

# Apollonius and the Hyperbolic Circle

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## Abstract

Given a positive number  $n > 1$  and two distinct planar focal points  $A$  and  $B$ , the locus of all points  $P$  such that  $|AP| = n|BP|$  is a circle of Apollonius of index  $n$ . Under what conditions is  $B$  the center of a hyperbolic circle in the Poincaré disk? In particular, for any circle  $\mathcal{E}$  in  $\mathbb{C}$  lying within the unit circle, where the Euclidean center  $u$  of  $\mathcal{E}$  is other than  $O$  and its Euclidean radius is  $\epsilon$ ,  $0 < \epsilon < 1$ , there exists another circle  $\mathcal{D}$  and a point  $B$  in  $\mathcal{E}$  where the center of  $\mathcal{D}$  is  $O$  and its Euclidean radius is  $\delta$ ,  $0 < \delta < 1$ , for which  $\mathcal{D}$  and  $\mathcal{E}$  are hyperbolic translates of one another, as are  $O$  and  $B$ ; by rotation symmetry of  $\mathbb{C}$  about  $O$ , we may take  $A$ ,  $B$ , and  $u$  as real numbers with  $v = B$ , and  $u + \epsilon n = A$ , with  $0 < u < v < 1$ , and we conclude that  $n\delta v = 1$ . That is,  $B$  is the hyperbolic center of  $\mathcal{E}$  as well as the inner Apollonian focus of  $\mathcal{E}$  whose Apollonian index is  $n$ , and  $A = u + \epsilon n$  is the outer focus of  $\mathcal{E}$ . Furthermore, if  $P$  and  $Q$  are distinct points on such a circle of Apollonius with index  $n$  we have the cross-ratio of the four points (not necessarily collinear) for which  $\frac{|AP||BQ|}{|AQ||BP|} = 1$ ; with  $X$  being an indeterminate point, this identity in turn means that  $\frac{|AP||BX|}{|AQ||BX|} = \frac{|AP||BQ|}{|AQ||BP|}$  is the equation of a circle, demonstrating why this old Apollonian algorithm forms the core idea for a hyperbolic metric.

**Keywords:** circle of Apollonius; Poincaré disk; hyperbolic geometry; Möbius transformation; cross-ratio; ideal points; ordinary points; focal points; Apollonius index

**MSC:** 51M09

## 1. Introduction

An old algorithm of Apollonius (c. 240–190 BCE) essentially characterizes what are now called *hyperbolic circles* within the Poincaré disk and provides the core idea for the hyperbolic metric. We rephrase this *circle of Apollonius* algorithm from his lost work *Plane Loci*, fortunately preserved by Pappus (c. 290–350 CE), ([1], pp. 540–543, locus 2 algorithm) as

Given a number  $n > 1$  and two planar focal points  $A$  and  $B$ , the locus of all points  $P$  in  $\mathbb{R}^2$  with  $|PA| = n|PB|$  is a circle  $\mathcal{E}$ , called the *circle of Apollonius with index  $n$* .

We say that  $A$  is the *outer* focus and  $B$  is the *inner* focus, since  $B$  lies within  $\mathcal{E}$  and  $A$  lies outside  $\mathcal{E}$ .

As a graphic that may induce continued reading, Figure 1a depicts ten circles of Apollonius with index  $n = 4$ , along with each inner focal point  $v \equiv B$ , which doubles as the hyperbolic center of that circle. For index  $n = 4$ , the permissible positive real values for  $v$ ,  $\frac{1}{n} < v < 1$ , appear as dots along the  $x$ -axis within the dashed unit-circle. Figure 1b depicts this infinite family of index 4 circles as a continuum in 3D, where the artificial third coordinate is  $v$ ; thus, when  $v$  is slightly larger than  $\frac{1}{4}$ , the circles have Euclidean radii slightly less than 1, and when  $v$  is slightly less than 1, the circles have Euclidean radii slightly more than 0.



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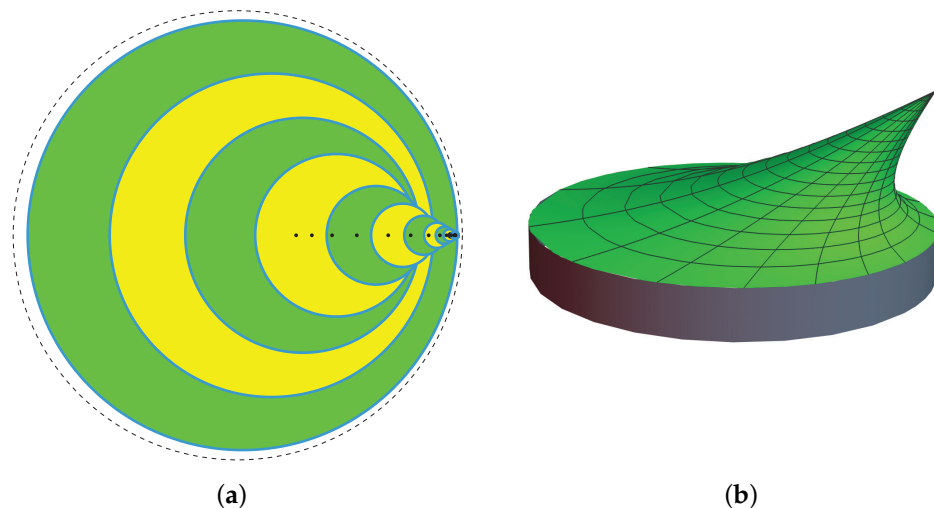
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**Figure 1.** The family of Apollonian circles with index  $n = 4$  within the hyperbolic Poincaré unit disk in  $\mathbb{C}$ , with positive real hyperbolic centers  $v$ ,  $\frac{1}{n} < v < 1$ . **(a)** Several members within the unit disk. The dots, from left to right, are, respectively, the centers of the circles, from left to right. **(b)** The family in 3D, with vertical component  $v$ .

To generalize this example, we imagine a recreational mathematical puzzle: Upon randomly selecting a circle  $\mathcal{E}$  within the Poincaré disk so that its Euclidean center lies along the positive  $x$ -axis, where is its hyperbolic center, and—upon viewing  $\mathcal{E}$  as a circle of Apollonius—what is its index?

