International Journal of *Topology*

MDPI

*Review*

# On Linear Operators in Hilbert Spaces and Their Applications in OFDM Wireless Networks

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**Abstract:** This paper explores the application of Hilbert topological spaces and linear operator algebra in the modelling and analysis of OFDM signals and wireless channels, where the channel is considered as a linear time-invariant (LTI) system. The wireless channel, when subjected to an input OFDM signal, can be described as a mapping from an input Hilbert space to an output Hilbert space, with the system response governed by linear operator theory. By employing the mathematical framework of Hilbert spaces, we formalise the representation of OFDM signals, which are interpreted as elements of an infinite-dimensional vector space endowed with an inner product. The LTI wireless channel is characterised by using bounded linear operators on these spaces, allowing for the decomposition of complex channel behaviour into a series of linear transformations. The channel's impulse response is treated as a kernel operator, facilitating a functional analysis approach to understanding the signal transmission process. This representation enables a more profound understanding of channel effects, such as fading and interference, through the eigenfunction expansion of the operator, leading to a spectral characterization of the channel. The algebraic properties of linear operators are leveraged to develop optimal solutions for mitigating channel distortion effects.

**Keywords:** OFDM; multi-carrier communications; wireless channel; topological spaces; C*-algebra; Hilbert space; linear operators; projection operators; eigen-vectors; unitary operators



**Citation:** Louvros, S. On Linear Operators in Hilbert Spaces and Their Applications in OFDM Wireless Networks. *Int. J. Topol.* **2024**, *1*, 27–54. https://doi.org/10.3390/ijt1010004

Academic Editor: Michel Planat

Received: 12 November 2024
Revised: 20 November 2024
Accepted: 25 November 2024
Published: 29 November 2024



## 1. Introduction

The use of Hilbert spaces and linear operators [1,2] introduces a powerful mathematical framework for modelling and analysing wireless communication systems. OFDM is widely used in modern telecommunications due to its efficiency in managing the effects of multipath fading and its ability to divide the available bandwidth into narrow, orthogonal sub-channels [3]. However, the complexity of OFDM signal structure and channel interactions calls for a rigorous approach that can accommodate the infinite-dimensional nature of signal spaces and the intricate transformations occurring during transmission. Here, Hilbert space theory, with its ability to handle infinite-dimensional spaces and inner product structures, provides an essential foundation for capturing these complexities [1]. The bounded linear operators defined on Hilbert spaces allow for the decomposition of the complex, time-varying behaviour of wireless channels into manageable linear transformations. This approach not only provides a rigorous mathematical framework for understanding the underlying processes, but also offers valuable insights into the interplay between signal properties and channel effects. In Hilbert space theory, signals can be viewed as elements within a vector space equipped with an inner product, which allows for the quantification of relationships like orthogonality and projection. For OFDM, this perspective is invaluable, because the signals are represented as a function that can be decomposed into orthogonal components aligned with the subcarriers of the system. Each subcarrier can be understood as an orthogonal basis function, and the overall OFDM signal can be seen as a superposition of these basis functions. This decomposition is facilitated by projection operators in Hilbert space, which enable us to map the signals onto specific subspaces spanned by the subcarriers. These projections provide a mathematical representation of how the signal is divided across different frequency components, formalizing the assignment of signal portions to individual frequencies.

The concept of unitary operators in Hilbert spaces further enriches our understanding of signal transformations, especially when dealing with channel effects [4]. Unitary operators preserve the inner product, meaning they maintain the signal's energy and orthogonality characteristics, which are essential for accurate signal reconstruction after transmission. When applied to OFDM, unitary operators can model transformations such as phase shifts and rotations in signal space, which are common in wireless channels. This allows for the clear mathematical description of the channel's impact on each component of the signal without the loss of information, which is critical for maintaining signal integrity. Additionally, convolution operators play a central role in channel modelling within this framework. Since wireless channels are often represented as linear time-invariant (LTI) systems, the action of the channel on a transmitted signal can be expressed as a convolution operation. In Hilbert spaces, convolution can be viewed as the application of a bounded linear operator to the signal, effectively representing the channel's impulse response [5]. By using convolution operators, we can model complex channel behaviours such as fading and dispersion in a systematic, mathematical way, capturing both time-domain and frequency-domain characteristics of the signal transformation.

The Hilbert space approach provides more than just a theoretical elegance—it also offers practical benefits for optimizing wireless communication. By framing signal transmission and channel effects within the structure of Hilbert spaces, we gain the ability to apply powerful tools from linear algebra and functional analysis to develop adaptive modulation schemes and error correction strategies. This is especially important for next-generation wireless networks, where high reliability and adaptability are required to meet diverse application needs. Techniques such as eigenfunction expansion and spectral analysis become available through this framework, enabling the decomposition of the channel into distinct frequency components, which can then be selectively processed or equalised to enhance performance. In summary, Hilbert spaces and linear operators offer a structured and rigorous methodology for modelling OFDM and other complex signal processing tasks in wireless communications [6]. The ability to represent signals as elements of infinite-dimensional vector spaces, to use projection operators for decomposing signals onto subcarrier spaces, and to apply unitary and convolution operators for channel effects, all contribute to a deeper understanding of signal behaviour and system performance. This approach not only strengthens the theoretical foundation of wireless communications, but also paves the way for innovations in signal processing that can drive the evolution of future wireless systems.

