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Von Neumann–Morgenstern Hypergraphs

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Abstract: A simple hypergraph H with vertex set X and edge set $\mathbf{E}$ is representable by Von Neumann–Morgenstern (VNM)-stable sets—or VNM—if there exists an irreflexive simple digraph D with vertex set X such that each edge of H is a VNM-stable set of D. It is shown that a simple hypergraph H is VNM if and only if each edge of H is a *maximal* clique of the *conjugation graph* of H. A related algorithm that identifies finite VNM hypergraphs is also provided.

Keywords: VNM-stable sets; kernels; hypergraphs; clutters; Sperner systems; games

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1. Introduction

A Von Neumann–Morgenstern solution or stable set can be described as a set of outcomes that do not provide effective mutual 'objections' while including at least one effective 'objection' against any outcome that does not belong to it. Von Neumann–Morgenstern-stable sets—henceforth *VNM-stable sets*—are a prominent solution rule for games in strategic, coalitional, and digraphic form (the so-called abstract games consisting of a loopless simple digraph), and can in fact be applied to virtually any sort of game. Indeed, (Greenberg, 1990) advances the notion that VNM-stable sets can be usefully deployed as an unifying solution paradigm for both cooperative and noncooperative game theory. Furthermore, following the very interpretation of VNM-stable sets as *standards of behavior* that was originally suggested by Von Neumann and Morgenstern, one may also regard them as sets of distinct *feasible* outcomes over which games can be defined and played, or indeed as maximal collections of mutually consistent *modes of behaviour* occupying niches of a given environment (the latter interpretation is proposed by (Wilson, 1972), and (Roth, 1984) in an ecological dynamic setting).

A considerable amount of work has been devoted to VNM-stable sets in the literature on game theory, and comparable attention has been attracted by *kernels*, their dual digraph-theoretic counterparts, in the literature on graph theory. A characteristic feature of VNM-stable sets and kernels is that they indeed consist of *sets* as opposed to *points*, and that a game may well have *many* VNM-stable sets or kernels. By definition, any set of VNM-stable sets of a game induces a simple hypergraph on the outcome set of that game. But then, *what simple hypergraphs can be realized in that manner as a set of VNM-stable sets of a game*? Or, equivalently, under the Wilson–Roth ecological interpretation mentioned above, *what families of collections of modes of behavior should be observed to fully occupy the niches of a given environment for a significant stretch of time if such collections are indeed VNM-stable sets of the same underlying game*?

A remarkable elementary property of VNM-stable sets (and kernels) is that they form a *clutter* or antichain with respect to set inclusion, i.e., any two distinct VNM-stable sets of a game are set-inclusion incomparable. That fact immediately reduces the foregoing issue on simple hypergraphs to the following more specialized problem: *what clutters can be* *represented as a set of VNM-stable sets of the irreflexive 'dominance' relation of an underlying game*? The present paper offers a solution to the foregoing representation problem by providing *a simple characterization of VNM hypergraphs or clutters in terms of maximal cliques of their conjugation graphs, and a related algorithm to identify them* among all simple hypergraphs on a fixed (finite) set. Such a characterization can also be regarded as a contribution to the study of those choice functions that can be represented as selections of VNM-stable sets of an underlying game on a fixed finite outcome set X, and of its restrictions to subsets of X (see Wilson, 1970; Vannucci, 2023). That is so precisely because the aforementioned algorithm allows one to recognize whether a given family of choice sets of a finite set can be represented as a collection of VNM-stable sets of an underlying game.