In what follows we give a solution to the puzzle, and, along the way, we'll see why this Apollonian algorithm suggests the core idea for the hyperbolic metric between any two ordinary points  $A$  and  $B$  lying within the Poincaré disk, namely,

$$d(A, B) = \left| \ln \frac{|AP| |BQ|}{|AQ| |BP|} \right|, \quad (1)$$

where  $P$  and  $Q$  are *ideal points*, points on the unit circle, which are also the endpoints of the hyperbolic line  $AB$ , and where  $|XY|$  is the Euclidean distance between points  $X$  and  $Y$ . We end with an electrostatics application due to Lord Kelvin (1824–1907). For completeness, we first prove Apollonius's algorithm, and establish useful notation.

We suspect that much of the material in this paper may be found somewhere here and there in the literature. What may be new, however, is Lemma 6; to rephrase it briefly, suppose that  $\mathcal{D}$  and  $\mathcal{E}$  are circles in the unit disk of  $\mathbb{C}$ , where  $O$  is the center of  $\mathcal{D}$  with Euclidean radius  $\delta > 0$ , and  $\mathcal{E}$  is a non-trivial hyperbolic translate of  $\mathcal{D}$  whose Euclidean radius is  $\epsilon > 0$  and Euclidean center is  $u$ . Then  $n\delta v = 1$ , where  $n$  is the Apollonian index of  $\mathcal{E}$ ,  $v$  is both the hyperbolic center of  $\mathcal{E}$  and the inner Apollonian focus of  $\mathcal{E}$ , and  $u + \epsilon n$  is its outer Apollonian focus.

## 2. The Circle of Apollonius

To prove that Apollonius's algorithm generates a circle, we need two lemmas. Proofs for these lemmas and theorem may be found in college geometry texts such as ([2], pp. 158–164).

**Lemma 1** (Proportionate triangle division). *For any triangle, its internal and external angle bisectors, respectively, partition the opposite side, internally and externally, into segments proportional to the adjacent sides of the angle (provided, in the latter case, when the angle being externally bisected fails to be the non-base angle of an isosceles triangle). Furthermore, the angle formed by the internal and external bisectors at a vertex is a right angle.*

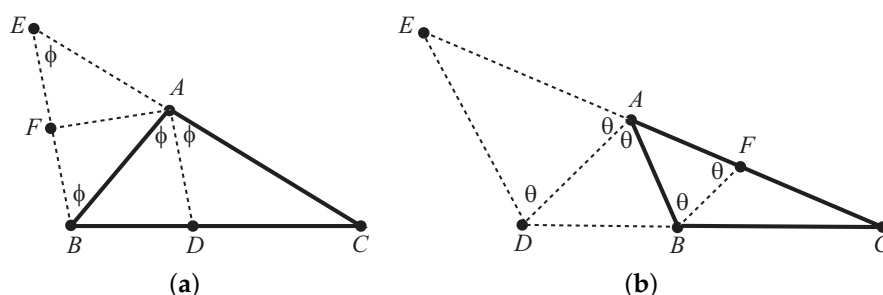
**Proof.** Consider the internal case. In  $\triangle ABC$ , let  $AD$  be a segment bisecting  $\angle A$  internally with  $D$  on side  $BC$  as shown in Figure 2a. Construct a line  $\mathcal{L}$  through  $B$  parallel to  $AD$  and extend side  $AC$  to intersect  $\mathcal{L}$  at  $E$ . Let  $\phi$  be the measure of  $\angle BAD$ . The measures of  $\angle ABE$  and  $\angle BAE$  are, respectively,  $\phi$  and  $\pi - 2\phi$ , forcing  $\angle AEB$  to have measure  $\phi$ , which means that  $\triangle ABE$  is isosceles. Observe that

$$\frac{|BD|}{|CD|} = \frac{|AE|}{|AC|} = \frac{|AB|}{|AC|}. \quad (2)$$

For the external case, in  $\triangle ABC$ , let  $AD$  be a segment bisecting  $\angle A$  (whose external angle measure is  $2\theta$ ) externally with  $D$  on side  $BC$  as shown in Figure 2b. Construct a line through  $B$  parallel to  $AD$ , intersecting  $AC$  internally at  $F$ , making  $\triangle ABF$  isosceles. Observe that

$$\frac{|AB|}{|AC|} = \frac{|AF|}{|AC|} = \frac{|DB|}{|DC|}.$$

Finally (when bisecting  $\angle A$  for a given  $\triangle ABC$ ), observe that  $2\theta + 2\phi = \pi$ , which means that  $\theta + \phi = \frac{\pi}{2}$ .  $\square$



**Figure 2.** Internal and external angle bisectors at  $\angle A$ . (a) Internal bisector. (b) External bisector.

**Lemma 2** (The converse of Lemma 1). For  $\triangle ABC$ , let point  $D$  on side  $BC$  partition the side internally or externally, respectively, so that the ratio of  $\triangle ABC$  side lengths  $AB$  and  $AC$  is the same, respectively, as the ratio of side lengths  $BD$  and  $DC$ , internally or externally. Then  $AD$  bisects  $\angle A$ .

**Proof.** Consider the internal partition case. Using the notation of Figure 3a, apply the law of sines, and observe that

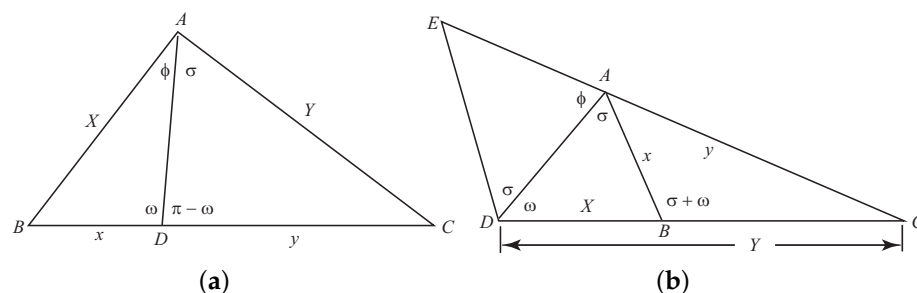
$$\frac{\sin \phi}{x} = \frac{\sin \omega}{X} \quad \text{and} \quad \frac{\sin \sigma}{y} = \frac{\sin(\pi - \omega)}{Y} = \frac{\sin \omega}{Y},$$

forcing  $\sin \phi = \sin \sigma$ . Since the measure of  $\angle BAC$  is less than  $\pi$ , we have  $\phi = \sigma$ .

