A Condorcet Jury Theorem for Large *Poisson* Elections with Multiple Alternatives

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Abstract: Herein, we prove a Condorcet jury theorem (CJT) for large elections with multiple alternatives. Voters have common interests that depend on an unknown state of nature. Each voter receives an imprecise private signal about the state of nature and then submits one vote (simple plurality rule). We also assume that this is a Poisson voting game with population uncertainty. The question is whether the simple plurality rule aggregates information efficiently so that the correct alternative is elected with probability tending to one when the number of voters tends to infinity. The previous literature shows that the CJT holds for large elections with two alternatives, but there is also an example of a large election with three alternatives that has an inefficient equilibrium. We show that there always exists an efficient equilibrium, independent of the number of alternatives. Under certain circumstances (informative types), it is unique in elections with two alternatives. The existence of inefficient equilibria in elections with more than two alternatives is generic.

Keywords: efficient information aggregation; simple plurality rule; Poisson games; condorcet jury theorem

JEL Classification: C72; D71; D72; D82

1. Introduction

Consider voters in a large election. Also assume that there is uncertainty about the state of nature. If voters knew the state of nature, they would all agree on which alternative was best (common interests). Each voter receives an imprecise private signal about the state of nature. As it is quite common in the recent literature on large elections, we assume that the setting is a large Poisson game in which voters are not quite sure about the actual number of voters in the game.[1]

Myerson [2] proved a generalised Condorcet jury theorem (CJT) for such elections with two alternatives and two states of nature: If voters use the simple plurality rule (one person, one vote) and vote strategically, there exists an (limit) equilibrium (as the number of voters converges to infinity), in which the probability of electing the correct alternative in each state of nature is one.[2] On the other hand, Goertz and Maniquet [11] provides an example of a limit equilibrium in the same type of election, but with three alternatives and three states of nature, which is inefficient because the correct

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[1] Poisson games were first introduced by Myerson [1].

[2] The original CJT, introduced by Condorcet [3], is as follows: In a two-alternative election of the above kind, in which voters simply vote for their private information (informative voting), the probability of the correct decision by a committee is always larger than that by a single voter alone, and this probability converges to one if the number of voters converges to infinity. Considering strategic voting, similar results have been shown for small and large two-alternative elections by Wit [4], Feddersen and Pesendorfer [5–8], Bouton and Castanheira [9], among others. Bhattacharya [10] shows that information aggregation is no longer efficient in two-alternative elections if voter preferences are sufficiently different.

alternative is selected with probability zero in one of the states of nature.

We seek to reconcile those two papers and answer the following questions: (1) Can we prove a general Condorcet jury theorem for all elections, independent of the number of alternatives? (2) Are there inefficient equilibria in all such elections, independent of the number of alternatives? (3) What can we say about the co-existence of efficient and inefficient equilibria in the same election?

Myerson [2] presents two examples of elections with two alternatives and hints at the fact that a certain type of inefficient equilibrium exists for one of the examples, but not for the other. Of course, given his CJT for two-alternative elections, we already know that an efficient equilibrium exists for both of his examples. We add the following important qualifications: We provide a condition under which a certain type of inefficient equilibrium (unresponsive equilibrium) exists in each election that satisfies a condition on prior and signal distributions (non-informative signals/types), independent of the number of alternatives. It turns out that one of Myerson's examples satisfies this condition (the one with the inefficient equilibrium), but the other does not. This condition has a quite natural interpretation and shows that these inefficient equilibria may be less likely than anticipated by reading Myerson's example. We also show that—besides unresponsive equilibria—no other inefficient equilibrium can exist in two-alternative elections. So, when signals/types are informative, all equilibria in two-alternative elections are efficient.

Our results add to the common wisdom that two-alternative elections are a "quite different animal" with properties that do not extend to more alternatives. Even if signals/types are informative, inefficient equilibria exist in elections with more than two alternatives (Goertz and Maniquet [11] shows an example of such). As we argue below (and as it has been done elsewhere), more general results for elections with more than two alternatives are currently beyond reach. However, given our results, we now know that an efficient equilibrium always exists, even if it may not be the only one.[3]

2. The Setting

2.1. The Model

We keep our notation similar to the most closely related previous work with more than two alternatives, Goertz and Maniquet [11], but extend it to any number of alternatives beyond three.