2. Von Neumann–Morgenstern Hypergraphs

A *simple hypergraph* is a pair $H = (X, \mathbf{E})$, where X is a nonempty set and $\mathbf{E}$ is a subset of $\mathcal{P}(X)$, the power set of X: X will also be referred to as the set of *vertices* of H, and $\mathbf{E}$ as the set of *edges* of H. A simple hypergraph $H = (X, \mathbf{E})$ is a (simple) *clutter* iff $\mathbf{E}$ is an $\subseteq$-antichain—i.e., $E \nsubseteq E'$ for any two *distinct* $E, E' \in \mathbf{E}$. Thus, the 'boundary' cases with $\mathbf{E} = \{\emptyset\}$, or $\mathbf{E} = \emptyset$ are allowed. A *simple digraph* is a pair $D = (X, \Delta)$ where X is a set and $\Delta \subseteq X \times X$; D is *irreflexive*—or loopless—iff $(x, x) \notin \Delta$ for any $x \in X$, and *symmetric* iff $(y, x) \in \Delta$ whenever $(x, y) \in \Delta$ for any $x, y \in X$. A *graph* is a symmetric digraph. A graph $D = (X, \Delta)$ is *complete* iff $(x, y) \in D$ for each $x, y \in X$. A *clique* of graph $D = (X, \Delta)$ is a set $Y \subseteq X$ such that (Y, Δ_Y)—where $\Delta_Y = \Delta \cap (Y \times Y)$—is a complete graph: it is, moreover, a *maximal* clique of D if for each Z such that $Y \subset Z \subseteq X$, (Z, Δ_Z) is not a complete graph. [Notice that such a terminology is slightly at variance with that employed by many authors, who sometimes denote as 'hypergraphs' only hypergraphs such that $\emptyset \notin \mathbf{E}$ and $\bigcup \mathbf{E} = X$, and 'Sperner systems' or 'Sperner families' those simple clutters as defined above that are also 'hypergraphs' in that sense: see, e.g., (Berge, 1989). Other authors denote as 'cliques' complete subgraphs or even *maximal* complete subgraphs: in that respect, it is Berge's usage that is followed here].

A *VNM-stable set* of an irreflexive simple digraph (X, Δ) is a set $S \subseteq X$ such that the following conditions hold: (i) *internal stability*, i.e., $[(x, y) \notin \Delta$ for any $x, y \in S]$, and (ii) *external stability*, i.e., [for any $z \in X \setminus S$, there exists $x \in S$ such that $(x, z) \in \Delta]$ (see, e.g., Von Neumann & Morgenstern, 1953; Richardson, 1953; Schmidt and Ströhlein, 1985; Lucas, 1992; Ghoshal et al., 1998).

The set of all VNM-stable sets of (X, Δ) will be denoted by $\mathcal{S}(X, \Delta)$.

The present note addresses the following issue, as made precise by the ensuing definition of VNM hypergraphs: what simple hypergraphs are VNM, i.e., can be realized as a collection of VNM-stable sets of a game in simple digraphic form, namely, of an irreflexive simple digraph?

Definition 1 (VNM hypergraphs)**.** *A simple hypergraph $H = (X, \mathbf{E})$ is representable by VNM-stable sets or* VNM *iff there exists an irreflexive simple digraph (X, Δ) such that $\mathbf{E} \subseteq \mathcal{S}(X, \Delta)$.*

To begin with, notice that $S_1 \nsubseteq S_2$ for any two *distinct* $S_1, S_2 \in \mathcal{S}(X, \Delta)$ (indeed, if $S_1 \subset S_2$, then the internal stability of S_2 and external stability of S_1 turn out to be mutually inconsistent): namely, $\mathcal{S}(X, \Delta)$ is an *antichain* with respect to set inclusion. Thus, for any irreflexive simple digraph (X, Δ), $(X, \mathcal{S}(X, \Delta))$ is a (simple) *clutter*. This elementary observation shows that a simple hypergraph can be realized as a set of VNM-stable sets of a game only if it is indeed a clutter.

However, a clutter may or may not be VNM, as made clear by the following examples.

