

Article

Markovian Analysis of a Small Population Evolutionary Prisoner's Dilemma Game

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Abstract

In this work we consider an *evolutionary game* in which a *small* population repeatedly plays the *Iterated Prisoner's Dilemma* (IPD) game using the *pairwise proportional imitation* (PPI) revision protocol. Since we are dealing with a small population, we can explicitly formulate a *Markov chain* (MC) model of the evolutionary game and study its properties. In particular, we identify the *absorbing states* and their *basins of attraction*. In addition, we briefly study the *replicator dynamics* of the same game for the case of a *large* population. We find that, as the size of the population increases, the Markovian dynamics approximate the replicator dynamics.

Keywords: Prisoner's Dilemma; evolutionary games; Markov chains

1. Introduction

In this paper, we investigate the *evolutionary dynamics* of a finite population engaged in repeated strategic interactions through the *Iterated Prisoner's Dilemma* (IPD). Individuals in the population are assumed to update their strategies according to the *pairwise proportional imitation* (PPI) revision protocol, whereby agents compare their performance with that of other individuals and adopt alternative strategies with a probability proportional to the observed payoff advantage. This update mechanism captures a simple form of social learning and adaptation that is commonly employed in evolutionary game-theoretic models.

As the population under consideration is relatively small, stochastic fluctuations arising from finite-size effects play a significant role in shaping the evolutionary outcomes. Consequently, a deterministic description based on average strategy frequencies may fail to accurately capture the system's behavior. To account for these effects, we formulate the evolutionary process as a *discrete-time Markov chain* whose states correspond to the possible compositions of the population. This framework enables a complete probabilistic characterization of the dynamics and allows us to study the long-term behavior of the system.

Using the Markov chain representation, we identify all *absorbing states* of the evolutionary process, namely those population configurations from which no further strategic changes are possible. We further characterize the corresponding *basins of attraction* by determining the sets of initial population states that can lead to each absorbing configuration. This analysis provides insight into the likelihood of different evolutionary outcomes and the role of stochasticity in selecting among competing strategic equilibria.

For comparison, we also examine the corresponding replicator dynamics, which describe the evolution of strategy frequencies in the limit of an infinitely large population.



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Unlike the Markovian formulation, the replicator dynamics constitute a deterministic system of ordinary differential equations that captures only the average effect of selection. This comparison highlights the relationship between finite-population evolutionary processes and their mean-field approximations while also clarifying the conditions under which deterministic models provide an accurate description of evolutionary behavior.

The study of *evolutionary game theory* is intended to provide a mathematical framework for the analysis of co-evolving populations. In this context, a particularly important question is how mutual cooperation can evolve and be sustained under conditions which favor selfish behavior. The classical description of evolutionary games appears in (Hofbauer & Sigmund, 1998; Sandholm, 2010a, 2010b). We are using a simplified version of the model presented in (Sandholm, 2010a), which reduces naturally to a MC. This reduction has been recognized in the literature, but the usual approaches to evolutionary games fall into two categories: (a) analyze the MC at a very general level (b) or take the mean field approximation, which holds when the number of players tends to infinity (for more details see Sections 2 and 4, respectively). We take a different approach: By looking at specific examples, we formulate a general approach for building the *probability transition matrix* P of the MC and use this to study its “low-level” properties.

The most popular model for the mathematical analysis of evolutionary games is the deterministic *replicator dynamics* (P. D. Taylor & Jonker, 1978). This is reflected in most classic treatments of the subject, either in book chapters (Barron, 2013; Peters, 2015) or full book treatments (Hofbauer & Sigmund, 1998; Maynard-Smith, 1982; Nowak, 2006; Nowak et al., 2004; Samuelson, 1997; Weibull, 1995).

Evolutionary game theory also employs stochastic population models to describe the evolution of competing strategies. In *finite populations*, stochastic processes such as the Moran and Wright–Fisher models provide a more realistic description than the deterministic models by accounting for demographic fluctuations at the individual level.

The Moran process is a discrete-state birth–death Markov chain that may be formulated in either discrete or continuous time, whereas the Wright–Fisher process evolves in discrete generations through multinomial sampling. For two-type populations, the state space is one-dimensional, allowing for explicit analysis of fixation probabilities, absorption times, and stationary distributions. Nowak (2006) is a good introduction to two-type Moran processes. A more advanced treatment appears in (de Souza et al., 2019); see also (Fudenberg et al., 2006; C. Taylor et al., 2006) for analyses which incorporate the effect of mutations. *Spatial* Moran processes (e.g., on graphs) are studied in (Pattni et al., 2015; Whigham & Dick, 2006). A good recent review of Moran, Wright–Fisher, and related two-type processes appears in (Chalub & Souza, 2017). In contrast, multi-type populations give rise to higher-dimensional Markov chains. These are particularly relevant for evolutionary games involving several strategies, mutation, or coexistence phenomena, but their analysis is considerably more challenging due to the rapid growth of the state space and the resulting complexity of the underlying stochastic dynamics. Examples of multi-type analysis are presented in (Ferreira & Neves, 2018; Hajihashemi & Samani, 2022); discrete time multi-type Moran processes on graphs are studied in (Goldberg et al., 2024; Schwarz, 2022). There are also *continuous*-time Moran processes, see for example (Braga et al., 2026).

The classical example used to model the emergence of cooperation is the Prisoner’s Dilemma (PD) game (Rapoport & Chammah, 1965). In the “classical”, one-shot PD game the unique Nash equilibrium is mutual defection. However, in the Iterated Prisoner’s Dilemma (Peters, 2015), which considers repeated interaction of a given pair of players, cooperation can emerge under certain conditions. This will also depend on the strategies used in the Iterated Prisoner’s Dilemma; these form a much larger set than that of the PD

strategies, since now the players can use *history-dependent* strategies. Most studies focus on a selected set of strategies, such as *always cooperate* (All-C), *always defect* (All-D) and *tit-for-tat* (TFT). All-C always cooperates in any iteration, and similarly All-D always defects. TFT cooperates in the first round and then copies its opponent's previous move. A particularly influential method for the analysis of iterated PD (IPD) was initiated by Axelrod (Axelrod, 1980a, 1980b, 1984; Axelrod & Dion, 1988; Axelrod & Hamilton, 1981) who introduced the use of *evolutionary tournaments*. An overview of such tournaments up to 2007 appears in (Kendall et al., 2007), and some book-length treatments are (Axelrod, 1984; Binmore, 2007).

The paper is organized as follows. In Section 2 we present some basic facts about evolutionary games and Markov chains. In Section 3 we present the evolutionary game based on the Iterated Prisoner's Dilemma, model the evolutionary game, under the *pairwise proportional imitation protocol*, by a *Markov chain* and characterize its absorbing states and basins of attraction. Also, we present some numerical experiments. In Section 4 we model the *large-population* approximation of the Markov chain by *replicator dynamics* and characterize its rest points. Finally, in Section 5 we summarize our results and propose some future research directions.

2. Evolutionary Games and Markov Chains

2.1. Evolutionary Games

Classical descriptions of evolutionary games appear in (Hofbauer & Sigmund, 1998; Sandholm, 2010a, 2010b). Our main starting point is (Sandholm, 2010a), but we use a simplified version of the model presented therein¹. We find that our results closely approximate those of (Sandholm, 2010a). We start with a two-player *symmetric* base game described by an $M \times M$ bimatrix (B, B^T) , i.e., each player uses a pure strategy from the set $\Sigma = \{\sigma_1, \dots, \sigma_M\}$, and we have

B_{mi} = "payoff received by player 1 when he uses σ_m against player 2 using σ_i ".

Assuming that the population consists of N players, we define the following:

1. A *population state* vector $\mathbf{z} = (z_1, \dots, z_N)$ indicates the strategy used by each player, i.e., $z_n = m$ means the n -th player is using the m -th strategy (we will often call such a player an " m -user"); $\mathbf{z}$ takes values in the

$$\text{Population Space: } \mathbf{Z}_N^M = \{(z_1, \dots, z_N) : \forall n : z_n \in \{1, \dots, M\}\}$$

2. A *frequency state* vector $\mathbf{s} = (s_1, \dots, s_M)$ indicates how many players are using each strategy, i.e., s_m is the number of m -users; $\mathbf{s}$ takes values in the

$$\text{Frequency Space: } \mathbf{S}_N^M = \left\{ (s_1, \dots, s_M) \in \{0, 1, \dots, N\} \text{ such that } \sum_{m=1}^M s_m = N \right\}.$$

In what follows we denote the cardinality of a set A by $|A|$. We will also use the following notation: For each $x \in \mathbb{R}$ we write the *positive part* of x as

$$[x]_+ = \begin{cases} x & \text{when } x > 0 \\ 0 & \text{when } x \leq 0 \end{cases}.$$

For given payoff matrix B , number of strategies M and number of players N , the evolutionary game can be described by the following *Evolutionary Game Protocol*. The update of the players' strategies according to Equations (2)–(4) is the so-called *pairwise proportional imitation* (PPI). The rationale of the PPI is the following: In every iteration a

single player is selected for a possible update of his strategy; if this player already plays the highest-scoring strategy, then he keeps it in the next iteration; otherwise, he selects the i -th strategy with probability proportional to the positive part of the payoff difference between the i -th strategy and his current strategy.