In Sections 2 and 3, a short but comprehensive introduction to Vector spaces and Hilbert spaces is presented, as a preliminary topic to the linear operators' formalism. Section 4 discusses the functionals and their importance in Hilbert space as a preliminary introduction to Hilbert Operators, where important theorems and proposals are properly proved. In Section 4.3, a thorough introduction to Dirac notation is presented in order to prepare for the typical formalism in wireless communication channels, similar to physical systems. In Sections 5 and 6, we further discuss the projection operators, emphasising their importance in the signal representation in physical systems, as well as in signals in wireless communications. This is an important component of this paper, and all-important theorems and proposals are proved in full detail, helping the reader to understand how projection operators could be used in the basic theory of wireless communications. Finally, in Section 7, we discuss the wireless communication signal operators. This is a unique section in this paper, where the prior theory of linear operators is fully applicable into the wireless communication channels as part of a general physical system. Sections 5–7 are, indeed, a novel contribution to the theory of linear operators, since they prove that a linear system, as a wireless communication system, can be easily modelled by linear operators and linear operations, using core mathematical formalism as an elegant way to represent signal processing and wireless signal transmission. As to our knowledge, no paper in the international literature has discussed this topic of interest with the proper mathematical formalism. In this paper, we aim to present a formal mathematical framework to start discussing and representing the basic wireless communication principles of channel response and signal analysis, using Hilbert spaces, projection, and unitary operators. This framework could be extended as a future paper contribution based on signal processing discussions (channel estimation, channel decomposition, and signal analysis).

## 2. Vector Spaces and Hilbert Spaces Review

A real vector space is a set $\boldsymbol{A}$ over $\mathbb{C}$ with elements $\boldsymbol{a} \in \boldsymbol{A}$ following the definition and the properties [1]:

**Definition 1:** *An algebra is a vector space $\boldsymbol{A}$ over $\mathbb{C}$, with the multiplication mapping: $(a,b) \to ab$ from $\boldsymbol{A} \times \boldsymbol{A}$ into $\boldsymbol{A}$ such that, for all $a,b,c \in \boldsymbol{A}$ and $\lambda \in C$, it satisfies the conditions:*

(i) $(ab)c = a(bc)$;
(ii) $a(b+c) = ab + ac$, and $(a+b)c = ac + bc$;
(iii) $(\lambda a)b = \lambda(ab) = a(\lambda b)$;
(iv) $\exists$ *a unit element* $e \in \boldsymbol{A}$: $e \neq 0$, *for which* $ea = ae = a \; \forall \; a \in \boldsymbol{A}$.

*The unit (or identity) element e of any algebra $\boldsymbol{A}$ is uniquely defined. Indeed, if there existed another such element $e'$, then by definition, $e = ee' = e'$.*

*Usually, for obvious reasons, the unit element is denoted by $1_{\boldsymbol{A}}$ , or by just 1 if there is no confusion involved.*

*An associative algebra $\boldsymbol{A}$ is an algebraic structure equipped with the compatible operations of addition, multiplication, and a scalar multiplication by elements in some field.*

*Define the following bilinear operations:*

$\forall\lambda,\ \mu \in \mathbb{C}, \forall a,b \in \boldsymbol{A}, (\lambda a + \mu b)c = \lambda(ac) + \mu(bc)$,
$\forall\lambda,\ \mu \in \mathbb{C}, \forall a,b \in \boldsymbol{A}, c(\lambda a + \mu b) = \lambda(ca) + \mu(cb)$,
*then $\boldsymbol{A}$ along with this bilinear multiplication and addition operation is a ring [7,8].*

**Definition 2:** *For a linear space $\boldsymbol{A}$ over C, an inner product on $\boldsymbol{A}$ is the functional mapping $(\cdot,\cdot) \equiv \langle\cdot,\cdot\rangle : A \times A \to \mathbb{C}$, satisfying the following axioms:*

1. $\forall a \in A:\ (a,a) \equiv \langle a,a\rangle \geq 0$.
2. $\forall a \in A:\ (a,a) \equiv \langle a,a\rangle = 0 \; iff \; a = 0$.
3. $\forall \lambda,\mu \in \mathbb{C}$ *and* $\forall a,b,c \in A:\ (\lambda a + \mu b, c) \equiv \langle \lambda a + \mu b, c\rangle = \lambda\langle a,b\rangle + \mu\langle b,c\rangle$.
4. $\forall a,b \in A : (a,b) \equiv \langle a,b\rangle = \overline{\langle b,a\rangle} \equiv \overline{(b,a)}$

*Following axioms 3&4:* $\forall\lambda,\mu \in \mathbb{C}$ and $\forall a,b,c \in A:\ \langle a,\ \lambda b + \mu c\rangle = \overline{\lambda}\langle a,b\rangle + \overline{\mu}\langle a,c\rangle$.

