

Article

School Choice with Hybrid Schedules

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Abstract: During the pandemic, school districts have adopted hybrid schedules to continue the education of the students while maintaining social distance. In a hybrid schedule, students in the same classroom are usually divided into two groups and students only in the same group can physically attend class together two days a week. School districts do not take preferences of the students/parents over the days they would like to come to school into account during this procedure. In this paper, we propose a solution that divides students into groups based on their preferences. Our solution respects the number of classrooms initially reserved for each grade and enables possible efficiency gains by swapping classrooms across grades. Moreover, when there are two alternative schedules provided for students, our solution is immune to preference manipulations.

Keywords: school choice; matching theory; Covid-19



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1. Introduction

In the midst of the pandemic, school districts consider many different options to continue the education of the children while maintaining social distance. Online learning is one of the options that has been considered and applied by school districts. However, online learning falls short in the intellectual and social development of children. Moreover, the inequality that school choice was designed to eliminate has resurfaced. Lower income families are lacking the technology hardware, access to stable internet, and adult supervision to assist children with course content. While districts have implemented plans to distribute technology to their students, having a stable internet connection creates another hurdle for families who live in low income and rural areas [1,2]. Moreover, under online learning, some parents need to stay home to watch younger children and assist them with accessing courses and content. Unfortunately, some jobs are not suitable to work from home. Furthermore, parents who can afford to work from home may find it difficult to balance their job and assisting their child with work that would normally be handled by in-person instruction [3].

Complaints from parents, students, and teachers have led to a need for an alternative to online learning. Subsequently, school districts in the US and other countries have adopted various plans to limit class size which enables them to maintain social distancing and allow students to attend class physically. Three most widely used plans that have been implemented are the AM/PM hybrid model, the A/B hybrid model, and the three week rotation model. In the AM/PM hybrid model, students are divided into two cohorts where one cohort attends school in the morning and the other in the afternoon four days a week. The A/B hybrid model divides students into two groups, where each cohort attends in person two to three times a week for a full day. The three week rotation model divides students into three cohorts. Students in each cohort attend school in-person for one week and virtually for two. In Table 1, we list some of the school districts using these hybrid models.

School districts usually run their chosen plan by placing students into cohorts without providing choice. Some districts assign students into cohorts based on bus routes, e.g.,

Charlotte-Mecklenburg Schools and McDowell County. (See <https://tinyurl.com/29my55y6> and <https://www.mcdowell.k12.nc.us/news/1685914/2020-2021-a-b-schedule>, accessed on 20 April 2021). Some school districts, including Herricks and Hilliard City, simply divide students alphabetically into cohorts. (See <https://www.herricks.org/cms/lib/NY02208178/Centricity/Domain/1544/Reopening%20Plan%20Final.pdf> and <https://www.hilliardschools.org/20-21/hybrid/>, accessed on 20 April 2021).

Table 1. School district and countries considered plans.

School District/Country	State	AM/PM	A/B	3 Week Rotation
Chico Unified School District	CA	X		
University High School ^a	IN		X	
Mid-Del Public School ^a	OK		X	
McDowell County School District ^a	NC		X	
Madison City Schools ^b	TN		X	
Charlotte-Mecklenburg Schools ^b	NC		X	
Pender County Public Schools ^b	NC		X	
Bullitt County Public Schools ^b	KY		X	
Cumberland County Public Schools ^b	NC		X	
Hamilton County Schools ^b	TN		X	
Edmonds School district ^b	WA		X	
New York City Public Schools ^{ck}	NY		X	X
Hilliard City Schools ^c	OH		X	
Toledo Public Schools ^d	OH		X	
Dobbs Ferry Schools ^d	NY		X	
Rockingham County Public Schools ^d	VA		X	
Herricks Public Schools ^e	NY		X	
Washoe County School District ^e	NV		X	
Valley Stream Central High School District ^e	NY		X	
Wilks County School ^e	NC		X	
Bellmore-Merrick Central High School District ^d	NY		X	
Greenville City Schools ^e	TN		X	
Poudre School District ^f	CO		X	
Mill Valley ^g	KS		X	
Olentangy Schools ^h	OH		X	
Columbia County School district ⁱ	GA		X	
Wake County Public Schools	NC			X
Clark County	NV			X
Turkey		X		
Germany		X		
Denmark		X		
Norway		X		
Scotland		X		
Belgium ^c			X	
Greece			X	
Switzerland ^e			X	
Japan ^e		X	X	

Notes: ^a One cohort attends in-person for a full week and thually for the following week. ^b Cohort A attends in-person on Monday and Tuesday, while Cohort B attends in-person learning on Thursday and Friday. All students are virtual on Wednesday to allow for the school to be thoroughly cleaned. ^c Each cohort attends three days a week every other week. ^d Cohort A attends in-person on Monday and Thursday, while cohort B is in-person Tuesday and Friday. All students are virtual on Wednesday. ^e Cohorts alternate attending school two to three times a week. ^f Cohort A attends in person Monday and Wednesday. While cohort B attends in person Tuesday and Thursday. All students are remote Friday. ^g Students attend two consecutive days in-person then the next two virtual. ^h Group A attends in-person on Monday, Thursday and every other Wednesday. Group B attends Tuesday, Friday and every other Wednesday. ⁱ Alternate cohort days except Friday all students have the option to attend with any earning below a C mandatory. ^j Students are divided into three cohorts and each cohort attends in-person twice a week on different days over a three week rotation. ^k New York City Public Schools' principles choose between two plans.