It is worth mentioning that all randomness in our model is generated during strategy-updating from the random selection of the next strategy-updating player. There is no randomness due to “erroneous playing”, behavioral deviations, etc. In other words, all users of the same strategy receive the exact same payoff. This is reflected in Equation (2).²

Evolutionary Game Protocol

1. Initialize the population vector at some $\mathbf{z}(0) = (z_1(0), \dots, z_N(0)) \in \mathbf{Z}_N^M$.
2. Repeat the following iteration for $j \in \{0, 1, \dots, J-1\}$.
 - (a) Compute the frequency vector $\mathbf{s}(j) = (s_1(j), \dots, s_M(j)) \in \mathbf{S}_N^M$ as follows:

$$\forall m : s_m(j) = |\{n : z_n(j) = m\}|. \quad (1)$$

- (b) Let the players play all possible matches between each other, i.e., for all the pairs (n_1, n_2) with $n_1, n_2 \in \{1, \dots, N\}$ and $n_1 \neq n_2$, play one round of the (B, B^T) game.
 - (c) Hence an m -user (for $m \in \{1, \dots, M\}$) collects a total payoff

$$Q_m(j) = \left(\sum_{i=1}^M s_i(j) B_{mi} \right) - B_{mm}. \quad (2)$$

- (d) Select n equiprobably from $\{1, \dots, N\}$ and let

$$z_n(j+1) = \begin{cases} z_n(j) & \text{w.p. 1} \quad \text{iff } Q_{z_n(j)}(j) = \max_m Q_m(j), \\ m & \text{w.p. prop. to } [Q_m(j) - Q_{z_n(j)}(j)]_+ \quad \text{else;} \end{cases} \quad (3)$$

and

$$\forall k \neq n : z_k(j+1) = z_k(j). \quad (4)$$

It is clear that the sequence $(\mathbf{z}(j))_{j=0}^\infty$ produced by the Evolutionary Game Protocol is a Markov chain; as we will show a little later, the same is true of the state process $(\mathbf{s}(j))_{j=0}^\infty$. It is worth emphasizing that, under the Evolutionary Game Protocol, once a strategy becomes extinct in an evolutionary game, it will never reappear. Hence, the “pure” states (i.e., those with all players using the same strategy) are absorbing states of the MC $(\mathbf{s}(j))_{j=0}^\infty$.

2.2. Markov Chains

The frequency space $\mathbf{S}_N^M$ is much smaller than the population space $\mathbf{Z}_N^M$. Clearly, $|\mathbf{Z}_N^M| = M^N$; it is not hard to show that, for every M and N , $|\mathbf{S}_N^M| = O(N^{M-1})$. We can summarize the relation between population vector and state vectors by a function $G : \mathbf{Z}_N^M \rightarrow \mathbf{S}_N^M$, where

$$G(\mathbf{z}) = \mathbf{s} = (s_1, s_2, \dots, s_M) \text{ with } s_m = |\{n : z_n = m\}| \text{ for all } m \in \{1, \dots, M\}.$$

For example, with $M = 3$ and $N = 5$, the population state $\mathbf{z} = (3, 2, 2, 1, 3)$ corresponds to the frequency state

$$\mathbf{s} = G(3, 2, 2, 1, 3) = (1, 2, 2).$$

We will study the evolutionary game by analyzing $(\mathbf{s}(j))_{j=0}^{\infty}$. For example, suppose that we want to know which strategies survive in the limit $J \rightarrow \infty$; then, we want to find the *absorbing states* (if such exist). As we will show next, $(\mathbf{s}(j))_{j=0}^{\infty}$ is a Markov chain, so we can answer questions of this type using the *transition probability matrix* of $\mathbf{s}(t)$, which is denoted by P and defined by

$$\forall \mathbf{s}, \mathbf{s}' \in \mathbf{S}_N^M : P_{\mathbf{s}, \mathbf{s}'} = \Pr(\mathbf{s}(j+1) = \mathbf{s}' | \mathbf{s}(j) = \mathbf{s}).$$

It turns out that we can compute P relatively easily by exploiting the relationship between $(\mathbf{z}(t))_{t=0}^{\infty}$ and $(\mathbf{s}(t))_{t=0}^{\infty}$. We have

$$\forall t : \mathbf{s}(t) = G(\mathbf{z}(t)).$$

We note that the inverse G^{-1} is a set-valued function:

$$\forall \mathbf{s} : G^{-1}(\mathbf{s}) = \{\mathbf{z} : G(\mathbf{z}) = \mathbf{s}\}.$$

The following Definition 1 and Proposition 1 are from (Kemeny & Snell, 1969).

Definition 1. Suppose we are given (a) a Markov chain $(\mathbf{x}(t))_{t=0}^{\infty}$ with state space $R = \{1, \dots, N\}$ and (b) a partition $\mathcal{S} = \{R_1, \dots, R_M\}$ of R . We construct a new stochastic process $(\mathbf{y}(t))_{t=0}^{\infty}$ with state space $\mathcal{S}$ by setting

$$\forall t : \mathbf{y}(t) = R_m \text{ iff } \mathbf{x}(t) \in R_m.$$

We say that $(\mathbf{x}(t))_{t=0}^{\infty}$ is lumpable with respect to $\mathcal{S}$ iff $(\mathbf{y}(t))_{t=0}^{\infty}$ is a MC with transition probabilities which do not depend on the initial probability $(\Pr(\mathbf{x}(0) = \mathbf{x}))_{\mathbf{x} \in R}$.

Define the function $F : R \rightarrow \mathcal{S}$ by: $F(\mathbf{x}) = R_m$ iff $\mathbf{x} \in R_m$.

Proposition 1. Given $(\mathbf{x}(t))_{t=0}^{\infty}$ and $(\mathbf{y}(t))_{t=0}^{\infty}$ as in Definition 1, $(\mathbf{x}(t))_{t=0}^{\infty}$ is lumpable with respect to $\{R_1, \dots, R_M\}$ if

$$\forall \mathbf{y}, \hat{\mathbf{y}} \in \mathcal{S}, \forall \mathbf{x}, \hat{\mathbf{x}} \in F^{-1}(\mathbf{y}) : \Pr(\mathbf{x} \rightarrow \hat{\mathbf{y}}) = \Pr(\hat{\mathbf{x}} \rightarrow \hat{\mathbf{y}}). \quad (\text{C1})$$

Then it is also true that

$$\forall \mathbf{y}, \hat{\mathbf{y}} \in \mathcal{S}, \forall \mathbf{x} \in F^{-1}(\mathbf{y}) : \Pr(\mathbf{y} \rightarrow \hat{\mathbf{y}}) = \Pr(\mathbf{x} \rightarrow \hat{\mathbf{y}}).$$

Now, regarding our $(\mathbf{z}(t))_{t=0}^{\infty}$ and $(\mathbf{s}(t))_{t=0}^{\infty}$ processes, we have the following:

Proposition 2. For every M, N , $(\mathbf{z}(t))_{t=0}^{\infty}$ is lumpable with respect to $\mathbf{S}_N^M$.

Proof. Take any $\mathbf{s}, \hat{\mathbf{s}} \in \mathbf{S}_N^M$ and any $\mathbf{z} = (z_1, \dots, z_N)$, $\hat{\mathbf{z}} = (\hat{z}_1, \hat{z}_2, \dots, \hat{z}_N) \in G^{-1}(\mathbf{s})$. We have $G(\mathbf{z}) = G(\hat{\mathbf{z}}) = \mathbf{s}$. This means that $\mathbf{z}$ and $\hat{\mathbf{z}}$ contain the same number of 1s, 2s, ...; in other words, $\hat{\mathbf{z}}$ is obtained by a permutation of the elements of $\mathbf{z}$:

$$(\hat{z}_1, \hat{z}_2, \dots, \hat{z}_N) = (z_{i_1}, z_{i_2}, \dots, z_{i_N})$$

where $(i_1, \dots, i_N)$ is a permutation of $(1, 2, \dots, N)$. Now

$$\begin{aligned} \Pr(\mathbf{z} \rightarrow \mathbf{s}) &= \Pr((z_1, z_2, \dots, z_N) \rightarrow \mathbf{s}) \\ &= \sum_{(z'_1, z'_2, \dots, z'_N) \in G(\mathbf{s})} \Pr((z_1, z_2, \dots, z_N) \rightarrow (z'_1, z'_2, \dots, z'_N)) \\ &= \sum_{(z'_1, z'_2, \dots, z'_N) \in G(\mathbf{s})} \Pr((z_{i_1}, z_{i_2}, \dots, z_{i_N}) \rightarrow (z'_{i_1}, z'_{i_2}, \dots, z'_{i_N})) \\ &= \sum_{(z'_{i_1}, z'_{i_2}, \dots, z'_{i_N}) \in F(\mathbf{s})} \Pr((\hat{z}_1, \hat{z}_2, \dots, \hat{z}_N) \rightarrow (z'_{i_1}, z'_{i_2}, \dots, z'_{i_N})) = \Pr(\hat{\mathbf{z}} \rightarrow \mathbf{s}). \end{aligned}$$

So, Condition C1 of Proposition 1 is satisfied, and $(\mathbf{z}(t))_{t=0}^{\infty}$ is lumpable. $\square$

From Proposition 2 we see that $(\mathbf{s}(t))_{t=0}^{\infty}$ is a Markov chain; the following proposition shows how to compute its transition probability matrix:

Proposition 3. For every M and N , we can compute the transition matrix $(P_{\mathbf{s}, \mathbf{s}'}_{\mathbf{s}, \mathbf{s}' \in S})$ of $(\mathbf{s}(t))_{t=0}^{\infty}$ in $O(N^{2M-1})$ time.

Proof. Let us define the function $H : \mathbf{S}_N^M \rightarrow \mathbf{Z}_N^M$ as follows:

$$\forall (s_1, s_2, \dots, s_M) \in \mathbf{S}_N^M : H(s_1, s_2, \dots, s_M) = \underbrace{1 \dots 1}_{s_1 \text{ terms}}, \underbrace{2 \dots 2}_{s_2 \text{ terms}}, \dots, \underbrace{M \dots M}_{s_M \text{ terms}}.$$

Clearly,

$$\forall (s_1, s_2, \dots, s_M) \in S : H(s_1, s_2, \dots, s_M) \in F^{-1}(s_1, s_2, \dots, s_M).$$

Then, from Proposition 1 we have

$$\forall \mathbf{s}, \hat{\mathbf{s}} \in S : \Pr(\mathbf{s} \rightarrow \hat{\mathbf{s}}) = \Pr(H^{-1}(\mathbf{s}) \rightarrow \hat{\mathbf{s}}). \quad (5)$$

To obtain the entire $P = (P_{\mathbf{s}, \hat{\mathbf{s}}})_{\mathbf{s}, \hat{\mathbf{s}} \in S}$ we must compute (5) for $|S_N|^2 = O(N^{M-1})^2 = O(N^{2M-2})$ times. For any pair $(\mathbf{s}, \hat{\mathbf{s}})$ the computation of $\Pr(G^{-1}(\mathbf{s}) \rightarrow \hat{\mathbf{s}})$ takes time $O(N)$, using (3). This completes the proof. $\square$

It is easy to compute that, for the case $M = 3$, the size of the strategy state space S_N^3 is

$$|S_N^3| = \frac{N^2 + 3N + 2}{2},$$

i.e., it grows quadratically with N . We have

ine N	3	4	5	10	20	50	100	200
ine $ S_N^3 $	10	15	21	66	231	1326	5151	20,301
ine								

Hence, computation time is feasible for N up to 200, at least.

