger the change in a company's workforce is. Furthermore, we restrict this drop in productivity not only to the marginal productivity of the new workers but also allow the "established" workers to be affected.

*Wage demands and aspiration levels* Every period $t+1$, each worker $j \in W$ holds a *wage demand* of $w^{t}_{ij}$ for working for any $i \in F$ such that $w^{t}_{ij} \geq d^{-}_{ij}$, where the inequality stands for individually rational demands. Note that it follows from Proposition 1 that $w^{t}_{ij}$ does not depend on the coalition. Further, assume every worker $j$ is agnostic to the specific company they are working for (beyond payoff) and therefore calculates, in any period $t$, their *aspiration level*

$$a^{t}_{j} = w^{t}_{ij} - d^{-}_{ij} \quad \forall\, i. \tag{6}$$

Let $\mathbf{a}^t$ be the vector of all workers' aspiration levels in period $t$.[4] Since wage demands are directly deducible from aspiration levels, we shall use these terms interchangeably.

*Payoffs* In any period $t$, for any given company $(i, C)$, if $M^t_{iC} = 1$, the payoff to each worker $j \in C$ is their demanded wage and thus aspiration level, $a^t_j$. The firm $i$ receives the residual profit $\Pi^t_i = \alpha_{iC} - \sum_{j \in C} a^t_j$. Any inactive firm $i$ ($m^t(i) = 0$) receives payoff zero. Similarly, any unemployed worker $j$ ($m^t(j) = 0$) receives payoff zero.

Let $\mathbf{\Pi}^t$ be the vector of profits of all firms. We assume that a company only forms if its firm's profit is non-negative; for every company $(i, C)$, therefore, $\Pi^t_i \geq 0$.

*Better companies* Given any $[\mathbf{M}^t, \mathbf{d}^t]$, any potential company $(i, C)$ with $M^t_{iC} = 0$ is *better* if

$$\bar{\alpha}_{iC} - \sum_{j \in C} a^t_j \quad > \quad \Pi^t_i, \text{ or} \tag{7}$$

$$\bar{\alpha}_{iC} - \sum_{j \in C} a^t_j \quad = \quad 0, \text{ and } i \text{ is inactive and all } j \in C \text{ are unemployed.} \tag{8}$$

Note that for a better company to form, one or more workers may leave their current companies, unemployed workers may get employed, and previously employed workers may be laid off.[5]

*Outcome* At the end of any period $t$, the *outcome* of the process is given by $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$, the matching, the aspiration levels, and the initial surpluses.

### *3.3. Solution Concepts*

*Optimal matching* In an *optimal matching* total surplus, $\sum_{i \in F, C \in \mathcal{P}(W)} M_{iC} \cdot \bar{\alpha}_{iC}$, is maximized.

*Stability* Any match and aspiration level vector $[\mathbf{M}, \mathbf{a}]$ is *stable* if, for all firms $i$ and for all subsets of workers $C \subseteq W$,

$$\Pi_i + \sum_{j \in C} a_j \cdot m(j) \geq \bar{\alpha}_{iC}. \tag{9}$$

If a match and aspiration level vector are not stable, a firm $i$ and a set of workers $C$ with $M_{iC} = 0$ exist who have a common incentive to deviate and form a new company.

*The Core* **C**, the *core* of the game consists of all outcomes, $[\mathbf{M}, \mathbf{a}, \boldsymbol{\alpha}]$, such that $[\mathbf{M}, \boldsymbol{\alpha}]$ is optimal, $\bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$, and $[\mathbf{M}, \mathbf{a}]$ is stable.

*Bargaining gains (for firms, workers, and unions of workers)* We relate firms', workers' and unions' of workers *bargaining gains* (from trade) to their best alternative gaps within a company $(i, C)$.

