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# School Choice in Guangzhou: Why High-Scoring Students Are Protected?

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**Abstract:** Each year, millions of middle school graduates in China take a standardized test and compete for high school positions. Unlike other cities, Guangzhou still uses the immediate acceptance mechanism but implements a policy that students in the high-scoring group receive their allocations before those in the low-scoring group. In this paper, we study a class of the Guangzhou mechanisms, including the immediate acceptance (IA) and the serial dictatorship (SD) mechanism. We show that, if a collection of groups is refined by splitting its groups into a larger number of smaller subgroups, then the Guangzhou mechanism will perform more stably and less manipulable than before. This result provides a tool for policy makers to improve the allocation outcome of the IA mechanism under homogeneous priorities and justifies the use of a high-scoring student protection policy in Guangzhou's high school admission.

**Keywords:** school choice; immediate acceptance mechanism; test score

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## 1. Introduction

School choice has been widely discussed in the matching literature. There are two well-known mechanisms provided for policy makers. The first one is the deferred acceptance (DA) mechanism [1], which produces stable matching and is strategy-proof. The second one is the immediate acceptance (IA) mechanism [2], which matches a maximal number of students to their first choices.

In China, high school admissions are centralized processes via standardized tests and the IA mechanism which is defined in Section 2 was the only mechanism used for a long time [3]. Since students do not necessarily report their true preferences over schools, it had been heavily criticized by the public for student manipulations. In 2003, a new mechanism was applied in Hunan's college admission. Since a good test score is much more important than a good strategy in the ranking of colleges, it reduces the incentives to manipulate, and later, it also became popularized in high school admissions.

The Chinese parallel (CP) mechanism (high school admission version):

> Students are ranked from top to bottom with respect to their test scores, and they are allowed to list at most $e$ schools within each choice band. For example, if the preferences of a student are given by an ordering $c_1, c_2, c_3, c_4$, and the length of choice band is $e = 2$, then the first choice band contains $c_1$ (top choice) and $c_2$ (second choice), and the second choice band contains $c_3$ (third choice) and $c_4$ (fourth choice). The CP mechanism works in rounds. In each round $i$, based on students' test scores and their $e$ choices in the $i$th choice band, the clearinghouse uses a serial dictatorship (SD) mechanism which is defined in Section 2 to determine an allocation. Note that the allocation is finalized each $e$ choices.

The above description is based on [4]. In their paper, they consider both high school and college admissions and study a class of the CP mechanisms, including the IA and the DA mechanism. In the context of high school admissions, schools have homogeneous priorities over students and the DA mechanism is equivalent to the SD mechanism. For this reason, the class of the CP mechanisms includes both the IA and the SD mechanism in the sense that it is equivalent to the former one when $e = 1$ and the latter one when $e = \infty$. A mechanism is more stable than another mechanism if, (i) at each problem, the other mechanism produces stable matching, then this mechanism will produce a stable matching too, but (ii) the converse is not always true. A mechanism is more manipulable than another mechanism if, (i) at each problem, the other mechanism is manipulable by some student, then this mechanisms will be manipulable by some student too, but (ii) the converse is not always true. They show that, if the length of choice band, $e$, is increased to $e'$, then the CP($e'$) mechanism is more stable and less manipulable and assigns a lower number of students to their first choices than the CP($e$) mechanism.

Their results provide a justification for the use of the CP mechanism in high school admissions. If policy makers use the CP($e = 1$) (=IA) mechanism, then they will fail to obtain a stable outcome and has to face the manipulation problem. On the other hand, if policy makers use the CP($e = \infty$) (=SD) mechanism, then they will not achieve "first choice maximal", i.e., a maximal number of students to their first choices. Since those two goals are not compatible, the CP mechanism is a good mechanism in the sense that it takes a balance between the IA (first choice maximal) and the SD (stable/strategy-proof) mechanism. Currently, almost all major cities in China have already transitioned from the IA mechanism to the CP mechanism.

The only exception is Guangzhou, which is the third largest city in China after Beijing and Shanghai. The policy maker in Guangzhou still insists on using the IA mechanism but implements an original policy, which makes the mechanism similar but not identical to the IA mechanism.

High-scoring student protect policy:

> Based on students' test scores, the clearinghouse divides the set of students into two groups: the "high-scoring group"and the "low-scoring group". Members in the high-scoring group receive their allocations earlier than those in the low-scoring group.

The Guangzhou (GZ) mechanism:

> Students submit their preferences for schools at the beginning of the mechanism, and these preference submissions are not allowed to be revised later. Given a high-scoring student protection policy, the GZ mechanism works in two rounds. The clearinghouse assigns the high-scoring group to schools in round 1, and then the low-scoring group to the remaining schools in round 2. In each round, based on students' test scores and their preferences, the IA mechanism is used to determine an allocation. Note that the allocation is finalized at each step of the IA mechanism.

The above descriptions are based on Article 4.1.1 and 4.1.3 of [5] and the formal ones are provided in Section 3. As noticed, the GZ mechanism is similar to the CP mechanism in the sense that it is also equivalent to the IA and the SD mechanism in extreme cases. If there is only one group, then all students enter the market simultaneously and the GZ mechanism works as the IA mechanism. On the other hand, if the number of groups is sufficiently large such that each group contains exactly one student, then students enter the market one by one and the GZ mechanism works as the SD mechanism.

Since Guangzhou has continued to use the GZ mechanism over 15 years [6], one may be curious about whether there is a special factor in this mechanism. More specifically, if some factor (e.g., the size of gthe roup) is changed, then one may want to know whether the GZ mechanism has a similar property to the CP mechanism.

Motivated by this question, we consider a school choice model with homogeneous priorities. The high-scoring student protection policy is described as a "score partition", which is a partition of students based on their test scores. Since students can be score-partitioned in many different ways, we study a class of GZ mechanisms in which each member is associated with a score partition of students. We find that, if the size of a group is either increased or decreased, then we may not be able to guarantee that the GZ mechanism with the new score partition still produces stable matching.

Due to this negative finding, we restrict our attention to a special kind of relation between two score partitions, which is called "refinement". For example, given a score partition $\{A, B\}$, if group $A$ is divided into two smaller subgroups, $A1$ and $A2$, then $\{A1, A2, B\}$ is a refinement of $\{A, B\}$. We show that, as the score partition of students becomes refined finer, the GZ mechanism becomes more stable (Theorem 1) and less manipulable (Theorem 2). Moreover, we show that the GZ mechanism may assign a lower number of students to their first choices than the SD mechanism (Theorem 3).

One may take away two implications from these results. First, policy makers can manipulate the score partitions via refinements to get better results than using the IA mechanism. Since students are divided into two groups, the mechanism practiced in Guangzhou is more stable and less manipulable than the IA mechanism. This justifies the use of the high-scoring student protection policy and explains why Guangzhou has continued to use a variant of the IA mechanism for many years.