Similarly, using the notation of Figure 3b, we have

$$\frac{\sin \omega}{x} = \frac{\sin \sigma}{X} \quad \text{and} \quad \frac{\sin \omega}{y} = \frac{\sin(\pi - \phi)}{Y} = \frac{\sin \phi}{Y},$$

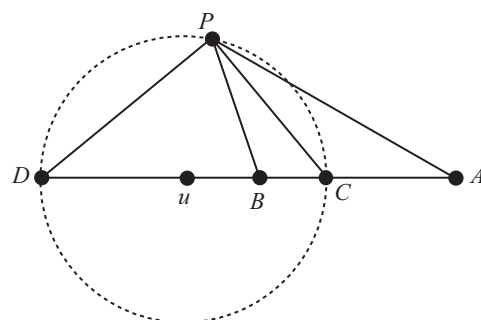
which, again, forces  $\phi = \sigma$ .  $\square$



**Figure 3.** The law of sines, showing that  $\phi = \sigma$ . (a) Internal partition. (b) External partition.

**Theorem 1** (The circle of Apollonius). *Given two focal points  $A$  and  $B$  and index  $n > 1$ , the circle of Apollonius is the locus of all points  $P$  for which the ratio of the distances from  $P$  to  $A$  and from  $P$  to  $B$  is  $n$ .*

**Proof.** Let  $\Delta = |AB|$ . For the moment, interpret  $A$  and  $B$  as real numbers with  $A > B$ . The equation  $(C - B) + n(C - B) = \Delta$ , gives the locus point  $C = B + \frac{\Delta}{n+1}$ . With  $\tau > 0$ , solving  $A - n\tau = B - \tau$  gives  $\tau = \frac{\Delta}{n-1}$  and the locus point  $D = B - \frac{\Delta}{n-1}$ . Figure 4 shows these four points arranged in the order  $D, B, C, A$ . Let  $P$  be a locus point not on line  $AB$ . By Lemma 2,  $PC$  and  $PD$  are, respectively, internal and external bisectors of  $\angle APB$ ; and by Lemma 1,  $\angle CPD$  is a right angle, which means that the set of all locus points forms a circle whose center  $u$  is the Euclidean midpoint of diameter  $CD$ , where  $u = B - \frac{\Delta}{n^2 - 1}$ .  $\square$



**Figure 4.** The circle of Apollonius when  $\frac{|AP|}{|BP|} = 2$ .

**Corollary 1** (Radius of the circle of Apollonius). *Let  $\mathcal{E}$  be the circle of Apollonius with radius  $\epsilon > 0$  corresponding to points  $A$  and  $B$  and index  $n > 1$ . With  $\Delta = |AB|$  and  $u$  as the Euclidean center of  $\mathcal{E}$ ,  $a = |uA|$ , and  $b = |uB|$ , we have*

$$ab = \epsilon^2, \quad \frac{a}{\epsilon} = n = \frac{\epsilon}{b}, \quad \epsilon = \frac{n\Delta}{n^2 - 1}, \quad a = \frac{n^2\Delta}{n^2 - 1}, \quad b = \frac{\Delta}{n^2 - 1}, \quad \text{and} \quad b < \epsilon < a. \quad (3)$$

**Proof.** With respect to the notation of Theorem 1, the equality

$$\frac{|AC|}{|BC|} = \frac{a - \epsilon}{\epsilon - b} = n = \frac{a + \epsilon}{b + \epsilon} = \frac{|AD|}{|CD|}$$

simplifies to  $ab = \epsilon^2$ . Observe also that  $n = \frac{a + \epsilon}{\frac{\epsilon^2}{a} + \epsilon}$  simplifies to  $n = \frac{a}{\epsilon}$ . Again, with respect to the notation of Theorem 1, we have  $C = B + \frac{\Delta}{n+1}$  and  $D = B - \frac{\Delta}{n-1}$ , which means that

$$2\epsilon = C - D = \frac{\Delta}{n+1} + \frac{\Delta}{n-1} = \frac{2n\Delta}{n^2 - 1},$$

which gives  $\epsilon = \frac{n\Delta}{n^2-1}$ ,  $a = n\epsilon = \frac{n^2\Delta}{n^2-1}$ , and  $b = \frac{\epsilon}{n} = \frac{\Delta}{n^2-1}$ . Finally, the mean of two proportionals or the geometric mean,  $\frac{a}{\epsilon} = \frac{\epsilon}{b}$ —which was first made popular, if not discovered, by Archytas (c. 428–345 BCE) ([3], p. 63)—yields the inequality  $0 < b < \epsilon < a$ .  $\square$

For example, with  $\Delta = |AB| = \frac{5}{6}$  and  $n = \frac{9}{2}$ , Equation (3) gives  $\epsilon = \frac{15}{77}$ ,  $b = \frac{10}{231}$ , and  $a = \frac{135}{154}$ . In anticipation of our advertised hyperbolic connection, it turns out that this circle  $\mathcal{E}$  corresponding to these values is the hyperbolic translate of a circle  $\mathcal{D}$  with Euclidean center  $O$  and Euclidean radius  $\delta = \frac{1}{3}$  so that the image of  $O$  is the point  $B = (\frac{2}{3}, 0) \equiv \frac{2}{3} \in \mathbb{C}$ . This point  $B$ , which we also refer to as  $v$ , is the hyperbolic center of  $\mathcal{E}$ , whereas the Euclidean center of  $\mathcal{E}$ , which we refer to as  $u$ , is  $u = \frac{48}{77}$ . As we will see, in Theorem 2, this relationship generalizes. See Figure 5; as an aid to interpret the diagram, we summarize the definitions of its variables.

