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# Assortative Matching by Lottery Contests

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**Abstract:** We study two-sided matching contests with two sets, $A$ and $B$, each of which includes a finite number of heterogeneous agents with commonly known types. The agents in each set compete in a lottery (Tullock) contest, and then are assortatively matched, namely, the winner of set $A$ is matched with the winner of set $B$ and so on until all the agents in the set with the smaller number of agents are matched. Each agent has a match value that depends on their own type and the type of their match. We assume that the agents' efforts do not affect their match values and that they have a positive effect on welfare. Therefore, an interior equilibrium in which at least some of the agents are active is welfare superior to a corner equilibrium in which the agents choose to be non-active. We analyze the conditions under which there exists a (partial) interior equilibrium where at least some of the agents compete against each other and exert positive efforts.

**Keywords:** two-sided matching; Tullock contest

**JEL Classification:** D44; J31; D72; D82

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## 1. Introduction

In two-sided matching contests, two contests take place independently within two groups. At the end of these contests, the agents in both groups are assortatively matched according to their efforts and the efforts of the other agents. Then, the prize of each pair who are matched is a function of their types. Two-sided matching contests can be observed, for example, in academic life, in which one of the groups includes universities that invest in hiring the best researchers and teachers as well as in providing the best learning conditions for the students. Such an investment improves its rank and thus will attract better candidates. The other group includes potential student candidates who aspire to be admitted to higher education universities and for this purpose put forth their best efforts in studying for entrance exams, acquiring recommendations, etc. Subsequently, candidates with the best qualities will be admitted to the highest ranked universities. Similar two-sided matching contests can be seen among accounting or law students on the one side and firms on the other, or among models, actors, and artists on the one side and talent agencies on the other.

Two-sided matching contests may involve incentive problems for the designer as well as for the agents. For example, Hoppe, Moldovanu, and Sela studied two-sided matching contests with incomplete information (marginal cost of effort), and compared random matching (without agents' efforts) to assortative matching (based on agents' wasteful efforts) in terms of total expected net welfare [1]. They showed that for distribution functions having a decreasing failure rate, assortative matching with wasteful efforts is welfare-superior, while for distribution functions having an increasing failure rate, random matching is superior. Furthermore, they also showed that each agent may be better off under random matching. We, on the other hand, assume that the agents' efforts are productive for the designer such that these efforts positively affect the net welfare. This means that the benefit from the agents' efforts is larger than their costs for the agents. For example, in Spence's model, investment in education represents the agents' efforts which are obviously not wasteful and have some benefit for the society's welfare [2]. Then, when the agents' efforts positively affect welfare, it is clear that assortative matching is welfare superior to random matching in which agents do not exert any effort at all. Therefore a designer's goal is to activate the agents in order to obtain assortative matching instead of random matching. In this paper, we examine for what reasons agents choose to be non-active in two-sided assortative matching contests and when and how the designer can motivate them to become active.

For this purpose, we study a matching model under complete information where there are two sets of agents, set $A$ with $m$ heterogeneous firms and set $B$ with $n, n \leq m$, heterogeneous workers, each of which has commonly known types. There is only one stage in which the two sets act simultaneously. The firms compete against each other in a Tullock contest [3], and at the same time, the workers compete against each other in another Tullock contest. The agents exert their efforts, and then are assortatively matched, namely, the winner in the contest of set $A$ is matched with the winner in the contest of set $B$, and so on until all the agents in the set with the smaller number of agents (workers) are matched. The agents have match-value functions that are monotonically increasing in both types of firms and workers. An agent who is matched has a payoff of his match-value minus the cost of their effort. It is worth noting that although the agents' types are commonly known, since the Tullock contest success function is stochastic, it is possible that a high-type firm from set $A$ is matched with a lower-type worker from set $B$ and vice versa, namely, a low-type firm will be matched with a high-type worker. This reflects real-life situations such as when the best university does not necessarily include all the best students nor all the best researchers. Likewise, students with lower ability may get higher grades than students with higher abilities. In other words, the ability (type) of an agent does not guarantee success in the matching contest.

We first claim that in the $n \times n$ assortative matching contest with any match-value functions, for every $n \geq 2$ there is a corner equilibrium in which the efforts of all the agents in both sets are zero, and therefore the agents are randomly matched and each firm (worker) has the same probability to be matched with each of the workers (firms). When the match-value functions are additive, we prove that the corner equilibrium in which all the agents are non-active is the unique symmetric equilibrium, and also every permutation of the vector of the agents' equilibrium effort in each set is also in equilibrium. The reason behind this is that the additive function has mixed second order derivatives that are equal to zero, which yields that each agent wins a minimum value that is equal to their own type. As such, the agents in the same set actually have the same prizes based on the agents' types in the other set, and therefore they are symmetric, and, furthermore, if they exert the same effort in equilibrium it is necessarily equal to zero. However, for different forms of the match-value functions, our matching contests may also have an interior equilibrium in which the agents are active and exert positive efforts, or, at least, a partial interior equilibrium in which some of the agents exert positive efforts. Since the explicit characterization of the agents' equilibrium efforts on both sides might be very complex, we focus on $2 \times 2$ assortative matching contests in which there are two agents on each side. We first establish that there is an interior equilibrium by providing necessary and sufficient conditions on the match-value functions.[1] In this case, there is no partial interior equilibrium in which some of the agents exert positive efforts. Then, for multiplicative match-value functions of the agents' types, we have a unique interior equilibrium for which we explicitly characterize the equilibrium efforts. Then, we show that the larger the type of the agent is, the larger is their equilibrium effort.

We proceed by analyzing $m \times n$ assortative matching contests where the number of firms $m$ is larger than the number of workers $n$. We claim that in a $m \times n$ assortative matching contest where $m > n$, at least $n$ firms exert positive efforts in equilibrium. Furthermore, with additive match-value functions where $m > n$, at least $n + 1$ firms exert positive efforts in equilibrium. We focus on $3 \times 2$ assortative matching contests for which we can explicitly characterize the agents' equilibrium efforts and show that it is possible that all the agents on both sides are active. On the other hand, we also show that it is possible that all the agents (firms) on the larger side are active while the agents (workers) in the other side are not active. These results establish the existence of an interior equilibrium with positive efforts in $m \times n$ assortative matching contests, and, in particular, the non-existence of a corner equilibrium without any efforts. Hence, by organizing assortative matching contests with different numbers of agents on both sides, the designer can ensure that, independent of the form of the match-value functions, at least $k$ agents, $k = (\min(m, n))$, will be active.

The rest of the paper is organized as follows: in Section 2, we present our assortative matching contest. In Section 3, we analyze $n \times n$ assortative matching contests, and in Section 4 we analyze $m \times n$ assortative matching contests. Section 5 concludes. Some of the proofs appear in the Appendix A.

*Related Literature*

There are several ways to award prizes in contests. One, which is the most common in the literature, is when there is one prize or several prizes which are identical to all the players, namely, the prize for the $i$'th place is the same independent of the type of player[2] [4–8]. The second, which is more complex, is when agents have heterogenous prizes but with the same order. Then it is usually assumed that the ratio of the values for every pair of prizes is the same for all the agents [9–12]. In our model of assortative matching contests, we study a complex case in which agents do not necessarily know the order of their prizes and even do not necessarily know their values.

In a matching model, efforts can be exerted by either one or both sides. One-sided activity has been modeled in the Tullock contest [3,13–20]; in the all-pay contest [21–23]; and in the rank-order tournament [24,25]. In these contests, there is one set of agents and one set of prizes, and the agents exert efforts to win the prizes. In such one-sided models, the higher the agent's effort is, the higher is their probability to win a larger prize. Some examples of one-sided models include [26,27] who considered a seller facing a continuum of customers differing in their private valuations of service quality. They showed how customers can be matched to different service qualities by offering them price menus that induce them to reveal their types. Likewise, Fernandez and Gali compared the performance of markets and tournaments in a model with a continuum of uniformly distributed agents on each side where only one side is active [28]. They found that despite wasteful signaling, tournaments may be welfare superior to markets if the active agents have budget constraints.