Example 1. *Consider $X = \{x, y, z\}$ with $x \neq y \neq z \neq x$, and $\mathbf{E} = \{\{x, y\}, \{x, z\}, \{y, z\}\}$. It is easily checked that $(X, \mathbf{E})$ is a clutter, but* not *a VNM one. Suppose to the contrary that there exists an irreflexive simple digraph (X, Δ) such that $\mathbf{E} \subseteq \mathcal{S}(X, \Delta)$. Then, by property (ii) of VNM-stable sets as applied to $\{x, y\}$, $\Delta \cap \{(x, z), (y, z)\} \neq \varnothing$, which contradicts the assumption that both $\{x, z\}$ and $\{y, z\}$ satisfy property (i) of VNM-stable sets.*

Example 2. *Consider now X as defined above in the previous example, $\mathbf{E}' = \{\{x, y\}, \{x, z\}\}$, and $\Delta' = \{(y, z), (z, y)\}$. Clearly, $\mathbf{E}' = \mathcal{S}(X, \Delta')$—i.e., $(X, \mathbf{E}')$ is indeed a VNM clutter.*

Example 3. *Let $(X, \mathbf{E})$ be a* matching*, namely a clutter such that $E \cap E' = \varnothing$ for any pair of* distinct $E, E' \in \mathbf{E}$.

Then, define $\Delta = \{(x, y) \in X \times X\colon \{x, y\} \nsubseteq E \text{ for all } E \in \mathbf{E}\}$.

Clearly, for any $E \in \mathbf{E}$ and any $x, y \in E$, $(x, y) \notin \Delta$ by definition. Moreover, suppose there exists $z \in X \setminus E$ such that $(x, z) \notin \Delta$ for all $x \in E$. Then, for any $x \in E$, there exists $E' \in \mathbf{E}$ such that $\{x, z\} \subseteq E'$. Since by construction $E \neq E'$ and $E \cap E' \neq \varnothing$, the existence of such an E' contradicts our starting hypothesis. It follows that E is indeed a VNM-stable set of (X, Δ). Therefore, $(X, \mathbf{E})$ is a VNM clutter.

Remark 1 (Kernel-representable hypergraphs)**.** *Let (X, Δ) be an irreflexive simple digraph, and $\Delta^{-1} \subseteq X \times X$ the* inverse *of Δ—namely, for any $x, y \in X$, $(x, y) \in \Delta^{-1}$ iff $(y, x) \in \Delta$. Of course, (X, Δ^{-1}) is also an irreflexive simple digraph. A subset of vertices $K \subseteq X$ is a* kernel *of (X, Δ)—written $K \in \mathcal{K}(X, \Delta)$—iff $K \in \mathcal{S}(X, \Delta^{-1})$. Now, consider any simple hypergraph $(X, \mathbf{E})$ and declare it* kernel-representable (KR) *iff there exists an irreflexive simple digraph such that $\mathbf{E} \subseteq \mathcal{K}(X, \Delta)$. Clearly, a simple hypergraph is a KR hypergraph iff it is VNM. In particular, again, a hypergraph is KR only if it is a clutter.*

The foregoing observations and examples make it clear that the task of characterizing VNM hypergraphs or KR hypergraphs is precisely the same as that of characterizing VNM clutters or KR clutters, but requires some further restrictions on the latter (in a somewhat similar vein, a representation problem concerning finite distributive lattices and stable matchings is addressed in (Blair, 1984)).

3. A Simple Characterization of VNM Hypergraphs

In order to state the result in a most concise manner, let us first introduce a few auxiliary notions.

Definition 2. *Let $H = (X, \mathbf{E})$ be a simple hypergraph. Then, the* conjugation *relation $C_H \subseteq X \times X$ of H is defined as follows: for any $x, y \in X$, $(x, y) \in C_H$ iff there exists an $E \in \mathbf{E}$ such that $\{x, y\} \subseteq E$. The pair (X, C_H) is the* conjugation graph *of H.*

An early definition of the conjugation relation, together with an acknowledgement of its connection to VNM-stable sets, is to be credited to (Wilson, 1970) (see also Vannucci, 2023). It is worth noticing here that the conjugation relation C_H as defined above is by construction *symmetric* (i.e., $(y, x) \in C_H$ whenever $(x, y) \in C_H$): hence, the conjugation graph may also be regarded as an undirected graph. Moreover, C_H is also *reflexive* (i.e., $(x, x) \in C_H$ for all $x \in X$), and thus in particular a *tolerance* relation, if and only if $X \subseteq \bigcup \mathbf{E}$ (i.e., whenever $\mathbf{E}$ is a *covering* of X). Clearly enough, any edge E of a simple hypergraph H is by definition a clique of the conjugation graph—or *conjugation clique*—of H, but it need not be a *maximal* conjugation clique.