- $\mathcal{D}$  and  $\delta$ :  $\mathcal{D}$  is a circle with Euclidean center  $O$  and Euclidean radius  $\delta$ ,  $0 < \delta < 1$ .
- $\mathcal{E}$ ,  $\epsilon$ ,  $u$ , and  $v$ :  $\mathcal{E}$  is a circle with Euclidean radius  $\epsilon$ , Euclidean center  $u$ , and hyperbolic center  $v$ , where  $0 < u < v < 1$  and  $0 < \epsilon < 1$ .
- $A$ ,  $B$ ,  $C$ ,  $D$ , and  $\Delta$ :  $\mathcal{E}$  is a circle of Apollonius with outer and inner Apollonian foci, respectively,  $A$  and  $B$ ;  $A > B$ ;  $C = u + \epsilon$ ;  $D = u - \epsilon$ ;  $\Delta = A - B$ ;  $B = v$ ;  $A = B + \Delta$ .
- $a$  and  $b$ :  $b = v - u$  and  $a = b + \Delta$ .

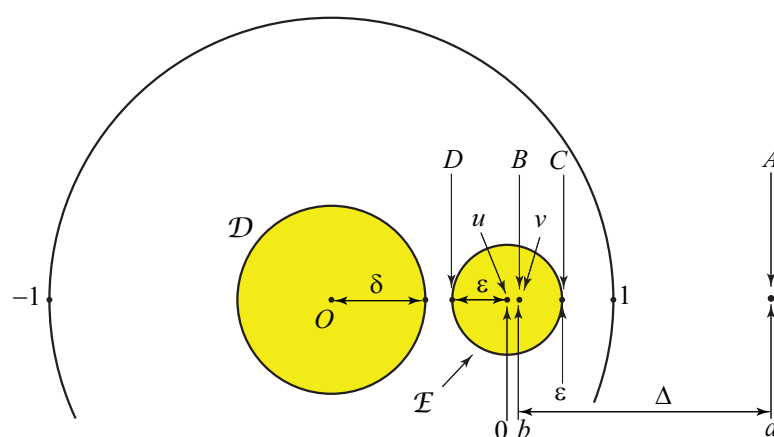


Figure 5. The circle of Apollonius with values  $\Delta = \frac{5}{6}$  and  $n = \frac{9}{2}$ .

### 3. The Apollonius Circle Equation

Although Apollonius lived well before algebraic equations, given focal points  $A$  and  $B$  and index  $n > 1$ , we can interpret the expression  $\frac{|AP|}{|BP|} = n = \frac{|AQ|}{|BQ|}$  as the equation of the circle of Apollonius. Simplifying it, as well as using functional notation  $F$  for the quotient, yields

$$F(A, B, P, Q) = \frac{|AP||BQ|}{|AQ||BP|} = 1, \quad (4)$$

which we may interpret as an equation of a circle that is the locus of all points  $Q$  given any points  $A$  and  $B$  and a specific point  $P$  on the circle. When the points  $A$ ,  $B$ ,  $P$ , and  $Q$  of the quotient expression of Equation (4) are collinear, it is known as the *cross-ratio* of the four points, which Pappus ascribed to Euclid (In his own *Book 7* which is his edited compilation of some of the lost works of Euclid, Apollonius, and Aristarchus, Pappus introduced Euclid's *Porisms* as “a very clever collection for the analysis of ... weighty problems; ... and Euclid put the proof of each that is most suggestive” ([1], p. 94)). Now we can use it to represent points on a circle.

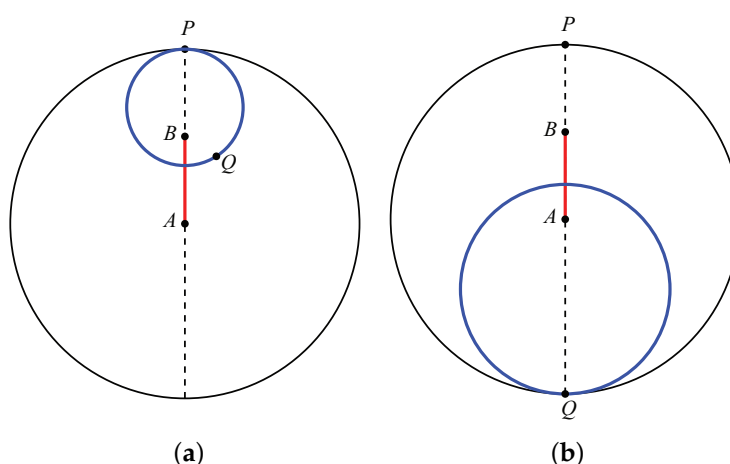
For example, Figure 6a displays the Apollonian circle with focal points  $A = (0, 0)$ ,  $B = (0, \frac{1}{2})$ , and with  $P = (0, 1)$  and  $Q = X = (x, y)$ . Note that its Apollonian equation, Equation (4),

$$1 = \frac{|AP||BX|}{|AX||BP|} = F(A, B, P, X) = \frac{\left(1\right)\left(\sqrt{x^2 + (y - \frac{1}{2})^2}\right)}{\left(\sqrt{x^2 + y^2}\right)\left(\frac{1}{2}\right)},$$

simplifies to  $x^2 + (y - \frac{2}{3})^2 = (\frac{1}{3})^2$ . To generalize this phenomenon, we have

$$F(A, B, P, X) = \frac{(1-a)\sqrt{x^2 + (y-b)^2}}{(1-b)\sqrt{x^2 + (y-a)^2}}, \quad (5)$$

where  $A = (0, a)$ ,  $B = (0, b)$ ,  $P = (0, 1)$ ,  $|a| < 1$ , and  $|b| < 1$ . When  $a < b$ , observe that  $F(a, b, x, y) = 1$  is the Apollonian circle with index  $n = \frac{|AP|}{|BP|} = \frac{1-a}{1-b} > 1$ .



**Figure 6.** An Apollonian circle and its dual. (a) The Apollonian circle through  $P$ . (b) Its dual circle through  $Q = (0, -1)$ .