Voters vote for one of a finite number of alternatives $\{A_1, A_2...\} = \mathbf{A}$. There is a finite number of states of nature $\{\omega_1, \omega_2 ...\} = \mathbf{\Omega}$, with $|\mathbf{A}| = |\mathbf{\Omega}|$. Voters are uncertain about the underlying state of nature; q_i is the prior probability of state ω_i, with $q_i \in (0, 1)$ for all i. Voters have common, state-dependent, dichotomous preferences. They prefer a particular alternative in each state of nature, and are indifferent between the remaining ones:

$$
\begin{aligned}
u(A_i|\omega_i) &= 1 \ \forall i, \\
u(A_j|\omega_i) &= 0 \ \forall j \neq i.
\end{aligned}
$$

Assume that dichotomous preferences simplifies the strategic environment for the voters. With dichotomous preferences, they care only about those election outcomes in which their vote changes the outcome from any of the $|\mathbf{A}| - 1$ disliked alternatives to the preferred alternative; they do not need to consider election outcomes in which they are pivotal between any of the disliked alternatives. This assumption has been made in the previous literature (e.g., Goertz and Maniquet [13]). Goertz and

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[3]   The familiar reader may wonder how this relates to McLennan [12], who shows for Bayesian games with common preferences and uncertainty about the state of nature, but no population uncertainty, that a strategy that maximises the common utility function is necessarily a Nash equilibrium of the game. McLennan is concerned with the relation between the solution to the maximisation problem and the concept of a Nash equilibrium. In our game, however, we are also interested in whether the strategy satisfies a certain benchmark: efficiency. Besides certain differences in the model (e.g., Bayesian versus Poisson game), McLennan makes no statement about the efficiency of the strategy. It is not necessarily true that the solution to the maximisation problem satisfies our efficiency benchmark.

Maniquet [11] do not assume dichotomous preferences in their example. We assume dichotomous preferences because it is common in the literature and because it makes the analysis of expected utilities much easier. None of our results depend on the assumption (i.e., it is not used in any of our proofs). They would all hold with a more general utility function as well.

We assume that voters are strategic and vote as a function of their expected impact on the outcome of the election. So, each voter should have a positive probability of being the pivotal voter because his/her vote never matters otherwise. Since we are interested in large electorates, we incorporate this feature by assuming that the actual number of voters is uncertain (population uncertainty).[4] This is a fairly common assumption in the literature (e.g., Feddersen and Pesendorfer [5,6,8], Myerson [1,2,14], and Goertz and Maniquet [13,15]). We follow Myerson [1,2,14] and assume that the population size is Poisson-distributed with parameter n. The probability that there are exactly N voters is

$$P(N|n) = \frac{e^{-n}n^N}{N!}. \tag{1}$$

If the actual number of voters is a random variable, then each voter has a strictly positive probability of being decisive in any equilibrium. This is true even in equilibria in which all voters vote for the same alternative because there is a strictly positive probability, albeit small, that only one or no other voter shows up.[5]

Another feature of a Poisson game is that strategies are defined type-by-type instead of agent-by agent because, in a Poisson game, "players" individual identities are not globally recognised (Myerson [2]). Thus, when we refer to a specific voter, we always mean a specific voter type. This corresponds to an assumption of symmetric equilibria in Bayesian games in which strategies are defined agent-by-agent.

Before the election, each voter receives a private signal about the state of nature which is informative but imprecise. Signals are independent and identically distributed. In this regard, our model is similar to Feddersen and Pesendorfer [7,8], Myerson [2], and Goertz and Maniquet [11,13].[6] The signal determines a voter's type in our Poisson voting game. For simplicity, we assume that there are only $|\Omega|$ types, so that $\mathbf{T} = \{t_1, ..., t_{|\Omega|}\}$. We denote by $r_i(t_j)$ the probability that a voter is of type t_j in state ω_i.

Myerson [2] shows: In a two-alternative election, an efficient equilibrium exists whenever there exist some types t such that $r_1(t) \neq r_2(t)$, which, by some appropriate relabelling of types, immediately implies that $r_i(t_i) > r_j(t_i)$ for all $i, i \neq j$ (i.e., type t_i is always more likely in state ω_i than in any other state, an important feature for information aggregation).The same feature is used in our Theorem 1 (existence of an efficient equilibrium). Unfortunately, simply assuming $r_i(t_i) \neq r_j(t_i)$ is not sufficient to lead to the desired feature in elections with more than two alternatives. We must assume explicitly:

$$r_i(t_i) > r_j(t_i) \ \ \forall i \neq j. \tag{2}$$

Notice, however, that this assumption is less restrictive than assuming that type t_i is the most likely type in state ω_i, i.e., $r_i(t_i) > r_i(t_j)$ for all $i, j, i \neq j$ which is, for example, assumed by the original Condorcet jury theorem and in Feddersen and Pesendorfer [7]. It simply suffices to assume that type t_i is more likely in state ω_i than in any other state. For the interested reader: This also implies that

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[4]   Population uncertainty implies that a voter does not know precisely how many other voters there are in the game. This is different from uncertainty about the number of voters that abstain; some of the voters that are in the game may decide to abstain.

[5]   This is an important difference between a Poisson voting game and voting games without population uncertainty: Equilibria in which all voters vote for the same alternative cannot be ruled out by eliminating weakly dominated strategies. So, they remain a viable possibility.