The *worker bargaining gain* of worker $j \in C$ with a *wage demand* of $x^{t+1}_j$ (we will specify how players choose their bargaining demand in Section 3.4) at the beginning of period $t+1$ is given by

$$\mu^{t+1}_j(x^{t+1}_j) = (x^{t+1}_j - b^{t+1}_j)_+, \tag{10}$$

where $b^{t+1}_j$ is worker $j$'s current best alternative,

$$b^{t+1}_j = \max_{i' \neq i, C'_j \subseteq W} \left\{ \bar{\alpha}_{i'C'_j} - \Pi^t_{i'} - \sum_{j' \in C'_{j'}, j' \neq j} a^t_{j'} \right\}. \tag{11}$$

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[4] Nax *et al.* [29] generalize this assumption of a one-dimensional aspiration vector to multi-dimensional demand vectors. A similar generalization may be applied in this framework.

[5] The equality in Equation (8) stems from the following tie-breaking rule: Firms (and workers) prefer to be active (employed) over being inactive (unemployed). If a worker is currently employed with a payoff of zero they prefer to remain with their current company over joining another company where their payoff will be zero.

We define the *average bargaining gain* of a worker in coalition $C$

$$\bar{\mu}_C^{t+1} = \frac{\sum_{j \in C} \mu_j^{t+1}}{|C|} \tag{12}$$

and we shall assume that $\bar{\mu}_C^{t+1} > 0$.

Analogously, firm $i$'s *firm bargaining gain* with *profit demand* $X_i^{t+1}$ (see Section 3.4) is given by

$$\mu_i^{t+1}(X_i^{t+1}) = (X_i^{t+1} - B_i^{t+1})_+, \tag{13}$$

where $B_i^{t+1} = \max_{C' \subseteq W} \left\{ \bar{\alpha}_{iC'} - \sum_{j \in C'} a_j^t \right\}$.

**Remark 2.** *Effectively, higher (lower) bargaining gains translate into decreased (increased) bargaining strengths for the respective party.*

Unions are represented by the coalition of workers in a given company and its bargaining gain as a coalition is an aggregation of the individual workers' bargaining gains. We consider the range of possibilities where the union's inner homogeneity (in terms of current wage payments) may, to different degrees, determine its aggregated bargaining strength. We therefore define the *union bargaining gain* of workforce $C$ of the company $(i, C)$ in accordance with the Atkinson [43] equality measure:[6]

$$\nu((\mu_j^{t+1})_{j \in C}) = \frac{|C| \cdot \bar{\mu}_C^{t+1}}{\mathcal{A}_\varepsilon((\mu_j^{t+1})_{j \in C})} \tag{14}$$

where the Atkinson measure $\mathcal{A}_\varepsilon((\mu_j^{t+1})_{j \in C}) \in [0, 1]$ is given by:

$$\mathcal{A}_\varepsilon((\mu_j^{t+1})_{j \in C}) = \begin{cases} \frac{1}{\bar{\mu}_C^{t+1}} \left( \frac{1}{|C|} \sum_{j \in C} (\mu_j^{t+1})^{1-\varepsilon} \right)^{1/(1-\varepsilon)} & \text{for } 0 \leq \varepsilon \neq 1 \\ \frac{1}{\bar{\mu}_C^{t+1}} \left( \prod_{j \in C} \mu_j^{t+1} \right)^{1/|C|} & \text{for } \varepsilon = 1 \end{cases} \tag{15}$$

where $\varepsilon \in [0, \infty)$ is a parameter guiding the disadvantage in bargaining gain through inequality (inequality aversion). For $\varepsilon = 0$, the joint bargaining gain is agnostic to distributional concerns and simply adds up the bargaining gains of workers. For $\varepsilon > 0$, more equality in individual bargaining gain (with the same average) leads to higher levels of the union bargaining gain. In particular, for perfect equality, the union bargaining gain is again the sum of the bargaining gains of the workers. Note that the qualitative feature of the model does not depend on the choice of the particular inequality measure, here of the Atkinson measure, as long as the measure, whichever one is chosen, has a parameter that smoothly guides between the two extreme cases.