**Lemma 3** (A standard Apollonian form). *The Cartesian coordinate equation equivalent to the Apollonian equation  $F(A, B, P, X) = 1$  where  $A = (0, a)$ ,  $B = (0, b)$ ,  $P = (0, 1)$ , and  $X = (x, y)$  is*

$$x^2 + \left(y - \frac{1-ab}{2-a-b}\right)^2 = \left(\frac{(1-a)(1-b)}{2-a-b}\right)^2$$

**Proof.** Simplify

$$(1-a)\sqrt{x^2 + (y-b)^2} = (1-b)\sqrt{x^2 + (y-a)^2}$$

and complete the square.  $\square$

We define the conic  $F(A, B, P, X) = F(A, B, P, Q)$  where  $Q = (0, -1)$  to be the *dual* of the Apollonian circle  $F(A, B, P, X) = 1$  of Lemma 3. This conic happens to be a circle lying within the unit disk. For example, Figure 6b displays the dual  $F\left((0, 0), (0, \frac{1}{2}), (0, 1), (x, y)\right) = F\left((0, 0), (0, \frac{1}{2}), (0, 1), (0, -1)\right) = 3$  for the Apollonian circle of Figure 6a. Its center and radius are respectively  $(0, -\frac{2}{5})$  and  $\frac{3}{5}$ . Note that every point on this dual circle satisfies its equation, not just point  $(0, -1)$ . As the reader might have anticipated, this dual circle is the Apollonian where  $A$  is the inner Apollonian focus,  $B$  is the outer Apollonian focus, and the Apollonian index is the reciprocal

of the Apollonian circle  $F(A, B, P, X) = 1$ . That is, the dual circle is the Apollonian circle  $F(A, B, Q, X) = 1$ . In general, we have the next lemma.

**Lemma 4** (Apollonian dual circles). *The Cartesian coordinate equation for the dual circle to the Apollonian circle with equation  $F(A, B, P, X) = 1$  is  $F(A, B, Q, X) = 1$  and is equivalent to*

$$x^2 + \left( y + \frac{1 - ab}{2 + a + b} \right)^2 = \left( \frac{(1 + a)(1 + b)}{2 + a + b} \right)^2,$$

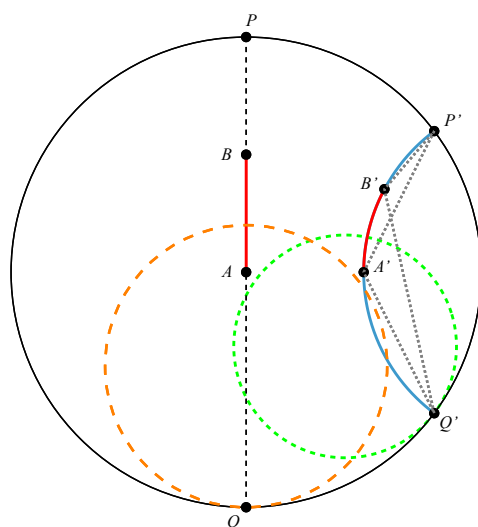
where  $P = (0, 1)$  and  $Q = (0, -1)$ .

**Proof.** Observe that  $F(A, B, P, X) = F(A, B, P, Q)$  simplifies to  $F(A, B, Q, X) = 1$ , which upon further simplification and completing the square gives the desired result.  $\square$

Because the cross-ratio of four points in  $\mathbb{C}$  is invariant under Möbius transformations, it was used as the core of the Cayley–Klein metric or the Klein–Hilbert metric [4] namely, the magnitude of the logarithm of the cross-ratio between any two *ordinary* points  $A$  and  $B$

$$d(A, B) = \left| \ln \frac{|AP||BQ|}{|AQ||BP|} \right|, \quad (6)$$

where  $P$  and  $Q$  are the *ideal* points along the hyperbolic line  $AB$  (Any positive scalar multiple of the right-hand side of Equation (6) gives a valid hyperbolic metric. For example, the metric for Felix Klein’s model of hyperbolic geometry in the unit disk uses the coefficient  $\frac{1}{2}$ ). Interpreting  $A, B, P$ , and  $Q$  as the respective complex numbers  $0, \frac{i}{2}, i$ , and  $-i$ , and using the Möbius transform  $f(z) = \frac{z + \frac{1}{2}}{\frac{z}{2} + 1}$  with  $z \in \mathbb{C}$ , the dual circle to the Apollonius circle of Figure 6 and the hyperbolic line  $AB$  are mapped to a new hyphenated dashed circle and a hyperbolic line  $A'B'$ ; see Figure 7. Of course, we have  $d(A, B) = d(A', B')$ . As an apt gesture ceding credit to Apollonius and his amazing algorithm, some authors refer to their various hyperbolic metrics as *Apollonian*, such as in [5,6].



**Figure 7.** Isometrically mapping hyperbolic line  $AB$  to hyperbolic line  $A'B'$ .

#### 4. An Apollonian Puzzle

Now for a first version of our puzzle: Drop a circle  $\mathcal{E}$  into the Poincaré disk; what is its hyperbolic center,  $B$ , and its Apollonian index  $n$ ? By rotation symmetry, we may assume that the Euclidean center  $u$ , the foci  $B$  (as the number  $v$ ) and  $A$ , with  $A > B$ , and a diameter of  $\mathcal{E}$  lie along the real axis with  $0 < v < 1$ . (The case when  $v = 0$  is uninteresting.)

Once  $\mathcal{E}$  settles into place, we know that the endpoints of its diameter along the  $x$ -axis lie at points  $p$  and  $q$  with  $|p| < q$ . Each circle  $\mathcal{E}$  must be a hyperbolic translate of a circle  $\mathcal{D}$  with Euclidean center  $O$  and some unknown Euclidean radius  $\delta > 0$ , as illustrated in Figure 5. Let  $v$  be the unknown hyperbolic center of  $\mathcal{E}$ ; the hyperbolic translation isometry taking  $O$  to  $v$  and  $-\delta$  to  $p$  and  $\delta$  to  $q$  is

$$f_v(z) = \frac{z + v}{vz + 1}. \quad (7)$$

**Lemma 5** (A hyperbolic center and a Euclidean radius). *The solution to  $f_v(-\delta) = p$  and  $f_v(\delta) = q$  for  $v$  and  $\delta$ , where  $|p| < q < 1$ ,  $p \in \mathbb{R}$ ,  $0 < \delta < 1$ , and  $0 < v < 1$ , is*

$$v = \frac{1 + pq - \sqrt{(1-p^2)(1-q^2)}}{p+q} \quad \text{and} \quad \delta = \frac{1 - pq - \sqrt{(1-p^2)(1-q^2)}}{q-p}. \quad (8)$$