Second, the CP mechanism is better than the GZ mechanism in the sense that it balances "stability/strategy-proofness"and "first-choice maximal". The CP mechanism is more stable and less manipulable than the IA mechanism and always assigns more students to their first choices than the SD mechanism. In contrast, the GZ mechanism cannot help the policy maker to balance those two goals because it may assign a lower number of students to their first choices than the SD mechanism. For this reason, the policy maker should consider using the CP mechanism in Guangzhou's high school admission.

The rest of this paper is organized as follows. Section 2 introduces the school choice model with homogeneous priorities and provides the formal descriptions of the SD and the IA mechanism. Section 3 introduces the GZ mechanism and presents the main results. Section 4 concludes. The Appendix A contains the proofs of the results in Section 3.

## 2. The Model and the Two Mechanisms

A school choice problem with homogeneous priorities consists of

1. a set of students, $S = \{s_1, \cdots, s_n\}$;
2. a set of schools, $C = \{c_1, \cdots, c_m\}$;
3. a vector of school quotas, $q = (q_{c_1}, \cdots, q_{c_m})$;
4. a list of strict student preferences, $P_S = (P_{s_1}, \cdots, P_{s_n})$; and
5. a strict common test score ordering, $\succ$.

The preference relation $P_s$ of student $s$ is defined over $C \cup \{\emptyset\}$, where $\emptyset$ is the option of being unmatched. For any $c, c' \in C \cup \{\emptyset\}$, we write $cR_sc'$ if and only if either $cP_sc'$ or $c = c'$. The test score ordering $\succ$ is derived from the result of standardized test. For any $s, s' \in S \cup \{\emptyset\}$, $s \succ s'$ if and only if $s$ has a higher test score than $s'$. We assume that the null student has the lowest test score. For any $s, s' \in S \cup \{\emptyset\}$, we write $s \succeq s'$ if and only if either $s \succ s'$ or $s = s'$.

Guangzhou's high school admission problem, or simply the problem, is denoted by $G = (S, C, q, P_S, \succ)$. Since $S$, $C$, and $q$ will be fixed, we also denote the problem by $G = (P_S, \succ)$. Note that this is a special case of the standard school choice problem [2].

A matching is an assignment of students to schools such that each student can be matched with at most one school and each school can admit no more students then its quota. Formally, a matching $\mu$ is a function from $S \cup C$ to subsets of $S \cup C$ such that, (i) for each $s \in S$, $\mu(s) \in C \cup \{\emptyset\}$; (ii) for each $c \in C$, $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$; and (iii) if $\mu(s) = c$, then $\mu(c) = s$.

A matching $\mu$ is individually rational if for each $s \in S$, $\mu(s)R_s\emptyset$. A matching $\mu$ is blocked by a student–school pair $(s, c)$ if (i) $cP_s\mu(s)$ and (ii) either (a) $|\mu(c)| < q_c$ and $s \succ \emptyset$ or (b) for some $s' \in \mu(c)$, $s \succ s'$. A matching $\mu$ is stable if it is individually rational and is not blocked by any student–school pair.

A mechanism $\phi$ is a systematic procedure to choose a matching for each problem. Let $\phi(P_S, \succ)$ be the matching chosen by mechanism $\phi$ for problem $(P_S, \succ)$ and $\phi_a(P_S, \succ)$ be the assignment of agent $a \in S \cup C$. A mechanism $\phi$ is said to be stable at problem $(P_S, \succ)$ if it chooses a stable matching at this problem. A mechanism $\phi$ is stable if it always chooses a stable matching.

To compare the degree of stability between two unstable mechanisms, we use the concept of "more stable", which is introduced by [4].

**Definition 1.** *Mechanism $\phi$ is more stable than mechanism $\phi'$ if, (i) at any problem, $\phi'$ is stable, then $\phi$ is also stable and, (ii) at some problem, $\phi$ is stable but $\phi'$ is not.*

A mechanism $\phi$ is strategy-proof for students if there exist no problem $(P_S, \succ)$, student $s$, and preferences $Q_s$ such that $\phi_s(Q_s, P_{-s}, \succ) P_s \phi_s(P_S, \succ)$. A mechanism $\phi$ is manipulable by student $s$ at problem $(P_S, \succ)$ if there exists $Q_s$ such that $\phi_s(Q_s, P_{-s}, \succ) P_s \phi_s(P_S, \succ)$. Thus, a mechanism $\phi$ is said to be manipulable at problem $(P_S, \succ)$ if there exists some student $s$ such that $\phi$ is manipulable by student $s$ at this problem.

To compare the degree of manipulability between two manipulable mechanisms, we use the concept of "more manipulable", which is introduced by [7].

**Definition 2.** *Mechanism $\phi$ is more manipulable than mechanism $\phi'$ if, (i) at any problem $\phi'$ is manipulable, then $\phi$ is also manipulable and, (ii) at some problem, $\phi$ is manipulable but $\phi'$ is not.*

Now, we describe two mechanisms, the SD and the IA mechanism. The first one is equivalent to the DA mechanism under homogeneous priorities, and the second one will be used to describe the Guangzhou mechanism in the next section.

The SD mechanism:

Step 1: The student with the highest test score is assigned their top choice.

Step $k$, $k \geq 2$: The student with the $k$th highest test score is assigned their top choice among all schools except the ones whose quotas have been filled.

End: The mechanism stops when all students have chosen a school or all schools have filled their quotas.

The IA mechanism (under homogeneous priorities):

Step 1: Each student proposes to their top choice. Each school (i) considers its applicants at this step; (ii) immediately accepts those applicants up to its quota, one at a time, following the test score ordering; and (iii) rejects the remaining applicants.

Step $k$, $k \geq 2$: Each student that has been rejected in the previous step proposes their $k$th choice. Each school (i) considers its applicants at this step; (ii) immediately accepts those applicants up to its remaining quota, one at a time, following the test score ordering; and (iii) rejects the remaining applicants.

End: The mechanism stops when no student is rejected or all schools have filled their quotas.

## 3. Guangzhou Mechanism

In this section, we introduce score partition to express the high-scoring student protection policy and investigate the Guangzhou mechanism with different score partitions. The notion presented below is meant to capture the idea that the set of students is partitioned into several groups based on their test scores.

**Definition 3.** *A partition of the set $S$ is a collection of nonempty disjoint subsets of $S$ for which the union is all of $S$. A score partition of the set $S$ is a collection of subsets $\mathbb{P} = \{S^i(\mathbb{P})\}_{i \in I}$ (where $I$ is a finite index set) such that (i) $\mathbb{P}$ is a partition of the set $S$ and, (ii) for each $s \in S^i(\mathbb{P})$ and $s' \in S^j(\mathbb{P})$ with $i < j$, we have $s \succ s'$.*

Score partition requires that students in the lower indexed group have higher test scores than those in the higher indexed group. Note that the indices are important because different groups will have different privileges in the admission. By using this notion, we describe the Guangzhou (GZ) mechanism with a score partition $\mathbb{P} = \{S^i(\mathbb{P})\}_{i \in I}$ as follows.