[6]   In a second set of papers on information aggregation in large elections, voters receive signals from signal technologies that are differently precise (e.g., Feddersen and Pesendorfer [5,6], and Goertz and Maniquet [15]).

informative voting (voting for one's type) is not automatically efficient because type t_i is not necessarily the most likely type in state ω_i.

If type distributions satisfy Equation (2), there is no aggregate uncertainty in the population. If all private information was public, the state of nature would be known and voters would unanimously vote for the correct alternative (the one that maximises every voter's utility). We use this as a benchmark and call an equilibrium informationally efficient if the elected alternative is the same as the one that would be selected if all information was public.

The voting rule is the plurality rule with the possibility of abstention. Each voter can vote for one alternative or abstain. So, the action space of a voter type is $\mathbf{C} = \mathbf{A} \cup \phi$, where ϕ denotes abstention. The alternative with the largest number of votes is elected. We assume a particular tie-breaking rule that is without loss of generality and has been used in the previous literature (e.g., Goertz and Maniquet [11,13]): Any tie involving alternatives A_i and A_j is broken in favor of A_i as long as $i < j$; otherwise, it is broken in favor of A_j.[7]

An economy in our model is defined by a list $(\mathbf{A}, \mathbf{\Omega}, q, \mathbf{T}, r, n, \mathbf{C})$ that satisfies Equation (2). For any expected size of the population n, a strategy is a function $\sigma_n : \mathbf{T} \to \Delta(\mathbf{C})$, associating a voter type with a probability distribution over $\mathbf{C}$. Let $\sigma_n^C(t) \geq 0$ denote the probability that a voter of type t chooses action $C \in \mathbf{C}$. It has to be true that $\sum_{C \in \mathbf{C}} \sigma_n^c(t) = 1 \forall t \in \mathbf{T}$. We simplify the notation slightly by denoting with $\sigma_n^i(t)$ the probability that a voter chooses to vote for alternative A_i. Suppose that σ_n^* is an equilibrium of the extended Poisson voting game with n expected voters.[8] We are interested in limit equilibria σ^* such that $\sigma_n^* \to \sigma^*$ as $n \to \infty$. Henceforth, we refer to a limit equilibrium whenever we use the term equilibrium.

We denote $P_{\sigma_n}(A_i|\omega_j)$ as the probability that alternative A_i is elected in state ω_j given strategy σ_n.

Definition 1. *Informationally Efficient Equilibrium.* *A limit equilibrium σ^* of an economy $\mathcal{E}$ is informationally efficient if it is true that $P_{\sigma^*}(A_i|\omega_i) = 1 \ \forall i$.*

Notice that this condition is quite strong: It requires that the correct alternative is elected with probability tending to one in each and every state of nature. However, this is a common assumption in the previous literature and is often possible because of large numbers at least in one equilibrium (see the proof of Theorem 1).

We need a few additional definitions to distinguish between different types of equilibria. Following Feddersen and Pesendorfer [7], we call an equilibrium responsive, if voters "change their vote as a function of their private information with positive probability" (p. 26). So, we call a limit equilibrium responsive if $\sigma^*(t_i) \neq \sigma^*(t_j)$ for at least two different $t_i, t_j \in \mathbf{T}$, so that not all voter types vote exactly the same way. We call an equilibrium unresponsive, if all voter types vote the same way, independent of their type. More formally, we call a limit equilibrium unresponsive if $\sigma^*(t_i) = \sigma^*(t_j)$ for all $t_i \neq t_j \in \mathbf{T}$.

The literature also tends to be interested in informative voting; i.e., voters voting for their type with probability one. If all types engage in informative voting, i.e., $\sigma^{i*}(t_i) = 1$ for all i, we call the equilibrium informative.

2.2. Voter Behaviour

Before we can prove any results, we need to consider how rational voters vote in a large Poisson voting game. We will only present overarching principles of voter behaviour in this section and relegate the specifics to the respective proofs and the Appendix A. In the Appendix A, we present the two most important tools for the analysis of a Poisson voting game—the magnitude theorem and the

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7   It is possible to assume another tie-breaking rule, such as a fair coin flip, but this makes computations of expected utilities much more cumbersome and computations unnecessarily long.
8   Extended Poisson voting games are first introduced in Myerson [2].

offset theorem—from Myerson [1].

Recall that voters derive utility from the outcome of the election alone. A rational voter considers pivotal events in which his or her vote changes the outcome of the election from a less to a more preferred alternative. Pivotal events depend on the particular ballot that a voter wants to submit, and to some extent, on the tie-breaking rule. A voter who considers ballot A_2, for example, is pivotal if alternative A_1 has the same number of votes as alternative A_2 and both have at least as many votes as any other alternative. If a voter considers ballot A_1, on the other hand, the voter is pivotal if some alternative has one more vote than alternative A_1 and any other alternative is sufficiently behind.