From P we can generate the *state transition graph* (STG).³ Two examples of STGs appear in the Figure 1; they are obtained from the IPD evolutionary game with the base game IPD, obtained from the PD with

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

(see Section 3 for details). The left figure is obtained when the number of players is $N = 5$ and the right one when $N = 10$. The plot is generated as follows: First, recall that every state (s_1, s_2, s_3) satisfies $s_1 + s_2 + s_3 = N$. Hence, given N and knowing two of s_1, s_2, s_3 uniquely specifies the third one. Then, we can map every state (s_1, s_2, s_3) to a unique point $W(s_1, s_2, s_3) = (s_1, s_2)$ in the simplex. A nonzero element $P_{(s_1, s_2, s_3), (s'_1, s'_2, s'_3)} > 0$ indicates a possible transition from (s_1, s_2, s_3) to (s'_1, s'_2, s'_3) , which we indicate by an arrow from (s_1, s_2) to (s'_1, s'_2) ; we omit self-loops. This plotting convention will be used in the sequel.

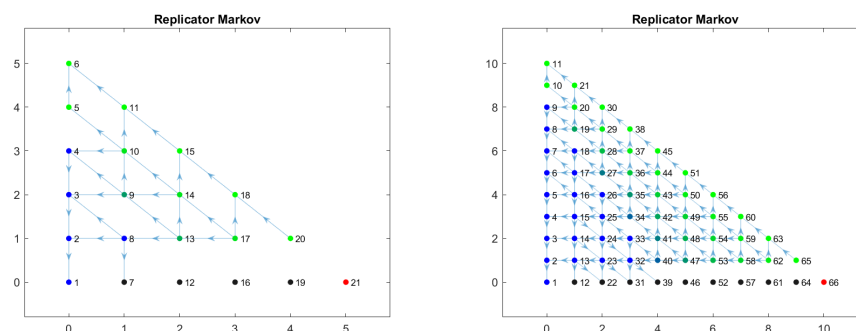


Figure 1. The state transition digraph for $N = 5$ (left) and $N = 10$ (right).

We will also use the following naming conventions: Since the states are arranged on an orthogonal grid, given a state (s_1, s_2, s_3) with $W(s_1, s_2, s_3) = (s_1, s_2)$, we can refer to the state (s'_1, s'_2, s'_3) such that $W(s'_1, s'_2, s'_3) = (s_1, s_2 + 1)$ as the “north neighbor” of (s_1, s_2, s_3) , to (s''_1, s''_2, s''_3) such that $W(s''_1, s''_2, s''_3) = (s_1 + 1, s_2 - 1)$ as its “southeast neighbor” etc. These are *geometric* neighbors; if they are also neighbors in the graph theoretic sense (i.e., connected by an edge), then we say they are *connected* neighbors. For example, in Figure 1, left, consider the state $(2, 1, 2)$. It corresponds to (s_1, s_2) point $W(2, 1, 2) = (2, 1)$ and is vertex no. 13. Its neighbors are as follows:

State	(s_1, s_2) point	Vertex No.	Direction	Connected
$(1, 1, 3)$	$(1, 1)$	8	West	Yes
$(1, 2, 2)$	$(1, 2)$	9	Northwest	Yes
$(2, 2, 1)$	$(2, 2)$	14	North	Yes
$(1, 0, 4)$	$(1, 0)$	7	Southwest	No
$(2, 0, 3)$	$(2, 0)$	12	South	No
...	...	...	...	...

3. Iterated Prisoner’s Dilemma: Markovian Dynamics

3.1. PD and IPD

In this section we will study the evolutionary game obtained from the *Iterated Prisoner’s Dilemma* (IPD). This consists of playing T rounds of the *Prisoner’s Dilemma* (PD) and is probably the most popular base game used in the literature of evolutionary games. Recall that PD is the bimatrix game

$$(A, A^T) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right)$$

where

$$a_{12} < a_{22} < a_{11} < a_{21}.$$

The IPD consists of playing T rounds of the PD. We will use the following three strategies (hence, in the rest of this paper, we always take $M = 3$):

Str. No. Strategy

- 1 *All-C*: always cooperate.
- 2 *All-D*: always defect.
- 3 *TfT*: cooperate in the first round, in every other round play what the other player played in the previous round.

Then it is easily seen that the B payoff matrix is

$$B = \begin{bmatrix} a_{11}T & a_{12}T & a_{11}T \\ a_{21}T & a_{22}T & a_{21} + a_{22}(T-1) \\ a_{11}T & a_{12} + a_{22}(T-1) & a_{11}T \end{bmatrix} \quad (6)$$

3.2. Markovian Dynamics

We will now study the structure of the transition probability matrix P where

$$\forall \mathbf{s}, \mathbf{s}' \in \mathbf{S}_N^3 : P_{\mathbf{s}, \mathbf{s}'} = \Pr(\mathbf{s}(j+1) = \mathbf{s}' | \mathbf{s}(j) = \mathbf{s}).$$

In the rest of this paper we will always take $M = 3$ and omit it from the notation. For $N = 3$ the population space and the frequency space are

$$\begin{aligned} \mathbf{Z}_3 &= \{(1, 1, 1), (1, 1, 2), (1, 1, 3), (2, 1, 1), \dots, (3, 3, 3)\}, \\ \mathbf{S}_3 &= \{(0, 0, 3), (0, 1, 2), (1, 0, 2), (0, 2, 1), (0, 3, 0), (1, 0, 2), \dots, (3, 0, 0)\}. \end{aligned}$$

Similarly, for any N we have

$$\mathbf{S}_N = \{(0, 0, N), (0, 1, N-1), (1, 0, N-1), (0, 2, N-2), \dots, (N, 0, 0)\}$$

and, in what follows, we will always order (and number) the strategy states as listed above. It will also be useful to partition $\mathbf{S}_N$ into the following components:

$$\begin{aligned} \text{pure states} & : \mathbf{S}_N^{(p)} = \{(N, 0, 0), (0, N, 0), (0, 0, N)\} \\ & \mathbf{S}_N^{(1)} = \{(0, 1, N-1), (0, 2, N-2), \dots, (0, N-1, 1)\} \\ \text{side states} & : \mathbf{S}_N^{(2)} = \{(1, 0, N-1), (2, 0, N-2), \dots, (N-1, 0, 1)\} \\ & \mathbf{S}_N^{(3)} = \{(1, N-1, 0), (2, N-2, 0), \dots, (N-1, 1, 0)\} \\ \text{interior states} & : \mathbf{S}_N^{(i)} = \{(s_1, s_2, s_3) \text{ s.t. } s_m \in \{1, 2, \dots, N-1\} \text{ for } m \in \{1, 2, 3\}\} \end{aligned}$$

Clearly, $\mathbf{S}_N = \mathbf{S}_N^{(p)} \cup \mathbf{S}_N^{(1)} \cup \mathbf{S}_N^{(2)} \cup \mathbf{S}_N^{(3)} \cup \mathbf{S}_N^{(i)}$. We will call $\mathbf{S}_N^{(p)} \cup \mathbf{S}_N^{(1)} \cup \mathbf{S}_N^{(2)} \cup \mathbf{S}_N^{(3)}$ the set of *boundary* states.

Let us now turn to the payoff function. Let $f_m(\mathbf{s})$ be the (state-dependent) m -user payoff:

$$\forall m, \forall \mathbf{s} : f_m(\mathbf{s}) : \text{total payoff obtained by user of } m\text{-th strategy at state } \mathbf{s}.$$

Hence, we have

$$\forall (s_1, s_2, s_3) \text{ with } s_1 \neq 0 : f_1(s_1, s_2, s_3) = (s_1 - 1)a_{11}T + s_2a_{12}T + s_3a_{11}T \quad (7)$$

$$\forall (s_1, s_2, s_3) \text{ with } s_2 \neq 0 : f_2(s_1, s_2, s_3) = s_1a_{21}T + (s_2 - 1)a_{22}T + s_3(a_{21} + a_{22}(T-1)) \quad (8)$$

$$\forall (s_1, s_2, s_3) \text{ with } s_3 \neq 0 : f_3(s_1, s_2, s_3) = s_1a_{11}T + s_2(a_{12} + a_{22}(T-1)) + (s_3 - 1)a_{11}T \quad (9)$$

For any (s_1, s_2, s_3) such that $s_m = 0$, we let (for $m \in \{1, 2, 3\}$)

$$f_m(s_1, s_2, s_3) = -\infty.$$

Letting $s_3 = N - s_1 - s_2$, (7)–(9) yield:

$$\begin{aligned} \forall (s_1, s_2) \text{ s.t. } s_1 \neq 0 : f_1(s_1, s_2) &= (s_1 - 1)a_{11}T + s_2a_{12}T + (N - s_1 - s_2)a_{11}T \\ \forall (s_1, s_2) \text{ s.t. } s_2 \neq 0 : f_2(s_1, s_2) &= s_1a_{21}T + (s_2 - 1)a_{22}T + (N - s_1 - s_2)(a_{21} + a_{22}(T - 1)) \\ \forall (s_1, s_2) \text{ s.t. } s_1 + s_2 \neq N : f_3(s_1, s_2) &= s_1a_{11}T + s_2(a_{12} + a_{22}(T - 1)) + (N - s_1 - s_2 - 1)a_{11}T \end{aligned}$$

In what follows we will examine the possible transitions from each (s_1, s_2, s_3) state. Exploiting the representation of states in the s_1 - s_2 plane, we will sometimes use the following self-explanatory terminology:

$$\begin{aligned} \text{North transition: } (s_1, s_2, s_3) &\rightarrow (s_1, s_2 + 1, s_3 - 1), \\ \text{South transition: } (s_1, s_2, s_3) &\rightarrow (s_1, s_2 - 1, s_3 + 1), \\ \text{West transition: } (s_1, s_2, s_3) &\rightarrow (s_1 - 1, s_2, s_3 + 1) \\ \text{Northwest transition: } (s_1, s_2, s_3) &\rightarrow (s_1 - 1, s_2 + 1, s_3), \\ &\text{etc.} \end{aligned}$$

