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On the Nash Equilibria of a Duel with Terminal Payoffs

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Abstract: We formulate and study a two-player *duel* game as a *terminal payoffs stochastic game.* Players P_1, P_2 are standing in place and, in every turn, each *may* shoot at the other (in other words, *abstention* is allowed). If P_n shoots P_m $(m \neq n)$, either they hit and kill them (with probability p_n) or they miss and P_m is unaffected (with probability $1 - p_n$). The process continues until at least one player dies; if no player ever dies, the game lasts an infinite number of turns. Each player receives a positive payoff upon killing their opponent and a negative payoff upon being killed. We show that the unique *stationary equilibrium* is for both players to always shoot at each other. In addition, we show that the game also possesses "*cooperative*" (i.e., non-shooting) *non-stationary equilibria*. We also discuss a certain similarity that the duel has to the *iterated Prisoner's Dilemma*.

Keywords: duel; Nash equilibrium; stochastic games

1. Introduction

In this paper, we study a two-player *duel* game played in turns. Players P_1, P_2 are standing in place and, in each turn, each player *may* shoots at the other; in other words, *abstention* is allowed. If P_n shoots at P_m $(m \neq n)$, either they hit and kill them or they miss and P_m is unaffected; the respective probabilities are p_n (P_n's *marksmanship*) and $1 - p_n$. The process continues until at least one player dies; if no player ever dies then the game lasts an infinite number of turns. We formulate the above as a stochastic game with *terminal payoffs*. The precise game rules and players' payoffs will be presented in Section 2.

Little work has been done on the duel. In fact, to the best of our knowledge, it has only been studied as a preliminary step in the study of the "*truel*", in which *three* stationary players shoot at each other. In early works on the truel [1–4], the postulated game rules guarantee the existence of *exactly one* survivor ("winner"). In an important early paper [5], the somewhat paradoxical result of "survival of the weakest" is established; namely for certain marksmanship combinations, the player with *lowest* marksmanship has the highest probability of survival. A more general analysis appears in a further study [6], which considers the possibility of "cooperation" between the players, in the sense that each player has the option of *abstaining*, i.e., not shooting at their opponent in one or more turns of the game. This idea is further studied by Kilgour (for the *simultaneous* truel) [7] and (the *sequential* truel) [8,9]. These papers are, to the best of our knowledge, the first to address the truel problem using a rigorous game theoretic analysis. Kilgour formulates both the simultaneous and sequential truel as *stochastic games* with terminal payoffs (i.e., the players receive a single payoff at the end of the game) and obtains *Nash equilibria*, under appropriate conditions. A similar analysis appears in a further study [10], where, however, the truel is formulated as a *discounted* stochastic game. Recent papers on the truel include: Refs. [11–14] where, among other innovations, the truel is formulated as an extensive form game; Refs. [15–18], where a Markov chain formulation of several truel variants is presented; and Refs. [19–21], in which truels among N players are studied, with each player being represented by a node in a scale-free network.[1]

Several applications of the duel and, more frequently, of the truel have been proposed in the above literature. The truel has been used to model behavior in confrontation situa-

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tions [25] and in political conflicts [26]. A truel variant has been used as a model of opinion dissemination [17]. Business applications have been presented in a further study [27], in which it is shown that, under certain conditions, weaker companies can grow stronger and stronger companies can grow weaker with all the parties eventually converging. In legal studies, the truel has been used to explore equality issues [28]. Last but not least, the *nuel* (an N-person generalization of the duel and truel) has been used in biology to explain the maintenance of variation in natural populations [29] and study marriage and reproduction mechanisms [30]. Furthermore, the truel is relevant to the existence of "suicidal strategies" employed by cells and bacteria [31,32].

A common characteristic of all the above-mentioned works is that they limit themselves to the study of *stationary* strategies. As we will show in the current paper, the duel also possesses Nash equilibria in non-stationary strategies and it is safe to assume that the same is true of the truel and the *nuel* (the N-player generalization of the duel and truel).

While the above papers focus on various forms of the truel, we believe that the duel is interesting in its own right and has not received the attention it deserves. In particular we will show that, under our formulation, the duel has a certain similarity to the *iterated Prisoner's Dilemma* (IPD) and possesses "*cooperative*" Nash equilibria in *non-stationary strategies*.

In this paper, we study two versions of the duel with terminal payoffs. The rest of the paper is structured as follows. In Section 2, we define the game rigorously. In Section 3 we establish the existence of equilibria in stationary strategies. In Section 4, we discuss some similarities between the game and the IPD. In Section 5, we prove that the duel also has equilibria in non-stationary strategies (namely *grim cooperation* and *Tit-for-Tat*). In Section 6, we summarize our results and propose some future research directions.