Generally, E_k^{ij} denotes the pivotal event in which one additional vote for alternative A_i changes the outcome of the election from alternative A_j to alternative A_i in state ω_k. And piv_k^{ij} denotes the probability of this pivotal event. The probability of a pivotal event depends on the underlying strategy. To save on notation, we will avoid this additional index, if it is not misleading.

Besides pivotal events, a rational voter also considers the posterior distribution of the states of nature conditional on his/her type. $q_i(t)$ denotes the posterior probability of state ω_i conditional on a voter being of type t.

If voter type t considers voting for alternative A_i rather than abstaining, the expected utility gain can be written as

$$EU(A_i|t) = \sum_{j \neq i} [q_i(t) piv_i^{ij} - q_j(t) piv_j^{ij}]. \tag{3}$$

If $EU(A_i|t)$ is larger than zero, the voter prefers voting for A_i to abstaining. If the expected utility gain of some other alternative A_j is larger, then the voter prefers voting for A_j instead.

To discuss equilibrium strategies, we have to be able to evaluate equations such as Equation (3). In large elections, the probability of a pivotal event converges to zero, and so do entire equations, such as Equation (3). However, Myerson [1] shows that probabilities of pivotal events do not converge to zero at the same speed. Events with probabilities that converge to zero faster than others are infinitely less likely, and can therefore be ignored. The difference in the speeds of convergence of pivotal probabilities makes comparisons between expected utilities from different ballots meaningful.

Myerson [1] proposes the magnitude as a measure for the speed of convergence in a large Poisson game. The magnitude μ of the probability of a pivotal event is defined as

$$\mu(E_k^{ij}) = lim_{n \to \infty} \frac{log(piv_k^{ij})}{n}.$$

Events with larger magnitude converge slower than those with smaller magnitude, and are therefore, infinitely more likely. The magnitude of an event is either zero or negative and can be calculated by solving a maximisation problem. In the Appendix A, we discuss in a little more detail, how magnitudes and precise probabilities of pivotal events are calculated (magnitude theorem and offset theorem from Myerson [1]) for the cases arising in our proofs. For a more detailed discussion, we would like to refer the reader to Myerson [1] or to Goertz and Maniquet [13,15].

3. Results

First, we show the existence of an efficient limit equilibrium for any economy, independent of the number of alternatives. So, indeed, the CJT extends to economies with more than two alternatives.

Theorem 1. *There exists an informationally efficient limit equilibrium for any economy $\mathcal{E}$ with multiple alternatives if type distributions satisfy Equation (2).*

Proof. The proof is divided into two steps. In step 1, we show that for any $\mathcal{E}$ with a finite number of alternatives that satisfies Equation (2), there exists a sequence of strategies σ_n such that $lim_{n \to \infty} P_{\sigma_n}(A_i|\omega_i) = 1$ for all i. To guarantee that this is true, it is sufficient (due to the law of large

numbers) to verify that the expected fraction of votes for alternative A_i in state ω_i is larger than the expected fraction of votes for each of the other alternatives in that state.[9] In step 2, we deduce from step 1 that there exists a limit equilibrium σ^* that aggregates information efficiently.

Step 1: Consider an economy $\mathcal{E}$ that satisfies Equation (2). Let ϵ_1, ϵ_2, ϵ_3, ...,$\epsilon_{|\mathbf{A}|}$ be small positive numbers such that

$$\epsilon_1 r_1(t_1) = \epsilon_2 r_2(t_2) = ... = \epsilon_K r_{|\mathbf{A}|}(t_{|\mathbf{A}|}).$$

Consider a sequence of strategies in which for each $t_i \in \mathbf{T}$, $\sigma_n^i(t_i) = \epsilon_i$ and $\sigma_n^\phi(t_i) = 1 - \epsilon_i$; i.e., a voter of type t_i mixes between voting for alternative A_i and abstention. Denote by λ_j^i, the expected fraction of votes for alternative A_i in state ω_j. Given the strategy, the expected fractions of votes for the different alternatives in state ω_i are

$$\begin{aligned} \lambda_i^i &= r_i(t_i)\epsilon_i \\ \lambda_i^j &= r_i(t_j)\epsilon_j \ \ \forall j \neq i. \end{aligned}$$

With Equation (2), it is true for each ω_i that $\lambda_i^i > \lambda_i^j \ \forall j \neq i$. So, by the law of large numbers, $lim_{n\to\infty} P_{\sigma_n}(A_i|\omega_i) = 1$ for all i as $n \to \infty$. Notice that σ_n does not depend on n.