The GZ mechanism with a score partition $\mathbb{P} = \{S^i(\mathbb{P})\}_{i \in I}$:

Round 1: The IA mechanism is applied to assign students in $S^1(\mathbb{P})$ to schools.

Round $i$, $i \geq 2$: The IA mechanism is applied to assign students in $S^i(\mathbb{P})$ to those remaining schools for which the quotas have been not filled.

End: The mechanism stops when all students have matched with a school or all schools have filled their quotas.

**Remark 1.** *All students submit their preferences at the beginning of the mechanism, and these preference submissions are not allowed to be revised later. Moreover, each student $s \in S^i(\mathbb{P})$ can be matched with a school only in Round $i$.*

As described above, the GZ mechanism is a multi-round mechanism. The clearing-house assigns students to schools sequentially and uses the IA mechanism for each round. If $\mathbb{P} = S$, then there is only one round and the mechanism works as the IA mechanism. On the other hand, if $\mathbb{P} = \{\{s_1\}, \cdots, \{s_n\}\}$, then there are $|S|$ rounds and the mechanism works as the SD mechanism. Thus, score partition plays a crucial role in determining the assignments and those two well-known mechanisms can be seen as the extreme cases of the GZ mechanism. Next, we provide an example to illustrate how this mechanism works.

**Example 1.** *Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3\}$. Each school has a quota of one. Students are assumed to truthfully reveal their preferences. The test score ordering and students' preferences are given by the following:*

$$\begin{array}{llll} P_{s_1}: & c_1, c_2, c_3, & \succ: & s_1, s_2, s_3, s_4, \\ P_{s_2}: & c_2, c_3, c_1, & & \\ P_{s_3}: & c_2, c_3, c_1, & & \\ P_{s_4}: & c_3, c_1, c_2. & & \end{array}$$

*Given a score partition $\mathbb{P}^1 = \{\{s_1, s_2\}, \{s_3, s_4\}\}$, the GZ mechanism works as follows. In round 1, $s_1$ and $s_2$ enter the market. The outcome produced by the IA mechanism is $\binom{s_1\ s_2}{c_1\ c_2}$. In round 2, $s_3$ and $s_4$ enter the market and the outcome is $\binom{s_3\ s_4}{\emptyset\ c_3}$. Thus, the outcome of $GZ(\mathbb{P}^1)$ is $\mu^1 = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ c_3}$. Similarly, we can calculate the outcomes of the GZ mechanism with different score partitions as follows (Table 1):*

**Table 1.** The GZ mechanism under different score partitions.

| Score Partition | Assignment |
|---|---|
| $\mathbb{P}^0 = \{\{s_1, s_2, s_3\}, \{s_4\}\}$ | $\mu^0 = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ c_3\ \emptyset}$ |
| $\mathbb{P}^1 = \{\{s_1, s_2\}, \{s_3, s_4\}\}$ | $\mu^1 = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ c_3}$ |
| $\mathbb{P}^2 = \{s_1, s_2, s_3, s_4\}$ | $\mu^2 = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ c_3}$ |
| $\mathbb{P}^3 = \{\{s_1, s_2\}, \{s_3\}, \{s_4\}\}$ | $\mu^3 = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ c_3\ \emptyset}$ |

As noticed, $\mu^0$ is stable. If we transfer $\mathbb{P}^0$ to $\mathbb{P}^1$, then $s_3$ is removed from the group $\{s_1, s_2, s_3\}$ and $\mu^1$ is not stable. On the other hand, if we transfer $\mathbb{P}^0$ to $\mathbb{P}^2$, then $s_4$ is added into the group $\{s_1, s_2, s_3\}$ and $\mu^2$ is also not stable. Thus, if the size of a group is either increased or decreased, then we may not able to guarantee that the GZ mechanism with the new score partition still produces stable matching at the same problem.

We then compare $\mathbb{P}^0$ and $\mathbb{P}^3$. Note that $\mathbb{P}^3$ is induced by $\mathbb{P}^0$ in the sense that we divide the group $\{s_1, s_2, s_3\}$ into two subgroups, $\{s_1, s_2\}$ and $\{s_3\}$ while keeping $s_4$ as the last one to enter the market. Since GZ($\mathbb{P}^3$) still produces stable matching $\mu^3$ at this problem, we may restrict our attention to a special kind of relation between two score partitions called "refinement".

**Definition 4.** *A score partition $\mathbb{Q}$ is a refinement of a score partition $\mathbb{P}$ if each element of $\mathbb{Q}$ is a subset of an element of $\mathbb{P}$.*

As the name suggests, a score partition is refined by splitting its groups into a larger number of smaller subgroups. We show that the clearinghouse can manipulate the score partitions via refinements to obtain better results from the viewpoint of stability.

**Theorem 1.** *Given two score partitions, $\mathbb{P}$ and $\mathbb{Q}$, if $\mathbb{Q}$ is a refinement of $\mathbb{P}$, then $GZ(\mathbb{Q})$ is more stable than $GZ(\mathbb{P})$.*

Theorem 1 indicates that, (i) at each problem in which the GZ mechanism with a given score partition produces a stable matching, the GZ mechanism with any corresponding refined one will also produce a stable matching and that, (ii) at some problem, the converse is not true.

Before discussing the strategic question, we first introduce a useful notation. Given a problem $G = (P_S, \succ)$, the problem $(Q_s, P_{-s}, \succ)$ in which $s$ reports their preferences as $Q_s$ is denoted by $G(Q_s)$. Note that, at both problems $G$ and $G(Q_s)$, the test score ordering is identical and students' preferences except $s$ are the same.

We begin by using the following example to consider the strategic question.

