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Article

Power Indices under Specific Multicriteria Status

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Abstract: By considering the maximal efficacy among allocation vectors, we define two power indices under specific multicriteria conditions. Additionally, we introduce a reduction approach to the axiomatic framework for these power indices. Furthermore, we propose an alternative formulation that focuses on discrepancy mapping. Based on reduction and discrepancy mapping, we also provide two dynamic procedures.

Keywords: multicriteria status; reduction; discrepancy mapping; dynamic procedure

1. Introduction

In standard systems, each contributor is either fully involved or not involved at all during operational processes with other contributors. Power indices have been employed to measure the effectiveness of all contributors within the system. For instance, contributors in a voting mechanism, such as political parties in a country or parliaments in a confederation, possess distinct amounts of votes, resulting in varied power. Several studies have investigated power indices, including Banzhaf [1], van den Brink and van der Laan [2], Dubey and Shapley [3], Haller [4], Lehrer [5], and Owen [6,7], among others.

A *multi-choice system* can be viewed as a logical extension of a standard system, where each contributor has multiple operational abilities. Power indices have been explored within the structure of multi-choice systems. Hwang and Liao [8], Liao [9,10], and van den Nouweland et al. [11] introduced several allocation concepts and related results by extending the core, the EANSC, and the Shapley value [12] and defining integrated values for specific contributors under multi-choice systems. In cooperative game theory, the term power index is normally a value for simple systems, i.e., for transferable-utility cooperative systems in which each coalition can either be winning or losing; see, e.g., Bertini et al. [13]. Thus, this study is not dealing with simple systems, but proposing and analyzing new values for multi-choice cooperative systems.

Consistency plays a crucial role in characterizing power indices within axiomatic frameworks. Consistency ensures that decisions made on any issue align with decisions made on sub-issues when the allocations of certain contributors are fixed. In addition to axiomatic procedures, *dynamic procedures* can also guide contributors towards a specific power index, starting from an arbitrarily useful allocation vector.

As mentioned previously, the following motivation can be taken into consideration:

- Exploring the possibility of incorporating multi-choice behavior and multicriteria status when considering power indexes.

Considering the above motivation, we proceed with the following steps and present related results.

1. We focus on the structure of *multicriteria multi-choice systems* in Section 2, which differs from that of multi-choice systems. We define a power index and its normalization for multicriteria multi-choice systems by utilizing the maximal efficacy among multi-choice allocation vectors.

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2. To validate the rationality of these power indices, we introduce a generalized reduction in Section 3, characterizing them. We further propose an alternative formulation that introduces dynamic procedures for the normalized power index through *discrepancy mapping*.
3. In Section 4, we demonstrate that contributors can achieve the normalized index by starting from an arbitrarily useful allocation vector, employing a specific reduced system and discrepancy mapping.

2. Preliminaries

Let UC be the universe of contributors. For $k \in UC$ and $a_k \in \mathbb{N}$, $A_k = \{0, 1, \cdots, a_k\}$ could be taken as the ability space of contributor k and $A_k^+ = A_i \setminus \{0\}$, where 0 denotes no operation. Let $A^C = \prod_{i \in C} A_i$ be the product set of the ability spaces of the total contributors of C. For all $T \subseteq C$, $\rho^T \in \omega^C$ is the vector with $\rho_k^T = 1$ if $k \in T$, and $\rho_k^T = 0$ if $k \in C \setminus T$. Define 0_C to be the zero vector in $\mathbb{R}^C$. For $m \in \mathbb{N}$, let 0_m be the zero vector in $\mathbb{R}^m$ and $\mathbb{N}_m = \{1, \cdots, m\}$.

Denote **the multi-choice system** as (C, a, d), where C is the collection of contributors, $a = (a_i)_{i \in C}$ is the vector that shows the amount of abilities for all contributors, and $d : A^C \to \mathbb{R}$ is an efficacy mapping with $d(0_C) = 0$ which allots to each $\mu = (\mu_k)_{k \in C} \in A^C$ the value that all contributors can receive if each contributor k adopts ability μ_k. Given a multi-choice system (C, a, d) and $\mu \in A^C$, one would define $N(\mu) = \{s \in C | \ \mu_s \neq 0\}$ and μ_K to be the restriction of μ at K for each $K \subseteq C$. Further, we define $d_*(K) = \max_{\mu \in A^C}\{d(\mu) | N(\mu) = K\}$ as the **maximal efficacy** among all ability vectors μ with $N(\mu) = K$. From now on, one should consider bounded multi-choice systems, defined as those systems (C, a, d) such that there exists $M_d \in \mathbb{R}$ such that $d(\mu) \leq M_d$ for all $\mu \in A^C$. One could apply this to ensure that $d_*(K)$ is well-defined. Denote **a multicriteria multi-choice system** to be (C, a, D^m), where $m \in \mathbb{N}$, $D^m = (d^t)_{t \in \mathbb{N}_m}$ and (C, a, d^t) is a multi-choice system for each $t \in \mathbb{N}_m$.