Step 2: Let σ_n^* be a sequence of strategies that maximises the ex-ante utility of the voters. Such strategies exist, as they maximise a continuous function on a compact set. We claim that they are equilibrium strategies. Indeed, the existence of a profitable deviation would contradict the fact that $\sigma_n^*(t)$ maximises expected utilities. Additionally, it is impossible that the expected utility from $\sigma_n^*(t)$ is lower than the expected utility from $\sigma_n(t)$ as defined above, because the expected utility of $\sigma_n(t)$ tends to 1. So, it has to be true that $P_{\sigma_n^*}(A_i|\omega_i) \to 1$ for all i as $n \to \infty$, and therefore, $lim_{n\to\infty}\sigma_n^*$ is a limit equilibrium that aggregates information efficiently.[10]   □

The CJT extends to all elections, independent of the number of alternatives. So, an efficient equilibrium also exists for the example in Goertz and Maniquet [11]. Our Theorem 1 and their example, together imply that efficient and inefficient equilibria may co-exist for the same economy/election. In Goertz and Maniquet [11], voters vote informatively in the inefficient equilibrium. Since it is only an example, we do not know whether this type of equilibrium exists for other economies as well. Our Theorem 2, however, establishes that a certain type of inefficient equilibrium exists for every economy, independent of the number of alternatives, if a certain condition on prior and posterior probabilities is satisfied. It is an unresponsive equilibrium in which all voters vote for the same alternative.

The existence of unresponsive equilibria in voting games is often thought of as trivially true. After all, who has an incentive to change their vote if—by construction—nobody is ever pivotal? Notice that this argument does not actually apply to Poisson voting games with population uncertainty. There is always a positive probability—however small—with which every single voter is the only one who shows up for the election. So, voting for the same alternative is not a weakly dominated strategy in a Poisson voting game. Theorem 2 shows that unresponsive equilibria exist only for certain elections that satisfy a condition on prior and posterior probabilities. Since voters do not play weakly dominated strategies, there is no reason to assume that they are less likely to coordinate on these types of equilibria than on other ones, especially other inefficient ones.

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[9]   According to the Law of Large Numbers, the whole mass of probability concentrates in arbitrarily close neighbourhoods around the expected outcomes as $n \to \infty$.

[10]   The second step of the proof is reminiscent of McLennan [12]'s statement (see footnote 3). The important step of our proof is Step 1 which establishes the existence of an efficient strategy. Once we establish that, we use an argument similar to McLennan [12] to argue that this implies the existence of an efficient equilibrium.

Theorem 2. *There exists an unresponsive limit equilibrium in which all voter types vote for the same alternative A_i for economies that satisfy Equation (2) if and only if $\mathcal{E}$ is such that $\frac{r_i(t)}{r_j(t)} \geq \frac{q_j}{q_i}$ for all $i \neq j$ and all $t \in \mathbf{T}$.*

Proof. Recall that each voter has a positive probability of being pivotal. So, we have to check whether voting for A_i is the best response for a voter when pivotal. According to the statement of the theorem, $\sigma^i(t_j) = 1 \ \forall j$, so that in all states of nature ω_j, $\lambda_j^i = 1$, and $\lambda_j^k = 0 \ \forall j$ and $\forall k \neq i$. This means that each respective pivotal event has the same probability in each state of nature. The pivotal events depend on which alternative is voted for, so let us consider the two possible cases one after the other.

Case 1: Suppose that $i = 1$. If all voters vote for alternative A_1, there is no pivotal event in which a single voter can change the outcome of the election from another alternative to A_1. So, $EU(A_1|t) = 0$. For any other alternative A_j, a voter is pivotal if no other voter shows up for voting. The probability of that event (avoiding unnecessary notation) is $piv^0 = e^{-n}$ (with Equation (1) and $N = 0$). With $EU(A_j|t) = q_j(t)piv^0 - q_1(t)piv^0$, voting A_1 is a best response for all t if and only if $q_j(t) \leq q_1(t)$, or if $\frac{r_i(t)}{r_j(t)} \geq \frac{q_j}{q_i}$, for all $j \neq i$.

Case 2: Suppose that $i > 1$. A voter is pivotal for A_i if no other voter shows up and A_1 wins otherwise. So, $EU(A_i|t) = q_i(t)piv^0 - q_1(t)piv^0$. A voter may deviate by voting for A_j with either $j < i$ or $j > i$. If $j > i$, then $EU(A_j|t) = q_j(t)piv^0 - q_1(t)piv^0$. So, voting for A_i is a best response if and only if $q_i(t) \geq q_j(t)$ for all t and all $j > i$.

If $j < i$, a voter is pivotal if either no voter shows up (in which case A_1 wins otherwise), or if one other voter shows up (in which case A_i wins otherwise). The probability of one other voter showing up is $piv^1 = e^{-n}n$ (again, with Equation (1) and $N = 1$). So, $EU(A_j|t) = q_j(t)piv^0 - q_1(t)piv^0 + q_j(t)piv^1 - q_i(t)piv^1$. Voting for A_i i a best response for all t if $piv^0(q_i(t) - q_1(t)) \geq q_j(t)piv^0 - q_1(t)piv^0 + q_j(t)piv^1 - q_i(t)piv^1$. Notice that $lim_{n \to \infty} \frac{piv^0}{piv^1} = 0$. [11] So, in the limit, the comparison of utility gains reduces to $0 \geq q_j(t) - q_i(t)$. Voting for A_i is a best response if and only if $q_i(t) \geq q_j(t)$ for all t and all $j < i$. $\square$

All unresponsive equilibria are, of course, inefficient because the same alternative is elected in all states of nature.