2. Game Description

Our duel game involves players P_1, P_2 and evolves in discrete time steps (*turns*) $t \in \{1, 2, ...\}$. The *state* at time t is

$$\mathbf{s}(t) = s_1(t)s_2(t) \in S = \{11, 10, 01, 00\}.$$

For $n \in \{1, 2\}$, $s_n(t)$ is P_n's *state* at $t \in \{0, 1, 2, ...\}$ and can be

$$\begin{aligned} s_n(t) = 1: &\quad \text{when } P_n \text{ is alive at the } t\text{-th turn;} \\ s_n(t) = 0: &\quad \text{when } P_n \text{ is dead at the } t\text{-th turn.} \end{aligned}$$

P_n's *action* at $t \in \{1, 2, ...\}$ is $f_n(t)$, which can be $\mathbf{F}$ (P_n is shooting) or $\mathbf{A}$ (P_n is not shooting). If $f_n(t) = \mathbf{F}$ then: (a) we have $s_{-n}(t) = 0$ (i.e., P_{-n} dies)[2] with probability $p_n \in (0, 1)$ and (b) $s_{-n}(t) = 1$ with probability $1 - p_n$. We set $\mathbf{f}(t) = f_1(t)f_2(t)$ and $\mathbf{p} = (p_1, p_2)$. We assume throughout the paper that for $n \in \{1, 2\}$, $p_n \in (0, 1)$, i.e., *it is strictly between zero and one*.

The game starts at an initial state $\mathbf{s}(0)$; obviously, the case of interest is $\mathbf{s}(0) = 11$. At times $t \in \{1, 2, ...\}$, the players simultaneously choose their actions $f_1(t)$, $f_2(t)$ and the game moves to state $\mathbf{s}(t)$ according to the conditional *state transition probability* $\Pr(\mathbf{s}(t)|\mathbf{s}(t-1), \mathbf{f}(t))$. If we number the states as follows

$$00 \to 1, \quad 01 \to 2, \quad 10 \to 3, \quad 11 \to 4,$$

then we get a "controlled" transition probability matrix $\Pi(\boldsymbol{\phi})$ where

$$\Pi_{ij}(\boldsymbol{\phi}) = \Pr(\mathbf{s}(t) = j | \mathbf{s}(t-1) = i, \mathbf{f} = \boldsymbol{\phi}).$$

For every action vector, a terminal state $\mathbf{s} \in \{1,2,3\}$ (or, equivalently, $\mathbf{s} \in \{00,01,10\}$) transits to itself with probability one; i.e., for all $i \in \{1,2,3\}$:

$$\Pi_{ii}(\mathbf{AA}) = \Pr(\mathbf{s}(t) = i|\mathbf{s}(t-1) = i, \mathbf{f} = \mathbf{AA}) = 1;$$
$$\Pi_{ii}(\mathbf{AF}) = \Pr(\mathbf{s}(t) = i|\mathbf{s}(t-1) = i, \mathbf{f} = \mathbf{AF}) = 1;$$
$$\Pi_{ii}(\mathbf{FA}) = \Pr(\mathbf{s}(t) = i|\mathbf{s}(t-1) = i, \mathbf{f} = \mathbf{FA}) = 1;$$
$$\Pi_{ii}(\mathbf{FF}) = \Pr(\mathbf{s}(t) = i|\mathbf{s}(t-1) = i, \mathbf{f} = \mathbf{FF}) = 1.$$

Transitions from the state $\mathbf{s} = 4$ (or $\mathbf{s} = 11$) are a little more complicated. Consider, for example, the case $\mathbf{f} = \mathbf{FF}$, i.e., when both players fire. Then, letting $\overline{p}_n = 1 - p_n$ for $n \in \{1,2\}$, we have:

$$\Pi_{41}(\mathbf{FF}) = \Pr(\mathbf{s}(t) = 00|\mathbf{s}(t-1) = 11, \mathbf{f} = \mathbf{FF}) = \Pr(\text{“}P_1 \text{ hits } P_2, P_2 \text{ hits } P_1\text{”}) = p_1 p_2;$$
$$\Pi_{42}(\mathbf{FF}) = \Pr(\mathbf{s}(t) = 01|\mathbf{s}(t-1) = 11, \mathbf{f} = \mathbf{FF}) = \Pr(\text{“}P_1 \text{ misses } P_2, P_2 \text{ hits } P_1\text{”}) = \overline{p}_1 p_2;$$
$$\Pi_{43}(\mathbf{FF}) = \Pr(\mathbf{s}(t) = 10|\mathbf{s}(t-1) = 11, \mathbf{f} = \mathbf{FF}) = \Pr(\text{“}P_1 \text{ hits } P_2, P_2 \text{ misses } P_1\text{”}) = p_1 \overline{p}_2;$$
$$\Pi_{44}(\mathbf{FF}) = \Pr(\mathbf{s}(t) = 11|\mathbf{s}(t-1) = 11, \mathbf{f} = \mathbf{FF}) = \Pr(\text{“}P_1 \text{ misses } P_2, P_2 \text{ misses } P_1\text{”}) = \overline{p}_1 \overline{p}_2.$$

The elements of the other matrices are computed similarly, yielding

$$\Pi(\mathbf{AA}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Pi(\mathbf{AF}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & p_2 & 0 & \overline{p}_2 \end{bmatrix}$$

$$\Pi(\mathbf{FA}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & p_1 & \overline{p}_1 \end{bmatrix} \qquad \Pi(\mathbf{FF}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_1 p_2 & \overline{p}_1 p_2 & p_1 \overline{p}_2 & \overline{p}_1 \overline{p}_2 \end{bmatrix}$$

From the above matrices (or from game rules), we see that there exist two possibilities.