A matching model in which efforts are exerted by agents on two sides with complete information has been studied by Bhaskar and Hopkins who considered a continuum of homogenous agents who are matched according to the tournament model of Lazear and Rosen [29,30]. As was already mentioned, Hoppe, Moldovanu, and Sela studied two-sided markets with incomplete information and a finite number of agents where the agents are matched according to the all-pay contest. Later, Hoppe, Moldovanu, and Ozdenoren studied that model where the agents on both sides compete in the all-pay contest, but with an infinite number of agents on each side [31]. Peters showed that equilibrium efforts in a very large finite two-sided matching model can be quite different from the equilibrium efforts in the continuum model [32]. Dizdar, Moldovanu, and Szech also studied a two-sided model with a finite number of agents where on each side the agents compete in the all-pay contest [33], but in contrast to Hoppe, Moldovanu, and Sela they assumed that the agents' efforts generate benefits for their partners that are increasing in the level of effort [1,34,35]. We, on the other hand, assume that agents' efforts are productive for the designer, but they do not affect the match values and therefore do not generate any benefit for their partners. Our model is the first to combine a two-sided matching model with the lottery (Tullock) contest. Although we assume that there is complete information, the stochastic lottery (Tullock) success function generates uncertainty in the matching between the two sides.

## 2. The Assortative Matching Contest

We consider a set $A = \{1, 2, \ldots, m\}$ of $m$ firms and a set $B = \{1, 2, \ldots, n\}$ of $n$ workers where $n \leq m$. Firm $i$'s type is $m_i$, where $m_i \geq m_{i+1}$, $i = 1, \ldots, m-1$. Worker $j$'s type is $w_j$, where $w_j \geq w_{j+1}$, $j = 1, \ldots, n-1$. All these types are commonly known. As we can see below a firm's utility function as well as a worker's utility function increase in their own types. The matching contest proceeds as follows: There is one stage in which both sets act simultaneously. Each firm $i, i = 1, 2, \ldots, m$ exerts an effort $x_i$, and each worker $j, j = 1, 2, \ldots, n$ exerts an effort $y_j$. Efforts are submitted simultaneously in each set. The order of the firms (workers) to be matched is determined according to the method of Clark and Riis (1998a) as follows: The first firm to be matched is determined by the probability success function which takes into account the efforts of all the firms. Formally, firm $i$, $i = 1, \ldots, m$ wins to be the first match with probability $\frac{x_i}{\sum_{k=1}^{m} x_k}$, where $x_k$ is firm $k$'s effort, $k = 1, \ldots, m$.[3] Then, the second firm to be matched is determined by the probability success function that is based on the efforts of all the firms excluding the effort of the first winner. Thus, firm $i$, $i = 1, \ldots, m$ wins to be the second match with probability $\sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{x_k}{\sum_{j=1}^{m} x_j} \frac{x_i}{\sum_{\substack{j=1 \\ j \neq k}}^{m} x_j}$, and so on until all the firms are ranked, and similarly, all the workers are ranked. Then, the firm and the worker who win first place in their sets are matched, those who win second place in their sets are matched and so on until all the workers are matched. If firm $i$ is matched with worker $j$ after exerting efforts of $x_i$ and $y_j$, correspondingly, the firm's utility is $f(m_i, w_j) - x_i$ and, similarly, the worker's utility is $g(m_i, w_j) - y_j$, where $f, g : R^2 \to R^1$ are the match-value functions which are monotonically increasing in the types of the firms and the workers who are matched. We say that a matching contest has an equilibrium if every agent chooses an effort that maximizes their expected utility given the efforts of the other agents in both sets.

## 3. The $n \times n$ Assortative Matching Contests

In our $n \times n$ assortative matching contests there is always a corner equilibrium in which all the agents do not exert any effort.

**Proposition 1.** *In the $n \times n$ assortative matching contest with any match-value functions, for every $n \geq 2$, there is a symmetric equilibrium in which the efforts of all the agents are zero and therefore the agents are randomly matched such that each firm (worker) has the same probability to be matched with each of the workers (firms).*

The reason behind this corner equilibrium is that if all the $n$ workers exert the same effort, all the workers have the same probability to be in first, second or, any other place. Thus, each firm actually faces $n$ identical prizes since it has the same probability to be matched with each of the workers. As such, the place of each firm is not important, and therefore each of the firms does not have an incentive to exert any effort at all. Likewise, each of the workers exerts an effort of zero, and we have an equilibrium in which all the firms as well all the workers do not exert efforts.

It is worth noting that the existence of a corner equilibrium holds in more general models independent of the preferences of the agents. Furthermore, the existence of the corner equilibrium holds for any contest success function that breaks ties with a fair lottery. For example, consider the assortative all-pay matching contest in which the agent with the highest effort wins for sure, but if there is more than one agent with the highest effort, all these agents win with the same probability [33]. Then, in this assortative all-pay matching contest there is always a corner (trivial) equilibrium in which all the agents in both sets do not exert efforts and the agents are randomly matched.

The corner equilibrium in $n \times n$ assortative matching contests is not necessarily unique and might also be an interior equilibrium in which all or some of the agents exert positive efforts or a partial interior equilibrium in which some of the agents are active. The following result provides sufficient conditions on the match-value functions such that if there is a symmetric equilibrium it is the corner one in which all the agents exert an effort of zero.

**Proposition 2.** *Consider $n \times n$ assortative matching contests with match-value functions that satisfy $f(m_i, w_j) = s(m_i) + t(w_j), g(m_i, w_j) = \widetilde{s}(m_i) + \widetilde{t}(w_j), g(m_i, w_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, n$ where $s, \widetilde{s}, t, \widetilde{t} : R^1 \to R^1$ are monotonically increasing functions. Then, for every $n \geq 2$, if there is a symmetric equilibrium then it is the corner equilibrium in which the efforts of all the agents are zero. If, on the other hand, there is an asymmetric equilibrium, then any permutations of the vectors of the agents' equilibrium efforts in both sides is also an asymmetric equilibrium.*

**Proof.** See Appendix A.1. □

The intuitive explanation to the above result is that in this assortative matching contest, the agents from each set face the same list of prizes which are the types in the other set. In other words, the agents in each set are actually symmetric with the same strategy. We prove that this symmetric strategy has to be an effort of zero. We conjecture that this result will hold for all match-value functions that have mixed second order derivatives that are equal to zero.

In the following, we demonstrate that there is an interior equilibrium in which firms and workers exert positive efforts or at least a partial interior equilibrium in which some of the firms and/or some of the workers exert positive efforts. A characterization of the equilibrium efforts in an assortative matching contest with a large number of agents on both sides is very complex, and therefore, for simplicity, we first focus on the smaller $n \times n$ matching contest with two firms and two workers.