Let us look at possible state transitions. The key observation is the following. According to the PPI, the only possible transitions are the ones satisfying the following:

$$\begin{aligned} \text{For } (s_1, s_2, s_3) \rightarrow (s_1 + 1, s_2 - 1, s_3): f_1(s_1, s_2, s_3) &> f_2(s_1, s_2, s_3) \\ \text{For } (s_1, s_2, s_3) \rightarrow (s_1 - 1, s_2 + 1, s_3): f_2(s_1, s_2, s_3) &> f_1(s_1, s_2, s_3) \\ \text{For } (s_1, s_2, s_3) \rightarrow (s_1 - 1, s_2, s_3 + 1): f_3(s_1, s_2, s_3) &> f_1(s_1, s_2, s_3) \\ \text{For } (s_1, s_2, s_3) \rightarrow (s_1 + 1, s_2, s_3 - 1): f_1(s_1, s_2, s_3) &> f_3(s_1, s_2, s_3) \\ \text{For } (s_1, s_2, s_3) \rightarrow (s_1, s_2 + 1, s_3 - 1): f_2(s_1, s_2, s_3) &> f_3(s_1, s_2, s_3) \\ \text{For } (s_1, s_2, s_3) \rightarrow (s_1, s_2 - 1, s_3 + 1): f_3(s_1, s_2, s_3) &> f_2(s_1, s_2, s_3) \end{aligned}$$

We will treat the boundary and interior of the (s_1, s_2) -simplex separately. For the boundary we have the following:

Proposition 4. *In the evolutionary game with A as base game, N players and T rounds, if*

$$a_{12} < a_{22} < a_{11} < a_{21},$$

then the following hold:

1. *For the pure states $\mathbf{S}_N^{(p)}$ only self-transitions are possible.*
2. *For the states in $\mathbf{S}_N^{(1)}$, except for self-transitions, the only possible transitions are*

$$\text{For } s_2 < N - 1 : (0, s_2, s_3) \rightarrow (0, s_2 - 1, s_3 + 1)$$

$$\text{For } s_2 = N - 1 : (0, s_2, s_3) \rightarrow (0, s_2 + 1, s_3 - 1)$$

3. *For the states in $\mathbf{S}_N^{(2)}$, only self-transitions are possible.*
4. *For the states in $\mathbf{S}_N^{(3)}$, except for self-transitions, the only possible transitions are*

$$\text{For } s_2 < N : (s_1, s_2, 0) \rightarrow (s_1 - 1, s_2 + 1, 0).$$

Proof. We prove each case separately.

1. These are the states: $(N, 0, 0)$, $(0, N, 0)$, $(0, 0, N)$; clearly, only self-transitions are possible.

2. These are the states of the form $(0, s_2, s_3)$ (with $s_2 > 0$, $s_3 > 0$). Since $s_1 = 0$, we have $s_2 + s_3 = N$. Furthermore, excluding self-transitions, the only possible transitions are

$$(0, s_2, s_3) \rightarrow (0, s_2 - 1, s_3 + 1) \text{ and } (0, s_2, s_3) \rightarrow (0, s_2 + 1, s_3 - 1).$$

To determine which of these have positive probabilities, we note that

$$\delta_{23} = f_2(s_1, s_2, s_3) - f_3(s_1, s_2, s_3) = T(s_2 - N + 1)(a_{11} - a_{22})$$

(where we have used $s_1 = 0$ and $s_3 = N - s_2$). We have

$$s_2 < N - 1 \Rightarrow \delta_{23} < 0$$

$$s_2 = N - 1 \Rightarrow \delta_{23} = 0$$

from which we conclude

For $s_2 < N - 1$: we can only have $(0, s_2, s_3) \rightarrow (0, s_2 - 1, s_3 + 1)$ transitions

For $s_2 = N - 1$: we can only have $(0, s_2, s_3) \rightarrow (0, s_2 + 1, s_3 - 1)$ transitions

3. These are the states of the form $(s_1, 0, s_3)$ (with $s_1 > 0$, $s_3 > 0$). When $s_2 = 0$, we have $s_1 + s_3 = N$. Furthermore, excluding self-transitions, the only possible transitions are

$$(s_1, 0, s_3) \rightarrow (s_1 - 1, 0, s_3 + 1) \text{ and } (s_1, 0, s_3) \rightarrow (s_1 + 1, 0, s_3 - 1).$$

To determine which of these have positive probabilities, we note that

$$\delta_{31} = f_3(s_1, s_2, s_3) - f_1(s_1, s_2, s_3) = 0$$

(where we have used $s_2 = 0$ and $s_3 = N - s_1$). It follows that both

$$(s_1, 0, s_3) \rightarrow (s_1 - 1, 0, s_3 + 1) \text{ and } (s_1, 0, s_3) \rightarrow (s_1 + 1, 0, s_3 - 1).$$

have zero probability, i.e., only self transitions are possible.

4. These are the states of the form $(s_1, s_2, 0)$ (with $s_1 > 0$, $s_2 > 0$). When $s_3 = 0$, we have $s_1 + s_2 = N$. Furthermore, excluding self-transitions, the only possible transitions are

$$(s_1, s_2, 0) \rightarrow (s_1 - 1, s_2 + 1, 0) \text{ and } (s_1, s_2, 0) \rightarrow (s_1 + 1, s_2 - 1, 0).$$

To determine which of these have positive probabilities, we note that

$$\delta_{12} = f_1(s_1, s_2, s_3) - f_2(s_1, s_2, s_3) = T(s_1(a_{11} - a_{21}) + s_2(a_{12} - a_{22}) + (a_{22} - a_{11})) < 0$$

(since $a_{12} < a_{22} < a_{11} < a_{21}$). Hence, excluding self-transitions, the only possible transitions are

$$(s_1, s_2, 0) \rightarrow (s_1 - 1, s_2 + 1, 0).$$

This completes the proof. $\square$

Now let us consider transitions of interior states. First we need to introduce the following geometrical elements. Consider the straight lines

$$l_{12} : (a_{21} - a_{22})s_1 + (a_{11} - a_{12})s_2 = (N - 1)(a_{11} - a_{22}) \quad (10)$$

$$l_{23} : (a_{21} - a_{22})s_1 + (a_{11} - a_{22})s_2 = (N - 1)(a_{11} - a_{22}) \quad (11)$$

1. The line l_{12} separates the s_1 - s_2 plane into two half-planes. Also, it intersects the s_1 axis at $\hat{s}_1 = (N - 1) \frac{(a_{11} - a_{22})}{(a_{21} - a_{22})}$ and the s_2 axis at $\hat{s}_2 = (N - 1) \frac{(a_{11} - a_{22})}{(a_{11} - a_{12})} < N - 1$.
2. The line l_{23} separates the s_1 - s_2 plane into two half-planes. Also, it intersects the s_1 axis at $\hat{s}_1 = (N - 1) \frac{(a_{11} - a_{22})}{(a_{21} - a_{22})}$ and the s_2 axis at $\hat{s}_2 = (N - 1)$.

Note that we will always have $\hat{s}_2 < \tilde{s}_2$. The combination of l_{12} and l_{13} separates $\mathbf{S}_N^{(i)}$ (the set of interior states) into three regions (as seen in Figure 2):

$$\begin{aligned} \mathcal{A}_1 : & \quad (a_{21} - a_{22})s_1 + (a_{11} - a_{12})s_2 < (N - 1)(a_{11} - a_{22}) \text{ (i.e., below } l_{12}) \\ & \quad (a_{21} - a_{22})s_1 + (a_{11} - a_{22})s_2 < (N - 1)(a_{11} - a_{22}) \text{ (i.e., below } l_{23}) \\ \mathcal{A}_2 : & \quad (a_{21} - a_{22})s_1 + (a_{11} - a_{12})s_2 > (N - 1)(a_{11} - a_{22}) \text{ (i.e., above } l_{12}) \\ & \quad (a_{21} - a_{22})s_1 + (a_{11} - a_{22})s_2 < (N - 1)(a_{11} - a_{22}) \text{ (i.e., below } l_{23}) \\ \mathcal{A}_3 : & \quad (a_{21} - a_{22})s_1 + (a_{11} - a_{12})s_2 > (N - 1)(a_{11} - a_{22}) \text{ (i.e., above } l_{12}) \\ & \quad (a_{21} - a_{22})s_1 + (a_{11} - a_{22})s_2 \geq (N - 1)(a_{11} - a_{22}) \text{ (i.e., above } l_{23}) \end{aligned}$$

Note that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ do not include states which fall exactly on the lines l_{12}, l_{13} . The separation of the plane can be seen in the following figure.

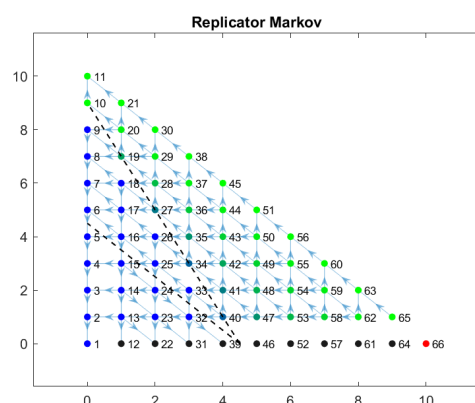


Figure 2. The (s_1, s_2) -simplex, the state transitions, and the separating lines l_{12} and l_{23} .

Now we can prove the following.

Proposition 5. In the evolutionary game with A as base game, N players and T rounds, if

$$a_{12} < a_{22} < a_{11} < a_{21}$$

and T is sufficiently large, then the allowable state transitions are as follows.