**Proof.** We know that  $f_v$  is an isometry on the interior of the unit disk in  $\mathbb{C}$ . Thus, the solution for  $v$  and  $\delta$  must be unique;  $v$  must lie between  $p$  and  $q$ ; and  $\delta$  must lie between 0 and 1. Suppose Equation (8) is not the solution. Then by the quadratic formula, the solution for  $v$  and  $\delta$  is

$$v = \frac{1 + pq + \sqrt{(1-p^2)(1-q^2)}}{p+q} \quad \text{and} \quad \delta = \frac{1 - pq + \sqrt{(1-p^2)(1-q^2)}}{q-p}. \quad (9)$$

Since  $(1-p^2) > (1-q^2)$ , and introducing helpful functional notation  $S(p, q)$  and  $T(p, q)$ , we have

$$v > \frac{2 + pq - q^2}{p+q} = S(p, q) \quad \text{and} \quad \delta > \frac{2 - pq - q^2}{q-p} = T(p, q). \quad (10)$$

Because

$$\frac{\partial S}{\partial p} = \frac{2q^2 - 2}{(p+q)^2} < 0 \quad \text{and} \quad \frac{\partial T}{\partial p} = \frac{2 - 2q^2}{(q-p)^2} > 0,$$

for any fixed  $q$ , we know that  $S(p, q)$  is a decreasing function and  $T(p, q)$  is an increasing function, for  $|p| < q$ . Since

$$\lim_{p \rightarrow q} S(p, q) = \frac{1}{q} > 1 \quad \text{and} \quad \lim_{p \rightarrow -q} T(p, q) = \frac{1}{q} > 1,$$

we know that  $S(p, q)$  and  $T(p, q)$  always exceed 1. By Equation (10), then both  $v$  and  $\delta$  exceed 1, making our supposition false, and we are done. (The reader may wish to explore further implications of Equations (8) and (9). The  $v$  of the former equation gives the inner Apollonian focus  $B$  and the  $v$  of the latter equation, surprisingly, gives the outer Apollonian focus  $A$ . To prove this result, a computer algebra system is helpful: in the notation of Corollary 1, the expression  $u + a = u + \frac{\epsilon^2}{b} = u + \frac{\epsilon^2}{v-u}$ , where  $v$  is from Equation (8), simplifies to  $v$  of Equation (10).)  $\square$

The Euclidean center  $u$  and radius  $\epsilon$  of  $\mathcal{E}$  are  $u = \frac{p+q}{2}$  and  $\epsilon = \frac{q-p}{2} > 0$ . They may also be obtained from  $u = \frac{f_v(\delta) + f_v(-\delta)}{2}$  and  $\epsilon = \frac{f_v(\delta) - f_v(-\delta)}{2}$ , which simplify as

$$u = \frac{v(1 - \delta^2)}{1 - v^2\delta^2} \quad \text{and} \quad \epsilon = \frac{\delta(1 - v^2)}{1 - v^2\delta^2}. \quad (11)$$

The reader may check that

$$d(v, p) = \ln \frac{\sqrt{(1+p^2)(1-q^2)}}{1+q-p-pq} = \ln \frac{1+\delta}{1-\delta} = d(v, q)$$

via Equation (6) where  $P$  and  $Q$  are  $\pm 1$ . Also note that  $0 < u < v$ .

To find the Apollonian index of circle  $\mathcal{E}$  where its inner focus  $B$  lies at  $v$ , let us use the notation of Corollary 1 and Equation (3). That is, let  $b = v - u$ , which means that  $n = \frac{\epsilon}{b}$ . Since  $0 < b < \epsilon$ ,  $n > 1$ , and we have proved the following.

**Theorem 2** (Puzzle solution.). *Given the circle  $\mathcal{E}$  with Euclidean center at  $u$  along the positive  $x$ -axis and Euclidean radius  $\epsilon$ , we can find its hyperbolic center  $v$ , whereupon, with  $b = v - u$ , its Apollonian index  $n$  is  $n = \frac{\epsilon}{b} > 1$ , and focal points  $B$  and  $A$  may be taken as  $B = v$  and  $A = u + \epsilon n$ .*

**Proof.** Using the notation of this section, we know that  $p = u - \epsilon$  and  $q = u + \epsilon$ . By Equation (8) we recover the hyperbolic center  $v$  of  $\mathcal{E}$ . Using the notation of Equation (3), with  $b = v - u$ , we have  $n = \frac{\epsilon}{b} > 1$ . Therefore, we may take the foci  $A$  and  $B$  to be  $B = v$  and  $A = u + n\epsilon$ .  $\square$

For example, let  $\mathcal{E}$  be the circle with diameter endpoints at  $p = \frac{1}{3}$  and  $q = \frac{3}{5}$ . Its Euclidean center is  $u = \frac{7}{15}$  and its Euclidean radius is  $\epsilon = \frac{2}{15}$ . By Equation (8), its hyperbolic center is  $v = \frac{9-4\sqrt{2}}{7} \approx 0.478$ . By Equation (3),  $b = v - u = \frac{86}{105} - \frac{4\sqrt{2}}{7} \approx 0.011$ ,  $n = \frac{\epsilon}{b} = \frac{43+30\sqrt{2}}{7} \approx 12.204$ , and  $a = \frac{86+60\sqrt{2}}{105} \approx 1.627$ . And the Apollonian foci  $A$  and  $B$  may be taken as  $B = v$  and  $A = a + u = \frac{9+4\sqrt{2}}{7} \approx 2.094$ . Finally,  $\delta = 3 - 2\sqrt{2} \approx 0.172$ .

Let us solve a second version of the puzzle. But first, a lemma.