Let $(C, a, D^m) \in \Gamma$. An **allotment vector** of (C, a, D^m) is a vector $x = (x^t)_{t \in \mathbb{N}_m}$ and $x^t = (x_k^t)_{k \in C} \in \mathbb{R}^C$, where x_k^t is the allotment to contributor i in (C, a, d^t) for each $t \in \mathbb{N}_m$ and for each $i \in C$. An allotment vector x of (C, a, D^m) is **multicriteria useful** if $\sum_{i \in C} x_i^t = d_*^t\big(N\big)$ for each $t \in \mathbb{N}_m$. The set of total multicriteria useful vectors of (C, a, D^m) is defined as $\Theta(C, a, D^m)$. An **index** is a mapping Φ assigning to each $(C, a, D^m) \in \Gamma$ an element

$$\Phi\big(C, a, D^m\big) = \big(\Phi^t\big(C, a, D^m\big)\big)_{t \in \mathbb{N}_m},$$

where $\Phi^t\big(C, a, D^m\big) = \big(\Phi_i^t\big(C, a, D^m\big)\big)_{i \in C} \in \mathbb{R}^C$ and $\Phi_i^t\big(C, a, D^m\big)$ is the allotment of the contributor i assigned by Φ in $\big(C, a, d^t\big)$.

Next, we provide the maximal individual index and the maximal normalized individual index.

Definition 1. *The* ***maximal individual index (MII)****, η, is defined by*

$$\eta_k^t(C, a, D^m) = d_*^t(\{k\})$$

for each $(C, a, D^m) \in \Gamma$, for each $t \in \mathbb{N}_m$, and for each $k \in C$. Under the index η, all contributors receive its ***m****aximal individual efficacy.*

An index Φ satisfies **multicriteria usefulness (MUSE)** if for all $(C, a, D^m) \in \Gamma$ and for all $t \in \mathbb{N}_m$, $\sum_{i \in C} \Phi_i(C, a, D^m) = d_*^t(C)$. The MUSE property means that the total contributors allocate the whole efficacy entirely. It is trivial to verify that the MII violates MUSE. Hence, one would like to consider a useful normalization.

Definition 2. *The **maximal normalized individual index (MNII)**, $\overline{\eta}$, is defined as follows. For each $(C, a, D^m) \in \Gamma'$, for each $t \in \mathbb{N}_m$, and for each $k \in C$,*

$$\overline{\eta_k^t}(C, a, D^m) = \frac{d_*^t(C)}{\sum\limits_{s \in C} \eta_s^t(C, a, D^m)} \cdot \eta_k^t(C, a, D^m),$$

where $\Gamma' = \{(C, a, D^m) \in \Gamma | \text{For all } t \in \mathbb{N}_m, \sum\limits_{i \in C} \eta_i^t(C, a, D^m) \neq 0\}$. Under the definition of $\overline{\eta}$, all contributors distribute the maximal efficacy of the grand coalition proportionally via maximal individual efficacy.

Lemma 1. *The MNII satisfies MUSE on Γ'.*

Proof. This proof can be finished easily via definitions of usefulness and the MNII. So it is omitted. □

3. Axiomatic Results

Here, one would like to present that there exists a relevant reduced system that could be introduced to axiomatize the MII and the MNII.

First, an alternative formulation for the MNII would be defined in terms of *discrepancy*. Given $(C, a, D^m) \in \Gamma$, $K \subseteq C$ and an allotment vector x, let $x^t(K) = \sum_{s \in K} x_s^t$ for all $t \in \mathbb{N}_m$. The **discrepancy** of a coalition $K \subseteq C$ under x is

$$\theta(K, D^m, x) = (\theta(K, d^t, x^t))_{t \in \mathbb{N}_m} \text{ and } \theta(K, d^t, x^t) = d_*^t(K) - x^t(K). \tag{1}$$

$\theta(K, d^t, x^t)$ can be taken as the **variation** among efficacy and total allotments in coalition K if all contributors to K receive their allotments from x^t in (C, a, d^t).

Lemma 2. *Let $(C, a, D^m) \in \Gamma'$, $t \in \mathbb{N}_m$, $x \in \Theta(C, a, D^m)$ and $\omega^t = \frac{d_*^t(C)}{\sum\limits_{k \in C} \eta_k^t(C, a, D^m)}$. Then,*

$$\theta(\{i\}, d^t, \frac{x^t}{\omega^t}) = \theta(\{j\}, d^t, \frac{x^t}{\omega^t}) \ \forall\, i, j \in C \iff x = \overline{\eta}(C, a, D^m).$$

Proof. Let $(C, a, D^m) \in \Gamma'$ and $x \in \Theta(C, a, D^m)$. For each $t \in \mathbb{N}_m$ and for every $i, j \in C$,

$$\begin{aligned} \theta(\{i\}, d^t, \tfrac{x^t}{\omega^t}) = \theta(\{j\}, d^t, \tfrac{x^t}{\omega^t}) &\iff d_*^t(\{i\}) - \tfrac{x_i^t}{\omega^t} = d_*^t(\{j\}) - \tfrac{x_j^t}{\omega^t} \\ &\iff x_i^t - x_j^t = \omega^t \cdot [d_*^t(\{i\}) - d_*^t(\{j\})]. \end{aligned} \tag{2}$$