1. For states $(s_1, s_2, s_3) \in \mathcal{A}_1$, excepting self-transitions, the only allowable transitions are

$$(s_1 + 1, s_2 - 1, s_3), (s_1 - 1, s_2, s_3 + 1), (s_1, s_2 - 1, s_3 + 1).$$

2. For states $(s_1, s_2, s_3) \in \mathcal{A}_2$, excepting self-transitions, the only allowable transitions are

$$(s_1 - 1, s_2 + 1, s_3), (s_1 - 1, s_2, s_3 + 1), (s_1, s_2 - 1, s_3 + 1).$$

3. For states $(s_1, s_2, s_3) \in \mathcal{A}_3$, excepting self-transitions, the only allowable transitions are

$$(s_1 - 1, s_2 + 1, s_3), (s_1 - 1, s_2, s_3 + 1), (s_1, s_2 + 1, s_3 - 1).$$

Proof. We have $s_3 = N - s_1 - s_2$ and

$$\begin{aligned} f_1(s_1, s_2, s_3) &= (s_1 - 1)a_{11}T + s_2a_{12}T + (N - s_1 - s_2)a_{11}T \\ f_2(s_1, s_2, s_3) &= s_1a_{21}T + (s_2 - 1)a_{22}T + (N - s_1 - s_2)(a_{21} + a_{22}(T - 1)) \\ f_3(s_1, s_2, s_3) &= s_1a_{11}T + s_2(a_{12} + a_{22}(T - 1)) + (N - s_1 - s_2 - 1)a_{11}T \end{aligned}$$

Then we compute the following differences:

$$\begin{aligned} \delta_{12} &= f_1(s_1, s_2, s_3) - f_2(s_1, s_2, s_3) = A_{12}(s_1, s_2, s_3)T + B_{12}(s_1, s_2, s_3) \\ \delta_{23} &= f_2(s_1, s_2, s_3) - f_3(s_1, s_2, s_3) = A_{23}(s_1, s_2, s_3)T + B_{23}(s_1, s_2, s_3) \\ \delta_{31} &= f_3(s_1, s_2, s_3) - f_1(s_1, s_2, s_3) = A_{31}(s_1, s_2, s_3)T + B_{31}(s_1, s_2, s_3) \end{aligned}$$

For large enough T , we can ignore the B_{ij} terms. Hence, after some algebra, we get

$$\delta_{12} \simeq T(-a_{11} + a_{12}s_2 + Na_{11} - s_2a_{11} - s_1a_{21} + a_{22} - Na_{22} + s_1a_{22}) \quad (12)$$

$$\delta_{23} \simeq T(s_1a_{21} - s_2a_{22} - a_{22} + Na_{22} - s_1a_{22} - Na_{11} + s_2a_{11} + a_{11}) \quad (13)$$

$$\delta_{31} \simeq Ts_2(a_{22} - a_{12}) \quad (14)$$

Since $a_{22} > a_{12}$, from (10) follows that we will have transitions $(s_1, s_2, s_3) \rightarrow (s_1, s_2 - 1, s_3 + 1)$ but not $(s_1, s_2, s_3) \rightarrow (s_1, s_2 + 1, s_3 - 1)$. After some algebra, we can rewrite (12) and (13) as follows:

$$\delta_{12} \simeq T((a_{22} - a_{21})s_1 + (a_{12} - a_{11})s_2 + (N - 1)(a_{11} - a_{22}))$$

$$\delta_{23} \simeq T((a_{21} - a_{22})s_1 + (a_{11} - a_{22})s_2 - (N - 1)(a_{11} - a_{22}))$$

$$\delta_{12} \simeq T((a_{21} - a_{22})s_1 + (a_{11} - a_{12})s_2 + (N - 1)(a_{22} - a_{11}))$$

$$\delta_{23} \simeq T((a_{21} - a_{22})s_1 + (a_{11} - a_{22})s_2 + (N - 1)(a_{22} - a_{11}))$$

Remember the straight lines l_{12} , l_{23} . The following facts are easily seen:

1. The line l_{12} separates the s_1 - s_2 plane into two half-planes.
 - (a) Below l_{12} we have transitions $(s_1, s_2, s_3) \rightarrow (s_1 + 1, s_2 - 1, s_3)$.
 - (b) Above l_{12} we have transitions $(s_1, s_2, s_3) \rightarrow (s_1 - 1, s_2 + 1, s_3)$.
2. The line l_{23} separates the s_1 - s_2 plane into two half-planes.
 - (a) Below l_{23} we have transitions $(s_1, s_2, s_3) \rightarrow (s_1, s_2 - 1, s_3 + 1)$.
 - (b) Above l_{23} we have transitions $(s_1, s_2, s_3) \rightarrow (s_1, s_2 + 1, s_3 - 1)$.

As remarked, the combination of l_{12} and l_{13} separates the interior of the (s_1, s_2) -simplex into the regions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and, obviously, the signs of δ_{12} and δ_{23} are as follows.

	l_{12}	δ_{12}	l_{23}	δ_{23}
In $\mathcal{A}_1$	below	< 0	below	< 0
In $\mathcal{A}_2$	above	> 0	below	< 0
In $\mathcal{A}_3$	above	> 0	above	> 0

Hence, excepting self-transitions, the allowable transitions in these regions are as follows:

$$\mathcal{A}_1 : (s_1, s_2, s_3) \rightarrow \{(s_1 + 1, s_2 - 1, s_3), (s_1 - 1, s_2, s_3 + 1), (s_1, s_2 - 1, s_3 + 1)\}$$

$$\mathcal{A}_2 : (s_1, s_2, s_3) \rightarrow \{(s_1 - 1, s_2 + 1, s_3), (s_1 - 1, s_2, s_3 + 1), (s_1, s_2 - 1, s_3 + 1)\}$$

$$\mathcal{A}_3 : (s_1, s_2, s_3) \rightarrow \{(s_1 - 1, s_2 + 1, s_3), (s_1 - 1, s_2, s_3 + 1), (s_1, s_2 + 1, s_3 - 1)\}$$

This completes the proof. $\square$

Using Propositions 4 and 5, the following proposition describes fully the absorbing states of the $(\mathbf{s}(t))_{t=0}^{\infty}$ and partly their basins of attraction.

Proposition 6. *In the evolutionary game with A as base game, N players and T rounds, if*

$$a_{12} < a_{22} < a_{11} < a_{21}$$

and T is sufficiently large, then

1. *Each state in the set $\{(0, N, 0)\} \cup \mathbf{S}_N^{(2)}$ is absorbing.*
2. *If $\mathbf{s}(0) = (0, N - 1, 1)$, then $\mathbf{s}(t)$, with probability one, will eventually be absorbed into $(0, N, 0)$.*
3. *If $\mathbf{s}(0) \in \{(0, N - 2, 2), \dots, (0, 1, N - 1)\}$, then $\mathbf{s}(t)$, with probability one, will eventually be absorbed into $(0, 0, N)$.*
4. *If $\mathbf{s}(0) \in \{(1, N - 1, 0), \dots, (N - 1, 1, 0)\}$, then $\mathbf{s}(t)$, with probability one, will eventually be absorbed into $(0, N, 0)$.*
5. *If $\mathbf{s}(0) \in \mathcal{A}_1 \cup \mathcal{A}_2$ then $\mathbf{s}(t)$, with probability one, will eventually be absorbed into one of the following states:*

$$(0, 0, N), \quad (1, 0, N - 1), \dots, (n^*, 0, N - n^*),$$

where

$$n^* = \text{“greatest integer less than } \frac{a_{11} - a_{22}}{a_{11} - a_{12}}(N - 1)\text{”}.$$

6. *If $\mathbf{s}(0) \in \mathcal{A}_3$ then $\mathbf{s}(t)$, with probability one, will eventually be absorbed into*

$$\text{either } (0, N, 0) \text{ or one of } (0, 0, N), \quad (1, 0, N - 1), \dots, (n^*, 0, N - n^*).$$

Proof. We will prove each part of the proposition separately.

1. The first part follows immediately from Proposition 6.
2. Suppose $\mathbf{s}(0) = (0, N - 1, 1)$. As seen from Proposition 4-2, the only transitions of positive probability are into $(0, N - 1, 1)$ and $(0, N, 0)$, so eventually $\mathbf{s}(t)$ will be absorbed into $(0, N, 0)$.
3. Suppose $\mathbf{s}(0) = (0, s_2, s_3) \in \{(0, N - 2, 2), \dots, (0, 1, N - 1)\}$. As seen from Proposition 4-2, the only transitions of positive probability are into $(0, s_2, s_3)$ and $(0, s_2 - 1, s_3 + 1)$, so eventually $\mathbf{s}(t)$ will be absorbed into $(0, s_2 - 1, s_3 + 1)$. Now the only transitions of positive probability are into $(0, s_2 - 1, s_3 + 1)$ and $(0, s_2 - 2, s_3 + 2)$. Continuing in this manner, eventually $\mathbf{s}(t)$ will be absorbed into $(0, 0, N)$.
4. Suppose $\mathbf{s}(0) = (s_1, s_2, 0) \in \{(1, N - 1, 0), \dots, (N - 1, 1, 0)\}$. Then, from Proposition 4-4, the only transitions of positive probability are into $(s_1, s_2, 0)$ and $(s_1 - 1, s_2 + 1, 0)$; so, by the same reasoning as in the previous case, eventually $\mathbf{s}(t)$ will be absorbed into $(0, N, 0)$.
5. Suppose $\mathbf{s}(0) \in \mathcal{A}_1 \cup \mathcal{A}_2$. From every such $\mathbf{s}(0)$ we can only have west, south or south east transitions. Hence $s_1(1) \leq s_1(0)$, $s_2(1) \leq s_2(0)$, and $\mathbf{s}(1) \in \mathcal{A}_1$. Continuing in this manner we see that $\mathbf{s}(2), \mathbf{s}(3), \dots \in \mathcal{A}_1$ until we reach some $\mathbf{s}(t)$ with (a) either

- $s_1(t) = 0$, in which case the process will eventually be absorbed into $(0, 0, N)$ (b) or $s_2(t) = 0$, in which case $\mathbf{s}(t)$ is an absorbing state. In the latter case, for every increase of s_1 by one, we will have a corresponding decrease of s_2 by one; hence, the maximum increase of s_1 cannot be larger than $s_2(0)$, which is less than the intersection of l_{12} with the s_2 axis, i.e., $\hat{s}_2 = \frac{a_{11}-a_{22}}{a_{11}-a_{12}}(N-1)$. Therefore, at absorption $s_2(t) \leq \frac{a_{11}-a_{22}}{a_{11}-a_{12}}(N-1)$.
6. Suppose $\mathbf{s}(0) \in \mathcal{A}_3$. From every such $\mathbf{s}(0)$, we can have west, northwest and north transitions. Consequently, we will move into some $\mathbf{s}(1)$ which will belong to either $\mathcal{A}_1$ or again to $\mathcal{A}_2$. A sequence of $\mathcal{A}_1$ transitions will eventually lead to absorption into one of $(0, 0, N)$, $(1, 0, N-1)$, ..., $(n^*, 0, N-n^*)$; meanwhile, an $\mathcal{A}_2$ transition will eventually lead to absorption into $(0, N, 0)$.