One of the examples in Myerson [2] satisfies the condition in Theorem 2; the other one does not. This is the reason that an unresponsive equilibrium exists for one of the examples, but not for the other. Myerson moves from one example to the other by changing the prior probabilities. However, Theorem 2 shows that the existence of an unresponsive equilibrium actually depends on the interplay between prior and type distributions.

In fact, the condition in Theorem 2 has quite an intuitive interpretation under which the voting behaviour appears to be natural: It implies that all voters consider the same state of nature the most likely, conditional on their type. The private information is not particularly informative, relative to the prior distribution of states.

We conclude that inefficiencies can arise in any election with two or more alternatives if types are not particularly informative. In such an equilibrium, the inefficiency is caused by what we call 'information failures' because the voters' private information is not particularly informative.

In what follows, we assume away such information failures by focusing on economies that do not satisfy the condition in Theorem 2. The "informative types" condition assures that not all voter types consider the same state of nature as the most likely, conditional on their type.

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[11] For the sake of this proof, this can easily be seen from the expressions for piv^0 and piv^1. In general, the offset-theorem (see Appendix A) can be used to evaluate these types of limits, especially when the different pivotal events have the same magnitude.

Definition 2. *Informative Types. For all ω_i for which there exists one t' such that $q_i(t) \geq q_j(t) \ \forall j, j \neq i$, and all $t \in \mathbf{T} \setminus t'$, there exists at least one ω_k such that $q_i(t') < q_k(t')$.*

So far, we know that an efficient equilibrium exists for each economy, independent of the number of alternatives. If types are un-informative, an unresponsive inefficient equilibrium exists as well. So, for economies with un-informative types, efficient and inefficient equilibria co-exist.

Theorem 3 shows that there is an important difference between economies with two and with more than two alternatives. If types are informative, all equilibria in two-alternative elections are efficient.

Theorem 3. *For any $\mathcal{E}$ with $|\mathbf{A}| = 2$ that satisfies Equation (2) and has informative types, all limit equilibria are efficient.*

Proof. Consider any economy with $|\mathbf{A}| = 2$ that has informative types. For alternative A_i and type t, we can write $EU(A_i|t) = q_i(t)piv_i^{ij} - q_j(t)piv_j^{ij}$. Because of Equation (2) and Bayes' Rule, $q_i(t_i) > q_i(t_j)$ and $q_j(t_i) < q_j(t_j)$. So, we can immediately conclude that type t_i is at least as likely to vote for A_i as type t_j and that type t_j is at least as likely to vote for A_j as type t_i (this is true in any two-alternative election). In addition, at most one of the two types mixes. Because types are informative, it cannot be the case that both types vote for the same alternative with probability one. So, with informative types and in a limit equilibrium, type t_i is, in fact, more likely to vote for alternative A_i than type t_j ($\sigma^{*i}(t_i) > \sigma^{*i}(t_j)$) and type t_j is more likely to vote for alternative A_j than type t_i ($\sigma^{*j}(t_j) > \sigma^{*j}(t_i)$). This does not, however, immediately imply that the correct alternative necessarily receives more votes in each state of nature because type t_i is not necessarily the most likely type in ω_i (see Equation (2)).

Let us consider a limit equilibrium that is inefficient because alternative A_j does not win in ω_j. Because of our special tie-breaking rule, this can mean either that A_i and A_j receive the same expected vote share in ω_j (if $j > 1$) or that A_i receives a larger expected vote share than A_j in ω_j (if $j = 1$). We do not need to make a distinction between those two cases because the proof can be one in one step. The difference in expected vote shares in the two different states of nature can be written as:

$$
\begin{aligned}
\lambda_i^i - \lambda_i^j &= r_i(t_i)(\sigma^{*i}(t_i) - \sigma^{*j}(t_i)) + r_i(t_j)(\sigma^{*i}(t_j) - \sigma^{*j}(t_j)) \\
\lambda_j^i - \lambda_j^j &= r_j(t_i)(\sigma^{*i}(t_i) - \sigma^{*j}(t_i)) + r_j(t_j)(\sigma^{*i}(t_j) - \sigma^{*j}(t_j)).
\end{aligned}
$$