**Lemma 6** (A hyperbolic-Euclidean twist.). *Let  $\mathcal{E}$  be a hyperbolic circle with hyperbolic center  $v$  as well as an Apollonian circle with index  $n$ . Let  $\delta$  be the radius of a circle  $\mathcal{D}$  with Euclidean center  $O$  for which  $f_v(\mathcal{D}) = \mathcal{E}$ . Then*

$$\delta vn = 1. \quad (12)$$

**Proof.** By Equation (7), Equation (3), Theorem 2, and Equation (11), we have

$$n = \frac{\epsilon}{b} = \frac{\delta(1-v^2)}{(1-v^2\delta^2)(v-u)} = \frac{\delta(1-v^2)}{(1-v^2\delta^2)\left(v - \frac{v(1-\delta^2)}{1-v^2\delta^2}\right)} = \frac{\delta(1-v^2)}{v((1-v^2\delta^2) - (1-\delta^2))} = \frac{1}{\delta v},$$

which is equivalent to  $\delta vn = 1$ .  $\square$

Here is the second puzzle: Characterize and depict in some way all those hyperbolic circles with a given Apollonian index  $n$ . By Corollary 4, we know that any two Apollonian circles with the same index are scaled versions of each other; that is, if  $\mathcal{E}$  is a circle of Apollonius with index  $n$  and Euclidean radius  $R$  with focal points  $A$  and  $B$  on the  $x$ -axis, where  $A > B$  and  $u$  is the Euclidean center of this Apollonian circle, then for each  $s > 0$ , the Apollonian circle with index  $n$  has parameters  $a = s(A - u)$ ,  $b = s(B - u)$ , and  $\epsilon = sR$ , namely, a scaled version of  $\mathcal{E}$ .

**Corollary 2** (Uniqueness of circle size versus hyperbolic center). *Of all those hyperbolic circles with hyperbolic centers  $v$  with  $0 < v < 1$  which also are Apollonian circles with index  $n$ , for each  $v$  with  $\frac{1}{n} < v < 1$ , the corresponding circle has a unique Euclidean radius  $\epsilon$  given by  $\epsilon = \frac{f_v(\delta) - f_v(-\delta)}{2}$  for a unique  $\delta$ ,  $\frac{1}{n} < \delta < 1$ .*

**Proof.** Since we know that  $0 < \delta < 1$  and  $0 < v < 1$ , then by Lemma 6, we have  $\delta = \frac{1}{vn} > \frac{1}{n}$ . Again, because  $\delta = \frac{1}{vn}$ ,  $\delta$  is unique for each  $v$ . Therefore, by Equation (11),  $\epsilon$  is unique for each  $\delta$ .  $\square$

Figure 1b displays all the Apollonian circles of index  $n = 4$  with positive real hyperbolic centers.

## 5. A Hyperbolic Application

To demonstrate that some of the preceding material is more than recreational, let us consider an application in electrostatics. The key idea is that the *electric potential*  $V$  at a point  $X$  in  $\mathbb{R}^3$  with respect to a point charge of  $\alpha$  at point  $P$  varies directly with  $\alpha$  and inversely with respect to distance  $|PX|$ . See Remark 1 for a concise overview of electrostatics.

**Remark 1** (A brief course in electrostatics). *Physical phenomena such as sound, light, gravity, and electric charge follow an inverse square law with respect to distance from the phenomenon's point-source. Why? For any of these phenomena, the intensity  $I(r)$  at any two points  $r$  units from a point-source should be the same; since the surface area of a sphere of radius  $r$  is  $4\pi r^2$ , then  $I(r) = \frac{c}{r^2}$  for some constant  $c$ . Furthermore, with respect to  $r$ , the antiderivative of  $I(r)$ , namely,  $-\frac{c}{r}$ , is arguably a simpler expression than  $I(r)$ .*

*Because of this simplicity, physicists define the electric potential  $V(X)$  of a point charge  $\alpha$  coulombs positioned at a point  $X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$  as*

$$V(X) = \frac{\alpha}{|X - X_0|} = \frac{\alpha}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}, \quad (13)$$

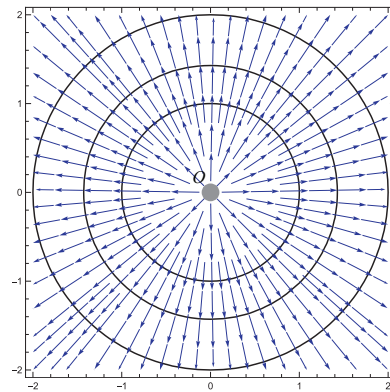
*for all points  $X \in \mathbb{R}^3$ . In particular, when  $X = (r, 0, 0)$  and  $X_0 = O$  with  $r > 0$ , we interpret Equation (13) as  $V(r) = \frac{\alpha}{r}$ ; then we have  $V'(r) = -\frac{\alpha}{r^2}$ , which is equivalent to the vector expression  $-\frac{\alpha(r, 0, 0)}{(r^2 + 0^2 + 0^2)^{\frac{3}{2}}}$ . (Technically, the definition is  $V(X) = \frac{Q}{4\pi\epsilon_0|X - X_0|}$  where  $\epsilon_0 \approx 8.85 \times 10^{-12}$ , the permittivity of free space, whose units are coulombs squared per newton meter squared; but to simplify matters, we take this proportionality constant to be unity. As a second convention, electric fields flow from positive to negative charges.) The vector-valued derivative of Equation (13) ([7], p. 61), called the electric field, denoted  $E(X)$ , is*

$$E(X) = -\nabla V(X) = \frac{\alpha(X - X_0)}{|X - X_0|^3} = \frac{\alpha(x - x_0, y - y_0, z - z_0)}{\left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right)^{\frac{3}{2}}}. \quad (14)$$

*Every two-dimensional cross section through  $X_0$  of the electric field flow lines and the equipotential surfaces generated by the point charge  $\alpha$  at  $X_0$  looks the same. See Figure 8. Formalizing Equation (14) ([7], p. 79), we define*

$$E(X) = -\nabla V(x, y, z) = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right), \quad (15)$$

*where  $\frac{\partial V}{\partial x}$ ,  $\frac{\partial V}{\partial y}$  and  $\frac{\partial V}{\partial z}$  are the partial derivatives of  $V$  with respect to the variables  $x, y, z$ .*



**Figure 8.** The electric field  $E$  (streams of arrows) and electric potential (equipotential concentric circles) generated by a point charge  $\alpha$  positioned at  $X_0 = O$ .