Finally, we define a *bargaining weight* $\zeta \in (0, \infty)$ determining the relative bargaining strength of a workforce *vis-à-vis* a firm. For $\zeta < 1$ ($> 1$), the firm (workforce) has an intrinsic advantage. For $\zeta = 1$ and $\varepsilon = 0$, bargaining strengths of firm and workforce are equal if the union bargaining gain of the workers equals the bargaining gain of the firm.

*Firm bargaining solutions* We define bargaining solutions in our many-to-one setup as follows:

*Firm bargaining solution* ($f$-BS) is a state such that, for any company $(i, C)$,

$$\zeta \cdot \mu_i = \nu((\mu_j)_{j \in C}). \tag{16}$$

[6] If $\mathcal{A}_\varepsilon((\mu_j^{t+1})_{j \in C}) = 0$, we set set $\nu((\mu_j^{t+1})_{j \in C}) = \infty$.

*Workforce bargaining solution* ($w$-BS) is a state such that, for any company $(i, C)$ and for all $j, j' \in C$,

$$\mu_j = \mu_{j'}. \tag{17}$$

*Firm-workforce bargaining solution* ($fw$-BS) is both a $f$-BS and a $w$-BS.[7]

### 3.4. Dynamics

At the beginning of any period $t+1$, a coin is tossed determining whether the period will be one of inter- or intra-company bargaining.

#### 3.4.1. Inter-Company Bargaining

Possible triggers for inter-company bargaining are a worker's incentive to find a better job and a firm's incentive to optimize its company structure in order to maximize profit.

*Job Market* If period $t+1$ constitutes an inter-company bargaining period, at the beginning of any such period, each unemployed worker goes to the *Job Market* with probability 1. Each employed worker goes to the Job Market with probability $p_M \in (0, 1]$.

Every period $t+1$, therefore, some $S^W \subseteq W$ is on the Job Market. For all firms $i \in F$, there exists, by consideration of the workers on the Job Market, a set of potential companies, $(i, C')$, such that $C' \subseteq (S^W \cup C)$, where $C$ is firm $i$'s last period workforce ($M^t_{iC} = 1$). Some of these potential companies may be better companies; denote by $\mathbf{C}^{t+1}$ the set of such better companies (see Equations (7) and (8)).

*Wage demands* Unemployed workers make demands equal to their aspiration level. Employed workers who do not go to the Job Market have aspiration level $a^t_j$. Employed workers who go to the Job Market make wage demands based on a temporarily-adjusted aspiration level $a^t_j + Y^t_j$, where $Y^t_j \in R^+$ are independent positive-valued random variables where 0 has positive probability (bounded from 0 by a constant).

Assume negotiations ensue over $\mathbf{C}^{t+1}$, the outcome of this negotiation round is modeled by a random draw of better companies, $\mathbf{C}^{t+1}_{comp} \subseteq \mathbf{C}^{t+1}$, such that companies in $\mathbf{C}^{t+1}_{comp}$ are mutually compatible *(i.e.*, at the end of the period, each worker is only employed by one firm, and each firm only employs a unique set of workers). These companies then form.

Note that firms may suffer a profit decrease when their workforce changes (due to the difference of the full surplus and the initial surplus of a new work arrangement). We assume that a firm stays in a company with its remaining workers unless it cannot satisfy the workers wage demands at a non-negative profit.

*New aspiration levels* At the end of period $t+1$, any worker $j$ who did not go to the Job Market has aspiration level $a^{t+1}_j = a^t_j$. Recall that every unemployed worker receives a payoff of zero, and every employed worker a payoff equal to their aspiration level.