By definition of $\overline{\eta}$,

$$\overline{\eta_i^t}(C, a, D^m) - \overline{\eta_j^t}(C, a, D^m) = \omega^t \cdot [d_*^t(\{i\}) - d_*^t(\{j\})]. \tag{3}$$

By (2) and (3), for every $i, j \in C$,

$$x_i^t - x_j^t = \overline{\eta_i^t}(C, a, D^m) - \overline{\eta_j^t}(C, a, D^m).$$

Hence,

$$\sum_{j \neq i} [x_i^t - x_j^t] = \sum_{j \neq i} [\overline{\eta_i^t}(C, a, D^m) - \overline{\eta_j^t}(C, a, D^m)].$$

That is, $(|C| - 1) \cdot x_i^t - \sum_{j \neq i} x_j^t = (|C| - 1) \cdot \overline{\eta_i^t}(C, a, D^m) - \sum_{j \neq i} \overline{\eta_j^t}(C, a, D^m)$. Since $x \in \Theta(C, a, D^m)$ and $\overline{\eta}$ satisfies MUSE, $|C| \cdot x_i^t - d_*^t(C) = |C| \cdot \overline{\eta_i^t}(C, a, D^m) - d_*^t(C)$. Therefore, $x_i^t = \overline{\eta_i^t}(C, a, D^m)$ for each $t \in \mathbb{N}_m$ and for each $i \in C$. That is, $x = \overline{\eta}(C, a, D^m)$. □

Remark 1. *It is trivial to verify that* $\theta(C \setminus \{i\}, D^m, \eta(C, a, D^m)) = \theta(C \setminus \{j\}, D^m, \eta(C, a, D^m))$ *for all* $(C, a, D^m) \in \Gamma$ *and for all* $i, j \in C$.

Inspired by the complement-reduced systems due to Hsieh and Liao [14] and Moulin [15], one would like to introduced a multi-choice analogue and relative consistency. Let Φ be an index, $(C, a, D^m) \in \Gamma$ and $H \subseteq C$. The **reduced system** $(H, a_H, D^m_{H,\Phi})$ is defined by $D^m_{H,\Phi} = (d^t_{H,\Phi})_{t \in \mathbb{N}_m}$ and

$$d^t_{H,\Phi}(\alpha) = \begin{cases} 0 & , \alpha = 0_H, \\ d^t_*(\{k\}) & , H \geq |2| \text{ and } N(\alpha) = \{k\}, \\ d^t_*\big(N(\alpha) \cup (C \setminus H)\big) - \sum\limits_{i \in C \setminus H} \Phi_i(C, a, d) & , \text{otherwise.} \end{cases}$$

Φ satisfies **consistency (CSY)** if $\Phi^t_k(H, a_H, D^m_{H,\Phi}) = \Phi^t_k(C, a, D^m)$ for each $(C, a, D^m) \in \Gamma$, for each $H \subseteq C$ with $|H| \leq 2$, for each $t \in \mathbb{N}_m$, and for each $k \in H$. Unfortunately, it is trivial to verify that $\sum_{k \in H} \eta^t_k(C, a, d) = 0$ for some $(C, a, D^m) \in \Gamma$, for some $t \in \mathbb{N}_m$, and for some $H \subseteq C$, i.e., $\overline{\eta}(H, a_H, D^m_{H,\Phi})$ does not exist for some $(C, a, D^m) \in \Gamma$ and for some $H \subseteq C$. Thus, one would consider the *resilient consistency* as follows. An index Φ satisfies **resilient consistency (RCSY)** if $(H, a_H, d^m_{H,\Phi})$ and $\Phi(H, a_H, d^m_{H,\Phi})$ exist for some $(C, a, D^m) \in \Gamma$ and for some $H \subseteq C$ with $|H| \leq 2$; it holds that $\Phi^t_i(H, a_H, d^m_{H,\Phi}) = \Phi^t_i(C, a, D^m)$ for all $t \in \mathbb{N}_m$ and for all $i \in H$.

Lemma 3.

1. *The MII satisfies CSY on* Γ.
2. *The MNII satisfies RCSY on* Γ'.

Proof. To analyze result 1, let $(C, a, D^m) \in \Gamma'$ and $H \subseteq C$. The proof is trivial if $|C| = 1$. Assume that $|C| \geq 2$ and $H = \{i, j\}$ for some $i, j \in C$. For each $t \in \mathbb{N}_m$ and for each $i \in H$,

$$\begin{aligned} \eta^t_i(H, a_H, D^m_{H,\eta}) &= (d^t_{H,\eta})_*(\{i\}) \\ &= d^t_*(\{i\}) \\ &= \eta^t_i(C, a, D^m). \end{aligned} \tag{4}$$

That is, the MII satisfies CSY.