Not allowing for parents' input on the cohort their child is assigned causes potential issues. One issue is daycare for younger children. Some parents may not have the flexibility to adjust their work schedules to accommodate when their child is not in school [4]. This could force a parent to give up working hours and potentially decrease their pay. In 2018, 41.2 million workers had children under the age of 18, representing 32 percent of the workforce [5]. As a result of schools closing from the pandemic, by June 2020 13 percent of parents were forced to cut hours or quit [6]. Another potential issue is a family that has students across different schools. For example, a family could have a child in elementary school and middle school and can end up in two different rotations, further straining parent work schedules and need for daycare.

Allocation procedures ignoring parents' choices yield inequalities among students and avoidable efficiency losses. In this paper, we introduce choice into hybrid schedules provided by school districts during the pandemic. We call each schedule option as a shift. For instance, the AM/PM hybrid model consists of two shifts, i.e., AM and PM. Through the introduction of parents' preferences over shifts, we aim to improve the current allocation procedure that assigns students to their reserved classrooms' shifts without asking their preferences. To achieve our objective, we model this problem by incorporating the matching with contracts framework into the school choice problem. In particular, students have preferences over the shifts and for each shift a certain number of classrooms are reserved for each grade as in the current practice. Then, we assign students to shift-classroom pairs by not allowing students from different grades to be assigned to the same shift-classroom. For a given shift, students have a higher claim on the classrooms that are reserved for their grades. However, our choice based approach allows some students to be assigned a classroom reserved for another grade as long as it does not hurt students from that grade. Notice that, the current practice allows students to be assigned to only the classrooms reserved for their grade in each shift without taking preferences into account. We introduce a mechanism which is based on the celebrated deferred acceptance mechanism [7]. Our proposed mechanism assigns students to shift-classroom pairs in a non-wasteful way by respecting the within-grade priorities and initial reservation of the classrooms. Moreover, when the number of shifts is two, which is the common practice, reporting true preferences is a (weakly) dominant strategy for the students. When the number of shifts is more than two, any mechanism inheriting desired features can be manipulated by students.

The application of matching theory to school choice was first introduced in Balinski and Sönmez [8] and Abdulkadiroğlu and Sönmez [9]. Balinski and Sönmez [8] show that the mechanism used in centralized college admission in Turkey is a variant of the college proposing deferred acceptance mechanism and it suffers severe deficiencies. They propose student proposing deferred acceptance (DA) mechanism, which is first introduced by Gale and Shapley [7] for marriage problem, as the first best solution in the centralized college admission. Abdulkadiroğlu and Sönmez [9] study the public school choice systems in the US. They propose two strategy-proof mechanisms, DA and Top Trading Cycles mechanisms, to replace a highly manipulable mechanism that is commonly used by school districts. The other key contributions on school choice include, but not limited to, are Erdil and Ergin [10], Abdulkadiroğlu et al. [11], Kesten [12], Pathak and Sönmez [13], Ehlers et al. [14], and Dur et al. [15]. We refer the readers to Pathak [16] and Pathak [17] for surveys in school choice.

We construct our model in the framework of matching with contracts introduced by Hatfield and Milgrom [18]. They show when the choice function of the multi-unit demand side satisfy the law of aggregate demand and are substitutes, there exists a stable and strategy proof mechanism. Hatfield and Kojima [19] and Hatfield and Kojima [20] introduce two weaker conditions sufficient for the existence of a stable matching. The matching with contracts framework has been used to model matching markets including cadet-branch matching [21], inter-district school choice [22] and school choice with sibling guarantee [23]. Despite the importance of the law of aggregate demand condition for

the strategy-proofness, as shown in the extant literature, we are able to obtain strategy-proofness without it for the case of two-shift.

There is a literature on the shift scheduling problem with days-off preferences, mostly appearing in Operation Research journals (see [24]). While the current problem sounds similar to that, there are significant differences. In shift scheduling, the primary objective is to optimize agents' day-off assignments with respect to their day-off preferences over a certain time-window. In our setting, each grade has a reserved classroom in each shift, entailing individual rationality constraints, which are not present in the shift scheduling problem. Moreover, students are prioritized, therefore, fairness is different from that considered in the shift scheduling problems, where agents are not prioritized. Partly because of these differences, the pursued lines of researches are quite different as well.