This completes the proof. $\square$

3.3. Experiments

We now present several experiments on the evolutionary games obtained from the IPD base game with finite population N . We use the PD payoff matrix

$$A = \begin{bmatrix} a & 1 \\ 4 & 2 \end{bmatrix}$$

with $a \in \{3.0, 3.1, \dots, 3.9\}$ being a game parameter. For the IPD game we have used $T = 1000$.

3.3.1. Case $N = 3$ Players

We compute the transition matrix P in the usual manner, and from this we construct the state transition graph (STG), which we plot in Figure 3 for the case $a = 3.0$ (the STG is the same for all $a \in \{3.0, 3.1, \dots, 3.9\}$).

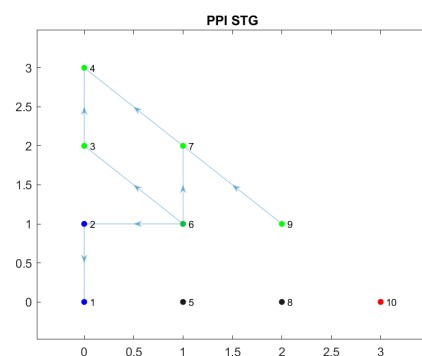


Figure 3. The state transition digraph with $a = 3.0$ and $N = 3$.

Looking at the STG we can summarize state absorptions as follows:

$$\begin{aligned} \text{absorbing states} &: (3, 0, 0), (2, 0, 1), (1, 0, 2), (0, 0, 3), (0, 3, 0) \\ (0, 0, 3) \text{ absorbed states} &: (0, 1, 2) \\ (0, 3, 0) \text{ absorbed states} &: (1, 1, 1), (0, 2, 1), (1, 2, 0), (2, 1, 0) \end{aligned}$$

It appears that, for the case of $N = 3$ players, the All-D strategy has an advantage over the cooperative strategies All-C and TfT, in the sense that a larger number of initial states are eventually absorbed into the pure All-D state $(0, 3, 0)$. The STG remains the same for all used values of $a \in \{3.0, 3.1, \dots, 3.9\}$.

The Markovian analysis results are corroborated by simulation. We have simulated the evolutionary game 100 times; in Figure 4 we plot the evolution of the strategy vector

across generations for starting state $(1, 1, 1)$. As can be seen we have very fast absorption to a pure state, for all starting states. The left panel of Figure 4 shows absorption to TtT and the right one to All-D. Again, results are essentially the same for other a values.

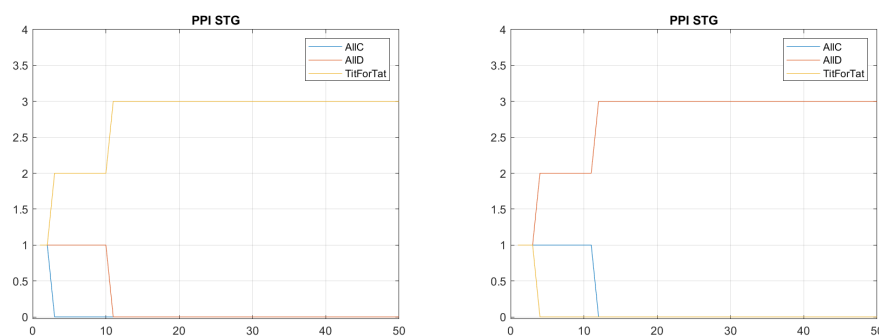


Figure 4. Two simulations of the strategy process of the evolutionary game with $a = 3.0$. Starting values $(1, 1, 1)$.

3.3.2. Case $N = 5$ Players

Now we repeat the analysis of Section 3.3.1 with $N = 5$ players. In this case the state space has 21 elements:

$$S = \{(0, 0, 5), (0, 1, 4), \dots, (5, 0, 0)\}.$$

Again, we compute the transition probability matrix P and the STG. Varying values of a can yield different STGs, as seen in Figure 5. The top left panel corresponds to $a = 3.0$, the top right to $a \in \{3.1, 3.2, 3.3\}$, the bottom left to $a \in \{3.4, 3.5, 3.6, 3.7\}$ and the bottom right to $a \in \{3.8, 3.9\}$. We can see that, as a increases, defection becomes less attractive, and the attraction basin of the pure All-D state $(0, 5, 0)$ shrinks. The Markovian analysis results are again corroborated by simulation.

In Figure 6 we plot the evolution of the strategy vector across generations, starting from state $(1, 2, 2)$. Note that, in the plotted experiments, we have absorption into mixed states, either $(1, 0, 4)$ or $(2, 0, 3)$.

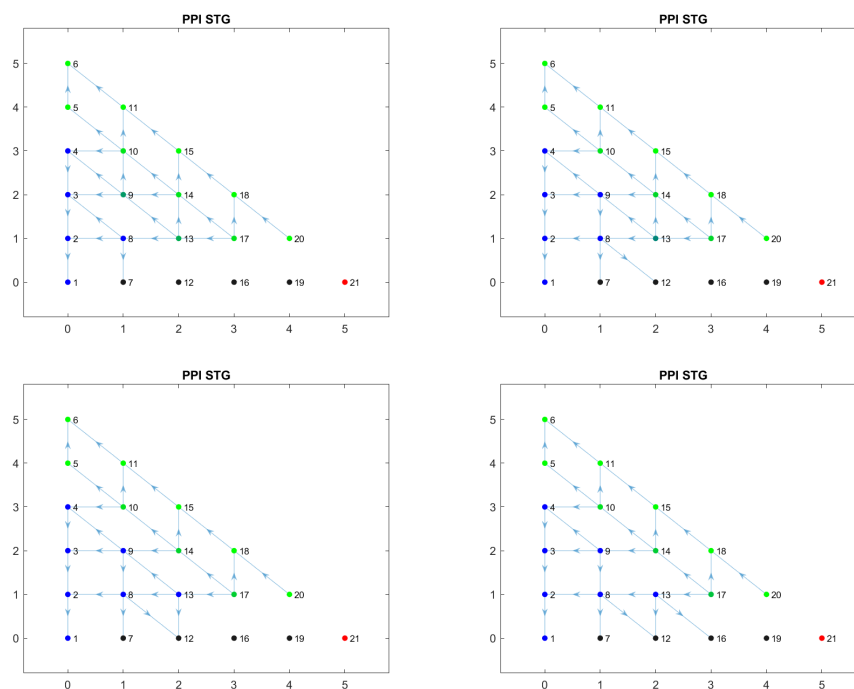


Figure 5. The state transition digraph with $a \in \{3.0, 3.1, 3.4, 3.8\}$ and $N = 5$.

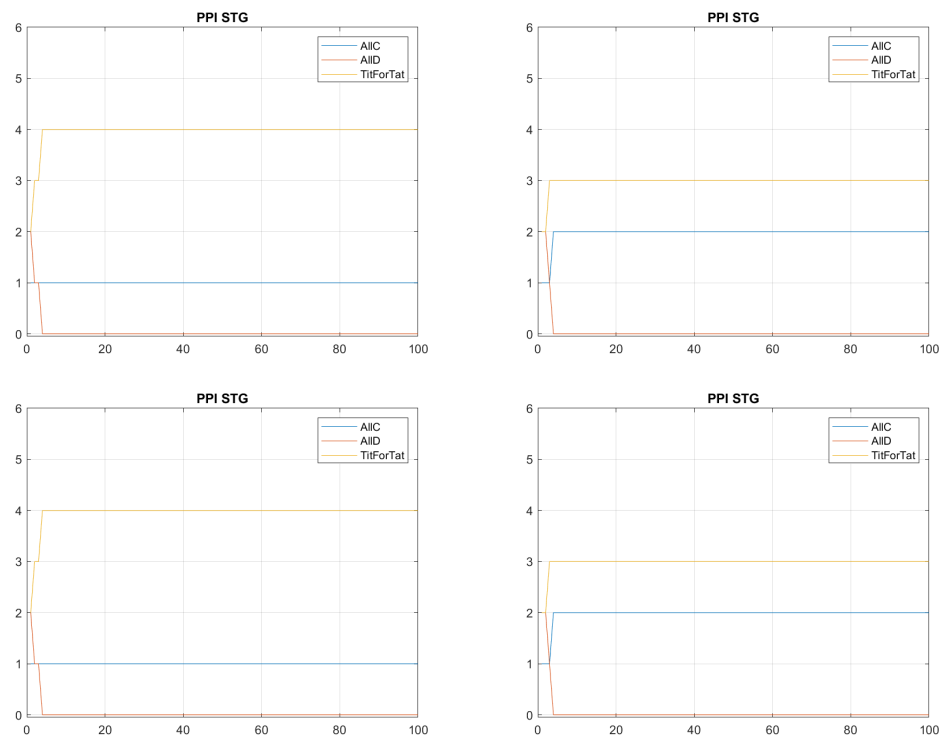


Figure 6. Simulations of the strategy process of the evolutionary game with $a = 3.4$. Starting value (1,2,2).

3.3.3. Cases $N = 10$, $N = 20$, $N = 50$ Players

We repeat our analysis for the cases of $N = 10$, $N = 20$ and $N = 50$ players and plot the corresponding STGs in Figures 7–9. The coloring, especially in Figure 9, indicates clearly the separation into regions of attraction $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$.