Any previously unemployed worker $j$ has aspiration level

$$a^{t+1}_j = \begin{cases} (a^t_j - Z^t_j)_+ & \text{if } j \text{ is unemployed in period } t+1, \\ a^t_j & \text{if } j \text{ is employed in period } t+1. \end{cases}$$

[7] Rochford [44] first introduced such a solution as a pairwise-bargained solution for the one-to-one assignment game.

where $Z_j^t \in \mathbf{R}^+$ are independent positive-valued random variables with positive expectation bounded below by a positive constant. Any previously employed worker $j$ who went to the Job Market has aspiration level

$$a_j^{t+1} = \begin{cases} a_j^t & \text{if } j \text{ is unemployed in period } t+1, \\ a_j^t + Y_j^t \cdot \Delta(i, C_j) & \text{if } j \text{ is employed (in } (i, C_j)) \text{ in period } t+1, \end{cases}$$

where $\Delta(i, C_j) = 1$ if the firms' additional revenue allows or, if not, $\Delta(i, C_j) < 1$ is a scaling factor such that the firm has revenue exactly zero. Note that, if a worker returns to their previous employer and the rest of the employer's workforce remained unchanged, $\Delta(i, C_j) = 0$.

#### 3.4.2. Intra-Company Bargaining

The firm's payoff is the residual of company value minus paid wages. Hence, possible payoff schedules lie on the intra-company Pareto frontier. Nevertheless, there may be re-bargaining of payoffs within existing firms, which occur in various ways. For example, the workforce of a firm could demand higher wages, and the firm accepts the resulting profit cut to avoid the risk of strike. Similarly, these "movements" on the intra-company Pareto frontier could result from negotiations over the inflation adjustment in the annual wage negotiations (a one-to-one adjustment of the nominal payoff/wage to inflation would mean "no movement" on the intra-company Pareto frontier in terms of real payoffs/wages).

*Appeal* If period $t+1$ constitutes an intra-company bargaining period, one company is drawn from any distribution with full support and enters *intra-company negotiations*. Let company $(i, C)$ be in intra-company negotiations.

At random, either workforce $C$ or firm $i$ makes an appeal. An appeal by either party is such that the appealing party, when it succeeds, is better-off (that is, no one is worse-off and at least someone is better off). This means that the other party ("passive" side) is worse-off when an appeal against it succeeds.

Whether or not an appealing party will win the negotiation depends on the appeal it makes (as we shall detail below). If an appeal fails, the old state is sustained.

*Workforce appeal* $C$ makes a joint appeal for bargaining demands $((x_j^{t+1})_{j \in C})$. Assume that the appeal is such that each worker is at least as well off as before if the appeal is successful; that is, $x_j^{t+1} \geq a_j^t$ for all $j \in C$ and at least one worker is strictly better off. If the workforce appeal leads to the firm having bargaining gain 0, the firm wins the negotiation. Else the probability that $C$ wins the negotiation in period $t+1$ is given by the logit function with *responsiveness* $\beta \in [0, \infty)$,[8] such that

$$p_C^{t+1} = \frac{e^{\beta(\zeta \cdot \mu_i^{t+1}) - \nu_C((\mu_j^{t+1})_{j \in C})}}{e^{\beta(\nu_C((\mu_j^{t+1})_{j \in C}) - \zeta \cdot \mu_i^{t+1})} + e^{\beta(\zeta \cdot \mu_i^{t+1} - \nu_C((\mu_j^{t+1})_{j \in C}))}}. \tag{18}$$

*Firm appeal* $i$ makes an appeal for profit demand $X_i^{t+1} > \Pi_i^t$, such that, if successful, $i$ is better off. Suppose that the response by the workforce is $((x_j^{t+1})_{j \in C})$ such that $\sum_{j \in C} x_j^{t+1} = \bar{\alpha}_{iC} - X_i^{t+1}$. Note that the strength of the response depends on the distribution among the workers. In particular, if $\varepsilon \neq 1$ in Equation (15), $\nu_C$ is uniquely maximized when workers split equally among themselves. Suppose that, with strictly positive probability bounded away from zero, workers respond with the $\nu_C$-maximizing bargaining demand vector.

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[8] Note that this function is often used for smooth-perturbed best-response modeling; other functions, such as probit, would yield qualitatively the same result.