*The 2 × 2 Assortative Matching Contests*

Consider a set $A = \{h, l\}$ of two firms and a set $B = \{h, l\}$ of two workers. We call the types $m_h$ and $w_h$ the high-type firm and worker, respectively, and the other types, $m_l$ and $w_l$, the low-type firm and worker, respectively. Suppose that firm $i, i = h, l$ exerts effort $x_i$ and worker $j, j = h, l$ exerts effort $y_j$, and the two firms compete against each other in a Tullock contest and the two workers compete against each other in another Tullock contest simultaneously. Then, if the agents exert positive efforts, they are assortatively matched, namely the firm that won the contest is matched with the worker who won the contest, and, similarly, the firm that lost the contest is matched with the worker who lost the contest. In this case, the maximization problem of the high-type firm is

$$\begin{aligned} \max_{x_h} f(m_h, w_h) &\left[ \frac{x_h}{x_h + x_l} \frac{y_h}{y_h + y_l} + \frac{x_l}{x_h + x_l} \frac{y_l}{y_h + y_l} \right] \\ + f(m_h, w_l) &\left[ \frac{x_h}{x_h + x_l} \frac{y_l}{y_h + y_l} + \frac{x_l}{x_h + x_l} \frac{y_h}{y_h + y_l} \right] - x_h, \end{aligned} \tag{1}$$

and that of the low-type firm is

$$\begin{aligned} \max_{x_l} f(m_l, w_h) &\left[ \frac{x_l}{x_h + x_l} \frac{y_h}{y_h + y_l} + \frac{x_h}{x_h + x_l} \frac{y_l}{y_h + y_l} \right] \\ + f(m_l, w_l) &\left[ \frac{x_l}{x_h + x_l} \frac{y_l}{y_h + y_l} + \frac{x_h}{x_h + x_l} \frac{y_h}{y_h + y_l} \right] - x_l. \end{aligned} \tag{2}$$

The maximization problem of the high-type worker is

$$\max_{y_h} g(m_h, w_h)\left[\frac{y_h}{y_h+y_l}\frac{x_h}{x_h+x_l}+\frac{y_l}{y_h+y_l}\frac{x_l}{x_h+x_l}\right] \tag{3}$$
$$+g(m_l, w_h)\left[\frac{y_h}{y_h+y_l}\frac{x_l}{x_h+x_l}+\frac{y_l}{y_h+y_l}\frac{x_h}{x_h+x_l}\right]-y_h,$$

and that of the low-type worker is

$$\max_{y_l} g(m_h, w_l)\left[\frac{y_h}{y_h+y_l}\frac{x_l}{x_h+x_l}+\frac{y_l}{y_h+y_l}\frac{x_h}{x_h+x_l}\right] \tag{4}$$
$$+g(m_l, w_l)\left[\frac{y_h}{y_h+y_l}\frac{x_h}{x_h+x_l}+\frac{y_l}{y_h+y_l}\frac{x_l}{x_h+x_l}\right]-y_l.$$

The first-order conditions (F.O.C.) of the maximization problems (1)–(4) are

$$\begin{aligned}
(f(m_h,w_h)-f(m_h,w_l))\frac{x_l}{(x_h+x_l)^2}\frac{y_h-y_l}{y_h+y_l} &\leq 1 \\
(f(m_l,w_h)-f(m_l,w_l))\frac{x_h}{(x_h+x_l)^2}\frac{y_h-y_l}{y_h+y_l} &\leq 1 \\
(g(m_h,w_h)-g(m_l,w_h))\frac{y_l}{(y_h+y_l)^2}\frac{x_h-x_l}{x_h+x_l} &\leq 1 \\
(g(m_h,w_l)-g(m_l,w_l))\frac{y_h}{(y_h+y_l)^2}\frac{x_h-x_l}{x_h+x_l} &\leq 1.
\end{aligned} \tag{5}$$

We focus on the analysis of the interior equilibrium in which all the agents are active. It is worth noting that a partial interior equilibrium in which some of the players exert positive efforts and others do not exert any effort at all is not possible in a $2 \times 2$ assortative matching contests. Thus, we have only a corner equilibrium as well as an interior equilibrium. In the case of an interior equilibrium, there is an equality between the LHS and the RHS of (5) and then we have

**Proposition 3.** *The agents' equilibrium efforts in the $2 \times 2$ assortative matching contest are obtained by the solution of the equations given in (5).*

**Proof.** See Appendix A.2. □

In an interior equilibrium, if we divide the LHS of the first two equations of (5) by each other, and also divide both RHS of these equations by each other, we obtain that

$$\frac{f(m_h,w_h)-f(m_h,w_l)}{f(m_l,w_h)-f(m_l,w_l)} = \frac{x_h}{x_l}. \tag{6}$$

Similarly, if we divide both LHS of the last two equations of (5) by each other, and divide the RHS of these equations by each other, we obtain that

$$\frac{g(m_h,w_h)-g(m_l,w_h)}{g(m_h,w_l)-g(m_l,w_l)} = \frac{y_h}{y_l}. \tag{7}$$

We assume now that all the agents have the same multiplicative match-value function, $f(m_i, w_j) = g(m_i, w_j) = m_i w_j$, $i = h, l$, $j = h, l$.[4] By (5), the agents' equilibrium efforts satisfy:

$$
\begin{aligned}
m_h(w_h - w_l)\frac{x_l}{(x_h + x_l)^2}\frac{y_h - y_l}{y_h + y_l} &\leq 1 \\
m_l(w_h - w_l)\frac{x_h}{(x_h + x_l)^2}\frac{y_h - y_l}{y_h + y_l} &\leq 1 \\
w_h(m_h - m_l)\frac{y_l}{(y_h + y_l)^2}\frac{x_h - x_l}{x_h + x_l} &\leq 1 \\
w_l(m_h - m_l)\frac{y_h}{(y_h + y_l)^2}\frac{x_h - x_l}{x_h + x_l} &\leq 1.
\end{aligned} \tag{8}
$$

In an interior equilibrium, by (6) and (7), we obtain

$$
\begin{aligned}
\frac{y_h}{y_l} &= \frac{w_h}{w_l} \\
\frac{x_h}{x_l} &= \frac{m_h}{m_l}.
\end{aligned} \tag{9}
$$

Thus, we have

**Proposition 4.** *In the $2 \times 2$ assortative matching contest with a multiplicative match-value function, there is either a corner equilibrium in which all the agents exert an effort of zero or a unique interior equilibrium in which the agents' equilibrium efforts are*

$$
\begin{aligned}
x_h &= \frac{m_h^2 m_l}{(m_l + m_h)^2}\frac{(w_h - w_l)^2}{(w_h + w_l)} \\
x_l &= \frac{m_h m_l^2}{(m_l + m_h)^2}\frac{(w_h - w_l)^2}{(w_h + w_l)} \\
y_h &= \frac{w_h^2 w_l}{(w_h + w_l)^2}\frac{(m_h - m_l)^2}{(m_h + m_l)} \\
y_l &= \frac{w_h w_l^2}{(w_h + w_l)^2}\frac{(m_h - m_l)^2}{(m_h + m_l)},
\end{aligned} \tag{10}
$$

*where the worker (firm) with the larger type exerts a larger effort than his opponent.*

In the one-sided standard Tullock contest between firms (workers) when their values of winning are $m_h, m_l$ $(w_h, w_l)$, the equilibrium efforts (see [3]) are

$$
\begin{aligned}
\widetilde{x}_h &= \frac{m_h^2 m_l}{(m_l + m_h)^2} \\
\widetilde{x}_l &= \frac{m_h m_l^2}{(m_l + m_h)^2},
\end{aligned}
$$

and the equilibrium efforts of the workers are

$$
\begin{aligned}
\widetilde{y}_h &= \frac{w_h^2 w_l}{(w_h + w_l)} \\
\widetilde{y}_l &= \frac{w_h w_l^2}{(w_h + w_l)}.
\end{aligned}
$$

If we compare the agents' equilibrium efforts in the (two-sided) assortative matching contest with the (one-sided) standard Tullock contest, we obtain that each firm's effort in the $2 \times 2$ assortative matching contest with a multiplicative match-value function is larger than in the standard Tullock contest iff

$$(w_h - w_l)^2 > (w_h + w_l).$$

Similarly, each worker's effort is larger than in the standard Tullock contest iff

$$(m_h - m_l)^2 > (m_h + m_l).$$

This comparison indicates that even if, for example, $w_h, w_l > 1$ such that the agents' values of winning in the $2 \times 2$ assortative matching contest are larger than the agents's values of winning in the standard Tullock contest, the agents' efforts in the $2 \times 2$ assortative matching contest are not necessarily larger than in the standard Tullock contest. As the above analysis indicates, the necessary condition that the agents' efforts are larger than their efforts in the Tullock contest is that the difference in their opponents' types be relatively larger with respect to their sum. The reason is that when the variance of the agents' types in one set is relatively large, the agents of the other set have a high incentive to compete against each other, while in the one-sided Tullock contest, similar to any other one-sided contest, if the variance of the agents' type is large, the competition between them is weak.