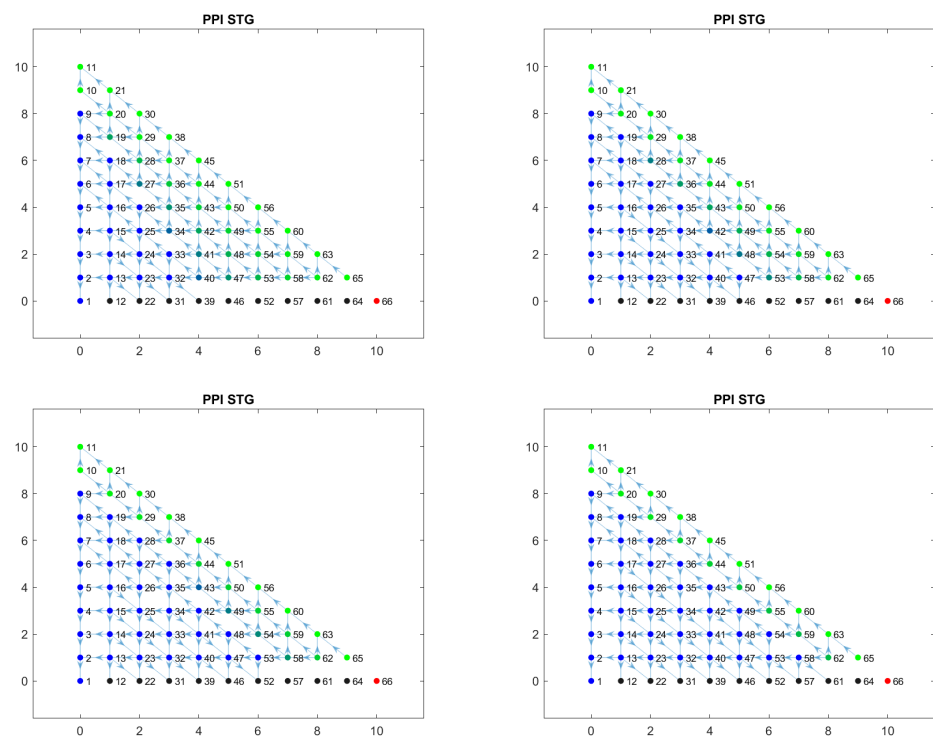


Figure 7. The state transition digraph with $a \in \{3.0, 3.3, 3.6, 3.9\}$ and $N = 10$.

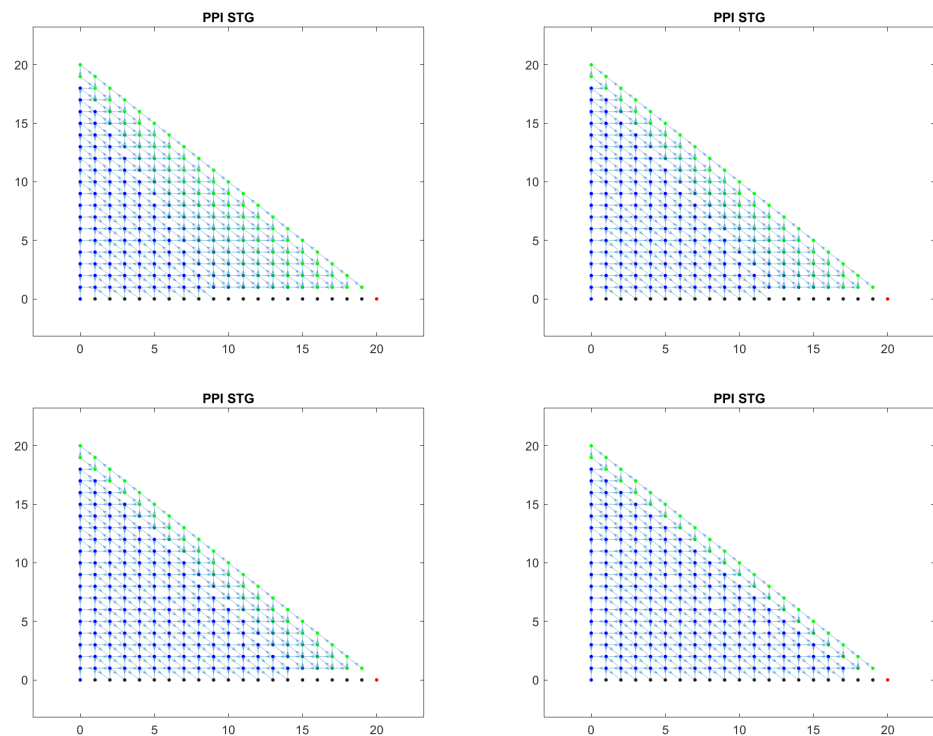


Figure 8. The state transition digraph with $a \in \{3.0, 3.3, 3.6, 3.9\}$ and $N = 20$.

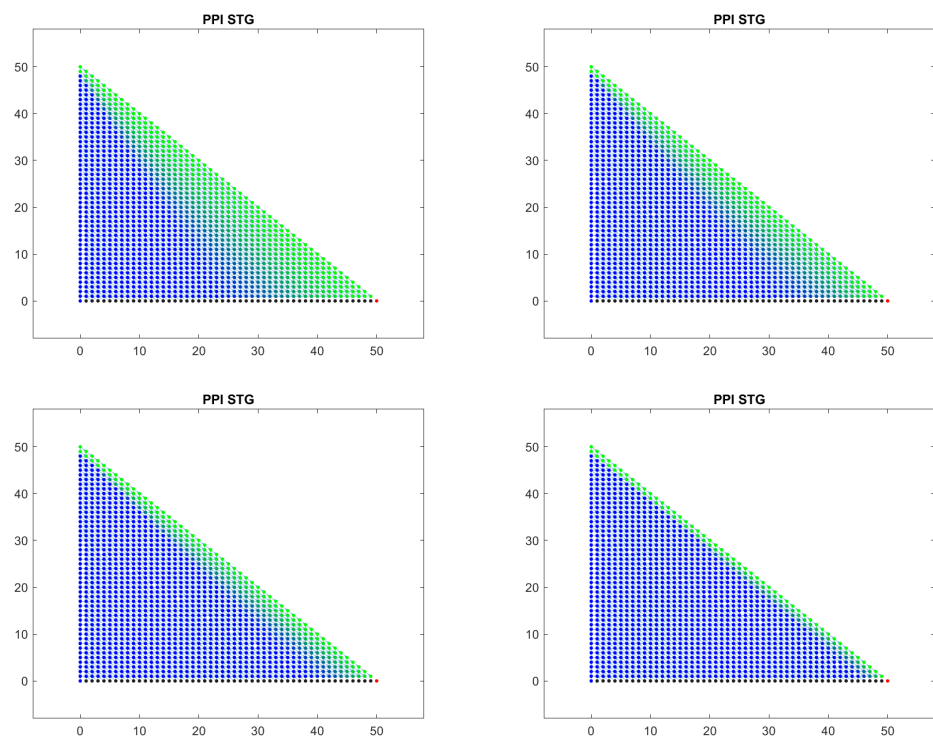


Figure 9. The state transition digraph with $a \in \{3.0, 3.3, 3.6, 3.9\}$ and $N = 50$.

3.4. Discussion

Looking at the STGs for varying a values and for all N values, we see that, as a increases, the attraction basin of the pure All-D noncooperative strategy shrinks. This is reasonable: as $a_{11} = a$ gets closer to $a_{21} = 4$, the temptation to defect decreases, and the tendency to cooperate increases. In addition, we can clearly see that the basins of attraction,

as evidenced by the STGs, conform to the division of the simplex by the lines l_{12} and l_{23} of (10) and (11).

4. Iterated Prisoner's Dilemma: Replicator Dynamics

When the number of players N tends to infinity, the Markovian PPI dynamics can be approximated by the replicator dynamics, which consists of the ODE system

$$\forall m : \frac{dx_m}{dt} = x_m \left((Bx)_m - x^T Bx \right) \quad (15)$$

with B given by (6). Using (15) we can prove the following.

Proposition 7. *In the evolutionary game with A as base game, N players, and T rounds, if*

$$a_{12} < a_{22} < a_{11} < a_{21},$$

N tends to infinity and T is sufficiently large, then the replicator dynamics have the following rest points:

1. *Every point of the set*

$$S_3^{(2)} = \{(1 - x_3, 0, x_3) \text{ with } x_3 \in [0, 1]\}$$

is a rest point. Furthermore:

$$\tilde{S}_3^{(2)} = \left\{ (1 - x_3, 0, x_3) \text{ with } x_3 \in \left(\frac{a_{21} - a_{11}}{a_{21} - a_{22}}, 1 \right] \right\}$$

is a stable set, and every point of the set $S_3^{(2)} \setminus \tilde{S}_3^{(2)}$ is an unstable rest point.

2. *The rest point $(0, 1, 0)$ is stable.*
3. *The rest point $\left(\frac{-a_{12} + a_{22}}{a_{11} + a_{22} - a_{12} - a_{21}}, \frac{-a_{21} + a_{11}}{a_{11} + a_{22} - a_{12} - a_{21}}, 0 \right)$ is unstable.*

Proof. We find the rest points by solving

$$\forall m : x_m \left((Bx)_m - x^T Bx \right) = 0.$$

The solutions are

$$\begin{array}{lll} x_1 = 1 - x_3, & x_2 = 0, & x_3 = x_3 \\ x_2 = 1, & x_1 = 0, & x_3 = 0 \\ x_1 = 0, & x_2 = \frac{Ta_{11} - a_{21} - Ta_{22} + a_{22}}{Ta_{11} - Ta_{22} + 2a_{22} - a_{21} - a_{12}}, & x_3 = \frac{-a_{12} + a_{22}}{Ta_{11} - Ta_{22} + 2a_{22} - a_{21} - a_{12}} \\ x_1 = \frac{-a_{12} + a_{22}}{a_{11} + a_{22} - a_{12} - a_{21}}, & x_2 = \frac{-a_{21} + a_{11}}{a_{11} + a_{22} - a_{12} - a_{21}}, & x_3 = 0 \end{array}$$

The first solution is admissible for all $x_3 \in [0, 1]$; the second solution is admissible. Regarding the third solution, for large enough T , we have

$$x_3 = \frac{-a_{12} + a_{22}}{Ta_{11} - Ta_{22} + 2a_{22} - a_{21} - a_{12}} \simeq \frac{a_{22} - a_{12}}{T(a_{11} - a_{22})} \in [0, 1] \Rightarrow x_1 = 1 - x_3 \in [0, 1];$$

hence, the third solution is admissible. Regarding the fourth solution, we have

$$a_{11} - a_{21} < 0 \Rightarrow \frac{-a_{12} + a_{22}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{a_{22} - a_{12}}{a_{22} - a_{12} + a_{11} - a_{21}} > 1;$$

hence, the fourth solution is inadmissible.