1. The game stays in state 11 ad infinitum (no player is ever killed);
2. At some t' the game moves to a state $\mathbf{s}(t') \in \{10,01,00\}$ (one or both players are killed). These are *terminal states*, i.e., as soon as they are reached, the game terminates.

When the game reaches a terminal state $\mathbf{s}$, P_n ($n \in \{1,2\}$) receives payoff $q_n(\mathbf{s})$ as follows:

$$\begin{array}{ll} q_1(10) = a_1 & q_2(01) = -b_2 \\ q_1(01) = -b_1 & q_2(01) = a_2 \\ q_1(00) = a_1 - b_1 & q_2(00) = a_2 - b_2 \end{array}$$

where we assume that for $n \in \{1,2\}$, $a_n > 0$ and $b_n > 0$. We set $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$.

A *finite history* is a sequence $h = \mathbf{s}(0)\mathbf{f}(1)\mathbf{s}(1)...\mathbf{f}(T)\mathbf{s}(T)$, a *non-terminal finite history* is an $h = \mathbf{s}(0)\mathbf{f}(1)\mathbf{s}(1)...\mathbf{f}(T)\mathbf{s}(T)$ where $\mathbf{s}(T) = 11$ and an *infinite history* is an $h = \mathbf{s}(0)\mathbf{f}(1)\mathbf{s}(1)...$. An *admissible* history is one which conforms to the game rules; the set of all admissible finite (resp. infinite) histories is denoted by H^* (resp. H^∞); $\overline{H}^*$ denotes the set of all non-terminal finite histories. The set of all histories is $H = H^* \cup H^\infty$. It will be useful to define payoff as a function $\mathsf{Q}_n : H \to \mathbb{R}$ as follows

$$\mathsf{Q}_n(h) = \begin{cases} q_n(\mathbf{s}(T)) & \text{if } h = \mathbf{s}(0)\mathbf{f}(1)\mathbf{s}(1)...\mathbf{f}(T)\mathbf{s}(T) \in H^*, \mathbf{s}(T) \text{ is terminal,} \\ 0 & \text{if } h = \mathbf{s}(0)\mathbf{f}(1)\mathbf{s}(1)...\mathbf{f}(T)\mathbf{s}(T) \in H^*, \mathbf{s}(T) \text{ is non-terminal,} \\ 0 & \text{if } h \in H^\infty \end{cases}$$

Note that if the game never terminates, both players receive zero payoff.

A *strategy* for P_n is a function $\sigma_n : \overline{H}^* \to [0,1]$; it corresponds to, for every non-terminal finite history h, the probability that, given that the current history is h, P_n will shoot P_{-n}:

$$\sigma_n(h) = \Pr(\text{"}P_n \text{ shoots } P_{-n}\text{"}).$$

A *stationary strategy* is a σ_n depending only on the current state $\mathbf{s}$, hence we simply write $\sigma_n(\mathbf{s})$. Since a stationary strategy σ_n depends only on the current state, it is fully determined by the values $\sigma_n(\mathbf{s})$ for $\mathbf{s} \in \{00, 01, 10, 11\}$, i.e., from

$$\sigma_n(00), \qquad \sigma_n(01), \qquad \sigma_n(10), \qquad \sigma_n(11).$$

But any *admissible* strategy (i.e., compatible with the game rules) must assign

$$\sigma_n(00) = \sigma_n(01) = \sigma_n(10) = 0.$$

Consequently, a stationary strategy is determined by a single number $x_n = \sigma_n(11)$.

A *strategy profile* is a vector $\sigma = (\sigma_1, \sigma_2)$. We denote the set of all *admissible* strategies by Σ and the set of all admissible stationary strategies by $\overline{\Sigma}$.