**Example 2.** *Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ and $C = \{c_1, c_2, c_3\}$. Each school has a quota of one. Students do not necessarily reveal their true preferences. The test score ordering and students' preferences are given by the following:*

$$\begin{array}{llll} P_{s_1}: & c_1, & \succ: & s_1, s_2, s_3, s_4, s_5, \\ P_{s_2}: & c_1, c_2, c_3, & & \\ P_{s_3}: & c_2, & & \\ P_{s_4}: & c_1, c_3, & & \\ P_{s_5}: & c_3. & & \end{array}$$

*We list the outcomes of the GZ mechanism with different score partitions as follows (Table 2):*

**Table 2.** The GZ mechanism under different score partitions.

| Score Partition | Assignment |
|---|---|
| $\mathbb{P}^1 = \{\{s_1, s_2, s_3, s_4, s_5\}\}$ | $\mu^1 = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ \emptyset\ c_2\ \emptyset\ c_3}$ |
| $\mathbb{P}^2 = \{\{s_1, s_2, s_3\}, \{s_4, s_5\}\}$ | $\mu^2 = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_3\ c_2\ \emptyset\ \emptyset}$ |
| $\mathbb{P}^3 = \{\{s_1, s_2\}, \{s_3\}, \{s_4, s_5\}\}$ | $\mu^3 = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_2\ \emptyset\ \emptyset\ c_3}$ |

*All three mechanisms are manipulable at this problem. First, we compare $GZ(\mathbb{P}^1)$ and $GZ(\mathbb{P}^2)$. $GZ(\mathbb{P}^2)$ is manipulable by $s_2$ because they can report $Q_{s_2} : c_2$ to obtain $c_2$. Note that $s_2$ is a student who prefers $\phi_{s_2}^{GZ(\mathbb{P}^2)}(G(Q_{s_2})) = c_2$ to $\phi_{s_2}^{GZ(\mathbb{P}^1)}(G) = \emptyset$. Now if we use a coarser score partition $\mathbb{P}^1$, then $GZ(\mathbb{P}^1)$ is also manipulable by $s_2$ because they can use the same strategy to obtain $c_2$. Then, we compare $GZ(\mathbb{P}^2)$ and $GZ(\mathbb{P}^3)$. $GZ(\mathbb{P}^3)$ is manipulable by $s_4$ because they can report $Q_{s_4} : c_3$ to obtain $c_3$. Note that $s_4$ is a student who prefers $\phi_{s_4}^{GZ(\mathbb{P}^3)}(G(Q_{s_4})) = c_3$ to $\phi_{s_4}^{GZ(\mathbb{P}^2)}(G) = \emptyset$. Now if we use a coarser score partition $\mathbb{P}^2$, then $GZ(\mathbb{P}^2)$ is manipulable, not by $s_4 \in S^2(\mathbb{P}^2)$ but by an earlier entered student $s_2 \in S^1(\mathbb{P}^2)$ because $s_2$ can report $Q_{s_2} : c_2$ to obtain $c_2$. Note that $s_2$ is a student who prefers $\phi_{s_2}^{GZ(\mathbb{P}^3)}(G) = c_2$ to $\phi_{s_2}^{GZ(\mathbb{P}^2)}(G) = \phi_{s_4}^{GZ(\mathbb{P}^3)}(G(Q_{s_4})) = c_3$.*

At this example, for given two score partitions, $\mathbb{P}$ and $\mathbb{Q}$, in which $\mathbb{Q}$ is a refinement of $\mathbb{P}$, if GZ($\mathbb{Q}$) is manipulable by some student $s$ such as $s_2$ or $s_4$, then this means that $s$ can misreport to obtain a better school $c$ in GZ($\mathbb{Q}$) and we will have two cases in GZ($\mathbb{P}$): (i) GZ($\mathbb{P}$) is manipulable by this student $s$ or (ii) GZ($\mathbb{P}$) is not manipulable by this student $s$, but we can find an earlier entered student $s'$ who is matched with $c$ in GZ($\mathbb{P}$) and prefers the outcome of GZ($\mathbb{Q}$) than $c$. Based on this observation, we have the following lemma.

**Lemma 1.** *Suppose that $\mathbb{Q}$ is a refinement of score partition $\mathbb{P}$. If there is a student $s \in S^p(\mathbb{P})$ and preference $Q_s$ such that $\phi_s^{GZ(\mathbb{Q})}(G(Q_s))P_s\phi_s^{GZ(\mathbb{P})}(G)$, then either (i) GZ($\mathbb{P}$) is manipulable by student $s$ at problem $G$ or (ii) there is a student $s' \in S^{p'}(\mathbb{P})$ (where $p' < p$) such that $\phi_{s'}^{GZ(\mathbb{P})}(G) = \phi_s^{GZ(\mathbb{Q})}(G(Q_s))$ and $\phi_{s'}^{GZ(\mathbb{Q})}(G)P_{s'}\phi_{s'}^{GZ(\mathbb{P})}(G)$.*

**Corollary 1.** *Suppose that $\mathbb{Q}$ is a refinement of score partition $\mathbb{P}$. If there is a student $s \in S^p(\mathbb{P})$ such that $\phi_s^{GZ(\mathbb{Q})}(G)P_s\phi_s^{GZ(\mathbb{P})}(G)$, then either (i) GZ($\mathbb{P}$) is manipulable by student $s$ at problem $G$ or (ii) there is a student $s' \in S^{p'}(\mathbb{P})$ (where $p' < p$) such that $\phi_{s'}^{GZ(\mathbb{P})}(G) = \phi_s^{GZ(\mathbb{Q})}(G)$ and $\phi_{s'}^{GZ(\mathbb{Q})}(G)P_{s'}\phi_{s'}^{GZ(\mathbb{P})}(G)$.*

By using Lemma 1 and Corollary 1, we show that the clearinghouse can manipulate the score partitions via refinements to obtain better results from the viewpoint of manipulability.

**Theorem 2.** *Given two score partitions, $\mathbb{P}$ and $\mathbb{Q}$, if $\mathbb{Q}$ is a refinement of $\mathbb{P}$, then GZ($\mathbb{P}$) is more manipulable than GZ($\mathbb{Q}$).*

Theorem 2 indicates that, (i) at each problem in which the GZ mechanism with a given score partition is manipulable by a student, the GZ mechanism with any corresponding coarser one will be also be manipulable either by this student or some student who enters to the market earlier and that, (ii) at some problem, the converse is not true.

**Remark 2.** *As mentioned in the Introduction, students in Guangzhou are divided into two groups based on their test scores. Since the set of students is score-partitioned, the above results justify the use of a high-scoring student protection policy and show that the mechanism practiced in Guangzhou is more stable and less manipulable than the IA mechanism.*