In the $2 \times 2$ assortative matching contest the agents' total effort is

$$\begin{aligned} TE &= x_h + x_l + y_h + y_l \\ &= m_h m_l \frac{(m_h - m_l)}{(m_l + m_h)^2} \frac{(w_h - w_l)^2}{(w_h + w_l)} \\ &\quad + w_h w_l \frac{(w_h - w_l)}{(w_h + w_l)^2} \frac{(m_h - m_l)^2}{(m_h + m_l)}. \end{aligned}$$

Thus, when the sum of the agents' types is constant on both sides, the larger the difference of the agents' types on both sides is, the larger is the equilibrium total effort.

In the next section, we show that if the number of firms and workers are not the same, in contrast to Proposition 2, there is at least a partial interior equilibrium according to which some of the agents compete in the contest and exert positive efforts.

## 4. The $m \times n$ Assortative Matching Contests

Consider now that the two sets do not necessarily have the same size such that there is a set $A = \{1, 2, \ldots, m\}$ of $m \geq 2$ firms and a set $B = \{1, 2, \ldots, n\}$ of $n \geq 2$ workers where $n < m$. The firms' types are $m_i$, where $m_i \geq m_{i+1}$, $i = 1, \ldots, m-1$. The workers' types are $w_j$, where $w_j > w_{j+1}$, $j = 1, \ldots, n-1$. We showed that in an equilibrium of the $n \times n$ assortative matching contest the agents from both sets may not exert efforts in equilibrium. However, this does not occur in the $m \times n$ matching contests if $m > n$ or vice versa. However, $m \times n$ matching contests might have a partial interior equilibrium in which some of the agents exert positive efforts but others do not exert any effort at all.

**Proposition 5.** *In an equilibrium of a $m \times n$ assortative matching contest where $m > n$, at least $n$ firms exert positive efforts.*

**Proof.** See Appendix A.3. □

The intuition behind Proposition 5 is as follows: if less than $n$ firms exert positive efforts, a firm that does not exert any effort will not be matched with a positive probability. Then, if such a non-active firm will choose to exert a sufficiently small effort, it will be matched for sure with one of the workers and then their expected payoff significantly increases.

We showed that in an equilibrium of the $2 \times 2$ assortative matching contest with an additive match-value function the agents from both sets do not exert efforts. By Proposition 5, this does not occur in the $m \times n$ matching contests if $m > n$ or vice versa. Furthermore, the minimum number of agents who exert positive efforts is even larger when the match-value function is additive.

**Proposition 6.** *In an equilibrium of a $m \times n$ assortative matching contest with an additive match-value function where $m > n$, at least $n+1$ firms exert positive efforts.*

**Proof.** See Appendix A.4. □

By Proposition 5 at least $n$ firms exert positive efforts. If exactly $n$ firms exert positive efforts we actually have a $n \times n$ assortative matching contest, and by Proposition 2, all the firms do not exert any effort. Then, similarly to the case in Proposition 5, any non-active firm has an incentive to exert a sufficiently small positive effort such that there will be at least $n+1$ firms that exert positive efforts.

Consider now that $n = 2$ such that the firms' types are $m_i$, where $m_i \geq m_{i+1}$, $i = 1, \ldots, m-1$, and the workers' types are $w_h$ and $w_l$, where $w_h \geq w_l$. Then, if the firms and the workers have a multiplicative match-value function, we have the following result:

**Proposition 7.** *In a $m \times 2$ assortative matching contest with multiplicative match-value functions $f(m_i, w_j) = g(m_i, w_j) = m_i w_j$, the efforts of the workers satisfy*

$$w_h y_l - w_l y_h = 0.$$

**Proof.** See Appendix A.5. □

Furthermore, if the firms and the workers have additive match-value functions we have

**Proposition 8.** *In a $m \times 2$ assortative matching contest with additive match-value functions $f(m_i, w_j) = g(m_i, w_j) = m_i + w_j$ the equilibrium workers' efforts satisfy*

$$y_h = y_l$$

**Proof.** See Appendix A.6. □

Propositions 7 and 8 indicate the relation between the two workers' efforts, but they do not exclude the option that these efforts are equal to zero. An explicit characterization of the equilibrium efforts in these matching contests with large numbers of agents on both sides is very complex and therefore, for simplicity, in the next subsection we focus on the smaller $m \times n$ matching contest with three firms and two workers.

*The 3 × 2 Assortative Matching Contest*

We now consider two sets with a different number of agents where in set $A = \{h, m, l\}$ there are three firms and in set $B = \{h, l\}$ there are two workers. The firms' types are $m_h, m_m$ and $m_l$, where $m_h \geq m_m \geq m_l$, and the workers' types are $w_h$ and $w_l$ where $w_h \geq w_l$. Suppose that firm $i, i = h, m, l$ exerts effort $x_i$ and worker $j, j = h, l$ exerts effort $y_j$. Then, the maximization problem of firm $h$ is

$$\begin{aligned} &\max_{x_h} f(m_h, w_h) \left[ \frac{1}{y_h + y_l} \left( \frac{y_h x_h}{x_h + x_m + x_l} + \frac{y_l x_l}{x_h + x_m + x_l} \frac{x_h}{x_h + x_m} + \frac{y_l x_m}{x_h + x_m + x_l} \frac{x_h}{x_h + x_l} \right) \right] \\ &+ f(m_h, w_l) \left[ \frac{1}{y_h + y_l} \left( \frac{y_l x_h}{x_h + x_m + x_l} + \frac{y_h x_l}{x_h + x_m + x_l} \frac{x_h}{x_h + x_m} + \frac{y_h x_m}{x_h + x_m + x_l} \frac{x_h}{x_h + x_l} \right) \right] - x_h, \end{aligned} \tag{11}$$

the maximization problem of firm $m$ is

$$\begin{aligned} &\max_{x_m} f(m_m, w_h) \left[ \frac{1}{y_h + y_l} \left( \frac{y_h x_m}{x_h + x_m + x_l} + \frac{y_l x_h}{x_h + x_m + x_l} \frac{x_m}{x_l + x_m} + \frac{y_l x_l}{x_h + x_m + x_l} \frac{x_m}{x_m + x_h} \right) \right] \\ &+ f(m_m, w_l) \left[ \frac{1}{y_h + y_l} \left( \frac{y_l x_m}{x_h + x_m + x_l} + \frac{y_h x_h}{x_h + x_m + x_l} \frac{x_m}{x_l + x_m} + \frac{y_h x_l}{x_h + x_m + x_l} \frac{x_m}{x_m + x_h} \right) \right] - x_m, \end{aligned} \tag{12}$$

and the maximization problem of firm $l$ is

$$\max_{x_l} f(m_l,w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_l}{x_h+x_m+x_l}+\frac{y_l x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_l x_m}{x_h+x_m+x_l}\frac{x_l}{x_l+x_h}\right)\right]$$
$$+f(m_l,w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_l}{x_h+x_m+x_l}+\frac{y_h x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_h x_m}{x_h+x_m+x_l}\frac{x_l}{x_l+x_h}\right)\right]-x_l. \tag{13}$$