An initial state $\mathbf{s}(0)$ and two strategies σ_1 and σ_2 (used, respectively, by P_1 and P_2) determine a probability measure on the set of all histories; hence we can define the *expected payoffs*

$$\forall n \in \{1,2\} : Q_n(\mathbf{s}(0), \sigma_1, \sigma_2) = \mathbb{E}_{\mathbf{s}(0),\sigma_1,\sigma_2}(\mathrm{Q}_n(h)).$$

We have, thus, formulated the terminal payoffs duel as a game. We are interested in the game that starts at $\mathbf{s}(0) = 11$, which we will denote by $\Gamma(\mathbf{p}, \mathbf{a}, \mathbf{b})$. We assume that P_1 and P_2 are looking for a *Nash equilibrium* (NE), i.e., a strategy profile $(\widehat{\sigma}_1, \widehat{\sigma}_2)$ such that

$$\forall n \in \{1,2\}, \forall \sigma_n \in \Sigma : Q_n((1,1), \widehat{\sigma}_n, \widehat{\sigma}_{-n}) \geq Q_n((1,1), \sigma_n, \widehat{\sigma}_{-n}).$$

3. Stationary Equilibria

As already noted, an admissible stationary strategy σ_1 for P_1 is fully determined by $x_1 = \sigma_1(11) = \Pr(P_1 \text{ shoots } P_2)$; i.e., σ_1 is determined by a single variable $x_1 \in [0,1]$. Similarly, every admissible stationary strategy σ_2 for P_2 is fully determined by a single variable $x_2 \in [0,1]$. Hence, we will often speak of the strategy x_n (rather than σ_n) and the strategy profile (x_1, x_2) (rather than (σ_1, σ_2)). When P_1 and P_2 use strategies x_1 and x_2, the state sequence is a Markov chain; using the previous numbering of states we have the transition probability matrix

$$\Pi(x_1,x_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_1p_1x_2p_2 & (x_1\overline{p}_1 + \overline{x}_1)x_2p_2 & x_1p_1(x_2\overline{p}_2 + \overline{x}_2) & (x_1\overline{p}_1 + \overline{x}_1)(x_2\overline{p}_2 + \overline{x}_2) \end{bmatrix}$$

Also, we can define

$$V_n(x_1, x_2) = Q_n(11, (x_1, x_2)).$$

If $(x_1, x_2) = (0,0)$ we obviously get $V_1(0,0) = 0$. Conversely, if $(x_1, x_2) \neq (0,0)$ then we have the following equation for V_1 (temporarily omitting the x_1, x_2 arguments for brevity of notation):

$$\begin{aligned} V_1 = {} & x_1p_1(x_2\overline{p}_2 + \overline{x}_2)a_1 - (x_1\overline{p}_1 + \overline{x}_1)x_2p_2b_1 \\ & + x_1p_1x_2p_2(a_1 - b_1) + (\overline{x}_1 + x_1\overline{p}_1)(x_2\overline{p}_2 + \overline{x}_2)V_1. \end{aligned}$$

The equation is obtained as follows: the expected payoff from state 11 is the sum of four terms:

1. The transition to state 10 gives payoff a_1 and takes place with probability x_1p_1 (P_1 shot and hit P_2) multiplied by $(x_2\overline{p}_2 + \overline{x}_2)$ (P_2 either shot and missed or did not shoot);

2. The transition to state 01 gives payoff $-b_1$ and takes place with probability $x_2 p_2$ (P_2 shot and hit P_1) multiplied by $(x_1 \overline{p}_1 + \overline{x}_1)$ (P_1 either shot and missed or did not shoot);
3. The transition to state 00 gives payoff $a_1 - b_1$ and takes place with probability $x_1 p_1$ (P_1 shot and hit P_2) multiplied by $x_2 p_2$ (P_2 shot and hit P_1);
4. The transition to state 11 gives payoff V_1 (it is as if the game starts from the beginning) and takes place with probability $(\overline{x}_1 + x_1 \overline{p}_1)$ (P_1 either shot and missed or did not shoot) multiplied by $(x_2 \overline{p}_2 + \overline{x}_2)$ (P_2 either shot and missed or did not shoot).

After some algebra, the V_1 equation is simplified to

$$V_1 = x_1 p_1 a_1 - x_2 p_2 b_1 + V_1 x_1 p_1 x_2 p_2 - V_1 x_1 p_1 - V_1 x_2 p_2 + V_1$$

and has the following solution[3]:

$$V_1(x_1, x_2) = \frac{x_1 p_1 a_1 - x_2 p_2 b_1}{1 - (1 - x_1 p_1)(1 - x_2 p_2)}.$$

By the same analysis for $V_2(x_1, x_2)$, we finally get the expressions

$$V_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 = 0 \\ \frac{x_1 p_1 a_1 - x_2 p_2 b_1}{1 - (1 - x_1 p_1)(1 - x_2 p_2)} & \text{otherwise} \end{cases} \quad (1)$$

$$V_2(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 = 0 \\ \frac{x_2 p_2 a_2 - x_1 p_1 b_2}{1 - (1 - x_1 p_1)(1 - x_2 p_2)} & \text{otherwise} \end{cases} \quad (2)$$

Proposition 1. *The only stationary NE of* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{p})$ *is* $(x_1, x_2) = (1, 1)$.

Proof. Suppose that P_1 and P_2 use the profile (x_1, x_2). To determine whether this is an NE, from P_1's point of view we have to check whether they have anything to gain by unilaterally deviating to some other strategy σ_1. A crucial fact is that we only have to check whether P_1 gains by switching to another *stationary* strategy. This is true because, if P_2 uses the stationary strategy x_2, then P_1 must solve an Markov Decision Process problem; it is well known that in this case he gains nothing by using non-stationary strategies [33].