**Remark 3.** *Ref. [4] show that, under general priority structure, as the length of choice band becomes longer, the CP mechanism becomes more stable and less manipulable. By using a similar approach, we show that, under homogeneous priority structure, as the score partition of students becomes refined finer, the GZ mechanism becomes more stable (Theorem 1) and less manipulable (Theorem 2). Our results heavily rely on the assumption of homogeneous priority structure and would not hold when this assumption is relaxed. The limitation of our results is illustrated by using the following example. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3\}$, in which each school has a quota of one. Students' preferences are given by $P_{s_1} : c_1; P_{s_2} : c_1, c_2; P_{s_3} : c_2, c_3; P_{s_4} : c_3$ and the priority structure is given by $\succ_{c_1}: s_1, s_2, s_3, s_4; \succ_{c_2}: s_3, s_4, s_1, s_2; \succ_{c_3}: s_3, s_4, s_1, s_2$. We consider two partitions of students, $\mathbb{P}^1 = \{s_1, s_2, s_3, s_4\}$ and $\mathbb{P}^2 = \{\{s_1, s_2\}, \{s_3, s_4\}\}$. If $\mathbb{P}^1$ is used, then the GZ mechanism works as the IA mechanism and the outcome is $\mu^1 = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ \emptyset\ c_2\ c_3}$, which is stable and is not manipulable by any student. If $\mathbb{P}^2$ is used, then $s_1, s_2$ are in Round 1, $s_3, s_4$ are in Round 2, and the outcome of the GZ mechanism is $\mu^2 = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ c_3}$. Note that $\mu^2$ is not stable because $c_3P_{s_3}\emptyset$ and $s_3 \succ_{c_4} s_4$ and is manipulable by $s_3$ because they can report $Q_{s_3} : c_3$ to obtain $c_3$. Thus, the GZ mechanism may not be more stable or less manipulable by refining the partition under the general priority structure.*

We have so far compared the GZ mechanism with different score partitions by using the criteria of stability and manipulability. In practice, policy makers may also care about "first-choice maximal", i.e., the number of students who are matched to their first choices. If they want to achieve this goal, then the IA mechanism would be a better choice than the SD mechanism. Since the GZ mechanism is closely related to the IA mechanism, one may expect that the GZ mechanism can always match a higher number of students to their first choices than the SD mechanism. However, the following example shows that this is not true.

**Example 3.** *Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ and $C = \{c_1, c_2, c_3, c_4\}$. Each school has a quota of one. Students are assumed to truthfully reveal their preferences. The test score ordering and students' preferences are given by the following:*

$$\begin{array}{llcl} P_{s_1}\!: & c_1, & \succ\!: & s_1, s_2, s_3, s_4, s_5, \\ P_{s_2}\!: & c_1, c_2, & & \\ P_{s_3}\!: & c_1, c_2, c_3, c_4, & & \\ P_{s_4}\!: & c_1, c_3, & & \\ P_{s_5}\!: & c_4. & & \end{array}$$

*If the score partition is given by $\mathbb{P}^1 = \{\{s_1, s_2, s_3, s_4\}, \{s_5\}\}$, then the outcome of the GZ mechanism is $\mu^1 = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_2\ c_4\ c_3\ \emptyset}$ in which only one student $s_1$ matches their first choice. However, if we use $\mathbb{P}^2 = \{\{s_1\}, \{s_2\}, \{s_3\}, \{s_4\}, \{s_5\}\}$, which is a refinement of $\mathbb{P}^1$, then the GZ mechanism is equivalent to the SD mechanism and the outcome is $\mu^2 = \binom{s_1\ s_2\ s_3\ s_4\ s_5}{c_1\ c_2\ c_3\ \emptyset\ c_4}$. Note that two students, $s_1$ and $s_5$, are matched to their first choices under $\mu^2$.*

**Theorem 3.** *Given a score partition $\mathbb{P}$, $GZ(\mathbb{P})$ may match a lower number of students to their first choices than the SD mechanism.*

**Remark 4.** *Ref. [4] show that, as the length of choice band becomes longer, the CP mechanism becomes to match lower number of students to their first choices. This means that the CP mechanism always matches more students to their first choices than the SD mechanism. Since the policy makers in China would like to balance the SD ("stability/strategy-proofness") and the IA ("first choice maximal") mechanisms, the CP mechanism can help them to achieve the ideal balance between those two goals. However, such a balance, due to the drawback in Theorem 3, cannot be achieved by using the GZ mechanism.*

In our previous discussion, we looked at the GZ mechanism, which has the feature that the IA mechanism is used for each round. One may consider using the IA-skip mechanism [8,9], which allows students to automatically skip exhausted schools. Since we use the IA-skip mechanism for each round, such a mechanism will be called the GZ-skip (GZS) mechanism.

At first glance, the GZS mechanism may performed better than the GZ mechanism because no student will propose an exhausted school. However, the following example shows that refinements may not help the policy maker improve the degree of stability in the GZS mechanism.

**Example 4.** *Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$. Each school has a quota of one. Students are assumed to truthfully reveal their preferences. The test score ordering and students' preferences are given by the following:*

$$\begin{array}{llcl} P_{s_1}\!: & c_1, & \succ\!: & s_1, s_2, s_3, s_4, s_5, s_6, \\ P_{s_2}\!: & c_1, c_2, & & \\ P_{s_3}\!: & c_1, c_2, c_3, & & \\ P_{s_4}\!: & c_1, c_4, c_2, c_3, c_5, c_6, & & \\ P_{s_5}\!: & c_1, c_2, c_3, c_4, c_5, c_6, & & \\ P_{s_6}\!: & c_1, c_2, c_3, c_5, c_4, c_6. & & \end{array}$$

*Given a score partition $\mathbb{P}^1 = \{\{s_1, s_2, s_3\}, \{s_4, s_5, s_6\}\}$, the GZS mechanism works as follows. In Round 1, $s_1$, $s_2$, and $s_3$ enter the market. The outcome produced by the IA-skip mechanism is*

*$\binom{s_1\ s_2\ s_3}{c_1\ c_2\ c_3}$. In Round 2, $s_3$, $s_4$, $s_5$ enter the market and the outcome is $\binom{s_4\ s_5\ s_6}{c_4\ c_6\ c_5}$. Thus, the outcome of $GZS(\mathbb{P}^2)$ is $\mu^1 = \binom{s_1\ s_2\ s_3\ s_4\ s_5\ s_6}{c_1\ c_2\ c_3\ c_4\ c_6\ c_5}$. Since the pair $(s_5, c_5)$ is a blocking pair, $\mu^1$ is not stable. On the other hand, given a score partition $\mathbb{P}^2 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, the outcome of $GZS(\mathbb{P}^2)$ is $\mu^2 = \binom{s_1\ s_2\ s_3\ s_4\ s_5\ s_6}{c_1\ c_2\ c_3\ c_4\ c_5\ c_6}$, which is stable.*

This negative result may provide an explanation for why the policy maker in Guangzhou uses the GZ mechanism instead of the GZS mechanism.

## 4. Conclusions

School choice in China has been experienced a lot of reforms. The latest reform is the evolution from the IA mechanism to the CP mechanism. In this paper, we addressed several questions: Why has Guangzhou continued to use the IA mechanism? What is the effect of the high-scoring student protection policy? How does one improve the current mechanism in Guangzhou?