So, if the equilibrium is inefficient because alternative A_j does not win (in expected terms), in state ω_j, it must be true that $\lambda_j^i - \lambda_j^j \geq 0$ (equal to or larger than 0 depending on our tie-breaking rule and the respective alternatives). From above, we know that $\sigma^{*i}(t_i) > \sigma^{*i}(t_j)$ and that $\sigma^{*j}(t_j) > \sigma^{*j}(t_i)$. Combining these, we also have $(\sigma^{*i}(t_i) - \sigma^{*j}(t_i)) > (\sigma^{*i}(t_j) - \sigma^{*j}(t_j))$. Now, we can compare the expected vote share difference in the two states of nature and must come to the conclusion that if $\lambda_j^i - \lambda_j^j \geq 0$, then it must also be true that $\lambda_i^i - \lambda_i^j > \lambda_j^i - \lambda_j^j$. So, $\mu(E_j^{ji}) > \mu(E_i^{ji})$ (see the magnitude theorem in the Appendix A). Then, however, both types should only vote for alternative A_j, a contradiction to above. So, an inefficient equilibrium does not exist if types are informative. □

In two-alternative elections, inefficiencies only arise because of 'information failures'; i.e., uninformative types. If types are informative, then there exists a mechanism that leads to an efficient equilibrium: if the expected vote shares are inefficient (and not all voters vote for the same alternative), the pivotal event in the 'inefficient' state always has the larger magnitude, so that all voter types would prefer to vote more for the correct alternative that is now losing. Inefficient vote shares are not stable as an equilibrium.

Unfortunately, this mechanism does not work in elections with more alternatives. It is possible, as the example in Goertz and Maniquet [11] shows, that pivotal events in the inefficient state do not have the largest magnitude, so they are infinitely less likely than pivotal events in the other state(s). As a result, voters do not have an incentive to vote more for the losing alternative and it is not elected

when it should be.

In elections with more than two alternatives, inefficient equilibria can arise due to 'information failures' (when types are uninformative), but also due to 'coordination failures' as in the example of Goertz and Maniquet [11].

One "unfortunate" feature of the example in Goertz and Maniquet [11] is that voters vote informatively—a particularly intuitive voting strategy that is maybe more likely to occur than other types of more complicated strategies—but that the resulting equilibrium is inefficient. The interested reader may wonder whether this is typical in elections with more than two alternatives. This, however, is not the case. If one changes the example slightly—the type distribution in state ω_1 to $r_1(t_1) = 0.4$, $r_1(t_2) = r_1(t_3) = 0.3$—then informative voting is an equilibrium and is efficient. While informative voting may nor may not be efficient in elections with more than two alternatives, the existence of responsive inefficient equilibria is generic (see Goertz and Maniquet [11] for a more detailed discussion).

4. Conclusions

We have proven a Condorcet jury theorem (CJT) for elections with common interests and uncertainty about the state of nature and about the number of players in the election (Poisson voting game): There always exists an efficient equilibrium, independent of the number of alternatives. We have also established the "informative-types" condition: When types are not informative, each of these elections also has an inefficient equilibrium in which all voters vote for the same alternative. Recall that voters in a Poisson voting game do not play weakly dominated strategies in these inefficient equilibria, so that they cannot be eliminated with the typical equilibrium-selection arguments.

If types are informative, we can also show that all equilibria in elections with two alternatives are efficient. As such, two-alternative elections remain "quite a different animal" than elections with more than two alternatives.[12] The mechanism that drives expected vote shares towards efficiency when types are informative does not work in elections with more than two alternatives. While we can show that these elections always have an efficient equilibrium, we know from previous work that they may also have inefficient ones even if types are informative.

Of course, general results about the co-existence of efficient and inefficient equilibria in elections with more than two alternatives (and constructive proofs that show the equilibrium strategies explicitly) would be very desirable. We have to agree with the previous literature that this seems, currently, beyond reach. We will just shortly outline the argument and would like to refer the interested reader to Goertz and Maniquet [13] for a more thorough discussion of this (although their model includes partisans and is more concerned with approval voting than the simple plurality rule).

Myerson [2], among others, shows that strategies in two-alternative elections are step strategies (i.e., in some sense monotonous in the voter's type). In two-alternative elections, a voter of type t_j is always at least as likely to vote for alternative A_j as a voter of type t_i.[13] This also implies that only a small set of strategies qualify as possible equilibria: (1) both voter types vote for the same alternative, or (2) one voter type mixes between the two alternatives and the other voter type votes informatively. So, it is not very hard to narrow down the set of equilibria precisely for a two-alternative election.

Unfortunately, this monotonicity argument no longer holds in elections with more than two alternatives.[14] So, we have nothing other than trial-and-error to guide us when finding possible equilibria for a particular election. This is the reason that some of the recent important findings were presented only as examples (e.g., Goertz and Maniquet [13]'s Theorem 2 and the example in Goertz

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[12]   In a model with partisans and simple scoring rules, Goertz and Maniquet [13] point in a similar direction. Of course, their model is different because it includes partisans. In their model, the partisans are the reason why no simple scoring rule aggregates information efficiently.