Let us first check whether $(0, 0)$ is a Nash equilibrium. If P_1 deviates to another stationary strategy x_1, we will have

$$V_1(0, 0) - V_1(x_1, 0) = 0 - \frac{x_1 p_1 a_1 - 0 p_2 b_1}{1 - (1 - x_1 p_1)(1 - 0 p_2)} = -a_1 < 0.$$

Hence, $(0, 0)$ cannot be an NE. Next, take any $(x_1, x_2) \neq (0, 0)$ and suppose P_1 deviates to y_1. Then

$$\begin{aligned} & V_1(x_1, x_2) - V_1(y_1, x_2) \\ & = \frac{x_1 p_1 a_1 - x_2 p_2 b_1}{1 - (1 - x_2 p_2)(1 - x_1 p_1)} - \frac{y_1 p_1 a_1 - x_2 p_2 b_1}{1 - (1 - x_2 p_2)(1 - y_1 p_1)} \\ & = \frac{p_1 x_2 p_2 (a_1 + b_1 (1 - x_2 p_2))(x_1 - y_1)}{(1 - (1 - x_1 p_1)(1 - x_2 p_2))(1 - (1 - y_1 p_1)(1 - x_2 p_2))} \end{aligned}$$

The denominator is positive. The numerator has the sign of $x_1 - y_1$. Hence, the sign of $V_1(x_1, x_2) - V_1(y_1, x_2)$ is the same as that of $x_1 - y_1$ and consequently, P_1 never (resp. always) has an incentive to deviate from x_1 to a smaller (resp. greater) y_1. The same arguments can be applied to P_2 and their strategy x_2. It follows that the only stationary NE is $(x_1, x_2) = (1, 1)$ and this completes the proof. □

4. Connection to the Iterated Prisoner's Dilemma

Applying Formulas (1) and (2) to $(x_1, x_2) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, we get

$$\begin{array}{ll} V_1(0,0) = 0 & V_2(0,0) = 0 \\ V_1(0,1) = -b_1 & V_2(0,1) = a_2 \\ V_1(1,0) = a_1 & V_2(1,0) = -b_2 \\ V_1(1,1) = \frac{p_1 a_1 - p_2 b_1}{1-(1-p_2)(1-p_1)} & V_2(1,1) = \frac{p_2 a_2 - p_1 b_2}{1-(1-p_2)(1-p_1)} \end{array}$$

It can immediately be seen that

$$-b_1 = V_1(0,1) < 0 = V_1(0,0) < a_1 = V_1(1,0)$$

and if we identify the strategy $x_n = 0$ (never shooting at the opponent) with "cooperation" and the strategy $x_n = 1$ (always shooting at the opponent) with "defection", the above inequalities remind us of the Prisoner's Dilemma (PD). The similarity would be complete if the additional inequalities

$$V_1(0,1) < V_1(1,1) < V_1(0,0)$$

also held; because in this case we would have

$$V_1(0,1) < V_1(1,1) < V_1(0,0) < V_1(1,0) \tag{3}$$

which corresponds exactly to the well known sequence of PD inequalities [22]:

$$S < P < R < T.$$

Now, (3) is equivalent to

$$-b_1 < \frac{p_1 a_1 - p_2 b_1}{1-(1-p_2)(1-p_1)} < 0 < a_1.$$

The first inequality is equivalent to

$$0 < \frac{p_1 a_1 - p_2 b_1}{1-(1-p_2)(1-p_1)} + b_1 = p_1 \frac{a_1 + b_1(1-p_2)}{p_1 + p_2(1-p_1)}$$

which is always satisfied. The second inequality is

$$\frac{p_1 a_1 - p_2 b_1}{1-(1-p_2)(1-p_1)} < 0,$$

which will be satisfied iff

$$p_1 a_1 < p_2 b_1$$

The third inequality is always satisfied. Similarly, the inequalities

$$V_2(0,1) < V_2(1,1) < V_2(0,0) < V_2(1,0) \tag{4}$$

will be satisfied iff

$$p_2 a_2 < p_1 b_2$$

Combining the above, we get the following "PD-like condition"

$$\frac{a_2}{b_2} < \frac{p_1}{p_2} < \frac{b_1}{a_1} \tag{5}$$

which is necessary and sufficient to have the following ordering of the payoffs

$$V_1(0,1) < V_1(1,1) < V_1(0,0) < V_1(1,0) \tag{6}$$

$$V_2(0,1) < V_2(1,1) < V_2(0,0) < V_2(1,0) \tag{7}$$

In light of (6) and (7), we will call the never-shooting strategy $x_n = 0$ (which henceforth will also be denoted by σ^C) the *cooperating strategy*, and the always-shooting strategy $x_n = 1$ (which henceforth will also be denoted by σ^D) the *defecting strategy*. The terminology is inspired by the analogy to the PD. Namely, in both the PD and the duel, both players would have a higher payoff if they adhered to (σ^C, σ^C); but this is not a NE and each player has incentive to switch to σ^D. Consequently, rational players will follow the strategy profile (σ^D, σ^D), which, while being an NE, yields lower payoff to both players.[4]