Definition 3. *A simple hypergraph $H = (X, \mathbf{E})$ is* conjugation-saturated (CS) *iff every $E \in \mathbf{E}$ is a maximal clique of the conjugation graph of H.*

Let us now state our simple characterization of VNM clutters.

Theorem 1. *Let $H = (X, \mathbf{E})$ be a simple hypergraph. Then, the following statements are equivalent:*

(i) H is conjugation-saturated;

(ii) H is a VNM hypergraph;

(iii) H is a VNM clutter.

Proof. $(i) \Rightarrow (ii)$ Suppose that $H = (X, \mathbf{E})$ is conjugation-saturated—i.e., for any $E \in \mathbf{E}$ and any $z \in X \smallsetminus E$, there exists $x_z \in E$ with $(z, x_z) \notin C_H$. Then, define $\Delta^H \subseteq X \times X$ by the following rule: for any $x, y \in X$, $\{(x, y), (y, x)\} \subseteq \Delta^H$ if $x \neq y$ and $(x, y) \notin C_H$, and $\{(x, y), (y, x)\} \cap \Delta^H = \varnothing$ if $(x, y) \in C_H$ or $x = y$—notice that by construction (X, Δ^H) is irreflexive. Therefore, for any $E \in \mathbf{E}$ and any $x, y \in E$, $(x, y) \in C_H$. Hence, $(x, y) \notin \Delta^H$, and E satisfies internal stability with respect to (X, Δ^H). Moreover, by hypothesis, for any $z \in X \setminus E$, there exists $x_z \in E$ with $(z, x_z) \notin C_H$, i.e., in particular, $(x_z, z) \in \Delta^H$, by definition of Δ^H. Thus, E also satisfies external stability with respect to (X, Δ^H). It follows that $\mathbf{E} \subseteq \mathcal{S}(X, \Delta^H)$ as required.

$(ii) \Longrightarrow (i)$ Observe that, by definition, H is conjugation-saturated iff for any $E \in \mathbf{E}$ and any $z \in X \smallsetminus E$, there exists $x_z \in E$ such that $(z, x_z) \notin C_H$.

Then, suppose $H = (X, \mathbf{E})$ is VNM, and let (X, Δ) be an irreflexive simple digraph such that $\mathbf{E} \subseteq \mathcal{S}(X, \Delta)$. Now, assume that there exist an edge $E \in \mathbf{E}$ and a vertex $z \in X \setminus E$ such that $(z, x) \in C_H$ for all $x \in E$. Thus, for any $x \in E$ there exists an $E_{zx} \in \mathbf{E}$ such that $\{x, z\} \subseteq E_{zx}$; therefore, by the internal stability of E_{zx}, $(x, z) \notin \Delta$—but then, the external stability of E in (X, Δ) is violated. Hence, $E \notin \mathcal{S}(X, \Delta)$, a contradiction.

$(ii) \Longrightarrow (iii)$ By the previous part of the proof, if H is VNM, then it must also be conjugation-saturated, i.e., its hyperedges are maximal conjugation cliques: hence, $S_1 \nsubseteq S_2$ for any two distinct $S_1, S_2 \in \mathbf{E}$.