2. Model

2.1. Basics

School choice with hybrid schedules is based on the standard school choice problem introduced by Balinski and Sönmez [8] and Abdulkadiroğlu and Sönmez [9]. We define this problem for a single school and focus on the assignment of students to the classroom-shift pairs in this single school. Shifts can be morning vs. afternoon or different days of a week. Usually, schools group students into two and allow the first group to come to school on two days of the week, say Monday and Tuesday, and allow the second group to come to school on the other two days of the week, say Thursday and Friday. See Table 1 in Introduction. Since the set of students enrolled to each school is predetermined, the assignment problem for each school is independent from the others. In particular, we have a single school and finite set of students denoted with s and I , respectively. Currently, all students in I are enrolled to the school s . We denote the classrooms in school s with C .

Let G be the set of grades. Let $\gamma(i) \in G$ be the grade of student i . We represent the set of students in grade g with I_g . Then, $I_g = \{i \in I : \gamma(i) = g\}$ for each $g \in G$. In the current practice, classrooms are reserved for the grades that they are assigned to the corresponding student. Let $\kappa(c) \in G$ be the grade classroom c is reserved for. We represent grade g 's reserved classrooms with C_g , i.e., $C_g = \{c \in C : \kappa(c) = g\}$.

We denote the set of available shifts with M . We would like to assign students to a classroom in a shift. We assume that students do not care about the exact classroom they are assigned but they care about the shift they are assigned. Each student i is equipped with (strict) preferences over M and an outside option, and it is denoted with P_i . Here, the outside option can be considered as online learning and it is denoted with m_0 . If $m_0 P_i m$, then we say shift m is unacceptable for student i .

School s has a strict priority order over the students in I_g for every $g \in G$. We denote the priority order over students in I_g with \succ_g . This priority order can be the priority order used to assign grade g students in the regular school assignment in the previous years. Alternatively, \succ_g can be an alphabetical order over students in I_g . Notice that, we do not have a priority order that will allow us to rank students in different grades. A school may have its own rules to rank all students and such a ranking can be incorporated to our model and solution easily. Let $\succ = (\succ_g)_{g \in G}$.

Let q_c be the capacity of classroom $c \in C$. For simplicity, we assume $q_c = q_{c'} = q$ for all $c, c' \in C$. By the feasibility, $|I_g| \leq q \times |C_g|$ for all $g \in G$.

In order to maintain the social distance amid the pandemic, schools need to limit the number of students in a classroom. Let \hat{q} be the reduced capacity of classrooms at school s . In order to be able to allocate all students to a classroom and shift pair, we assume $\hat{q} \geq \frac{|I_g|}{|M| \times |C_g|}$ for all $g \in G$.

We denote the maximum number of classrooms that can be used for grade g at school s during shift m with $k_{g,m}$. Let $k = (k_{g,m})_{g \in G, m \in M}$. It is clear to see that $k_{g,m} \leq |C|$ and it can be further limited due to the number of teachers who can teach grade g . Given $|C_g|$ classrooms are reserved for grade g at school s , we further assume $|C_g| \leq k_{g,m}$.

In the rest of the paper, we fix all other elements but P and represent a problem $(I, C, G, M, q, \hat{q}, k, \succ, P)$ with P .

2.2. Notions

In the current practice, schools usually reserve $|C_g|$ classrooms for each grade g at each shift m and assign students to the shift-classroom pairs accordingly. We would like to assign students to the shifts by taking their preferences into account and provide an improvement compared to the current practice. We can relate this problem to the standard school choice problem such that students are assigned to shifts instead of different schools. Differently, in this problem, students are further assigned to the classrooms in each shift and only the same grade students can be assigned to the same classroom in a given shift. To address these points, we work in the matching with contracts framework introduced by Hatfield and Milgrom [18]. Let $X = (I \times M \times C) \cup \{x_0\}$ be the set of contracts such that $x = (i, m, c)$ means student i is assigned to classroom c in shift m . Contract x_0 represents the online learning option. We denote student, classroom, and shift associated with contract $x \in I \times M \times C$ with $i(x)$, $c(x)$, and $m(x)$, respectively. With slight abuse of notation, we will use $m(x_0) = m_0$. Given a subset of contracts $Y \subseteq X$, Y_i , Y_c and Y_m represent the contracts associated with student i , classroom c and shift m , respectively. Let $Y_{m,c} = Y_m \cap Y_c$ for any $Y \subseteq X$. Let $Y_{\bar{I}} = \cup_{i \in \bar{I}} Y_i$ for any $\bar{I} \subseteq I$.

Each student i ranks contracts in $X_i \cup \{x_0\}$ based on the shift she is assigned to. With slight abuse of notation, we denote student i 's preferences over contracts in X_i with P_i such that $m P_i m'$ and $c \neq c'$ implies $(i, m, c) P_i (i, m', c')$. Moreover, student i is indifferent between contracts including the same shift. That is, student i is indifferent between (i, m, c) and (i, m, c') for all $m \in M$ and $c, c' \in C$. We use R_i to represent the associated weak relation order over $X_i \cup \{x_0\}$ and $x R_i x'$ if and only if $x P_i x'$ or $m(x) = m(x')$. If student i prefers online learning over shift m , then all contracts in $X_i \cap X_m$ are ranked below x_0 . Let $P = (P_i)_{i \in I}$ and $R = (R_i)_{i \in I}$.