As is well known, cooperative NE do exist for the *iterated* PD, and these involve the use of *non-stationary* strategies, such as *grim-cooperation* and *Tit-for-Tat* (TfT). Hence, in the next section, we will show that there exist corresponding non-stationary cooperative strategies which are NE of $\Gamma(\mathbf{p}, \mathbf{a}, \mathbf{b})$.

Before concluding this section, it is worth discussing in what ways our duel game $\Gamma(\mathbf{p}, \mathbf{a}, \mathbf{b})$ differs from the IPD. Three obvious differences are:

1. The IPD is a deterministic game, while $\Gamma(\mathbf{p}, \mathbf{a}, \mathbf{b})$ involves randomness;
2. In the IPD, each player receives a payoff in every turn and the total payoff is the *discounted* (by a discount factor γ) sum of turn payoffs, while in $\Gamma(\mathbf{p}, \mathbf{a}, \mathbf{b})$, payoff is obtained only at the final turn and is undiscounted;
3. The IPD will last an infinite number of turns, while $\Gamma(\mathbf{p}, \mathbf{a}, \mathbf{b})$ may (depending on the p values and the strategy used) terminate in a finite number of turns (in fact, it may be the case that it will terminate in a finite number of terms with probability one).

However, there is an formulation of the IPD in which the payoffs are not discounted but the game may terminate in every turn with a positive probability $p = 1 - \gamma > 0$. In this formulation, the IPD is also a random game and will terminate in a finite number of turns with probability one; the total *expected* payoff of each player equals the discounted payoff of the deterministic IPD version.

5. Non-Stationary Equilibria

Drawing upon similar results for the IPD, we will now show that the duel has *cooperative* NE in non-stationary strategies. The first such strategy we introduce is the *grim cooperation* strategy σ^G, which is defined as follows for P_n ($n \in \{1, 2\}$):

$$\sigma^G : \quad \begin{array}{l} \text{As long as } P_{-n} \text{ does not shoot } P_n,\ P_n \text{ never shoots } P_{-n}; \\ \text{if } P_{-n} \text{ shoots } P_n \text{ at round } t, \text{ then } P_n \text{ shoots } P_{-n} \text{ at all rounds } t' > t. \end{array}$$

This strategy was originally used in the analysis of the IPD.

Proposition 2. (σ^G, σ^G) *is an NE of* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{p})$ *iff*

$$\frac{b_1}{a_1} > \frac{1 + p_2(1 - p_1)}{1 - p_1} \cdot \frac{p_1}{p_2} \qquad \text{and} \qquad \frac{b_2}{a_2} > \frac{1 + p_1(1 - p_2)}{1 - p_2} \cdot \frac{p_2}{p_1}. \tag{8}$$

Proof. We have

$$V_1^G = Q_1\left(11, \left(\sigma^G, \sigma^G\right)\right) = 0$$

since, if both players adhere to σ^G, nobody will ever get killed. Next, let us consider possible P_1 strategies σ_1 deviating from σ^G. It is easy to see that it suffices to consider the strategy σ^D, because, as soon as P_1 deviates from σ^G, P_2 will shoot at P_1 on every turn and hence, P_1 has no incentive to not shoot; furthermore, if P_1 deviates from σ^G, they might as well deviate on the first turn. Now, let us compute

$$V_1^R = Q_1\left(11, \left(\sigma^D, \sigma^G\right)\right).$$

If P_1 uses σ^D at $t = 1$, then P_2 will also revert to σ^D at times $t \in \{2, 3, ...\}$. Hence, P_1's expected payoff will be

$$\begin{aligned} V_1^R &= p_1 a_1 + (1 - p_1)\left(0 + V_1^D\right) \\ &= p_1 a_1 + (1 - p_1)\frac{p_1 a_1 - p_2 b_1}{1 - (1 - p_2)(1 - p_1)} \\ &= \frac{p_2(1 - p_1)(p_1 a_1 - b_1) + p_1 a_1}{1 - (1 - p_1)(1 - p_2)}. \end{aligned}$$