$(iii) \Rightarrow (ii)$ Trivial. □

Thus, in particular, whenever X is finite, whether the simple hypergraph $H = (X, \mathbf{E})$ is VNM or not can be established using the following Algorithm 1:

Algorithm 1 VNM Hypergraph Algorithm

(1) Set $\mathbf{E}^* := \mathbf{E}$.
(2) If $\mathbf{E}^* = \varnothing$, write 'yes' and stop.
(3) Choose $E \in \mathbf{E}^*$.
(4) Set $E^+ := E$.
(5) Set $E^\circ := \varnothing$.
(6) If $E^+ = X$, write 'yes' and stop.
(7) If $E^+ = \varnothing$, write 'no' and stop.
(8) If $X \smallsetminus (E^+ \cup E^\circ) = \varnothing$, set $\mathbf{E}^* := \mathbf{E}^* \smallsetminus \{E^+\}$ and go to (2).
(9) Choose $z \in X \setminus (E^+ \cup E^\circ)$.
(10) Choose $x \in E^+$.
(11) If $(x, z) \notin \bigcup_{E \in \mathbf{E}} E \times E$, set $E^\circ := E^\circ \cup \{z\}$ and go to (8).
(12) Set $E^+ := E^+ \setminus \{x\}$.
(13) If $E^+ = \varnothing$, write 'no' and stop.
(14) Go to (10).

Comment: Clearly, if the algorithm stops immediately after executing either (2) or (6), then hypergraph (indeed, clutter) H is VNM because it is conjugation-saturated (hence, the written output is 'yes'), while if it stops immediately after executing (7) or (13), then hypergraph H has an edge that is not a maximal clique of its conjugation graph, and is therefore *not* VNM (hence, the written output is 'no'). Notice that the algorithm is linear in the size of its input.

Remark 2. *Notice that, as can be easily checked, the trivial clutter $(X, \emptyset)$ is VNM while clutter $(X, \{\emptyset\})$ is not VNM. Moreover, it is worth emphasizing that the foregoing theorem also implies that clutter $H = (X, \mathbf{E})$ of Example 2 is not VNM because it is clearly not conjugation-saturated (indeed,* no *edge of H is a maximal clique of H's conjugation graph). On the contrary, its* odd-cyclicity *is* not *key to its being* not *VNM. To see this, consider clutter $(\{x, y, z, u, v\}, \mathbf{E})$ with $\mathbf{E} = \{\{x, y\}, \{y, z\}, \{z, u\}, \{u, v\}, \{v, x\}\}$, which is also* odd-cyclic *(namely, there exist a positive integer k and $2k + 1$ distinct edges $E_i \in \mathbf{E}$ and vertices x_i, $i = 1, ..., 2k + 1$ such that $\{x_i, x_{i+1}\} \subseteq E_i$, $i = 1, ..., 2k$, and $\{x_1, x_{2k+1}\} \subseteq E_{2k+1}$). However, $(\{x, y, z, u, v\}, \mathbf{E})$ is conjugation-saturated and thus a VNM clutter by Theorem 1. To confirm the latter statement, it need only be checked that there is no $E_i \in \mathbf{E}$ comprising one of the following pairs: $\{x, z\}, \{x, u\}, \{y, v\}, \{y, u\}, \{z, v\}$.*

Remark 3. *In view of the theorem presented above, it is easily checked that a remarkable class of clutters which are not VNM is provided by (nontrivial)* Steiner triple systems *i.e., clutters $H = (X, \mathbf{E})$ such that $\#X \geq 4$, $\mathbf{E} \subseteq \{Y \subseteq X : \#Y = 3\}$, and for any two distinct $x, y \in X$, there exists precisely* one *$E \in \mathbf{E}$ with $\{x, y\} \subseteq E$. Indeed, no edge of such a clutter is a maximal clique of the conjugation graph of H: to check the latter statement, take any $E \in \mathbf{E}$ and observe that $\emptyset \subset E \subset X$—hence, there exist $x \in X \setminus E$ and $y \in E$, and for any such x and y, there exists by assumption an $E' \in \mathbf{E}$ with $\{x, y\} \subseteq E'$.*

Next, let us denote a simple hypergraph $H = (X, \mathbf{E})$ as *VNM-complete* if it is representable as the set of *all* VNM-stable sets of a simple irreflexive digraph—i.e., $\mathbf{E} = \mathcal{S}(X, \Delta)$ for some simple irreflexive digraph (X, Δ).