In 1848, when he was twenty-four years old, Lord Kelvin solved a peculiar electrostatics problem ([8], pp. 192–195):

Position a charged particle beyond the center of a grounded sphere. (A *grounded* figure in electrostatics is an object for which  $V(X) = 0$  for all  $X$  on the object.) What is the potential  $V(X)$  for any  $X$  outside the sphere?

Here is how Lord Kelvin attacked the problem. He knew that, in response to being in the electric field generated by a point source at  $A$  of positive charge  $\alpha$ , the grounded sphere nullifies the electric potential at its surface by importing electrons onto its surface, thus forcing  $V \equiv 0$  on the spherical surface. Because he had studied Apollonius's proposition, he also realized that an artificial way to produce that field (and its accompanying potential  $V$ ) outside the sphere, so as to ground the sphere, is to remove the sphere and position a pseudo-point with a specific negative charge  $\beta$  at a special location  $B = (b, 0, 0)$  within the sphere so that the spherical cross section in the  $x$ - $z$  plane is the circle of Apollonius associated with focal points  $A$  and  $B$ . (Lord Kelvin knew that the solution to the partial differential equation  $\nabla^2 V = 0$ , called *Laplace's equation*, in the presence of quantified charges, is unique. Thus, if one were to find a solution which satisfied the given conditions, then it must be the solution.) That is, we may interpret the sphere as an *Apollonian sphere* with focal points  $A$  and  $B$ , namely, the set of all points  $P \in \mathbb{R}^3$  with  $|AP| = n|BP|$ , for some fixed  $n > 1$ . When we view any two-dimensional cross section of the electric-field flow lines  $E$  and the equipotential surfaces  $V$  through the points  $A$  and  $B$ , the spherical potential surface  $V \equiv 0$  will appear as a circle of Apollonius with inner and outer Apollonian foci at  $B$  and  $A$ .

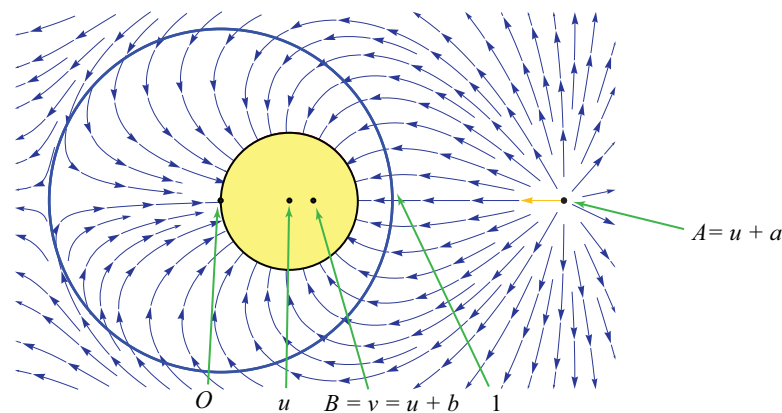
Let us attack Lord Kelvin's problem in reverse. Start with a sphere  $S$  inside the unit sphere so that its cross-section in the  $x$ - $y$  plane is a circle  $\mathcal{E}$  with its hyperbolic center at  $v \equiv B, v > 0$  with  $B = (v, 0, 0)$ . At what point  $A$  can we position a point charge  $\alpha$  coulombs and with what charge  $\beta$  can we assign to  $B$  so that  $S$  is grounded? In particular, let  $S$  have Euclidean center  $u = \frac{2}{5} \equiv (\frac{2}{5}, 0, 0)$  and radius  $\epsilon = \frac{2}{5}$ , and let  $\alpha = 1$  coulomb. By Equation (8) with  $p = 0 = u - \epsilon$  and  $q = \frac{4}{5} = u + \epsilon$ , we have  $v = \frac{1}{2} = \delta$ . By Corollary 1, we have  $b = v - u = \frac{1}{10}$  and  $ab = \epsilon^2$ , which means that  $a = \frac{8}{5}$ , and that  $n = \frac{\epsilon}{b} = 4$ . Finally, for any point  $P$  on the sphere we know that

$$\frac{1}{|AP|} + \frac{\beta}{|BP|} = \frac{\alpha}{|AP|} + \frac{\beta}{|BP|} = 0,$$

which means that

$$\frac{1}{4} = \frac{1}{n} = \frac{|BP|}{|AP|} = -\beta.$$

That is, the charge at  $v$ , the inner focus of Apollonian circle  $\mathcal{E}$  is  $\beta = -\frac{1}{4}$  coulombs. Utilizing Equation (14), we recover the electric field flow lines in the  $x$ - $y$  plane, as in Figure 9. Note that the flow lines intercept the surface of  $\mathcal{E}$  normally—a good indication that the problem has been solved successfully.



**Figure 9.** A grounded hyperbolic (shaded) sphere centered at  $B$  within the unit sphere, with a point charge of  $-\frac{1}{4}$  at  $B$  and a point charge of 1 at  $A$  outside the unit disk. (The arrows indicate that the electric field streams from a positive charge to a negative charge, a convention attributed to Benjamin Franklin (1706–1790)).

## 6. Final Comments

Henri Poincaré (1854–1912), while reminiscing about his work on hyperbolic geometry within the unit disk, claimed ([9], p. 60)

I have been able to describe the fantastic worlds [in this hyperbolic realm] without ceasing to employ the language of ordinary geometry.

This claim—that hyperbolic geometry in the unit disk can be understood using only the insight gained from ordinary geometry—is astonishing for many. Poincaré offered his explanations from a perspective of a lifetime of juggling differential equations. My goal herein is to make sense of the hyperbolic metric, Equation (1): from where did it come, and how could one describe this fantastic formula using only the language of ordinary geometry?

The ins and outs of basic hyperbolic geometry can be elusive. As another example, besides the hyperbolic metric, we outlined why the trochoid family of curves in  $\mathbb{R}^2$  can be equivalently generated via two different methods in [10]: following a tack in wheels rolling about circular tracks and following a tack in a bungee cord whose ends are held by runners at constant relative rates about a circular track. However, in hyperbolic geometry—using what is mostly undergraduate mathematics—we showed that the two methods generate different curves. This result, at least to me, seems counterintuitive. In light of the mind experiment of Lord Kelvin in Section 5, my question to the mathematical community is, what other applications of hyperbolic geometry reasoning are there?

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