For $\left(\sigma^G, \sigma^G\right)$ to be an NE, we must have $V_1^R < V_1^G$, which is equivalent to

$$p_2(1 - p_1)(p_1 a_1 - b_1) + p_1 a_1 < 0. \tag{9}$$

By assumption

$$\begin{aligned} \frac{1 + p_2(1 - p_1)}{p_2(1 - p_1)} p_1 a_1 - b_1 < 0 \Leftrightarrow \\ \frac{-b_1 p_2 + b_1 p_2 p_1 + p_1 a_1 + p_1 a_1 p_2 - p_1^2 a_1 p_2}{p_2(1 - p_1)} < 0 \Leftrightarrow \\ -b_1 p_2 + b_1 p_2 p_1 + p_1 a_1 + p_1 a_1 p_2 - p_1^2 a_1 p_2 < 0 \Leftrightarrow \\ p_2(1 - p_1)(p_1 a_1 - b_1) + p_1 a_1 < 0. \end{aligned}$$

Hence, (9) holds and P_1 has no incentive to deviate from σ^G. By a similar analysis, we can also show that P_2 has no incentive to deviate from σ^G. This completes the proof. □

Remark 1. *The duel NE conditions (8) imply*

$$\begin{aligned} \frac{b_1}{a_1} > \frac{1 + p_2(1 - p_1)}{1 - p_1} \cdot \frac{p_1}{p_2} \Rightarrow \frac{p_1}{p_2} < \frac{1 - p_1}{1 + p_2(1 - p_1)} \cdot \frac{b_1}{a_1} < \frac{b_1}{a_1}, \\ \frac{b_2}{a_2} > \frac{1 + p_1(1 - p_2)}{1 - p_2} \cdot \frac{p_2}{p_1} \Rightarrow \frac{p_2}{p_1} < \frac{1 - p_2}{1 + p_1(1 - p_2)} \cdot \frac{b_2}{a_2} < \frac{b_2}{a_2}. \end{aligned}$$

Hence, the conditions (8) are stronger than the originally postulated condition (5) for the existence of a "PD-like" ordering in the duel.

Now, we will define another non-stationary cooperative strategy, which will turn out to be an NE of the duel. This is the *Tit-for-Tat* strategy σ^{TfT}, defined for P_n ($n \in \{1, 2\}$) as follows:

σ^{TfT} : In the first turn P_n does not shoot P_{-n};
at every other turn P_n performs the same action (shooting or not shooting) that P_{-n} performed in the previous round.

This strategy was also originally used in the analysis of the iterated PD.

Proposition 3. $\left(\sigma^{TfT}, \sigma^{TfT}\right)$ *is an NE of* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{p})$ *iff*

$$\frac{b_1}{a_1} > \frac{1 + p_2(1 - p_1)}{1 - p_1} \cdot \frac{p_1}{p_2} \quad \textit{and} \quad \frac{b_2}{a_2} > \frac{1 + p_1(1 - p_2)}{1 - p_2} \cdot \frac{p_2}{p_1}. \tag{10}$$

Proof. If both players play the strategy σ^{TfT}, then they never shoot at each other and their payoffs are

$$\forall n \in \{1, 2\} : V_n^{TfT} = Q_n\left(11, \sigma^{TfT}, \sigma^{TfT}\right) = Q_n\left(11, \sigma^C, \sigma^C\right) = 0.$$

Now, suppose that P_2 adheres to σ^{TfT} but P_1 deviates. If P_1 gains by deviating from σ^{TfT} at *some* turn, then they must also gain by shooting at P_2 in the *first* turn. If they do so, then P_2 shoots at P_1 for all subsequent turns, until P_1 reverts to not firing. Thus, P_1 has two options after their first deviation.

1. They can continue shooting in all subsequent turns, in which case, so will P_2;
2. They can revert to not shooting, in which case, in the next turn, they are in the same situation as at the start of the game.

Consequently, if P_1 can increase their payoff by deviating, then they can do so, either (a) by shooting in every turn, or (b) by alternating between shooting and not shooting. If we find conditions under which P_1 *cannot* increase their payoff by either of the above strategies, then, under the same conditions, P_1 cannot increase their payoff by deviating, which implies that $\left(\sigma^{TfT}, \sigma^{TfT}\right)$ is an NE.

1. Consider first the case in which P_1 adopts the strategy σ^D of shooting in each turn. Then we have
$$Q_n\left(11, \sigma^D, \sigma^{TfT}\right) = Q_1\left(11, \left(\sigma^D, \sigma^G\right)\right) = V_1^R$$
and, by the same analysis as in the proof of Proposition 2, we know that $V_1^C - V_1^R > 0$ iff
$$p_2(1-p_1)(p_1a_1 - b_1) + p_1a_1 < 0$$
which is equivalent to our assumption
$$\frac{b_1}{a_1} > \frac{1+p_2(1-p_1)}{1-p_1} \cdot \frac{p_1}{p_2} .$$

2. Next consider the case in which P_1 alternates between shooting and not shooting. Then their payoff will be
$$V_1^S = p_1a_1 + (1-p_1)\left(0 + p_2(-b_1) + (1-p_2)V_1^S\right).$$
The above equation holds because the expected payoff V_1^S is computed by summing the following possibilities. P_1 will certainly shoot and then:
   (a) With probability p_1, P_2 will kill P_2 and hence, receive payoff a_1;
   (b) With probability $1-p_1$, P_2 will miss (and receive zero payoff) and in the next turn P_2 will shoot and kill P_1; this combination has probability $(1-p_1)p_2$ and gives to P_1 payoff $-b_1$;
   (c) With probability $1-p_1$, P_1 will miss and in the next turn P_2 will shoot and miss P_1; this combination has probability $(1-p_1)(1-p_2)$ and returns the game to the original state, in which P_1 receives payoff V_1^S.