[13]   This can easily be derived from Equation (3).

[14]   This immediately follows from an inspection of Equation (3).

and Maniquet [11]). The example in Goertz and Maniquet [11], in which voters vote informatively, shows that inefficient equilibria in elections with more than two alternatives are generic and can also be quite stable (i.e., cannot easily be ruled out with the typical equilibrium selection arguments).

There are other previous works that have investigated other rules than the simple plurality rule for common-interest elections with more than two alternatives and found that there are other rules that may perform better than the simple plurality rule (e.g., Ahn and Oliveros [16] and Goertz and Maniquet [11] found that approval voting has some desirable properties). However, the simple plurality rule remains the most common voting rule in reality. Thus, our focus on that particular rule.

Appendix A. Magnitude Theorem and Offset Theorem

The Magnitude Theorem from Myerson [1] allows us to calculate the magnitude of pivotal events. Since the actual number of voters is uncertain, there are many, many subevents that make up a certain pivotal event—each characterised by the fact that two (or more) alternatives tie or are one vote apart, but each having a different number of actual voters. The magnitude theorem states that the magnitude of the entire pivotal event is, at the limit, equal to the magnitude of its most likely subevent. When calculating the magnitude of an event, one first computes this most likely subevent by solving a maximisation problem and then uses it to calculate the size of the magnitude.

The offset theorem allows us to compare the actual probabilities of two pivotal events that occur in the same state of nature and have the same magnitude.[15] The offset theorem only applies to pivotal events that occur in the same state of nature. There are no results on comparing the probabilities of pivotal events that occur in different states of nature. If, however, two pivotal events occur in two different states of nature in which the distributions of votes for the different alternatives are exactly the same, we can use the theorem.

In this Appendix A, we only show how to use the magnitude theorem and the offset theorem for the cases needed in our proofs. For a more general exposition, we would like to refer the reader to Myerson [1] or Goertz and Maniquet [13].

Appendix A.1. Magnitude Theorem

Consider some pivotal event E_k^{ij}. Recall that $n\lambda_j^i$ denotes the expected number of votes for alternative A_i in state ω_j. N_j^i denotes the actual number of votes for alternative A_i in state ω_j. The most likely subevent of E_k^{ij} is $N_k^i = N_k^j = n\sqrt{\lambda_k^i \lambda_k^j}$ (or, $N_k^i = n\sqrt{\lambda_k^i \lambda_k^j}$ and $N_k^j = N_k^i + 1$ if the pivotal event requires a one-vote difference with our tie-breaking rule), and $N_k^l = n\lambda_k^l$ for all remaining l.[16] Let us now calculate the magnitude of the pivotal event. For our purposes, it is sufficient to consider the following two distinct cases.

Case 1: Only two alternatives are involved in the close race determining pivotal event E_k^{ij}; i.e., there is no alternative A_s such that $n\lambda_k^s > n\sqrt{\lambda_k^i \lambda_k^j}$.

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[15]   The ratio of the probabilities of two pivotal events that occur in the same state of nature but have different magnitudes converges to zero or to infinity because one event is infinitely more likely than the other.

[16]   There are cases in which the most likely subevent does not take the simple form stated above, but these cases do not arise in our analysis.

In this case, the magnitude of the probability of E_k^{ij} can be calculated using the following formula:

$$\mu(E_k^{ij}) = 2\sqrt{\lambda_k^i \lambda_k^j} - (\lambda_k^i + \lambda_k^j). \tag{A1}$$

Case 2: There exists some alternative A_s, such that $n\lambda_k^s > n\sqrt{\lambda_k^i \lambda_k^j}$

In this case, the magnitude of event E_k^{ij} has to be calculated using the formula:

$$\mu(E_k^{ij}) = 3\sqrt[3]{\lambda_k^i \lambda_k^j \lambda_k^s} - (\lambda_k^i + \lambda_k^j + \lambda_k^s). \tag{A2}$$

The magnitude is now smaller because E_k^{ij} is less likely. The reason is that the most likely outcome for N_k^s is $n\lambda_k^s$. If this occurs, then a voter will no longer be pivotal between A_i and A_j. So, A_s needs to receive less votes.

Appendix A.2. Offset Theorem

Consider two pivotal events E_k^{ij} and E_k^{lm}, such that $E_k^{ij} = E_k^{lm} - w$, where $w = (w(A_i))_{A_i \in \mathbf{A}}$ is a vector of finite numbers of votes. So, the two pivotal events differ only by a finite number of votes. That is why they have the same magnitude. We get

$$lim_{n \to \infty} \frac{piv_k^{ij}}{piv_k^{lm}} = \prod_i lim_{n \to \infty} (\lambda_k^i)^{-w(A_i)}. \tag{A3}$$

The offset Theorem implies that $\mu(E_k^{ij}) = \mu(E_k^{ji})$.

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