Similarly, the maximization problem of worker $h$ is

$$\max_{y_h} g(m_h,w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_h}{x_h+x_m+x_l}+\frac{y_l x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{y_l x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right]$$
$$+g(m_m,w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_m}{x_h+x_m+x_l}+\frac{y_l x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{y_l x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m}\right)\right]$$
$$+g(m_l,w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h x_l}{x_h+x_m+x_l}+\frac{y_l x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_l x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l}\right)\right]-y_h, \tag{14}$$

and the maximization problem of worker $l$ is

$$\max_{y_l} g(m_h,w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_h}{x_h+x_m+x_l}+\frac{y_h x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{y_h x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right]$$
$$+g(m_m,w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_m}{x_h+x_m+x_l}+\frac{y_h x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{y_h x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m}\right)\right]$$
$$+g(m_l,w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l x_l}{x_h+x_m+x_l}+\frac{y_h x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{y_h x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l}\right)\right]-y_l. \tag{15}$$

The F.O.C. of firm $h$'s maximization problems is

$$f(m_h,w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h(x_m+x_l)}{(x_h+x_m+x_l)^2}-\frac{y_l x_l}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_m}-\frac{y_l x_m}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_l}\right)\right]$$
$$+f(m_h,w_h)\left[\frac{y_l}{y_h+y_l}\left(\frac{x_m}{(x_h+x_m)^2}\frac{x_l}{x_h+x_m+x_l}+\frac{x_l}{(x_h+x_l)^2}\frac{x_m}{x_h+x_m+x_l}\right)\right]$$
$$+f(m_h,w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l(x_m+x_l)}{(x_h+x_m+x_l)^2}-\frac{y_h x_l}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_m}-\frac{y_h x_m}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_l}\right)\right]$$
$$+f(m_h,w_l)\left[\frac{y_h}{y_h+y_l}\left(\frac{x_m}{(x_h+x_m)^2}\frac{x_l}{x_h+x_m+x_l}+\frac{x_l}{(x_h+x_l)^2}\frac{x_m}{x_h+x_m+x_l}\right)\right]\le 1. \tag{16}$$

The F.O.C. of firm $m$'s maximization problem is

$$f(m_m,w_h)\left[\frac{1}{y_h+y_l}\left(\frac{y_h(x_l+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_l x_h}{(x_h+x_m+x_l)^2}\frac{x_m}{x_l+x_m}-\frac{y_l x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{x_m+x_h}\right)\right]$$
$$+f(m_m,w_h)\left[\frac{y_l}{y_h+y_l}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{(x_l+x_m)^2}+\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{(x_m+x_h)^2}\right)\right]$$
$$+f(m_m,w_l)\left[\frac{1}{y_h+y_l}\left(\frac{y_l(x_l+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_h x_h}{(x_h+x_m+x_l)^2}\frac{x_m}{x_l+x_m}-\frac{y_h x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{x_m+x_h}\right)\right]$$
$$+f(m_m,w_l)\left[\frac{y_h}{y_h+y_l}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{(x_l+x_m)^2}+\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{(x_m+x_h)^2}\right)\right]$$
$$\le\ 1, \tag{17}$$

and the F.O.C. of firm $l$'s maximization problem is

$$
\begin{aligned}
& f(m_l,w_h)\left[\frac{1}{y_h+y_l}(\frac{y_h(x_m+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_lx_h}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_m}-\frac{y_lx_m}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_h})\right] \\
& +f(m_l,w_h)\left[\frac{y_l}{y_h+y_l}(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{(x_l+x_m)^2}+\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{(x_l+x_h)^2})\right] \\
& +f(m_l,w_l)\left[\frac{1}{y_h+y_l}(\frac{y_l(x_m+x_h)}{(x_h+x_m+x_l)^2}-\frac{y_hx_h}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_m}-\frac{y_hx_m}{(x_h+x_m+x_l)^2}\frac{x_l}{x_l+x_h})\right] \\
& +f(m_l,w_l)\left[\frac{y_h}{y_h+y_l}(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{(x_l+x_m)^2}+\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{(x_l+x_h)^2})\right] \\
\leq \; & 1
\end{aligned}
\tag{18}
$$

Similarly, the F.O.C of worker $h$'s maximization problems is

$$
\begin{aligned}
& g(m_h,w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{x_h}{x_h+x_m+x_l}-\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l})\right] \\
& +g(m_m,w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{x_m}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}-\frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m})\right] \\
& +g(m_l,w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{x_l}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l})\right]\leq 1
\end{aligned}
\tag{19}
$$

and the F.O.C. of worker $l$'s maximization problems is

$$
\begin{aligned}
& g(m_h,w_l)\left[\frac{y_h}{(y_h+y_l)^2}(\frac{x_h}{x_h+x_m+x_l}-\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l})\right] \\
& +g(m_m,w_l)\left[\frac{y_h}{(y_h+y_l)^2}(\frac{x_m}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}-\frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m})\right] \\
& +g(m_l,w_l)\left[\frac{y_h}{(y_h+y_l)^2}(\frac{x_l}{x_h+x_m+x_l}-\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}-\frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l})\right]\leq 1.
\end{aligned}
\tag{20}
$$

Then, we have the following interior equilibrium:

**Proposition 9.** *The equilibrium efforts of the $3 \times 2$ assortative matching contest are obtained by the solution of the equations given in (16)–(20).*

**Proof.** See Appendix A.7. □

By Proposition 5, there is no corner equilibrium in the $3 \times 2$ assortative matching contest in which all the three firms exert an effort of zero. However, the following example shows that there is a partial interior equilibrium in which both workers exert an effort of zero.

**Example 1.** *Assume a $3 \times 2$ matching contest with three symmetric firms where $m = m_h = m_m = m_l$ and two asymmetric workers where $w_h \geq w_l$. By symmetry of the firms, assume that every firm exerts the same effort $x$ and worker $j, j = h, l$ exerts effort $y_j$. By (16)–(18), the firms have the same F.O.C. which is given by*

$$
\begin{aligned}
& f(m,w_h)\left[\frac{2}{9x}\frac{y_h}{y_h+y_l}+\frac{1}{18x}\frac{y_l}{y_h+y_l}\right] \\
& +f(m,w_l)\left[\frac{2}{9x}\frac{y_l}{y_h+y_l}+\frac{1}{18}\frac{y_h}{y_h+y_l}\right] \\
= \; & 1.
\end{aligned}
$$

*By symmetry of the firms, the workers' F.O.C. (19) and (20) are*

$$\begin{aligned} 3g(m,w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{1}{3}-(\frac{1}{6}+\frac{1}{6}))\right]-1 &< 0 \\ 3g(m,w_l)\left[\frac{y_h}{(y_h+y_l)^2}(\frac{1}{3}-(\frac{1}{6}+\frac{1}{6}))\right]-1 &< 0. \end{aligned}$$

*Thus, the equilibrium efforts of the workers are* $y_l = y_h = 0$, *and* $\frac{y_l}{y_h+y_l} = \frac{y_h}{y_h+y_l} = \frac{1}{2}$. *Then, the identical effort of all three firms is* $x = \frac{5}{18}\frac{f(m,w_h)+f(m,w_l)}{2}$.

By Proposition 5, in any $3 \times 2$ assortative matching contest at least two firms exert positive efforts in equilibrium. The following example shows that in a $3 \times 2$ assortative matching contest with a multiplicative match-value function, it is possible that exactly two firms exert positive efforts and the third one exerts an effort of zero, or, alternatively, stays out of the contest.