   Simplifying the above equation and solving we obtain
$$V_1^S = \frac{p_1a_1 - p_2b_1 + p_2b_1p_1}{p_1 + p_2 - p_2p_1}.$$
For an NE we must have $V_1^C - V_1^S > 0$ and this will hold when
$$0 > p_1a_1 - p_2b_1 + p_2b_1p_1 \Leftrightarrow p_2b_1(1-p_1) > p_1a_1 \Leftrightarrow \frac{b_1}{a_1} > \frac{1}{1-p_1} \cdot \frac{p_1}{p_2}.$$
However, from our assumption (10), we have
$$\frac{b_1}{a_1} > \frac{1+p_2(1-p_1)}{1-p_1} \cdot \frac{p_1}{p_2} > \frac{1}{1-p_1} \cdot \frac{p_1}{p_2} .$$

Hence $V_1^C - V_1^S > 0$.

Combining 1 and 2, we see that P_1 has no advantage in deviating from σ^{TfT}; by a similar analysis, the same holds for P_2 and hence, the proof is completed. □

Corollary 1. *The duel NE conditions (8) and (10) are the same. In other words,* (σ^D, σ^D) *is an NE of* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{p})$ *iff* $\left(\sigma^{TfT}, \sigma^{TfT}\right)$ *is an NE of* $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{p})$.

Let us compare the stationary and non-stationary NE. Initially, we made no assumption regarding the relative size of a_n and b_n (although we did assume they are both positive). In other words, $a_n - b_n$ may be positive (P_n sets more value in surviving), negative (P_n hates their opponent so much that they value killing them more than surviving) or zero. However, even when $a_n > b_n$ for both n, if the players limit themselves to using stationary strategies, then the only Nash equilibrium consists of both players shooting at each other with probability one (by Proposition 1); for the more desirable outcome of both players surviving to be (another) Nash equilibrium, they must use non-stationary strategies.

6. Conclusions

We have defined a turn-based duel game with terminal payoffs and shown that it has both stationary and non-stationary Nash equilibria. The non-stationary equilibria that we have established are the *grim cooperation* and *Tit-for-Tat* pairs. These are of the same form as the synonymous strategies used in the *iterated Prisoner's Dilemma*; we were motivated to use these in the duel by the previously explained similarity between the payoff structure of our duel game and that of the IPD.

In addition to their independent interest, the above results have potential application to the truel and nuel problems. As we have pointed out, to the best of our knowledge, the literature on truel and nuel is limited to the study of *stationary* strategies. In the case of the duel, in addition to stationary NE, we also have non-stationary NE. We reported here two such non-stationary NE ((σ^G, σ^G) and $\left(\sigma^{TfT}, \sigma^{TfT}\right)$), and it is not hard to construct additional ones, using an approach similar to that used in the study of repeated games [35]. We conjecture that, using the methods of the current paper, it is also possible to establish a plethora of non-stationary NE for the general, N-player nuel; we intend to pursue this research direction in the future.

Several variants of the duel can be formulated and are worth exploring. In addition to the variant described in this paper, we have explored a variant in which each player receives some *discounted* payoff for every turn in which they stay alive. Including those results (and the techniques required for their proof) would increase the size of the current paper inordinately; hence, they will be reported in a separate publication. Further variants to be explored in the future include:

1. *sequential play*, in which a single player is allowed to shoot in each turn;
2. *random play*, in which the player allowed to shoot in each turn is chosen randomly and equi-probably.

In addition, in the future we intend to study the use of non-stationary strategies in truels and nuels.

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Notes

[1] It should also be noted that an extensive literature on a quite different type of duel games exists, which essentially are *games of timing* [22–24]. However, this literature is not relevant to the game studied in this paper.

[2] In the sequel we use the standard game theoretic notation by which $s_{-1} = s_2$, $s_{-2} = s_1$. The same notation is used for players, actions etc.

[3] Several parts of this paper require rather involved algebraic calculations. We have always performed these using the *computer algebra system* Maple and afterwards verified the results by hand.

[4] We should clarify at this point that, despite the use of the terms "cooperation" and "cooperative", the duel is *not* a cooperative game in Shapley's sense [34]. In other words, it does not involve external *enforcement* of cooperative behavior. Instead, the duel is a non-cooperative game and "cooperation" is used in the same sense as in the Prisoner's Dilemma literature; i.e., "cooperation" is understood as a spontaneous emergence of coordinated moves due to the players' selfish behavior, rather than due to an explicit alliance mechanism.

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