If the firm appeal leads to any worker having bargaining gain 0, the workforce wins the negotiation. Else, the firm appeal succeeds with probability

$$p_i^{t+1} = 1 - p_C^{t+1}. \tag{19}$$

Note that the above appeal success functions depend on $\beta$: for $\beta \to \infty$, a party only wins the negotiation if it has the higher weighted bargaining gain (taking the bargaining weight $\zeta$ into account). On the other hand, for $\beta = 0$, the success of a negotiation is drawn at random, independent of relative bargaining gains.

*New aspiration levels* At the end of period $t+1$, all matchings are preserved. Any worker who was not part of an intra-company negotiation has aspiration level $a_j^{t+1} = a_j^t$. Given company $(i, C)$, if an appeal was successful, each worker $j \in C$ adjusts their aspiration level to

$$a_j^{t+1} = \delta a_j^t + (1-\delta) x_j^{t+1}, \tag{20}$$

with $\delta \in (0,1)$ constant. Note that $\delta$ accounts for the fact that a negotiation outcome also depends on the previous state.

## 4. Analysis

Where non-ambiguous, we omit the time superscript $t$. First note that in a many-to-one transferable utility game, generically, there exists a unique optimal matching.

**Theorem 3.** *Given a generic many-to-one game with non-empty core, starting at any initial state,*

- *the process is absorbed into the core with probability 1 in finite time,*
- *for $\beta \to \infty$ and $\varepsilon > 0$, the process converges to $fw$-BS in the core.*

We will prove the first part of the theorem in Section 4.1, the second in Section 4.2.

### 4.1. Optimality and Stability

**Lemma 4.** *The unique absorbing set of the dynamic $[\mathbf{M}, \mathbf{a}, \boldsymbol{\alpha}]$ is a subset of the core of the underlying cooperative game. Therefore, the core is non-empty if and only if there exists an absorbing set.*

**Proof.** Recall that, for $[\mathbf{M}, \mathbf{a}, \boldsymbol{\alpha}] \in \mathbf{C}$ to be a core outcome, $[\mathbf{M}, \boldsymbol{\alpha}]$ need be optimal, $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$, and $[\mathbf{M}, \mathbf{a}]$ stable. Note that, with positive probability, companies "survive" their initial period of existence. Hence in an absorbing matching $\bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$.

First, suppose that $[\mathbf{M}, \mathbf{a}, \boldsymbol{\alpha}]$ is absorbing and$[\mathbf{M}, \mathbf{a}]$ is not stable. Then there exists a firm $i$ and a group of workers $C$ such that

$$\Pi_i + \sum_{j \in C} a_j < \bar{\alpha}_{iC}. \tag{21}$$

With positive probability, all workers in $C$ are on the Job Market in the same period. Hence, $(i, C)$ form a company with positive probability—a contradiction to the assumption that $[\mathbf{M}, \mathbf{a}, \boldsymbol{\alpha}]$ is absorbing.

Next, suppose $[\mathbf{M}, \mathbf{a}, \boldsymbol{\alpha}]$ is absorbing, but $\mathbf{M}$ is not optimal. Thus, we have, for all $i \in F$ and for all $C \subseteq W$,

$$\bar{\alpha}_{iC} \leq \Pi_i + \sum_{j \in C} a_j. \tag{22}$$

We know that unemployed workers must have aspiration level zero, else they destroy the absorption property. If they demand a positive amount, they remain single and reduce their aspiration levels with positive probability in future periods. Therefore, all aspiration levels must be materializing:

$$\sum_{i \in F} \Pi_i + \sum_{j \in W} a_j = \sum_{(i,C) \in \mathbf{M}} \bar{\alpha}_{iC}. \tag{23}$$

Given that, by assumption, the matching is not optimal, there exists $[\tilde{\mathbf{M}}, \tilde{\mathbf{a}}]$ with higher aspiration levels, for example when $[\tilde{\mathbf{M}}, \tilde{\mathbf{a}}]$ is an optimal matching. Thus:

$$RHS\ (23) < \sum_{(i,C)\in\tilde{\mathbf{M}}} \tilde{\alpha}_{iC}. \tag{24}$$

Since the initial state is stable by assumption, we must in particular have that Equation (22) holds for $(i, C) \in \tilde{\mathbf{M}}$. But then:

$$RHS(24) = \sum_{i\in F:\sum_{C\subseteq W}\tilde{M}_{iC}=1} \Pi_i + \sum_{j\in W:\sum_{i\in F}\tilde{M}_{iCj}=1} a_j \leq LHS(23). \tag{25}$$

This constitutes a contradiction with Equation (24).