**Example 2.** *Suppose that in the $3 \times 2$ assortative matching contest, firms h and m have the same type, and firm l exerts an effort of* $x_l = 0$. *Then, by the the equilibrium efforts in the $2 \times 2$ assortative matching contest given by (10), we obtain that the equilibrium efforts of the workers satisfy* $\frac{y_h}{y_l} = \frac{w_h}{w_l}$ *and that the equilibrium efforts of the firms that participate are*

$$x_m = x_h = x = \frac{m_h}{4}\frac{(w_h-w_l)^2}{(w_h+w_l)}.$$

*By (18), the F.O.C. of firm l's maximization problem is*

$$\begin{aligned} FOC_3 \;=\; & \frac{x_m+x_h}{(x_h+x_m)^2}\frac{1}{y_h+y_l}(f(m_l,w_h)y_h+f(m_l,w_l)y_l) \\ & +(\frac{x_h}{x_h+x_m}\frac{1}{x_m}+\frac{x_m}{x_h+x_m}\frac{1}{x_h})\frac{1}{y_h+y_l}(f(m_l,w_h)y_l+f(m_l,w_l)y_h) \\ & -1. \end{aligned}$$

*Inserting the equilibrium efforts of the other agents yields*

$$\begin{aligned} FOC_3 &= \frac{1}{2x}\frac{1}{1+\frac{w_l}{w_h}}m_l(w_h+\frac{w_l^2}{w_h})+\frac{1}{x}\frac{1}{1+\frac{w_l}{w_h}}m_l(2w_l) \\ &= \frac{1}{x}\frac{w_h}{w_h+w_l}m_l(2w_l+w_h+\frac{w_l^2}{w_h}) = \frac{4m_l}{m_h}\frac{w_h}{(w_h-w_l)^2}(2w_l+w_h+\frac{w_l^2}{w_h})-1 \\ &= \frac{4m_l}{m_h}\frac{(w_h+w_l)^2}{(w_h-w_l)^2}-1. \end{aligned}$$

*Thus, if* $\frac{m_l}{m_h}$ *is sufficiently small,* $FOC_3$ *is negative, which implies that firm l stays out of the contest.*

## 5. Concluding Remarks

We studied assortative matching contests in which there are two sets of agents. In each set, the agents compete against each other in a Tullock contest, and then, according to the results of both Tullock contests, if the agents exert positive efforts, the agents from both sets are assortatively matched such that the first agents from both sets are matched, the second agents are matched, and so on until all the agents from the smaller set are matched. Every two agents who are matched win a reward according to match-value functions that depend on both agents' types. Our findings for this simultaneous competition are summarized in the following Table 1:

**Table 1.** The simultaneous competition of different agents.

| Size | Multiplicative Form | Additive Form |
|---|---|---|
| $2 \times 2$ | Agents exert efforts | Agents do not exert efforts |
| $3 \times 2$ | At least two firms exert efforts | All the three firms exert efforts |
| $m \times 2$ | Both types of worker may exert efforts | Both types of worker exert the same effort |
| $m \times n$ | At least $n$ firms exert efforts | At least $n + 1$ firms exert efforts |
| $n \times n$ | All agents might not exert efforts | All agents might not exert efforts |

We can see that in symmetric assortative matching contests in which the number of agents in both sets is the same ($n \times n$), there is always an equilibrium in which all the agents in both sets do not exert efforts and as such the agents are randomly matched. However, these matching contests may also have an interior equilibrium in which the equilibrium strategies depend on the form of the match-value function. It is important to note that when the number of agents in both sets is not the same ($m \times n$), independent of the form of the match-value function, there is no equilibrium in which all the agents do not exert efforts.

In sum, the reason for non-activity of the agents in assortative matching contests is that the match-value functions have the same property of the additive function according to which the mixed second derivatives are equal to zero. Then, if the number of agents in both sets are the same, agents from the same set face the same differences between their possible prizes (values of winning) and so that they have the same strategy to be non-active. Thus, a contest designer has two ways to activate the agents: the first is to choose the correct match-value functions and the second is to organize two sets with different sizes. Since in many situations the forms of the match-value function are exogenous such that a designer does not necessarily have control over their forms, by choosing the correct number of agents on each side of the matching contest, independent of the form of the match-value functions, he can ensure that the results of the contest will not be random and the agents will exert positive efforts in order to find their best match.

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## Appendix A

### *Appendix A.1. Proof of Proposition 2*

The maximization problem of firm $i, i = 1, \ldots, n$ is

$$\max_{x_i} \sum_{j=1}^{n} s(m_i) + t(w_j) \sum_{k=1}^{n} Pr(\text{firm } i \text{ wins } k\text{-th place}) \cdot Pr(\text{worker } j \text{ wins } k\text{-th place}) - x_i$$

which is equivalent to the maximization problem

$$\max_{x_i} s(m_i) + \sum_{j=1}^{n} t(w_j) \sum_{k=1}^{n} Pr(\text{firm } i \text{ wins } k\text{-th place}) \cdot Pr(\text{worker } j \text{ wins } k\text{-th place}) - x_i$$

where $Pr$(firm $i$ wins $k$-th place) is the probability that firm $i$ wins $k$-th place, and $Pr$(worker $j$ wins $k$-th place) is the probability that worker $j$ wins $k$-th place. The first-order conditions (F.O.C.) of firm $i$'s maximization problems is

$$\sum_{j=1}^{n} t(w_j) \sum_{k=1}^{n} \frac{dPr(\text{firm } i \text{ wins the } k\text{-th place})}{dx_i} \cdot Pr(\text{worker } j \text{ wins } k\text{-th place}) = 1$$

Since the F.O.C. of each firms' maximization problem does not depend on the type of the firm, it is clear that if there is an asymmetric equilibrium, then any permutation of the vectors of the agents' equilibrium efforts is also an asymmetric equilibrium. In addition, it is clear that there is a symmetric solution that satisfies $x_1 = x_2 =, \ldots = x_n$. Given that all the firms' equilibrium strategies are the same, every worker has the same probability to be matched with each firm such that the best response for each worker $j$ is to exert an effort of $y_j = 0$, $j = 1, \ldots, n$. Now, since all the workers' equilibrium strategies are the same, every firm has the same probability to be matched with each worker so that the best response for each firm $i$ is to exert an effort of $x_i = 0$, $i = 1, \ldots, n$. *Q.E.D.*

*Appendix A.2. Proof of Proposition 3*

The S.O.C. of the maximization problems (1)–(4) are

$$\begin{aligned}
&(f(m_h, w_h) - f(m_h, w_l)) \frac{-2x_l}{(x_h + x_l)^3} \frac{y_h - y_l}{y_h + y_l} \\
&(f(m_l, w_h) - f(m_l, w_l)) \frac{-2x_h}{(x_h + x_l)^3} \frac{y_h - y_l}{y_h + y_l} \\
&(g(m_h, w_h) - g(m_l, w_h)) \frac{-2y_l}{(y_h + y_l)^3} \frac{x_h - x_l}{x_h + x_l} \\
&(g(m_h, w_h) - g(m_l, w_h)) \frac{-2y_h}{(y_h + y_l)^3} \frac{x_h - x_l}{x_h + x_l},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{-2}{(x_h + x_l)} \left[ (f(m_h, w_h) - f(m_h, w_l)) \frac{x_l}{(x_h + x_l)^2} \frac{y_h - y_l}{y_h + y_l} \right] \\
&\frac{-2}{(x_h + x_l)} \left[ (f(m_l, w_h) - f(m_l, w_l)) \frac{x_h}{(x_h + x_l)^2} \frac{y_h - y_l}{y_h + y_l} \right] \\
&\frac{-2}{(y_h + y_l)} \left[ (g(m_h, w_h) - g(m_l, w_h)) \frac{y_l}{(y_h + y_l)^2} \frac{x_h - x_l}{x_h + x_l} \right] \\
&\frac{-2}{(y_h + y_l)} \left[ (g(m_h, w_l) - g(m_l, w_l)) \frac{y_h}{(y_h + y_l)^2} \frac{x_h - x_l}{x_h + x_l} \right].
\end{aligned}$$

Since according to the F.O.C. (5), in an interior equilibrium each of the terms inside the parentheses is positive, we obtain that each of the equations of the S.O.C. is negative and therefore the solution obtained by the equations of the F.O.C is an equilibrium.