An **assignment** $Y \subset X$ is a subset of contracts such that

- $|Y_i| \leq 1$ for all $i \in I$ and $|Y_{m,c}| \leq \hat{q}$ for all $c \in C$, and $m \in M$;
- $\gamma(i(x)) = \gamma(i(x'))$ for all $x, x' \in Y_{m,c}$, $c \in C$, and $m \in M$; and
- $|\{c \in C : x \in Y_{m,c} \text{ and } \gamma(i(x)) = g\}| \leq k_{g,m}$ for all $g \in G$, and $m \in M$.

First bullet guarantees that capacity constraints are satisfied. The second bullet requires only students at the same grade can attend a classroom. The last bullet restricts the number of classrooms at each shift m assigned to the grade g students to be less than or equal to $k_{g,m}$. If $Y_i = \emptyset$, then student i is assigned to contract x_0 . In that case, with slight abuse of notation, we will use $Y_i = x_0$.

For a given assignment Y , we denote the set of classrooms in shift m in which grade g students are assigned with $C_{m,g}(Y)$, i.e., $C_{m,g}(Y) = \{c \in C : x \in Y_{m,c} \text{ and } \gamma(i(x)) = g\}$.

A **mechanism** ψ is a procedure which selects an assignment for any problem. We denote the assignment selected by mechanism ψ for a given problem P with $\psi(P)$ and assignment of student i with $\psi_i(P)$.

Our assumptions on capacities guarantee the existence of an assignment in which each student $i \in I$ is assigned to a contract $x \neq x_0$. The assignment procedure in practice usually works as follows: for each grade g , students in I_g are split into $|M|$ subgroups each with size less than or equal to \hat{q} , and they are assigned to classrooms initially reserved to grade g in each acceptable shift according to \succ_g .

Next, we introduce the axioms that we use in our analysis. The first one requires each grade g student to have a guarantee for at least $|C_g|$ classrooms (i.e., the number of reserved classrooms) in each shift and no student is assigned to a contract worse than the outside option.

An assignment Y is **individually rational** if for each $i \in I$, $Y_i R_i x_0$ and whenever there exists $\tilde{x} \in X_i$ such that $\tilde{x} P_i Y_i$ then $|\{c \in C_{m(\tilde{x}), \gamma(i)}(Y) : |Y_{m(\tilde{x}), c}| = \hat{q}\}| \geq |C_{\gamma(i)}|$. Since students only care about the shift, $\tilde{x} P_i Y_i$ implies that i prefers shift $m(\tilde{x})$ to $m(Y_i)$, and therefore $m(\tilde{x}) \neq m(Y_i)$. That is, whenever a student i strictly prefers contract \tilde{x} to

her assignment, then the number of classrooms fully filled with grade $\gamma(i)$ students at shift $m(\tilde{x})$ is at least the number of classrooms reserved for grade $\gamma(i)$. The second axiom requires no seat to be wasted.

An assignment Y is **non-wasteful** if for each $i \in I$ whenever there exists $\tilde{x} \in X_i$ such that $\tilde{x} P_i Y_i$ then the following is true:

- There does not exist $c' \in C_{m(\tilde{x}), \gamma(i)}(Y)$ such that $|Y_{m(\tilde{x}), c'}| < \hat{q}$; and
- If there exists $c' \in C$ such that $|Y_{m(\tilde{x}), c'}| = 0$, then $|C_{m(\tilde{x}), \gamma(i)}(Y)| = k_{\gamma(i), m(\tilde{x})}$.

The first bullet says that all the seats in classrooms in $C_{m(\tilde{x}), \gamma(i)}(Y)$ are filled. The second bullet says if there is a classroom which has all its seats available in shift $m(\tilde{x})$ then grade $\gamma(i)$ has reached its allowed number of classrooms for shift $m(\tilde{x})$. Our next axiom requires that the priority order within each grade should be respected.

An assignment Y is **within-grade-fair** if whenever there exists $\tilde{x} \in X_i$ such that $\tilde{x} P_i Y_i$, then $j \succ_{\gamma(i)} i$ for every $j \in (i(Y_{m(\tilde{x})}) \cap I_{\gamma(i)})$.

A mechanism ψ is individually rational (non-wasteful) [within-grade-fair], if it selects an individually rational (non-wasteful) [within-grade-fair] assignment for any problem.

A mechanism ψ is **strategy-proof** if a student i cannot be better off by misreporting her true preferences in any problem P , i.e., $\psi_i(P) R_i \psi_i(P'_i, P_{-i})$ for all P'_i .

3. Results

We would like to introduce a mechanism which is individually rational, non-wasteful, within-grade-fair, and strategy-proof.