To analyze result 2, let $(C, a, D^m) \in \Gamma'$ and $H \subseteq C$. The proof is trivial if $|C| = 1$. Assume that $|C| \geq 2$. If $H = \{i, j\}$ for some $i, j \in C$ and $(H, a_H, d^m_{H,\overline{\eta}}) \in \Gamma'$. Similar to (4), for each $t \in \mathbb{N}_m$ and for each $i \in H$,

$$\eta^t_i(H, a_H, d^m_{H,\overline{\eta}}) = \eta^t_i(C, a, D^m). \tag{5}$$

By definition of $\overline{\eta}$ and Equation (5),

$$\begin{aligned} \overline{\eta^t_i}(H, a_H, d^m_{H,\overline{\eta}}) &= \frac{(d^t_{H,\overline{\eta}})_*(H)}{\sum\limits_{k \in H} \eta^t_k(H, a_H, d^m_{H,\overline{\eta}})} \cdot \eta^t_i(H, a_H, d^m_{H,\overline{\eta}}) \\ &= \frac{d^t_*(C) - \sum\limits_{k \in C \setminus H} \overline{\eta^t_k}(C, a, D^m)}{\sum\limits_{k \in H} \eta^t_k(C, a, D^m)} \cdot \eta^t_i(C, a, D^m) \\ &= \frac{\sum\limits_{k \in H} \overline{\eta^t_k}(C, a, D^m)}{\sum\limits_{k \in H} \eta^t_k(C, a, D^m)} \cdot \eta^t_i(C, a, D^m) \\ &= \omega^t \cdot \eta^t_i(C, a, D^m), \text{ where } \omega^t = \frac{d^t_*(C)}{\sum\limits_{k \in C} \eta^t_k(C, a, D^m)} \\ &= \overline{\eta^t_i}(C, a, D^m). \end{aligned}$$

Thus, the MNII satisfies RCSY on Γ'. $\square$

In the following, the MII and the MNII would be characterized via CSY and RCSY.

- An index Φ satisfies **individual-standard for systems (ISS)** if $\Phi(C,a,d) = \eta(C,a,d)$ for all $(C,a,d) \in \Gamma$ with $|C| \leq 2$.
- An index Φ satisfies **normalized-standard for systems (NSS)** if $\Phi(C,a,d) = \overline{\eta}(C,a,d)$ for all $(C,a,d) \in \Gamma'$ with $|C| \leq 2$.

Lemma 4. *If an index Φ satisfies NSS and RCSY on Γ', then it satisfies MUSE on Γ'.*

Proof. Let $(C,a,D^m) \in \Gamma'$. By NSS, Φ satisfies MUSE on Γ' if $|C| \leq 2$. Assume that $|C| > 2$. Suppose, on the contrary, that there is $(C,a,D^m) \in \Gamma'$ such that $\sum_{i \in C} \Phi_i^t(C,a,D^m) \neq d_*^t(C)$ for some $t \in \mathbb{N}_m$. This means that there exist $i,j \in C$ such that

$$d_*^t(C) - \sum_{k \in C \setminus \{i,j\}} \Phi_k^t(C,a,D^m)] \neq [\Phi_i^t(C,a,D^m) + \Phi_j^t(C,a,D^m).$$

By RCSY and Φ satisfies MUSE for two-person systems, this contradicts with

$$\begin{aligned} \Phi_i^t(C,a,D^m) + \Phi_j^t(C,a,D^m) &= \Phi_i^t(\{i,j\}, d^m_{\{i,j\},\Phi}) + \Phi_j^t(\{i,j\}, d^m_{\{i,j\},\Phi}) \\ &= d_*^t(C) - \sum_{k \in C \setminus \{i,j\}} \Phi_k^t(C,a,D^m). \end{aligned}$$

Hence, Φ satisfies MUSE. □

Theorem 1.

1. *On Γ, the MII is the only index satisfying ISS and CSY.*
2. *On Γ', the MNII is the only index satisfying NSS and RCSY.*

Proof. By Lemma 3, η and $\overline{\eta}$ satisfy CSY and RCSY on Γ and Γ', respectively. Absolutely, η and $\overline{\eta}$ satisfy ISS and NSS on Γ and Γ', respectively.

To analyze the uniqueness of statement 1, suppose Φ satisfies CSY and ISS on Γ. Let $(C,a,D^m) \in \Gamma$. If $|C| \leq 2$, then $\Phi(C,a,D^m) = \eta(C,a,D^m)$ by ISS. Assume that $|C| > 2$. Suppose that $H \subseteq C$ with $|H| = 2$. Let $t \in \mathbb{N}_m$ and $i \in S$.

$$\begin{aligned} \Phi_i^t(C,a,D^m) &= \Phi_i^t(H, a_H, d^m_{H,\Phi}) \quad \textbf{(By CSY of } \boldsymbol{\Phi}\textbf{)} \\ &= \eta_i^t(H, a_H, d^m_{H,\Phi}) \quad \textbf{(By ISS of } \boldsymbol{\Phi}\textbf{)} \\ &= (d^t_{H,\Phi})_*(\{i\}) \\ &= d_*^t(\{i\}) \\ &= \eta_i^t(C,a,D^m). \end{aligned}$$

Hence, $\Phi(C,a,D^m) = \eta(C,a,D^m)$ for all $(C,a,D^m)\Gamma$.