It is quite easy to see that the system of the F.O.C. given by (8) has a solution only when all the F.O.C. are equations, namely, there is not a partial interior equilibrium in which some of the equilibrium efforts are positive and others are zero. Then, if we insert the necessary Equation (9) into the F.O.C. we obtain that each of the F.O.C. is a first order equation of one of the agents' effort such that the solution is unique. In other words, the F.O.C given by (10) have a unique solution. *Q.E.D.*

*Appendix A.3. Proof of Proposition 5*

Suppose first that in the $m \times n$ assortative matching contests all the $m$ firms do not exert any effort. In such a case, it is obvious that also the $n$ workers do not have an incentive to exert positive efforts. Therefore, every firm is matched with each of the $n$ workers with a probability of $\frac{1}{m}$ and then a firm has a positive expected payoff. In addition, a firm is not matched at all with a probability of $\frac{m-n}{m}$ and then it has an expected payoff of zero.

Thus, if one firm exerts a sufficiently small effort, given that its opponents do not exert any effort, its expected payoff significantly increases since it is matched with a probability of $\frac{1}{n}$ with each of the workers who exert an effort of zero such that each has the same chance to win as well as to lose. Therefore there is no equilibrium in which all the firms do not exert any effort.

Now, suppose that $k, k < n$, firms exert positive efforts and all the other $m-k$ firms do not exert any effort. Then, each of the $k$ firms that exert positive efforts is matched for sure, while each of the $m-k$ firms that do not exert any effort is matched with one of the $n-k$ workers with the lowest efforts with a probability of $\frac{n-k}{m-k}$ and is not matched at all with a probability of $\frac{m-n}{m-k}$ in which case he has an expected payoff of zero. Thus, if a firm that does not exert any effort will choose to exert a sufficiently small effort he will be matched for sure with one of the workers and then their expected payoff significantly increases. Therefore there is no equilibrium in which the number of firms that exert positive efforts is smaller than the number of workers $n$. *Q.E.D*.

*Appendix A.4. Proof of Proposition 6*

By Proposition 5, suppose that exactly $n$ firms exert positive efforts. Then, we actually have a $n \times n$ assortative matching contest, and by Proposition 2, all the firms do not exert any effort. In that case, each of the firms has a probability of $\frac{1}{m}$ to be matched with each of the $n$ workers and a probability of $\frac{m-n}{m}$ not to be matched at all. Thus, if one of the firms exerts a sufficiently small effort it significantly increases its expected payoff since then it is matched for sure with a probability of $\frac{1}{n}$ with each of the workers. Consequently, in any equilibrium, at least $n+1$ firms participate and exert positive efforts. *Q.E.D*.

*Appendix A.5. Proof of Proposition 7*

The maximization problem of worker $h$ is

$$\max_{y_h} \sum_{i=1}^{m} m_i w_h \left[ \begin{array}{l} \frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{array} \right],$$

where $Pr$(firm $i$ wins first place) is the probability that firm $i$ wins first place, and $Pr$(firm $i$ wins second place) is the probability that firm $i$ wins second place.

Similarly, the maximization problem of worker $l$ is

$$\max_{y_l} \sum_{i=1}^{m} m_i w_l \left[ \begin{array}{l} \frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{array} \right].$$

If we subtract the F.O.C. of these workers' maximization problems from each other we obtain that

$$\begin{aligned} \Delta FOC \quad &= \sum_{i=1}^{m} \frac{m_i(w_h y_l - w_l y_h)}{(y_h+y_l)^2} Pr(\text{firm } i \text{ wins first place}) \\ &\quad - \sum_{i=1}^{m} \frac{m_i(w_h y_l - w_l y_h)}{(y_h+y_l)^2} Pr(\text{firm } i \text{ wins second place}). \end{aligned}$$

Thus, when $w_h y_l - w_l y_h = 0$, we obtain that $\Delta FOC = 0$, which implies that in equilibrium $w_h y_l = w_l y_h$. *Q.E.D*.

*Appendix A.6. Proof of Proposition 8*

The maximization problem of worker $h$ is

$$\max_{y_h} \sum_{i=1}^{m} (m_i + w_h) \left[ \begin{array}{l} \frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{array} \right],$$

where $Pr$(firm $i$ wins first place) is the probability that firm $i$ wins first place, and $Pr$(firm $i$ wins second place) is the probability that firm $i$ wins second place. Similarly, the maximization problem of worker $l$ is

$$\max_{y_l} \sum_{i=1}^{m} (m_i + w_l) \left[ \begin{array}{c} \frac{y_l}{y_h+y_l} Pr(\text{firm } i \text{ wins first place}) \\ +\frac{y_h}{y_h+y_l} Pr(\text{firm } i \text{ wins second place}) \end{array} \right].$$

If we subtract the F.O.C. of these workers' maximization problems from each other we obtain that

$$\begin{aligned} \Delta FOC \quad = \quad & \sum_{i=1}^{m} (m_i + w_h) \frac{y_l}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins first place}) \\ & - \sum_{i=1}^{m} (m_i + w_h) \frac{y_l}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins second place}) \\ & - \sum_{i=1}^{m} (m_i + w_l) \frac{y_h}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins first place}) \\ & + \sum_{i=1}^{m} (m_i + w_l) \frac{y_h}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins second place}). \end{aligned}$$

Since for $j = l, h$ we have

$$\begin{aligned} \sum_{i=1}^{m} w_j Pr(\text{firm } i \text{ wins the first place}) \quad &= \quad w_j \\ \sum_{i=1}^{m} w_j Pr(\text{firm } i \text{ wins the second place}) \quad &= \quad w_j, \end{aligned}$$

we obtain that

$$\begin{aligned} \Delta FOC \quad = \quad & \sum_{i=1}^{m} m_i \frac{y_l - y_h}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins the first place}) \\ & - \sum_{i=1}^{m} m_i \frac{y_l - y_h}{(y_h + y_l)^2} Pr(\text{firm } i \text{ wins the second place}). \end{aligned}$$

Thus, when $y_l = y_h$, we obtain that $\Delta FOC = 0$, which implies that in equilibrium $y_l = y_h$. *Q.E.D.*

*Appendix A.7. Proof of Proposition 9*

In an interior equilibrium, by (16), the F.O.C. of firm $h$'s maximization problem is

$$\begin{aligned} FOC_h \quad &= \quad f(m_h, w_h)(foc_1 + foc_2 + foc_3 + foc_4 + foc_5) \\ & \quad\quad +f(m_h, w_l)(foc_6 + foc_7 + foc_8 + foc_9 + foc_{10}) \\ &= \quad 1, \end{aligned}$$

where

$$\begin{aligned}
foc_1 &= \frac{x_m+x_l}{(x_h+x_m+x_l)^2}\frac{y_h}{y_h+y_l} \\
foc_2 &= -\frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_m}\frac{y_l}{y_h+y_l} \\
foc_3 &= -\frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_l}\frac{y_l}{y_h+y_l} \\
foc_4 &= \frac{x_m}{(x_h+x_m)^2}\frac{x_l}{x_h+x_m+x_l}\frac{y_l}{y_h+y_l} \\
foc_5 &= \frac{x_l}{(x_h+x_l)^2}\frac{x_m}{x_h+x_m+x_l}\frac{y_l}{y_h+y_l} \\
foc_6 &= \frac{x_m+x_l}{(x_h+x_m+x_l)^2}\frac{y_l}{y_h+y_l} \\
foc_7 &= -\frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_m}\frac{y_h}{y_h+y_l} \\
foc_8 &= -\frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_h}{x_h+x_l}\frac{y_h}{y_h+y_l} \\
foc_9 &= \frac{x_m}{(x_h+x_m)^2}\frac{x_l}{x_h+x_m+x_l}\frac{y_h}{y_h+y_l} \\
foc_{10} &= \frac{x_l}{(x_h+x_l)^2}\frac{x_m}{x_h+x_m+x_l}\frac{y_h}{y_h+y_l}.
\end{aligned}$$