Finally, note that intra-company bargaining preserves current matches. In particular, aspiration levels cannot change beyond outside options, and thus the absorbing set is a subset of the core. However, note that within core constraints aspiration levels may still change. □

**Lemma 5.** *Given a non-core state $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$, then either*

1a. *there exists a positive probability transition such that the number of unemployed workers increases and no previously unemployed worker is employed,*

   *and/or*

1b. *there exists a positive probability transition such that the sum of aspiration levels decreases,*

   *or*

2. *$[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$ is such that any worker either has aspiration level zero and is unemployed or he is employed in an optimal matching with stable matching and aspiration levels.*

**Proof.** We shall show that, given any state $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$, not 1a and not 1b implies 2. This suffices to show that either 1a and/or 1b or 2 must hold whenever outcome $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$ is not in the core.

Suppose not 1a and not 1b, and $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$ is not in the core.

*Part 1*: Not 1a and not 1b implies that all unemployed workers have aspiration level zero.

To establish a contradiction, suppose that $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$ is such that an unemployed worker $j$ has a positive aspiration level $a^t_j > 0$. Then, with positive probability, they remain unemployed (even if there exists a better company for them, since with positive probability the initial surplus is too small to support the forming of the company) and reduces their aspiration level at the end of the period by $Z^t_j$. This contradicts the assumption that 1b does not hold.

*Part 2*: Not 1a and not 1b implies that all employed workers and all active firms have stable matching and aspiration levels.

Suppose $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$ is such that there exists a firm $i$, currently in company $(i, C)$, that could form a better company with $C' \neq C$. In this case, with positive probability, $(i, C')$ forms and all previously employed workers $j \in C$ are now singles. Note that, given $(i, C')$ constitutes a newly formed work arrangement, with positive probability, the inital surplus $\alpha^t_{iC'} = I^t_{iC'} \cdot \tilde{\alpha}_{iC'}$ is smaller than the sum of aspiration levels, $\sum_{j\in C'} a^t_j$, hence the aspiration levels cannot be satisfied at non-negative profit and the company goes out of business in the next period of inter-company bargaining. Thus, the number of unemployed workers increased by at least one. This contradicts the assumption that 1a does not hold.

To summarize, independent of the unemployed workers' aspiration levels, the matching of the employed workers and active firms are stable. Therefore, such aspiration levels must also be supported in a core allocation; in particular in a state with any nonzero aspiration levels of currently unemployed workers. Hence, existing companies must be optimal. □

**Lemma 6.** *For any state $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$, there exists a finite positive probability path that terminates in a state as in Lemma 5, Case 2.*

**Proof.** Following the proof of Lemma 5, it is easy to see how to construct such a path. Given that in 1a and/or 1b either the number of unemployed workers increases or the sum of aspiration levels decreases (recall that the expectation of the decrease $Z_j$ is bounded below by a positive constant) it is clear that such a path must terminate in a state as in 2 after finite time. □

**Lemma 7.** *Starting at any state $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$ with $[\mathbf{M}^t, \mathbf{a}^t]$ as in Lemma 5, Case 2 there exists a finite positive probability path that terminates to the core.*