Recall that, students only care about the shift they are assigned to and indifferent between the classrooms. Hence, each student i is indifferent between any two contracts x and x' such that $i(x) = i(x')$ and $m(x) = m(x')$. Moreover, it is not clear how students from different grades will be ranked and assigned to a shift m when it is not possible to assign all students asking for shift m . Under these two features of our model, it is even not clear how students will be assigned to classrooms when there is only one shift. To this end, we define a choice function for each shift m which determines the set of chosen students for shift m when a subset of students \bar{I} is considered, and we denote it with $Ch^m(\bar{I}) \subseteq \bar{I}$. In order to calculate $Ch^m(\bar{I})$, we need a (strict) priority order over grades, denoted with π^m and a (strict) precedence order over classrooms, denoted with \triangleright^m . Priority order π^m determines which grade will be favored and precedence order \triangleright^m determines which classroom will be filled before the other ones in the selection procedure. Let $\pi = (\pi^m)_{m \in M}$ and $\triangleright = (\triangleright^m)_{m \in M}$ and we fix these profiles throughout the paper.

Now, we are ready to describe how $Ch^m(\bar{I})$ is calculated in two steps when we consider $\bar{I} \subseteq I$ and $c_k \triangleright^m c_{k+1}$ for all $k \in \{1, \dots, |C| - 1\}$.

Step A: Assigning students to reserved classrooms:

Step A.1: If $\bar{I} \cap I_{\kappa(c_1)} \neq \emptyset$, then we select up to \hat{q} students from $\bar{I} \cap I_{\kappa(c_1)}$ for classroom c_1 according to $\succ_{\kappa(c_1)}$. Each selected student i for c_1 is added to $Ch^m(\bar{I})$ and removed from \bar{I} . We remove c_1 if some student is assigned to it.

In general, for each $k \leq |C|$:

Step A.k: If $\bar{I} \cap I_{\kappa(c_k)} \neq \emptyset$, then we select up to \hat{q} students from $\bar{I} \cap I_{\kappa(c_k)}$ for classroom c_k according to $\succ_{\kappa(c_k)}$. Each selected student i for c_k is added to $Ch^m(\bar{I})$ and removed from \bar{I} . We remove c_k if some student is assigned to it.

Step B: Assigning students to unfilled classrooms:

Step B.0 If the number of classrooms filled with grade $g \in G$ students is $k_{g,m}$, then all remaining students in $\bar{I} \cap I_g$ are removed.

Step B.1: If c_1 is removed, then we move to the next step. Otherwise, we consider grade g , if there is any, such that $I_g \cap \bar{I} \neq \emptyset$ and $g \pi^m g'$ for all $g' \neq g$ with $I_{g'} \cap \bar{I} \neq \emptyset$. Then, we select up to \hat{q} students from $\bar{I} \cap I_g$ for c_1 according to \succ_g . Each selected student i is added to $Ch^m(\bar{I})$ and removed from \bar{I} . If the number of classrooms filled with grade g students is $k_{g,m}$, then all students in $\bar{I} \cap I_g$ are removed.

In general, for each $k \leq |C|$:

Step B.k: If c_k is removed, then we move to the next step. Otherwise, we consider grade g , if there is any, such that $I_g \cap \bar{I} \neq \emptyset$ and $g \succ_m g'$ for all $g' \neq g$ with $I_{g'} \cap \bar{I} \neq \emptyset$. Then, we select up to \hat{q} students from $\bar{I} \cap I_g$ for c_k according to \succ_g . Each selected student i is added to $Ch^m(\bar{I})$ and removed from \bar{I} . If the number of classrooms filled with grade g students is $k_{g,m}$, then all students in $\bar{I} \cap I_g$ are removed.

We first highlight the following observations on the choice function for shifts. If $i \in \bar{I} \setminus Ch^m(\bar{I})$, then

- a. There does not exist a student $j \in Ch^m(\bar{I})$ such that $\gamma(j) = \gamma(i)$ and $i \succ_{\gamma(i)} j$;
- b. Existence of a classroom c with no filled seat implies that the number of classrooms filled with grade $\gamma(i)$ students is $k_{\gamma(i),m}$;
- c. No classroom, in which some grade $\gamma(i)$ student assigned, has fewer grade $\gamma(i)$ students than \hat{q} ; and
- d. There are at least $|C_{\gamma(i)}|$ classrooms filled with grade $\gamma(i)$ students.

Observation a. follows from the fact that grade $\gamma(i)$ students are assigned to classrooms one by one according to $\succ_{\gamma(i)}$. Observations b. and c. follow from the fact that choice function Ch^m considers all available students for each classroom c until that classroom is full or there does not exist any remaining student whose assignment to that classroom violates feasibility conditions. Observation d. follows from the fact that in Step A all classroom in $C_{\gamma(i)}$ are filled by students in $\bar{I}_{\gamma(i)}$.

Notice that, the choice function of shift m determines the set of students selected and it assigns each selected student to a specific classroom. In particular, when Ch^m is applied to a subset of students \bar{I} and student i is selected in Step A.k or Step B.k, then i is selected for classroom c_k . Therefore, we can consider the outcome of Ch^m not only as a subset of selected students but also a set of contracts such that selection of a student i for classroom c implies the selection of contract (i, m, c) .