To analyze the uniqueness of statement 2, assume that Φ satisfies RCSY and NSS on Γ'. Further, Φ satisfies MUSE on Γ' by Lemma 4. Let $(C,a,D^m) \in \Gamma'$. The proof will be completed by induction on $|C|$. By NSS it is trivial that $\Phi(C,a,D^m) = \overline{\eta}(C,a,D^m)$ if $|C| \leq 2$. Assume that it holds if $|C| \leq r-1$, $r \geq 3$. The condition $|C| = r$: Let $t \in \mathbb{N}_m$ and $i,j \in C$ with $i \neq j$. By Definition 2, $\overline{\eta_k^t}(C,a,D^m) = \frac{d_*^t(C)}{\sum_{h \in C} \eta_h^t(C,a,D^m)} \cdot \eta_k^t(C,a,D^m)$ for all $k \in C$. Assume that $\beta_k^t = \frac{\eta_k^t(C,a,d)}{\sum_{h \in C} \eta_h^t(C,a,d)}$ for all $k \in C$. Therefore,

$$\begin{aligned}
\Phi_i^t(C,a,D^m) &= \Phi_i^t\big(C\setminus\{j\}, d^m_{C\setminus\{j\},\Phi}\big)\\
&= \overline{\eta_i^t}\big(C\setminus\{j\}, d^m_{C\setminus\{j\},\Phi}\big)\\
&= \frac{(d^t_{C\setminus\{j\},\Phi})_*\big(C\setminus\{j\}\big)}{\sum\limits_{k\in C\setminus\{j\}}\eta_k^t\big(C\setminus\{j\}, d^m_{C\setminus\{j\},\Phi}\big)}\cdot \eta_i^t\big(C\setminus\{j\}, d^m_{C\setminus\{j\},\Phi}\big)\\
&= \frac{d_*^t(C)-\Phi_i^t(C,a,D^m)}{\sum\limits_{k\in C\setminus\{j\}}\eta_k^t(C,a,D^m)}\cdot \eta_i^t(C,a,D^m)\\
&= \frac{d_*^t(C)-\Phi_i^t(C,a,D^m)}{-\eta_j^t(C,a,D^m)+\sum\limits_{k\in C}\eta_k^t(C,a,D^m)}\cdot \eta_i^t(C,a,D^m).
\end{aligned} \tag{6}$$

By Equation (6),

$$\begin{aligned}
& \Phi_i^t(C,a,D^m)\cdot[1-\beta_j^t] = [d_*^t(C)-\Phi_j^t(C,a,D^m)]\cdot\beta_j^t\\
\Longrightarrow\ & \sum_{i\in C}\Phi_i^t(C,a,D^m)\cdot[1-\beta_j^t] = [d_*^t(C)-\Phi_j^t(C,a,D^m)]\cdot\sum_{i\in C}\beta_j^t\\
\Longrightarrow\ & d_*^t(C)\cdot[1-\beta_j^t] = [d_*^t(C)-\Phi_j^t(C,a,D^m)]\cdot 1\\
\Longrightarrow\ & d_*^t(C)-d_*^t(C)\cdot\beta_j^t = d_*^t(C)-\Phi_j^t(C,a,D^m)\\
\Longrightarrow\ & \overline{\eta_j^t}(C,a,D^m) = \Phi_j^t(C,a,D^m).
\end{aligned}$$

The proof is completed. □

4. Dynamic Results

Here, one would like to adopt discrepancy mapping and a specific reduction to provide dynamic results for the MNII.

To establish dynamic results for the MNII, we define a switch mapping using discrepancy mappings. The switch mapping is based on the idea that each contributor minimizes the variation related to its own and others' non-cooperation, applying these regulations to switch the original allocation.

Definition 3. *Let $(C,a,D^m)\in\Gamma'$. The* **switch mapping** *is defined to be $S=(S^t)_{t\in\mathbb{N}_m}$, where $S^t=(S_i^t)_{i\in C}$ and $S_i^t:\Theta(C,a,D^m)\to\mathbb{R}$ is defined by*

$$S_i^t(x) = x_i^t + z\sum_{j\in C\setminus\{i\}}\omega^t\cdot\Big(\theta(\{i\},d^t,\frac{x^t}{\omega^t}) - \theta(\{j\},d^t,\frac{x^t}{\omega^t})\Big),$$

where $\omega^t=\frac{d_*^t(C)}{\sum_{k\in C}\eta_k^t(C,a,D^m)}$ and $z\in(0,\infty)$, which incarnates the assumption that contributor i does not ask for sufficient switch (if $z=1$) but only (often) a fraction of it. Define that $[x]^0=x,\ [x]^1=S([x]^0),\cdots,\ [x]^q=S([x]^{q-1})$ for all $q\in\mathbb{N}$.