**Proof.** Starting in a state $[\mathbf{M}^t, \mathbf{a}^t, \boldsymbol{\alpha}^t]$ with $[\mathbf{M}^t, \mathbf{a}^t]$ as in Lemma 5, Case 2, any currently matched company constitutes an optimal company with stable matching and aspiration levels. Let $(i, C)$ be an optimal company such that $i$ is currently inactive and all $j \in C$ are currently unemployed. With positive probability, this company forms in the next period of inter-company bargaining. The aspiration levels might not be supported in the core. In a subsequent intra-company bargaining period, with positive probability, the aspiration levels are supported in some core allocation. This is the case with positive probability, since we assumed that the core is non-degenerate and thus there exist open sets for each aspiration level which are supported in the core. Reiterating these steps for each un-matched optimal company $(i, C)$ yields a core state in a finite number of steps. □

Lemmas 4–7 together prove the first assertion of Theorem 3.

*4.2. Equity*

We now turn to the conditions under which drifts to the generalized bargaining solutions exist. We shall first show under which conditions a given company engaged in intra-company bargaining approaches the $f$-BS (Lemma 8) and the $fw$-BS (Lemma 9). In a next step we will show that indeed the whole dynamic converges to the $fw$-BS in the core.

**Lemma 8.** *For $\beta \to \infty$ the $f$-BS is implemented in any company $(i, C)$.*

**Proof.** First suppose that we are already in the core (see Lemma 4). Then the state only changes by intra-company bargaining, and in particular only $\mathbf{a}^t$ may change. In particular, this sustains the core by the fact that negotiations can only be won if the other party has no "better alternative". Now, for $\beta \to \infty$ we have for a workforce appeal

$$\lim_{\beta\to\infty} p_C^{t+1} = \begin{cases} 1 & \text{if } \nu((\mu_j^{t+1})_{j\in C}) < \zeta \cdot \mu_i^{t+1}, \\ 0.5 & \text{if } \nu((\mu_j^{t+1})_{j\in C}) = \zeta \cdot \mu_i^{t+1}, \\ 0 & \text{else.} \end{cases} \tag{26}$$

and analogously for a firm appeal

$$\lim_{\beta\to\infty} p_i^{t+1} = \begin{cases} 0 & \text{if } \nu((\mu_j^{t+1})_{j\in C}) < \zeta \cdot \mu_i^{t+1}, \\ 0.5 & \text{if } \nu((\mu_j^{t+1})_{j\in C}) = \zeta \cdot \mu_i^{t+1}, \\ 1 & \text{else.} \end{cases} \tag{27}$$

If a negotiation is successful, the next appeal in the future by the same party will have strictly less absolute negotiation power. This holds true since we assumed the actively-appealing party is "individually rational", that is, every member of the appealing party is at least as well off if the appeal succeeds.

Therefore, the difference"

$$|\nu((\mu_j)_{j\in C}) - \zeta \cdot \mu_i| \tag{28}$$

is strictly decreasing after each round where company $(i, C)$ successfully engaged in intra-company bargaining (such a sequence is called non-expansive). Convergence to

$$\nu((\mu_j)_{j\in C}) = \zeta \cdot \mu_i \tag{29}$$

follows. This is exactly the $f$-BS. □

**Lemma 9.** *If $\varepsilon > 0$ for $\beta \to \infty$, the $w$-BS is implemented in any company $(i, C)$. The $f$-BS remains implemented as in Lemma 8.*

**Proof.** First suppose that we are already in the core (see Lemma 4). Suppose $(i, C)$ is in a period of intra-company bargaining and the firm makes an appeal. Then, for $\varepsilon > 0$ the unique maximizer of $\nu$ is the $w$-BS. First suppose that the $f$-BS has not yet been reached. Then, with positive probability, this negotiation is successful. If a sequence of such negotiations are successful, eventually the only remaining appeal for the workforce which leads to

$$\nu((\mu_j)_{j\in C}) > \zeta \cdot \mu_i \tag{30}$$

is such that the $w$-BS holds (since $\nu$ is maximized by the egalitarian solution for $\varepsilon > 0$). Else, if the $f$-BS solution is already reached, we must also be in the $w$-BS, for otherwise there exists an appeal by the workforce which would succeed against the current split. This concludes the proof. □

Note that while Lemmas 8 and 9 hold for $\beta \to \infty$, it is also the case, by continuity, that these drifts are observed for finite $\beta > 0$. Then, an increase in $\varepsilon$ accentuates the drift.