Next, we define two properties for the choice functions which are considered as desirable in the literature. These two properties guarantee existence of strategy-proof and stable mechanisms in the two-sided matching with contracts framework (see [18]). We say a choice function for shift m , Ch^m , is **substitutable** if for any $\bar{I} \subset I$ and $i, j \notin \bar{I}$ $i \notin Ch^m(\bar{I} \cup \{i\})$ implies $i \notin Ch^m(\bar{I} \cup \{i, j\})$. We say a choice function for shift m , Ch^m , satisfies **law of aggregate demand (LAD)** if $|Ch^m(\bar{I})| \leq |Ch^m(\hat{I})|$ for any $\bar{I} \subset \hat{I} \subseteq I$.

In Proposition 1, we show that choice function Ch^m satisfies substitutability but not law of aggregate demand.

Proposition 1. *Choice function Ch^m is substitutable but does not satisfy law of aggregate demand.*

Proof. We start with substitutability. On the contrary, suppose Ch^m is not substitutable. Then, there exists $\bar{I} \subseteq I$, $i, j \notin \bar{I}$ such that $i \notin Ch^m(\bar{I} \cup \{i\})$ but $i \in Ch^m(\bar{I} \cup \{i, j\})$. First of all, since $i \notin Ch^m(\bar{I} \cup \{i\})$, in Step A all classrooms in $\gamma(i)$ are filled by students in $\bar{I}_{\gamma(i)} \setminus \{i\}$. That is, there are at least $\hat{q} \times |C_{\gamma(i)}|$ students in $\bar{I}_{\gamma(i)}$ who are ranked above i under $\succ_{\gamma(i)}$. Hence, the number of grade $\gamma(i)$ students in $\bar{I} \cup \{j\}$ who are ranked above i under $\succ_{\gamma(i)}$ is at least $\hat{q} \times |C_{\gamma(i)}|$ and i cannot be assigned to a classroom in Step A when Ch^m is applied to $\bar{I} \cup \{i, j\}$.

Let \bar{I}' and \bar{I}'' be the set of remaining students for Step B (i.e., after Step A is implemented) when Ch^m is applied to $\bar{I} \cup \{i\}$ and $\bar{I} \cup \{i, j\}$, respectively. As explained above, $i \in \bar{I}' \cap \bar{I}''$. Moreover, since inclusion of j to the set of students does not result in an extra student to be selected in Step A, we have $\bar{I}' \subseteq \bar{I}''$. Let \bar{C}' and \bar{C}'' be the set of remaining classrooms for Step B (i.e., after Step A is implemented) when Ch^m is applied to $\bar{I} \cup \{i\}$ and $\bar{I} \cup \{i, j\}$, respectively. By the definition of Step A, we have $\bar{C}'' \subseteq \bar{C}'$. As a result, students who are assigned to classroom $c \in \bar{C}'$ under $Ch^m(\bar{I} \cup \{i\})$ will be available when we consider that classroom in Step B of $Ch^m(\bar{I} \cup \{i, j\})$. Hence, $i \notin Ch^m(\bar{I} \cup \{i\})$ implies that $i \notin Ch^m(\bar{I} \cup \{i, j\})$. This is a contradiction.

We prove Ch^m does not satisfy LAD by an example. Let $\bar{I} = \{h, i, j, k, \ell\}$ such that $\gamma(h) = \gamma(i) = \gamma(j) = \gamma(k) \neq \gamma(\ell)$. Let $C = \{c_1, c_2\}$, $\hat{q} = 2$, $\kappa(c_1) = \gamma(h)$ and $\kappa(c_2) = \gamma(\ell)$. Let $h \succ_{\gamma(h)} i \succ_{\gamma(h)} j \succ_{\gamma(h)} k$. Then, $Ch^m(\{h, i, j, k\}) = \{h, i, j, k\}$ but $Ch^m(\{h, i, j, k, \ell\}) = \{h, i, \ell\}$. \square

Now, we are ready to define our proposal. Our proposed mechanism is the celebrated (generalized) DA [7,9] in which we use the choice functions introduced for shifts. Given a problem P , (generalized) DA mechanism selects its outcome by first selecting students for the shifts and then determining the corresponding contract as follows.

Generalized DA Mechanism:

Step 1: Each student $i \in I$ proposes to the best shift under P_i , possibly m_o . Let A_1^m be the set of applicants for shift $m \in M$ in Step 1. Shift $m \in M$ tentatively holds students in $Ch^m(A_1^m)$ and rejects all students in $A_1^m \setminus Ch^m(A_1^m)$.

In general for all $h > 1$,

Step h : Each student $i \in I$ proposes to the best shift under P_i , possibly m_o , which has not rejected her yet. Let A_h^m be the set of applicants for shift $m \in M$ in Step h . Shift $m \in M$ tentatively holds students in $Ch^m(A_h^m)$ and rejects all students in $A_h^m \setminus Ch^m(A_h^m)$.

The selection procedure of students terminates when no student is rejected from the shift she proposes. Let \bar{h} be the terminal step. For each $i \in I$, if $i \in A_{\bar{h}}^m = Ch^m(A_{\bar{h}}^m)$ and $Ch^m(A_{\bar{h}}^m)$ assigns i to classroom c , then the assignment of student i under the generalized DA mechanism is (i, m, c) . If $i \notin A_{\bar{h}}^m$ for any $m \in M$, then the assignment of student i under the generalized DA mechanism is x_o .