Lemma 5. *$S(x)\in\Theta(C,a,D^m)$ for all $(C,a,D^m)\in\Gamma'$ and for all $x\in\Theta(C,a,D^m)$.*

Proof. Let $(C,a,D^m)\in\Gamma'$, $t\in\mathbb{N}_m$, $i,j\in C$, and $x\in\Theta(C,a,D^m)$.

$$\sum_{j\in C\setminus\{i\}}\omega^t\cdot\Big(\theta(\{i\},d^t,\frac{x^t}{\omega^t}) - \theta(\{j\},d^t,\frac{x^t}{\omega^t})\Big) = \sum_{j\in C\setminus\{i\}}\omega^t\cdot\Big(d^t(\{i\})-d^t(\{j\})-\frac{x_i^t}{\omega^t}+\frac{x_j^t}{\omega^t}\Big). \tag{7}$$

By definition of $\overline{\eta}$,

$$\overline{\eta_i^t}(C,a,D^m)-\overline{\eta_j^t}(C,a,D^m) = \omega^t\cdot\Big(d^t(\{i\})-d^t(\{j\})\Big). \tag{8}$$

Based on (7) and (8),

$$\begin{aligned}
& \sum_{j\in C\setminus\{i\}} \omega^t \cdot \big(\theta(\{i\}, d^t, \tfrac{x^t}{\omega^t}) - \theta(\{j\}, d, \tfrac{x^t}{\omega^t})\big) \\
= & \sum_{j\in C\setminus\{i\}} \big(\overline{\eta_i^t}(C,a,D^m) - \overline{\eta_j^t}(C,a,D^m) - x_i^t + x_j^t\big) \\
= & \big((|C|-1)\big(\overline{\eta_i^t}(C,a,D^m) - x_i^t\big) - \sum_{j\in C\setminus\{i\}} \overline{\eta_j^t}(C,a,D^m) + \sum_{j\in C\setminus\{i\}} x_j^t\big) \\
= & \big(|C|\big(\overline{\eta_i^t}(C,a,D^m) - x_i^t\big) - d_*^t(C) + d_*^t(C)\big) \\
= & |C| \cdot \big(\overline{\eta_i^t}(C,a,D^m) - x_i^t\big).
\end{aligned} \tag{9}$$

Moreover,

$$\begin{aligned}
& \sum_{i\in C}\sum_{j\in C\setminus\{i\}} \omega^t \cdot \big(\theta(\{i\}, d^t, \tfrac{x^t}{\omega^t}) - \theta(\{j\}, d^t, \tfrac{x^t}{\omega^t})\big) \\
= & \sum_{i\in C} |C| \cdot \big(\overline{\eta_i^t}(C,a,D^m) - x_i^t\big) \\
= & |C| \cdot \big(\sum_{i\in C} \overline{\eta_i^t}(C,a,D^m) - \sum_{i\in C} x_i^t\big) \\
= & |C| \cdot \big(d_*^t(C) - d_*^t(C)\big) \\
= & 0.
\end{aligned} \tag{10}$$

So, we have that

$$\begin{aligned}
\sum_{i\in C} S_i^t(x) &= \sum_{i\in C} \Big[x_i^t + z \sum_{j\in C\setminus\{i\}} \omega^t \cdot \big(\theta(\{i\}, d^t, \tfrac{x^t}{\omega^t}) - \theta(\{j\}, d^t, \tfrac{x^t}{\omega^t})\big)\Big] \\
&= \sum_{i\in C} x_i^t + z \sum_{i\in C}\sum_{j\in C\setminus\{i\}} \omega^t \cdot \big(\theta(\{i\}, d^t, \tfrac{x^t}{\omega^t}) - \theta(\{j\}, d^t, \tfrac{x^t}{\omega^t})\big) \\
&= d_*^t(C).
\end{aligned}$$

Hence, $S(x) \in \Theta(C,a,D^m)$ if $x \in \Theta(C,a,D^m)$. $\square$

Theorem 2. *Let $(C,a,D^m) \in \Gamma'$. If $0 < z < \frac{2}{|C|}$, then $\{[x]^q\}_{q=1}^{\infty}$ converges geometrically to $\overline{\eta}(C,a,D^m)$ for each $x \in \Theta(C,a,D^m)$.*