Therefore, we have one family of admissible rest points and two isolated ones. To examine the stability of these, let $J(x_1, x_2, x_3)$ be the Jacobian matrix of the system, and let us evaluate the eigenvalues of $J(x_1, x_2, x_3)$ at each rest point (x_1, x_2, x_3) .

1. For the rest point family $(1 - x_3, 0, x_3)$, the eigenvalues turn out to be

$$\lambda_1 = -Ta_{11}, \quad \lambda_2 = -Ta_{21}x_3 + Ta_{22}x_3 - Ta_{11} + Ta_{21} + a_{21}x_3 - a_{22}x_3, \quad \lambda_3 = 0.$$

- (a) Clearly, $\lambda_1 < 0$.
- (b) For large enough T , the sign of λ_2 is determined by the terms linear in T , i.e., by

$$-T(x_3a_{21} - a_{22}x_3 + a_{11} - a_{21})$$

For this to be negative, we must have

$$x_3(a_{21} - a_{22}) + a_{11} - a_{21} > 0$$

Since $a_{21} > a_{22}$, the above holds

$$\text{for all } x_3 \in \left(\frac{a_{21} - a_{11}}{a_{21} - a_{22}}, 1 \right].$$

Hence, for $x_3 \notin \left(\frac{a_{21} - a_{11}}{a_{21} - a_{22}}, 1 \right]$, the rest point is unstable.

- (c) Since $\lambda_3 = 0$, to determine the stability for $x_3 \in \left(\frac{a_{21} - a_{11}}{a_{21} - a_{22}}, 1 \right]$, we will consider perturbations from $(1 - x_3, 0, x_3)$. Substituting $x_1 = 1 - x_2 - x_3$ we get the equations

$$\begin{aligned} \frac{dx_2}{dt} &= \hat{F}_2(x_2, x_3) \\ \frac{dx_3}{dt} &= \hat{F}_3(x_2, x_3) \end{aligned}$$

Now suppose $x_2 > 0$ and small. Expanding $\hat{F}_2(x_2, x_3)$ around $(0, x_3)$, we get

$$\begin{aligned} \frac{dx_2}{dt} &= \hat{F}_2(x_2, x_3) = (((a_{22} - a_{21})x_3 - a_{11} + a_{21})T - a_{22}x_3)x_2 + O(x_2^2) \\ \frac{dx_3}{dt} &= \hat{F}_3(x_2, x_3) \end{aligned}$$

For large T , the sign of $\frac{dx_2}{dt}$ will be the same as that of

$$(((a_{22} - a_{21})x_3 - a_{11} + a_{21})T - a_{22}x_3)x_2 \simeq ((a_{22} - a_{21})x_3 - a_{11} + a_{21})Tx_2$$

which is the same as that of $(a_{22} - a_{21})x_3 - a_{11} + a_{21}$. If we have

$$(a_{22} - a_{21})x_3 - a_{11} + a_{21} < 0 \Leftrightarrow (a_{21} - a_{22})x_3 - a_{21} + a_{11} > 0 \Leftrightarrow x_3 > \frac{a_{21} - a_{11}}{a_{21} - a_{22}},$$

then $\frac{dx_2}{dt} < 0$, and x_2 is decreasing towards zero. Hence, if $x_3 > \frac{a_{21} - a_{11}}{a_{21} - a_{22}}$ and the system is perturbed from $(1 - x_3, 0, x_3)$, then it will return to some $(1 - x'_3, 0, x'_3)$. Therefore,

$$\tilde{S}_3^{(2)} = \left\{ (1 - x_3, 0, x_3) : \text{with } x_3 > \frac{a_{21} - a_{11}}{a_{21} - a_{22}} \right\}$$

is a stable set. On the other hand, clearly no point of $\tilde{S}_3^{(2)}$ can be a stable rest point; a perturbation from $(1 - x_3, 0, x_3)$ to some $(1 - x'_3, 0, x'_3)$ will stay at $(1 - x'_3, 0, x'_3)$, since this is also a rest point.

In short, $\tilde{S}_3^{(2)}$ is a stable set.

2. For the rest point $(0, 1, 0)$, the eigenvalues turn out to be

$$\lambda_1 = a_{12} - a_{22} < 0, \quad \lambda_2 = -Ta_{22} < 0, \quad \lambda_3 = T(a_{12} - a_{22}) < 0.$$

Hence, the rest point is stable.

3. For the rest point $(0, \frac{Ta_{11}-a_{21}-Ta_{22}+a_{22}}{Ta_{11}-Ta_{22}+2a_{22}-a_{21}-a_{12}}, \frac{-a_{12}+a_{22}}{Ta_{11}-Ta_{22}+2a_{22}-a_{21}-a_{12}})$, it turns out that the first eigenvalue is

$$\lambda_1 = -\frac{Ta_{11}a_{12} - Ta_{11}a_{22} - Ta_{12}a_{22} + Ta_{22}^2 - a_{12}a_{21} + a_{12}a_{22} + a_{21}a_{22} - a_{22}^2}{Ta_{11} - Ta_{22} - a_{12} - a_{21} + 2a_{22}}$$

For large enough T , the sign of the denominator is determined by $T(a_{11} - a_{22}) > 0$. The sign of the numerator is determined by

$$Ta_{11}a_{12} - Ta_{11}a_{22} - Ta_{12}a_{22} + Ta_{22}^2 = -T(a_{22} - a_{12})(a_{11} - a_{22}) < 0$$

Hence, $\lambda_1 > 0$, and the rest point is unstable.

This completes the proof. $\square$

The rest points and their stability can be seen in the following Figure 10 for the A matrices

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 3.5 & 1 \\ 4 & 2 \end{bmatrix}.$$

Figure 10 has a remarkable similarity to the STGs of Section 3. Compare, for example, with the upper left panels of Figures 7–9. The stable state $(0, 1, 0)$ of Figure 10 corresponds to the absorbing pure state $(0, N, 0)$ of Figures 7–9. Similarly, the stable set $(x_1, 0, 1 - x_1)$ of Figure 10 corresponds to the set of absorbing states $(n, 0, N - n)$ of Figures 7–9. Note, in particular, that a perturbation from a state $(n, 0, N - n)$ will not return to $(n, 0, N - n)$ but to another state $(n', 0, N - n')$.

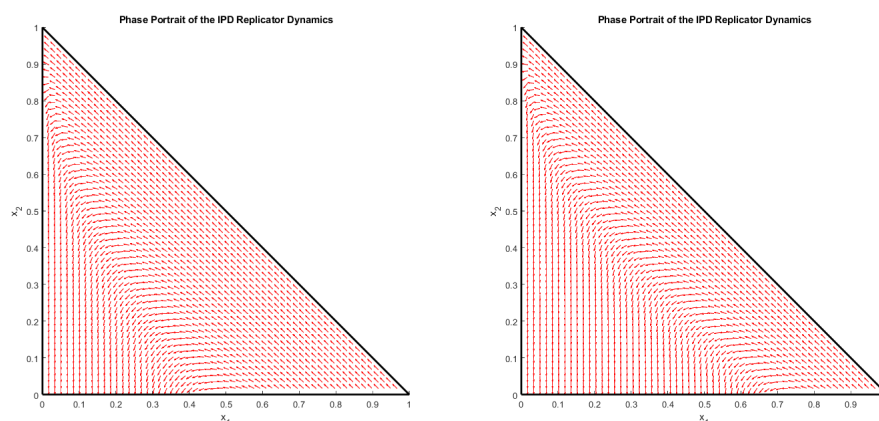


Figure 10. The phase portrait of the replicator dynamics for two different A matrices.

5. Conclusions

Using Markov chain analysis, we have studied evolutionary games with a finite number of players. We have performed this reduction for the evolutionary game using the

Iterated Prisoner's Dilemma as base game, concentrating on a finite strategy set consisting of a purely cooperative strategy (All-C), purely defective one (All-D), and a conditionally cooperative one (TfT). For this case we have determined the absorbing states and (partly) the basins of attraction. Our Markovian analysis is in agreement with the large population analysis by replicator dynamics.

The distinction between stochastic *finite-population* models and deterministic *infinite-population* models (such as replicator dynamics) is fundamental in evolutionary game theory. Markov chain models, such as the one presented in the current paper, provide a more realistic description of finite populations and allow direct computation of fixation and extinction probabilities. In addition, continuous dynamics predicts only the average deterministic direction of selection and therefore neglects the effects of drift. In short, continuous dynamics describes the *average* behavior of an idealized infinite population, whereas discrete Markov chain models describe the *full stochastic dynamics* of finite populations, including the possibility of fixation, extinction, and random evolutionary outcomes.

Let us conclude by proposing some future research directions.

1. Our model has ample scope for incorporating different strategy update mechanisms. For example, we could use different imitation protocols (Fermi-type protocols, best response, etc.). It is easy to incorporate these into our model, requiring only a small alteration in the computation of the state transition probabilities. The same approach can be used to incorporate the multi-type Moran process. The Wright–Fisher process, on the other hand, would result in a more complicated transition probability matrix because at every time step a state s can transit to any other state s' .
2. As already mentioned, randomness is introduced in our model only through the selection of the strategy-updating player. Another source of randomness considered in the literature is the “faulty execution” of strategies. For example, an “All-C” player may, on every turn, defect with a small probability. It is interesting and feasible to extend our Markovian model to handle such cases.

We intend to pursue these variants in future publications.

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Conflicts of Interest: The author declares no conflicts of interest.

Notes

- ¹ Let us note that, while the term “evolutionary games” is widely used in the literature, they are *not* games in the game theoretic sense, since their evolution is not governed by the players’ strategic considerations. Instead, an evolutionary game is a stochastic dynamical system.
- ² The expression “w.p.” is shorthand for “with probability”.
- ³ It is actually a *digraph* (directed graph), but we will always call it a graph for brevity.

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