Definition 4. *A simple hypergraph $H = (X, \mathbf{E})$ is* maximal conjugation-saturated *iff it is CS and for any clutter $H' = (X, \mathbf{E}')$ such that $\mathbf{E} \subset \mathbf{E}'$, H' is not CS.*

Then, we have the following corollary to the previous theorem.

Corollary 1. *A simple hypergraph $H = (X, \mathbf{E})$ is VNM-complete if it is maximal conjugation-saturated.*

Proof. Suppose $H = (X, \mathbf{E})$ is maximal CS but not VNM-complete. Since H is CS, it is VNM by Theorem 1. In particular, $\mathbf{E} \subseteq \mathcal{S}(X, \Delta^H)$ by the proof of Theorem 1 whence $\mathbf{E} \subset \mathcal{S}(X, \Delta^H) = \mathbf{E}'$ since H is not VNM-complete. But then, $H' = (X, \mathbf{E}')$ is also VNM. Therefore, by Theorem 1, H' must also be CS, a contradiction. □

As an example, consider for any set X with $\#X \geq 3$ the $2-$*uniform star-clutter* with centre $x \in X$, i.e., the clutter $H_x^2 = (X, \mathbf{E})$ such that $\bigcup \mathbf{E} = X$, $\#E = 2$ for every $E \in \mathbf{E}$, and $E \cap E' = \{x\}$ for any two distinct $E, E' \in \mathbf{E}$: by construction, H_x^2 is not conjugation-saturated since for any two distinct $E, E' \in \mathbf{E}$ and any $y \in E \setminus \{x\}$, $z \in E' \setminus \{x\}$, it turns out that $\{(y, z), (z, y)\} \cap C_{H_x^2} = \emptyset$. Moreover, H_x^2 is in particular a *maximal* CS clutter since for any clutter $H' = (X, \mathbf{E}')$ such that $\mathbf{E} \subset \mathbf{E}' \subseteq \mathcal{P}(X)$, there exists an $E' \in \mathbf{E}'$ with $\#E' \geq 2$ and $x \notin E'$. Hence, $(y, x) \in C_{H'}$ for all $y \in E'$—i.e., E' is a conjugation-unsaturated edge. Indeed, H_x^2 is VNM-complete as implied by Corollary 1. To see this, consider (X, Δ) where $\Delta = \{(y, z) : y, z \in X \setminus \{x\}\}$: clearly, $\mathcal{S}(X, \Delta^H) = \mathbf{E}$.

On the other hand, observe that clutter $H = (\{x, y, z\}, \{\{x, y\}\})$ where $x \neq y \neq z \neq x$ is clearly CS but *not* maximal CS since clutter $H' = (\{x, y, z\}, \{\{x, y\}, \{z\}\})$

is also CS. However, H *is* VNM-complete: to check this, consider $(\{x, y, z\}, \Delta)$, with $\Delta = \{(x, z), (y, z)\}$, and notice that

$$\mathcal{S}(\{x, y, z\}, \Delta) = \{\{x, y\}\}.$$

Thus, generally speaking, being maximal conjugation-saturated is a sufficient but not a necessary condition for a simple hypergraph (clutter) to be VNM-complete.

Finally, it should be noticed that, while we have defined VNM hypergraphs (and clutters) as a subclass of *simple* hypergraphs (i.e., hypergraphs with no multiple edges), that is clearly due to our choice to focus on *sets* of VNM-stable sets. Allowing multiple copies of VNN-stable sets, i.e., defining VNM hypergraphs via *multisets* of VNM-stable sets of possibly nonsimple irreflexive digraphs, would enable a straightforward reformulation of our characterization result holding for general, possibly nonsimple, hypergraphs and clutters.

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