Proof. Let $(C,a,D^m) \in \Gamma'$, $t \in \mathbb{N}_m$, $i \in C$ and $x \in \Theta(C,a,D^m)$. By Equation (9) and the definition of f,

$$\begin{aligned}
S_i^t(x) - x_i^t &= z \sum_{j\in C\setminus\{i\}} \omega^t \cdot \big(\theta(\{i\}, d^t, \tfrac{x^t}{\omega^t}) - \theta(\{j\}, d^t, \tfrac{x^t}{\omega^t})\big) \\
&= z \cdot |C| \cdot \big(\overline{\eta_i^t}(C,a,D^m) - x_i^t\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{\eta_i^t}(C,a,D^m) - S_i^t(x) &= \overline{\eta_i^t}(C,a,D^m) - x_i^t + x_i^t - S_i^t(x) \\
&= \overline{\eta_i^t}(C,a,D^m) - x_i^t - z \cdot |C| \cdot (\overline{\eta_i^t}(C,a,D^m) - x_i^t) \\
&= \big(1 - z\cdot|C|\big)\Big[\overline{\eta_i^t}(C,a,D^m) - x_i^t\Big].
\end{aligned}$$

So, for all $q \in \mathbb{N}$,

$$\overline{\eta}(C,a,D^m) - [x]^q = \big(1 - z\cdot|C|\big)^q \Big[\overline{\eta}(C,a,D^m) - x\Big].$$

If $0 < z < \frac{2}{|C|}$, then $-1 < \big(1 - z\cdot|C|\big) < 1$ and $\{[x]^q\}_{q=1}^{\infty}$ converges to $\overline{\eta}(C,a,D^m)$. $\square$

Inspired by Maschler and Owen [16], one would like to define a dynamic procedure under reductions.

Definition 4. *Let Φ be an index, $(C, a, D^m) \in \Gamma'$, $H \subseteq C$ and $x \in \Theta(C, a, D^m)$. The (x, Φ)-* **reduced system** *$(H, a_H, D^m_{\Phi,H,x})$ is given by $D^m_{\Phi,H,x} = (d^t_{\Phi,H,x})_{t\in\mathbb{N}_m}$ and for all $\alpha \in A^H$,*

$$d^t_{\Phi,H,x}(\alpha) = \begin{cases} d^t_*(C) - \sum_{i\in C\setminus H} x^t_i & , N(\alpha) = H, \\ d^t_{S,\Phi}(\alpha) & , \text{otherwise.} \end{cases}$$

Inspired by Maschler and Owen [16], different switch mapping could be defined as follows. The **R-switch mapping** is $P = (P^t)_{t\in\mathbb{N}_m}$, where $P^t = (P^t_i)_{i\in C}$ and $P^t_i : \Theta(C, a, D^m) \to \mathbb{R}$ is defined by

$$P^t_i(x) = x^t_i + z \sum_{k\in C\setminus\{i\}} \Big(\overline{\eta^t_i}(\{i,k\}, d^t_{\overline{\eta},\{i,k\},x}) - x^t_i\Big).$$

Define $[\Xi]^0 = x,\ [\Xi]^1 = P([\Xi]^0), \cdots,\ [\Xi]^q = P([\Xi]^{q-1})$ for all $q \in \mathbb{N}$.

Unlike the previous concept of switch mapping, the R-switch mapping in this study operates based on the mechanism of reduced systems. It allows participants who have concerns about the allocation to seek re-participation from all other participants in the most advantageous manner before redistributing the resources. The R-switch mapping takes into account all differences between the original allocation and the new allocation obtained after participants have revisited their participation, and it subsequently corrects the original allocation accordingly.

Lemma 6. *$P\big(x\big) \in \Theta(C, a, D^m)$ for all $(C, a, D^m) \in \Gamma'$ and for all $x \in \Theta(C, a, D^m)$.*

Proof. Let $(C, a, D^m) \in \Gamma'$, $t \in \mathbb{N}_m$, $i, k \in C$ and $x \in \Theta(C, a, D^m)$. Let $H = \{i, k\}$, by MUSE of $\overline{\eta}$ and Definition 4,

$$\overline{\eta^t_i}(H, a_H, D^m_{\overline{\eta},H,x}) + \overline{\eta^t_k}(H, a_H, D^m_{\overline{\eta},H,x}) = x^t_i + x^t_k.$$

By RCSY and NSS of $\overline{\eta}$,

$$\begin{aligned} \overline{\eta^t_i}(H, a_H, D^{com}_{\overline{\eta},H,x}) - \overline{\eta^t_k}(H, a_H, D^m_{\overline{\eta},H,x}) &= (d^t_{\overline{\eta},H,x})_*(\{i\}) - (d^t_{\overline{\eta},H,x})_*(\{k\}) \\ &= (d^t_{S,\overline{\eta}})_*(\{i\}) - (d^t_{S,\overline{\eta}})_*(\{k\}) \\ &= \overline{\eta^t_i}(H, a_H, D^m_{H,\overline{\eta}}) - \overline{\eta^t_k}(H, a_H, D^m_{H,\overline{\eta}}) \\ &= \overline{\eta^t_i}(C, a, D^m) - \overline{\eta^t_k}(C, a, D^m). \end{aligned}$$