We first show that generalized DA mechanism is individually rational, non-wasteful and within-grade-fair.

Proposition 2. *Generalized DA mechanism is individually rational, non-wasteful and within-grade-fair.*

Proof. Let μ be the assignment selected by generalized DA under an arbitrary problem P . Let $A^m = i(\mu_m)$, i.e., the set of students signing a contract with shift m under assignment μ . Aygün and Sönmez [25] show that irrelevance of rejected contracts plays crucial role in the existence of stable assignment in matching with contracts framework. It is easy to verify that shift choice function satisfies this property. Proposition 1 and Hatfield and Milgrom [18] imply that

- (i) $Ch^m(A^m) = A^m$, and
- (ii) if $m P_i m(\mu_i)$, then $i \notin Ch^m(A^m \cup \{i\})$.

In the rest of the proof, we use these conditions.

Individual rationality: On the contrary, suppose μ is not individually rational. Since no student proposes to an unacceptable shift, all students are assigned to either x_o or some acceptable contract. Then, there exist $i \in I$ and $\bar{x} \in X_i$ such that $\bar{x} P_i \mu_i$ and $|\{c \in C_{m(\bar{x}), \gamma(i)}(\mu) : |\mu_{c, m(\bar{x})}| = \hat{q}\}| < |C_{\gamma(i)}|$. During the application of the generalized DA mechanism, i has applied to shift $m(\bar{x})$ and she has been rejected. We consider $A^{m(\bar{x})} \cup \{i\}$ under choice function $Ch^{m(\bar{x})}$. Condition $|\{c \in C_{m(\bar{x}), \gamma(i)}(\mu) : |\mu_{c, m(\bar{x})}| = \hat{q}\}| < |C_{\gamma(i)}|$ implies that either there exists $c' \in C_{m(\bar{x}), \gamma(i)}(\mu)$ such that $|\mu_{c', m(\bar{x})}| < \hat{q}$ or $|C_{m(\bar{x}), \gamma(i)}(\mu)| < |C_{\gamma(i)}|$. In either case, by the definition of choice function $Ch^{m(\bar{x})}$, we have $i \in Ch^{m(\bar{x})}(A^{m(\bar{x})} \cup \{i\})$. However, this contradicts condition (ii) above.

Non-wastefulness: On the contrary, suppose μ is wasteful. Then, there exist $i \in I$ and $\bar{x} \in X_i$ such that $\bar{x} P_i \mu_i$ and either

- a. There exists $c' \in C_{m(\bar{x}), \gamma(i)}(\mu)$ such that $|\mu_{c', m(\bar{x})}| < \hat{q}$, or
- b. There exists $c' \in C$ such that $|\mu_{c', m(\bar{x})}| = 0$ and $|C_{m(\bar{x}), \gamma(i)}(\mu)| < k_{\gamma(i), m(\bar{x})}$.

During the application of the generalized DA mechanism, i has applied to shift $m(\bar{x})$, and she has been rejected. We consider $A^{m(\bar{x})} \cup \{i\}$ under choice function $Ch^{m(\bar{x})}$. If either condition holds, by the definition of choice function $Ch^{m(\bar{x})}$, we have $i \in Ch^{m(\bar{x})}(A^{m(\bar{x})} \cup \{i\})$. This follows from the fact that choice function Ch^m fills the seats of all classrooms as long as feasibility requirements (e.g., not assigning different grade students to the same classroom and not using more than $k_{g,m}$ classroom for each grade g) are not violated. However, this, $i \in Ch^{m(\bar{x})}(A^{m(\bar{x})} \cup \{i\})$, contradicts condition (ii).

Within-grade-fairness: On the contrary suppose μ is not within-grade-fair. Then, there exist $i, j \in I$ and $\bar{x} \in X_i$ such that $\bar{x} P_i \mu_i, \gamma(i) = \gamma(j), i \succ_{\gamma(i)} j$ and $m(\mu_j) = m(\bar{x})$. Then, $m(\bar{x}) P_i m(\mu_i), j \in A^{m(\bar{x})}, i \notin A^{m(\bar{x})}$ and $i \in Ch^{m(\bar{x})}(A^{m(\bar{x})} \cup \{i\})$. This follows from the fact that choice function Ch^m selects students from grade $\gamma(i)$ according to $\succ_{\gamma(i)}$. However, this contradicts condition (ii). \square

Unfortunately, generalized DA fails to be strategy-proof without restricting the environment. In fact, next we show that when the number of shifts is strictly more than two, then there does not exist an individually rational, non-wasteful, strategy-proof and within-grade-fair mechanism.

Proposition 3. *When $|M| > 2$, there does not exist an individually rational, non-wasteful and within-grade-fair mechanism which is strategy-proof.*

Proof. We prove this result by taking $|M| = 3$ and $|G| = 2$ and constructing an example. One can easily construct an example for higher number of shifts and grades similarly.