To answer these questions, we analyzed Guangzhou's high school admission and studied a class of GZ mechanisms, including the IA and the SD mechanisms. Although the GZ mechanism is neither stable nor strategy-proof, policy makers can manipulate the score partitions via refinements to improve the allocation outcome of this mechanism. The analysis in this paper showed that the mechanism practiced in Guangzhou is more stable and less manipulable than the IA mechanism. This result justifies the use of the high-scoring protection policy and explains why Guangzhou has continued to use a variant of the IA mechanism. The drawback of the GZ mechanism is that it may assign a lower number of students to their first choices than the SD mechanism. For this reason, we recommend using the CP mechanism to improve the current mechanism.

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## Appendix A

### *Appendix A.1. Proof of Theorem 1*

**Proof.** We show that, at any problem where GZ($\mathbb{P}$) is stable, GZ($\mathbb{Q}$) is also stable. Suppose that there is a problem $G = (P_S, \succ)$ such that GZ($\mathbb{P}$) is stable but GZ($\mathbb{Q}$) is not. We show that $\phi^{GZ(\mathbb{Q})}(G) = \phi^{GZ(\mathbb{P})}(G)$. If we suppose not, then the set $A = \{s \in S : \phi_s^{GZ(\mathbb{Q})}(G) \neq \phi_s^{GZ(\mathbb{P})}(G)\}$ is not empty. Let $s$ be the student who has the highest test score in the set $A$. We consider two cases:

Case 1: $\phi_s^{GZ(\mathbb{Q})}(G) P_s \phi_s^{GZ(\mathbb{P})}(G)$. This is the case that $s$ fails to be matched with $\phi_s^{GZ(\mathbb{Q})}(G)$ in GZ($\mathbb{P}$). Let $x = \phi_s^{GZ(\mathbb{Q})}(G)$. Since (i) for each $s'$ with $s' \succ s$, $\phi_{s'}^{GZ(\mathbb{Q})}(G) = \phi_{s'}^{GZ(\mathbb{P})}(G)$ and (ii) $x \neq \phi_s^{GZ(\mathbb{P})}(G)$, we have either (a) $|\phi_x^{GZ(\mathbb{P})}(G)| < q_x$ or (b) for some $s'' \in \phi_x^{GZ(\mathbb{P})}(G)$, $s \succ s''$. For either case, $\phi^{GZ(\mathbb{P})}(G)$ is blocked by the pair $(s, x)$, a contradiction.

Case 2: $\phi_s^{GZ(\mathbb{P})}(G) P_s \phi_s^{GZ(\mathbb{Q})}(G)$. This is the case that $s$ fails to be matched with $\phi_s^{GZ(\mathbb{P})}(G)$ in GZ($\mathbb{Q}$). Let $y = \phi_s^{GZ(\mathbb{P})}(G)$. Since (i) for each $s'$ with $s' \succ s$, $\phi_{s'}^{GZ(\mathbb{Q})}(G) = \phi_{s'}^{GZ(\mathbb{P})}(G)$ and (ii) $y \neq \phi_s^{GZ(\mathbb{Q})}(G)$, we have either (a) $|\phi_y^{GZ(\mathbb{Q})}(G)| < q_y$ or (b) for some $s'' \in \phi_y^{GZ(\mathbb{Q})}(G)$, $s \succ s''$. Assume that $s \in S^q(\mathbb{Q})$ and $s \in S^p(\mathbb{P})$.

For the case 2(a), $y$ must considered $s$ at some step of round $q$ in GZ($\mathbb{Q}$). Since $y$ had at least one vacant position at that step, $s$ should be accepted by $y$ and thus $\phi_s^{GZ(\mathbb{Q})}(G) = y$, a contradiction.

For the case 2(b), assume that $s'' \in S^{q''}(\mathbb{Q})$. If $q'' > q$, then the assignment of $s''$ is determined at round $q''$, which is later than round $q$ in GZ($\mathbb{Q}$). Since $s$ fails to be matched with $y$, $s''$ will also fail to be matched with $y$ in GZ($\mathbb{Q}$) and thus $\phi_{s''}^{GZ(\mathbb{Q})}(G) \neq y$, a contradiction. Thus, $s''$ cannot enter the market after round $q$ in GZ($\mathbb{Q}$), i.e., $q'' \leq q$. Since $s \succ s''$, we have $q'' = q$. Suppose that there are $\tilde{q}_y$ vacant positions of $y$ at the beginning of round $q$ in GZ($\mathbb{Q}$), then since $\phi_s^{GZ(\mathbb{Q})}(G) \neq y$, we have at least $\tilde{q}_y$ students such as $s''$ in the round $q$ of GZ($\mathbb{Q}$). Denote these students by the set $B$. For each $s'' \in B$, we have $s'' \in S^q(\mathbb{Q})$, $s \succ s''$, and $\phi_{s''}^{GZ(\mathbb{Q})}(G) = y$. Since $\phi_s^{GZ(\mathbb{Q})}(G) \neq y$, we have for each $s'' \in B$, $\text{rank}_{s''}(y) < \text{rank}_s(y)$, where $\text{rank}_s(c)$ is the rank of school $c$ in student $s$'s preferences. Since $\mathbb{Q}$ is a refinement of $\mathbb{P}$, $B \subset S^p(\mathbb{P})$. Since for each $s'$ with $s' \succ s$, $\phi_{s'}^{GZ(\mathbb{Q})}(G) = \phi_{s'}^{GZ(\mathbb{P})}(G)$ and $|B| \geq \tilde{q}_y$, $s$ fails to be matched with $y$ in GZ($\mathbb{P}$) and thus $\phi_s^{GZ(\mathbb{P})}(G) \neq y$, a contradiction.

For these reasons, the set $A$ is empty and we have that $\phi^{GZ(\mathbb{Q})}(G) = \phi^{GZ(\mathbb{P})}(G)$. Thus, if $\phi^{GZ(\mathbb{P})}(P_S, \succ)$ is stable, then $\phi^{GZ(\mathbb{Q})}(G)$ should be also stable.

Next, we show that, for some problem, GZ($\mathbb{Q}$) is stable but GZ($\mathbb{P}$) is not. We consider the following example. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3, c_4\}$. Each school has a quota of one. The test score ordering is $s_1 \succ s_2 \succ s_3 \succ s_4$ and students' preferences are given by the following: $P_{s_1} : c_1$; $P_{s_2} : c_1, c_2, c_3$; $P_{s_3} : c_2$; $P_{s_4} : c_4$. Given two score partitions, $\mathbb{P} = \{s_1, s_2, s_3, s_4\}$ and $\mathbb{Q} = \{\{s_1, s_2\}, \{s_3, s_4\}\}$, the matching $\phi^{GZ(\mathbb{Q})}(G) = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ c_4}$ is stable whereas $\phi^{GZ(\mathbb{P})}(G) = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_3\ c_2\ c_4}$ is not. □