Therefore,

$$2 \cdot \Big[\overline{\eta^t_i}(H, a_H, D^m_{\overline{\eta},H,x}) - x^t_i\Big] = \overline{\eta^t_i}(C, a, D^m) - \overline{\eta^t_k}(C, a, D^m) - x^t_i + x^t_k. \quad (11)$$

By definition of g and Equation (11),

$$\begin{aligned}
P_i^t(x) &= x_i^t + \tfrac{z}{2} \cdot \Big[\sum_{k \in C\setminus\{i\}} \overline{\eta_i^t}(C,a,D^m) - \sum_{k \in C\setminus\{i\}} x_i^t \\
&\qquad - \sum_{k \in C\setminus\{i\}} \overline{\eta_k^t}(C,a,D^m) + \sum_{k \in C\setminus\{i\}} x_k^t \Big] \\
&= x_i^t + \tfrac{z}{2} \cdot \Big[\sum_{k \in C\setminus\{i\}} \overline{\eta_i^t}(C,a,D^m) - \big(|C|-1\big) x_i^t \\
&\qquad - \sum_{k \in C\setminus\{i\}} \overline{\eta_k^t}(C,a,d) + \big(d_*^t(C) - x_i^t\big) \Big] \\
&= x_i^t + \tfrac{z}{2} \cdot \Big[\big(|C|-1\big)\overline{\eta_i^t}(C,a,D^m) - \big(|C|-1\big) x_i^t \\
&\qquad - \big(d_*^t(C) - \overline{\eta_i^t}(C,a,D^m)\big) + \big(d_*^t(C) - x_i^t\big) \Big] \\
&= x_i^t + \tfrac{|C|\cdot z}{2} \cdot \Big[\overline{\eta_i^t}(C,a,D^m) - x_i^t \Big].
\end{aligned} \tag{12}$$

So, we have that

$$\begin{aligned}
\sum_{i \in C} P_i^t(x) &= \sum_{i \in C} x_i^t + \tfrac{|C|\cdot z}{2} \cdot \Big[\sum_{i \in C} \overline{\eta_i^t}(C,a,D^m) - \sum_{i \in C} x_i^t \Big] \\
&= d_*^t(C) + \tfrac{|C|\cdot z}{2} \cdot \big[d_*^t(C) - d_*^t(C) \big] \\
&= d_*^t(C).
\end{aligned}$$

Thus, $P\big(x\big) \in \Theta(C,a,D^m)$ for all $x \in \Theta(C,a,D^m)$. □

Theorem 3. *Let $(C,a,D^m) \in \Gamma'$. If $0 < z < \frac{4}{|C|}$, then $\{[\Xi]^q\}_{q=1}^{\infty}$ converges to $\overline{\eta}(C,a,D^m)$ for each $x \in \Theta(C,a,D^m)$.*

Proof. Let $(C,a,D^m) \in \Gamma'$, $t \in \mathbb{N}_m$ and $x \in \Theta(C,a,D^m)$. By Equation (12), $P_i^t(x) = x_i^t + \frac{|C|\cdot z}{2} \cdot \Big[\overline{\eta_i^t}(C,a,D^m) - x_i^t \Big]$ for all $i \in C$. Therefore,

$$\Big(1 - \frac{|C| \cdot z}{2}\Big) \cdot \big[\overline{\eta_i^t}(C,a,D^m) - x_i^t \big] = \big[\overline{\eta_i^t}(C,a,D^m) - P_i^t(x) \big]$$

So, for all $q \in \mathbb{N}$,

$$\overline{\eta}(C,a,D^m) - [\Xi]^q = \Big(1 - \frac{|C| \cdot z}{2}\Big)^q \Big[\overline{\eta}(C,a,D^m) - x \Big].$$

If $0 < z < \frac{4}{|C|}$, then $-1 < \Big(1 - \frac{|C|\cdot z}{2}\Big) < 1$ and $\{[\Xi]^q\}_{q=1}^{\infty}$ converges to $\overline{\eta}(C,a,d)$ for all $(C,a,D^m) \in \Gamma'$, for all $t \in \mathbb{N}_m$ and for each $i \in C$. □

5. Concluding Remarks

This study introduces the maximal individual index and the maximal normalized individual index, two new values for multi-choice systems. We present several axiomatic results for these indices based on reduction. Additionally, we provide alternative formulations and relative dynamic procedures for the maximal normalized individual index using reduction and discrepancy mapping. A comparison can be made between the results of this study and related existing findings:

- The maximal individual index and the maximal normalized individual index are initially defined within the structure of multicriteria multi-choice systems.
- The switch mappings in Definitions 3 and 4, along with the related dynamic procedures, draw inspiration from Maschler and Owen's [16] dynamic procedures for the Shapley value [12]. The main difference is that the switch mappings in Definition 4 rely on "discrepancy mapping", while Maschler and Owen's [16] switch mapping relies on "reduced systems".

As mentioned earlier, the following question arises:

- Are there additional power indices, their normalizations, and related results applicable to multicriteria multi-choice systems?

To the best of our knowledge, these issues remain open questions.

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