The S.O.C. of firm $h$'s maximization problem is

$$\begin{aligned}
SOC_h &= f(m_h,w_h)(soc_1+soc_2+soc_3+soc_4+soc_5) \\
&\quad +f(m_h,w_l)(soc_6+soc_7+soc_8+soc_9+soc_{10})
\end{aligned}$$

where

$$\begin{aligned}
soc_1 &= \frac{-2(x_h+x_m+x_l)(x_l+x_m)}{(x_h+x_m+x_l)^4}\frac{y_h}{y_h+y_l} \\
soc_2 &= (\frac{2(x_h+x_m+x_l)x_l}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_m} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_l}{y_h+y_l} \\
soc_3 &= (\frac{2(x_h+x_m+x_l)x_m}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_l}{y_h+y_l} \\
soc_4 &= (\frac{-2(x_h+x_m)x_m}{(x_h+x_m)^4}\frac{x_l}{x_h+x_m+x_l} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_l}{y_h+y_l} \\
soc_5 &= (\frac{-2(x_h+x_l)x_l}{(x_h+x_l)^4}\frac{x_m}{x_h+x_m+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_l}{y_h+y_l} \\
soc_6 &= \frac{-2(x_h+x_m+x_l)(x_l+x_m)}{(x_h+x_m+x_l)^4}\frac{y_l}{y_h+y_l} \\
soc_7 &= (\frac{2(x_h+x_m+x_l)x_l}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_m} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_h}{y_h+y_l} \\
soc_8 &= (\frac{2(x_h+x_m+x_l)x_m}{(x_h+x_m+x_l)^4}\frac{x_h}{x_h+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_h}{y_h+y_l} \\
soc_9 &= (\frac{-2(x_h+x_m)x_m}{(x_h+x_m)^4}\frac{x_l}{x_h+x_m+x_l} - \frac{x_l}{(x_h+x_m+x_l)^2}\frac{x_m}{(x_h+x_m)^2})\frac{y_h}{y_h+y_l} \\
soc_{10} &= (\frac{-2(x_h+x_l)x_l}{(x_h+x_l)^4}\frac{x_m}{x_h+x_m+x_l} - \frac{x_m}{(x_h+x_m+x_l)^2}\frac{x_l}{(x_h+x_l)^2})\frac{y_h}{y_h+y_l}
\end{aligned}$$

We have the following relations among the elements of the $FOC_h$ and those of the $SOC_h$:

$$\begin{aligned}
soc_1 &= \frac{-2}{x_h+x_m+x_l}foc_1 \\
soc_2 &= (\frac{-2}{x_h+x_m+x_l}+\frac{x_m}{x_h(x_h+x_m)})foc_2 > \frac{-2}{x_h+x_m+x_l}foc_2 \\
soc_3 &= (\frac{-2}{x_h+x_m+x_l}+\frac{x_l}{x_h(x_h+x_l)})foc_3 > \frac{-2}{x_h+x_m+x_l}foc_3 \\
soc_4 &= (\frac{-2}{x_h+x_m}-\frac{1}{x_h+x_m+x_l})foc_4 < \frac{-2}{x_h+x_m+x_l}foc_4 \\
soc_5 &= (\frac{-2}{x_h+x_l}-\frac{1}{x_h+x_m+x_l})foc_5 < \frac{-2}{x_h+x_m+x_l}foc_4 \\
soc_6 &= \frac{-2}{x_h+x_m+x_l}foc_6 \\
soc_7 &= (\frac{-2}{x_h+x_m+x_l}+\frac{x_m}{x_h(x_h+x_m)})foc_7 > \frac{-2}{x_h+x_m+x_l}foc_7 \\
soc_8 &= (\frac{-2}{x_h+x_m+x_l}+\frac{x_l}{x_h(x_h+x_l)})foc_8 > \frac{-2}{x_h+x_m+x_l}foc_8 \\
soc_9 &= (\frac{-2}{x_h+x_m}-\frac{1}{x_h+x_m+x_l})foc_9 < \frac{-2}{x_h+x_m+x_l}foc_9 \\
soc_{10} &= (\frac{-2}{x_h+x_l}-\frac{1}{x_h+x_m+x_l})foc_{10} < \frac{-2}{x_h+x_m+x_l}foc_{10}.
\end{aligned}$$

Since $foc_j, j = 2,3,7,8$ are negative and $foc_j, j = 1,4,5,6,9,10$ are positive, we obtain that

$$SOC_h < \frac{-2}{x_h+x_m+x_l}FOC_h < 0.$$

Similarly, it can be shown that the S.O.C. of the maximization problems of firms $m$ and $l$ are negative as well.

Now, in an interior equilibrium, by (19), the F.O.C. of worker $h$'s maximization problem is

$$\begin{aligned}
foc_h = {} & g(m_h,w_h)\left[\frac{y_l}{(y_h+y_l)^2}\frac{x_h}{x_h+x_m+x_l}\right] \\
& -g(m_h,w_h)\left[(\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m}+\frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l})\right] \\
& +g(m_m,w_h)\left[\frac{y_l}{(y_h+y_l)^2}\frac{x_m}{x_h+x_m+x_l}\right] \\
& -g(m_m,w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m}+\frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m})\right] \\
& +g(m_l,w_h)\left[\frac{y_l}{(y_h+y_l)^2}\frac{x_l}{x_h+x_m+x_l}\right] \\
& -g(m_l,w_h)\left[\frac{y_l}{(y_h+y_l)^2}(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m}+\frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l})\right] = 1.
\end{aligned}$$

And, the S.O.C. of worker $h$'s maximization problem is

$$
\begin{aligned}
soc_h = - \quad & g(m_h, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\frac{x_h}{x_h+x_m+x_l}\right] \\
& + g(m_h, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\left(\frac{x_l}{x_h+x_m+x_l}\frac{x_h}{x_h+x_m} + \frac{x_m}{x_h+x_m+x_l}\frac{x_h}{x_h+x_l}\right)\right] \\
& - g(m_m, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\frac{x_m}{x_h+x_m+x_l}\right] \\
& + g(m_m, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_m}{x_l+x_m} + \frac{x_l}{x_h+x_m+x_l}\frac{x_m}{x_h+x_m}\right)\right] \\
& - g(m_l, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\frac{x_l}{x_h+x_m+x_l}\right] \\
& + g(m_l, w_h)\left[\frac{2(y_h+y_l)y_l}{(y_h+y_l)^4}\left(\frac{x_h}{x_h+x_m+x_l}\frac{x_l}{x_l+x_m} + \frac{x_m}{x_h+x_m+x_l}\frac{x_l}{x_h+x_l}\right)\right].
\end{aligned}
$$

Then, we obtain that

$$soc_h = -foc_h\frac{2(y_h+y_l)}{(y_h+y_l)^2} < 0.$$

Similarly, it can be shown that the S.O.C. of the maximization problem of worker $l$ is negative as well. $Q.E.D.$

## Notes

[1] In 2$x$2 matching contests there is no partial interior equilibrium in which only some of the agents exert positive efforts.

[2] The agents may be different by their marginal costs.

[3] If $x_i = 0$ for all $1 \leq i \leq m$, each firm's probability of winning is assumed to be $1/m$. Similarly, if $y_j = 0$ for all $1 \leq j \leq n$, each worker's probability of winning is assumed to be $1/n$.

[4] Note that our results in this section can be immediately extended to match-value functions of the form $f(m_i, w_j) = \delta(m_i)\rho(w_j)$, where $\delta$ and $\rho$ are strictly increasing and differentiable.

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