**Lemma 10.** *For a many-to-one game with non-empty core, starting in any initial state, for $\beta \to \infty$ and $\varepsilon > 0$, the process converges to the $fw$-BS in the core.*

**Proof.** First we know by the analysis in Section 4.1 that the dynamics is absorbed in the core in finite time. Thus, suppose a core state is reached. Consider the bargaining gain for the current aspiration levels of the workers and the current payoff for the firm. We define the vector of bargaining gain differences for all companies in the core:

$$v^t = (|\nu((\mu_j^t(a_j^t))_{j\in C}) - \zeta \cdot \mu_i^t(\Pi_i^t)|)_{C\in \mathbf{C}^t} \tag{31}$$

Let $T$ be the random mapping which describes a period of intra-company bargaining, randomly picks one company to be the one to bargain in the given period, and choosing an appeal which for $\beta \to \infty$ will succeed with probability 1. The mapping is given by

$$Tv^t = (\nu((x_j^{t+1})_{j\in C}) - \zeta \cdot \mu_i^{t+1}(\bar{\alpha}_{iC} - \sum_{j\in C} x_j^{t+1}))_{C\in \mathbf{C}^t} \tag{32}$$

By our update rule, the new aspiration level for a worker $j \in W$ is given by

$$a_j^{t+1} = \delta a_j^t + (1-\delta) x_j^{t+1} \tag{33}$$

Thus, we have in the next period

$$v^t = |\delta v^t + (1-\delta) T v^t| \tag{34}$$

This iteration rule is known as the "Mann iteration" [45]. Ishikawa [46] proved that if the sequence $\{v^t\}_{t\geq 0}$ is bounded and non-expansive (see Equation 28), then $v^t \to v^*$ with $v^*$ being a fixed point of $T$. A fixed point of $T$ indeed implies that each component of $v^t$ does not change. However, by Lemmas 8 and 9 we know that if convergence occurs for any given company, it must be to the $fw$-BS (given $\varepsilon > 0$ and $\beta \to \infty$). This concludes the proof. □

## 5. Conclusions

In a labor market with free hire and fire, a decentralized process of firm-worker recontracting and wage renegotiating ensues. We study the long-run convergence properties of a stylized model of such a market. Our results are summarized as follows. First, when the core of the underlying market is non-empty, core implementation is long-run stable. Moreover, equitable allocations are selected within the core, provided workers succeed in coordinated bargaining via union formation and that there is some (possibly weak) strengthening of the union through equality. These allocations can be characterized by what we term a "weighted firm-workforce Nash bargaining solution", a solution that combines within-workforce equity with overall revenue sharing reflecting relative bargaining gains of the firm compared to the workforce.

In terms of behavioral and empirical foundations, our model mirrors real-world labor market phenomena as, for example, compiled in Bewley [3,4], where trends of persistent stability and growing intra-company equity (regarding workers' relative wages) have also been observed. Our long-run predictions promise that a decentralized labor market would reach equitable and stable market outcomes. Consequently, one may be tempted to conclude that there is no need to interfere with the labor market. One must be cautious drawing such conclusions, however. Obvious reasons include frictions. Moreover, and perhaps the most important limitation of our results, is that we have only considered nonempty-core cases. Furthermore, we do not know whether convergence occurs in reasonable time.[9] In terms of theory, important avenues for future research, therefore, include analysis of empty core markets, as well as studying whether a typical real-world market has a nonempty core and what role taxation and market interventions could play in relation to this. Empirically, our modeling foundations can be refined using behavioral experiments, and our stylized (and rather optimistic) predictions can be tested empirically.

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[9] Some work in these directions exists for other kinds of matching markets (e.g., [30,47]).

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