*Appendix A.2. Proof of Lemma 1*

**Proof.** Suppose that there is a student $s$ such that $\phi_s^{GZ(\mathbb{Q})}(G(Q_s)) P_s \phi_s^{GZ(\mathbb{P})}(G)$. Assume that $s \in S^p(\mathbb{P})$ and $s \in S^q(\mathbb{Q})$. Since GZ($\mathbb{P}$) is individually rational, $\phi_s^{GZ(\mathbb{P})}(G) R_s \emptyset$. Hence, $\phi_s^{GZ(\mathbb{Q})}(G(Q_s)) P_s \emptyset$. Let $c(s) = \phi_s^{GZ(\mathbb{Q})}(G(Q_s))$. Since $\phi_s^{GZ(\mathbb{P})}(G) \neq c(s)$ and $c(s) P_s \phi_s^{GZ(\mathbb{P})}(G)$, we know that $|\phi_{c(s)}^{GZ(\mathbb{P})}(G)| = q_{c(s)}$. We consider two cases:

Case 1: for each $s' \in \phi_{c(s)}^{GZ(\mathbb{P})}(G)$, $s' \in \cup_{i=1}^{p-1} S^i(\mathbb{P})$. Since $\mathbb{Q}$ is a refinement of $\mathbb{P}$, for each $s' \in \phi_{c(s)}^{GZ(\mathbb{P})}(G)$, $s' \in \cup_{j=1}^{q-1} S^j(\mathbb{Q})$. If for each $s' \in \phi_{c(s)}^{GZ(\mathbb{P})}(G)$, $\phi_{s'}^{GZ(\mathbb{Q})}(G) = c(s)$, then all positions of $c(s)$ are filled before round $q$ in GZ($\mathbb{Q}$). Hence, for any $Q_s$, $\phi_s^{GZ(\mathbb{Q})}(G(Q_s)) \neq c(s)$, a contradiction. Therefore, we consider that, for some $s' \in \phi_{c(s)}^{GZ(\mathbb{P})}(G)$, $\phi_{s'}^{GZ(\mathbb{Q})}(G) \neq c(s)$. Assume that $s' \in S^{q'}(\mathbb{Q})$. Note that $q' \leq q - 1 < q$. We show that $\phi_{s'}^{GZ(\mathbb{Q})}(G) P_{s'} c(s)$. If we suppose not, then $c(s) P_{s'} \phi_{s'}^{GZ(\mathbb{Q})}(G)$ implies that $|\phi_{c(s)}^{GZ(\mathbb{Q})}(G)| = q_{c(s)}$ and for each $s'' \in \phi_{c(s)}^{GZ(\mathbb{Q})}(G)$, $s'' \in \cup_{j=1}^{q'} S^j(\mathbb{Q})$. Since all positions of $c(s)$ are filled before round $q$ in GZ($\mathbb{Q}$), for any $Q_s$, $\phi_s^{GZ(\mathbb{Q})}(G(Q_s)) \neq c(s)$, a contradiction.

Case 2: for some $s' \in \phi_{c(s)}^{GZ(\mathbb{P})}(G)$, $s' \notin \cup_{i=1}^{p-1} S^i(\mathbb{P})$. Let $A = \{s' \in S^p(\mathbb{P}) : \phi_{s'}^{GZ(\mathbb{P})}(G) = c(s)\}$. We consider that $s$ reports $Q_s^{c(s)}$ that they ranked $c(s)$ as their top choice. If $\phi_s^{GZ(\mathbb{P})}(G(Q_s^{c(s)})) = c(s)$, then GZ($\mathbb{P}$) is manipulable by $s$ at $G$. If $\phi_s^{GZ(\mathbb{P})}(G(Q_s^{c(s)})) \neq c(s)$, then this means that, for each $s' \in A$, $s' \succ s$ and $\text{rank}_{s'}(c(s)) = 1$. Let $q^*$ be a number such that $q^* < q$ and $\cup_{i=1}^{p-1} S^i(\mathbb{P}) = \cup_{j=1}^{q^*} S^j(\mathbb{Q})$. Note that we can find such a number $q^*$ because $\mathbb{Q}$ is a refinement of $\mathbb{P}$. Let $B = \{s'' \in \cup_{j=1}^{q^*} S^j(\mathbb{Q}) : \phi_{s''}^{GZ(\mathbb{Q})}(G) = c(s)\}$. If $|B| \geq q_{c(s)} - |A|$, then there are $|A|$ students who have higher scores than $s$ and who rank $c(s)$ as their top choice, competing for only $q_{c(s)} - |B|$ vacant positions of $c(s)$. Thus for any $Q_s$, $\phi_s^{GZ(\mathbb{Q})}(G(Q_s)) \neq c(s)$, a contradiction. If $|B| < q_{c(s)} - |A|$, then there is a student $s'' \in \cup_{i=1}^{p-1} S^i(\mathbb{P})$ such that $\phi_{s''}^{GZ(\mathbb{P})}(G) = c(s)$ but $\phi_{s''}^{GZ(\mathbb{Q})}(G) \neq c(s)$. Assume that $s'' \in S^{q''}(\mathbb{Q})$. Note that $q'' \leq q^* < q$. We show that $\phi_{s''}^{GZ(\mathbb{Q})}(G) P_{s''} c(s)$. If we suppose not, then $c(s) P_{s''} \phi_{s''}^{GZ(\mathbb{Q})}(G)$ implies that $|\phi_{c(s)}^{GZ(\mathbb{Q})}(G)| = q_{c(s)}$ and for each $s''' \in \phi_{c(s)}^{GZ(\mathbb{Q})}(G)$, $s''' \in \cup_{j=1}^{q''} S^j(\mathbb{Q})$. Since all positions of $c(s)$ are filled before round $q$ in GZ($\mathbb{Q}$), for any $Q_s$, $\phi_s^{GZ(\mathbb{Q})}(G(Q_s)) \neq c(s)$, a contradiction. □

*Appendix A.3. Proof of Theorem 2*

**Proof.** We show that at any problem GZ($\mathbb{Q}$) is manipulable, GZ($\mathbb{P}$) is manipulable. Suppose there is a problem $G$ that GZ($\mathbb{Q}$) is manipulable but GZ($\mathbb{P}$) is not. Then there is a student $s$ and preference $Q_s$ such that $\phi_s^{GZ(\mathbb{Q})}(G(Q_s)) P_s \phi_s^{GZ(\mathbb{Q})}(G)$. Assume that $s \in S^p(\mathbb{P})$ and $s \in S^q(\mathbb{Q})$. Let $c(s) = \phi_s^{GZ(\mathbb{Q})}(G(Q_s))$. We consider two cases:

Case 1: $c(s) P_s \phi_s^{GZ(\mathbb{P})}(G)$. By lemma 1, we know that either GZ($\mathbb{P}$) is manipulable by $s$ at $G$ or there is a student $s' \in S^{p'}(\mathbb{P})$ (where $p' < p$) such that $\phi_{s'}^{GZ(\mathbb{P})}(G) = c(s)$ and $\phi_{s'}^{GZ(\mathbb{Q})}(G) P_{s'} c(s)$. In the former case, we are done. In the latter case, let $c(s') = \phi_{s'}^{GZ(\mathbb{Q})}(G)$. By the corollary, we know that either GZ($\mathbb{P}$) is manipulable by $s'$ at $G$ or there is a student $s'' \in S^{p'}(\mathbb{P})$ (where $p'' < p'$) such that $\phi_{s''}^{GZ(\mathbb{P})}(G) = c(s')$ and $\phi_{s''}^{GZ(\mathbb{Q})}(G) P_{s''} c(s')$. Continue in this way if we can find that GZ($\mathbb{P}$) is manipulable by some student at $G$; then, we are done. If not, then there is a chain $(s, s', s'', \cdots, s^2, s^1)$ such that $s^1 \in S^1(\mathbb{P})$, $\phi_{s^1}^{GZ(\mathbb{P})}(G) = c(s^2)$ and $\phi_{s^1}^{GZ(\mathbb{Q})}(G) P_{s^1} c(s^2)$. Let $c(s^1) = \phi_{s^1}^{GZ(\mathbb{Q})}(G)$. We show that, if $s^1$ reports $Q_{s^1}^{c(s^1)}$ that she ranks $c(s^1)$ as her top choice, then GZ($\mathbb{P}$) is manipulable by $s^1$ at $G$. If we suppose not, then $|\phi_{c(s^1)}^{GZ(\mathbb{P})}(G)| = q_{c(s^1)}$ and for each $\tilde{s} \in \phi_{c(s^1)}^{GZ(\mathbb{P})}(G)$, $\tilde{s} \in S^1(\mathbb{P})$, $\tilde{s} \succ s^1$, and $\text{rank}_{\tilde{s}}(c(s^1)) = 1$. Assume that $s^1 \in S^{q^1}(\mathbb{Q})$. Then, we have that, for each $\tilde{s} \in \phi_{c(s^1)}^{GZ(\mathbb{P})}(G)$, $\tilde{s} \in \cup_{j=1}^{q^1} S^j(\mathbb{Q})$, $\tilde{s} \succ s^1$, and $\text{rank}_{\tilde{s}}(c(s^1)) = 1$. By the description of GZ($\mathbb{Q}$), this means that $\phi_{s^1}^{GZ(\mathbb{Q})}(G) \neq c(s^1)$, a contradiction.

Case 2: $\phi_s^{GZ(\mathbb{P})}(G) R_s c(s)$. We show that, for each $s' \in S$, $\phi_{s'}^{GZ(\mathbb{P})}(G) R_{s'} \phi_{s'}^{GZ(\mathbb{Q})}(G)$. If we suppose not, then for some $s'' \in S$, $\phi_{s''}^{GZ(\mathbb{Q})}(G) P_{s''} \phi_{s''}^{GZ(\mathbb{P})}(G)$. By using the corollary and the argument in case 1, we can show that GZ($\mathbb{P}$) is manipulable by some student at $G$, a contradiction. Moreover, we have $\phi_s^{GZ(\mathbb{P})}(G) P_s \phi_s^{GZ(\mathbb{Q})}(G)$ because $\phi_s^{GZ(\mathbb{P})}(G) R_s c(s)$ and $c(s) P_s \phi_s^{GZ(\mathbb{Q})}(G)$. Thus, $\phi^{GZ(\mathbb{P})}(G)$ Pareto dominates $\phi^{GZ(\mathbb{Q})}(G)$.

Next, we denote the set of all students who prefer $\phi^{GZ(\mathbb{P})}(G)$ to $\phi^{GZ(\mathbb{Q})}(G)$ by $D$, i.e., $D = \{s' \in S : \phi_{s'}^{GZ(\mathbb{P})}(G) P_{s'} \phi_{s'}^{GZ(\mathbb{Q})}(G)\}$. We consider the student $s^*$ who has the highest test score in $D$. Assume that $s^* \in S^{q^*}(\mathbb{Q})$. Note that, for each $s'' \in \cup_{j=1}^{q^*-1} S^j(\mathbb{Q})$, $\phi_{s''}^{GZ(\mathbb{P})}(G) = \phi_{s''}^{GZ(\mathbb{Q})}(G)$. Let $c(s^*) = \phi_{s^*}^{GZ(\mathbb{P})}(G)$. Since $\phi_{s^*}^{GZ(\mathbb{Q})}(G) \neq c(s^*)$, there is a student $s^{**} \in S^{q^*}(\mathbb{Q})$ such that $\phi_{s^{**}}^{GZ(\mathbb{Q})}(G) = c(s^*)$ and $\phi_{s^{**}}^{GZ(\mathbb{P})}(G) \neq c(s^*)$. Since $\phi^{GZ(\mathbb{P})}(G)$ Pareto dominates $\phi^{GZ(\mathbb{Q})}(G)$, we have $\phi_{s^{**}}^{GZ(\mathbb{P})}(G) P_{s^{**}} c(s^*)$. Let $c(s^{**}) = \phi_{s^{**}}^{GZ(\mathbb{P})}(G)$. Since $\phi_{s^{**}}^{GZ(\mathbb{Q})}(G) \neq c(s^{**})$, there is a student $s^{***} \in S^{q^*}(\mathbb{Q})$ such that $\phi_{s^{***}}^{GZ(\mathbb{Q})}(G) = c(s^{**})$ and $\phi_{s^{***}}^{GZ(\mathbb{P})}(G) \neq c(s^{**})$. If we continue in this way, we can find an infinite number of students in $S^{q^*}(\mathbb{Q})$, a contradiction.

Next, we show that, for some problem, GZ($\mathbb{Q}$) is not manipulable but GZ($\mathbb{P}$) is. We consider the following example. Let $S = \{s_1, s_2, s_3, s_4\}$ and $C = \{c_1, c_2, c_3, c_4\}$. Each school has a quota of one. The test score ordering is $s_1 \succ s_2 \succ s_3 \succ s_4$ and students' preferences are given by the following: $P_{s_1} : c_1$; $P_{s_2} : c_1, c_2, c_3$; $P_{s_3} : c_2$; $P_{s_4} : c_4$. Given two score partition, $\mathbb{P} = \{s_1, s_2, s_3, s_4\}$ and $\mathbb{Q} = \{\{s_1, s_2\}, \{s_3, s_4\}\}$, the matching $\phi^{GZ(\mathbb{Q})}(G) = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_2\ \emptyset\ c_4}$ is not manipulated by any student whereas $\phi^{GZ(\mathbb{P})}(G) = \binom{s_1\ s_2\ s_3\ s_4}{c_1\ c_3\ c_2\ c_4}$ is manipulated by student $s_2$. □

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