Let $M = \{m_1, m_2, m_3\}$ and $G = \{1, 2\}$. There are six students in Grade 1, $I_1 = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, and four students in Grade 2, $I_2 = \{j_1, j_2, j_3, j_4\}$. Let $C = \{c, c'\}$ where $C_1 = \{c\}$ and $C_2 = \{c'\}$. Let $\hat{q} = 2$. Student preferences over shifts and within-grade priority orders are given as:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{j_1}	P_{j_2}	P_{j_3}	P_{j_4}	\succ_1	\succ_2
m_1	m_1	m_1	m_2	m_1	m_1	m_3	m_3	m_3	m_3	i_1	j_1
m_2	m_2	m_2	m_1	m_2	m_3	m_2	m_1	m_2	m_1	i_2	j_2
m_3	m_3	m_3	m_3	m_3	m_2	m_1	m_2	m_1	m_2	i_3	j_3
m_0	m_0	m_0	m_0	m_0	m_0	m_0	m_0	m_0	m_0	i_4	j_4
										i_5	
										i_6	

In any individually rational, non-wasteful and within-grade-fair matching under problem P , i_1, i_2 and j_4 are matched with contracts including m_1, i_3, i_4 and j_3 are matched with contracts including m_2 , and i_5, i_6, j_1 and j_2 are matched with contracts including m_3 .

Now, suppose i_5 reports her preferences over shifts as follows: $P'_{i_5} : m_2 P'_{i_5} m_0 P'_{i_5} m_1 P'_{i_5} m_3$. Let $P' = (P'_{i_5}, P_{-i_5})$. In any individually rational, non-wasteful and within-grade-fair matching under problem P' , either

- a. i_1, i_2, i_3 and i_6 are matched with contracts including m_1, i_4 and i_5 are matched with contracts including m_2 , and j_1, j_2, j_3 and j_4 are matched with contracts including m_3 ; or
- b. i_1, i_2 and j_4 are matched with contracts including m_1, i_3, i_4 and j_3 are matched with contracts including m_2 , and j_1, j_2 , and i_6 are matched with contracts including m_3 .

If bullet a. holds, then under problem P i_5 can profitably manipulate any individually rational, non-wasteful and within-grade-fair mechanism. Suppose bullet b. holds. Let $P''_{i_6} : m_1 P''_{i_6} m_0 P''_{i_6} m_3 P''_{i_6} m_2$ and $P'' = (P''_{i_6}, P'_{-i_6})$. In any individually rational, non-wasteful and within-grade-fair matching under problem P'' , i_1, i_2, i_3 and i_6 are matched with contracts including m_1, i_4 and i_5 are matched with contracts including m_2 , and j_1, j_2, j_3 and j_4 are matched with contracts including m_3 . Then, under problem P' i_6 can profitably

manipulate any individually rational, non-wasteful and within-grade-fair mechanism selecting the allocation in bullet b. This completes the proof. \square

In current practice, the number of shifts is usually two. In the following proposition, we show that when $|M| = 2$ generalized DA cannot be manipulated by a student.

Proposition 4. *When $|M| = 2$, then generalized DA cannot be manipulated by a student.*

Proof. Let μ be the outcome of generalized DA under some problem P . By individual rationality and non-wastefulness, every student is assigned to a classroom in an acceptable shift or m_o (i.e., contract x_o) in μ . In particular, a student can be assigned to m_o , $\mu_i = x_o$, in the case she ranks at least one shift unacceptable. That is, every student is assigned to a contract with either her top or second choice shift, possibly m_o . A student who is assigned to her top choice shift does not need to misreport her preference order. By individual rationality, any student ranking m_o as top choice is assigned to m_o . Consider a student i who is assigned to her second choice shift, possibly m_o . By non-wastefulness, individual rationality and within-grade-fairness of μ , in the top choice shift of i , all the classrooms in $C_{\gamma(i)}$ are filled with students who are ranked above i under $\succ_{\gamma(i)}$. In particular, the number of students in $I_{\gamma(i)}$ top ranking i 's top choice shift and having higher priority than i is at least $|C_{\gamma(i)}| \times \hat{q}$. Since $|I_{\gamma(i)}| \leq |C_{\gamma(i)}| \times \hat{q} \times |M|$, if i top ranks any shift other than her true top choice, she will be assigned to a contract with that shift. Moreover, if she swaps her second and third choices, then she is either assigned to x_o or a contract with an unacceptable shift. As a result, i does not have profitable deviation strategy under generalized DA. \square

4. Conclusions

During the pandemic, school districts provide different options to the families for the continuation of students' education while maintaining social distancing mandated by states and governments. These options usually divide students in each grade into two and allow each group to attend school physically two days a week or either morning or afternoon four days a week. However, in this process, families preferences are not taken into account. In this paper, we provide a solution which is based on the celebrated deferred acceptance mechanism. Our solution respects the number of classrooms reserved to each grade in the current practice and it assigns students to the schedules (or shifts) in a non-wasteful way by taking the within-grade priority orders into account. Moreover, it is immune to preference manipulations when there are only two schedules (which is a common situation in practice).

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