*Moreover* $\forall a,b \in A:\ (a,b) \equiv \langle a,b\rangle = 0$ *which eventually implies that a, b are* ***orthogonal vectors*** *to each other.*

*The corresponding space $\boldsymbol{A}$ equipped with the inner product is called* ***inner product space****.*

**Definition 3:** *$\forall\ a \in \boldsymbol{A}$, where $\boldsymbol{A}$ is an inner product space with a defined norm $\|a\| := (a,a)^{1/2} \equiv \langle a,a\rangle^{1/2}$, it is known as* ***inner product space A*** *equipped with a* ***norm****. Then $\forall\ a,b \in \boldsymbol{A}$ and $\lambda \in \mathbb{C}$, the following properties hold [9,10]:*

(i) $\|\lambda a\| = |\lambda| \cdot \|a\|$;
(ii) $\|a+b\| \leq \|a\| + \|b\|$;
(iii) $\|ab\| \leq \|a\| \cdot \|b\|$;
(iv) $\|e\| = 1$;
(v) $\|a\| = 0$ iff $a = 0$.

*The pair $(\boldsymbol{A}, \|\cdot\|)$ is thus a* ***normed algebra*** *and, obviously, any normed algebra is also a* ***normed vector space****.*

**Definition 4:** *If a normed algebra* $\boldsymbol{A}$ *is complete in the topology defined by its norm, then this algebra is called a* ***Banach algebra****; that is to say that the normed linear space* $(\boldsymbol{A}, \|\cdot\|)$ *is a Banach space [9].*

*Moreover, a* ***commutative algebra*** $\boldsymbol{A}$ *is a* ***Banach algebra*** *is a with the extra property* $ab = ba$ *for all* $a, b \in \boldsymbol{A}$.

*A Banach algebra* $\boldsymbol{A}$ *may or may not contain a (unique) unit element. In the case where there exists a unit element* $e \in A$ *and* $\boldsymbol{A} = \{\forall\, a \in A,\ \exists\, b \in A : ab = ba = e\}$*, then, the algebra* $\boldsymbol{A}$ *with its unit element is known as a* ***unital Banach algebra*** *where each element* $a \in \boldsymbol{A}$ *is invertible, i.e., if* $\forall a, b \in \boldsymbol{A}\ \exists b = a^{-1} : ab = ba = e$*, then* $b \in \boldsymbol{A}$ *is unique.*

*The set of elements* $a, b, e$ *fitted with the operation* $\{a, b, e \in \boldsymbol{A} : ab = ba = e\}$ *defines a* ***multiplication operation*** *under the multiplication group, by the definition set as* $\boldsymbol{GL}(\boldsymbol{A}) = \{a \in \boldsymbol{A} : a \text{ has inverse}\}$. *The unit element e, then, can be any element satisfying the condition* $\{a, b, e \in \boldsymbol{A} : ab = ba = e\}$ *with the extra optional condition* $\|e\| = 1$ *[10].*

**Theorem 1:** $\forall a, b \in \boldsymbol{A}$ *where* $\boldsymbol{A}$ *is a normed vector space:*

(i) *Vector addition holds, i.e.,* $(a, b) \mapsto a + b$
(ii) *In a Banach algebra* $\boldsymbol{A}$ *considering the Cartesian product space* $\boldsymbol{A} \times \boldsymbol{A}$*, then the projection mapping of the Cartesian product* $\boldsymbol{A} \times \boldsymbol{A}$ *into* $\boldsymbol{A}$ *is defined in formal presentation as* $\boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A} = \{(a, b)/\forall a,\ b\ \in A\ \mapsto ab\}$ *is continuous, i.e.,* $(a, b) \mapsto a{\cdot}b$.

**Proof of Theorem 1:** Indeed, $\forall a_n, b_n \in \boldsymbol{A}$ with the following limit properties $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} (a_n + b_n) = a + b$.

Moreover $\forall\ a,\ a_n,\ b,\ b_n \in \boldsymbol{A}$ with the properties $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, consider the following identity: $(a - a_n){\cdot}(b - b_n) = ab - ab_n - a_n b + a_n b_n$ $= = ab - ab_n - a_n b + a_n b_n + a_n b_n - a_n b_n = (a_n - a)b_n + (b_n - b)a_n + ab - a_n b_n \Rightarrow ab - a_n b_n$ $= (a - a_n){\cdot}(b - b_n) + (a - a_n)b_n + (b - b_n)a_n$

Then

$\|ab - a_n b_n\| = \|(a - a_n){\cdot}(b - b_n) + (a - a_n)b_n + (b - b_n)a_n\| \le \|(a - a_n){\cdot}(b - b_n)\| + \|(a - a_n)b_n\| + \|(b - b_n)a_n\|$.

And using property (ii) from Definition 3 it results in

$\|ab - a_n b_n\| \le \|(a - a_n)\|{\cdot}\|(b - b_n)\| + \|(a - a_n)\|{\cdot}\|b_n\| + \|(b - b_n)\|{\cdot}\|a_n\|$

Finally,

$\lim_{n\to\infty} \|ab - a_n b_n\| \le \lim_{n\to\infty} (\|(a - a_n)\|{\cdot}\|(b - b_n)\| + \|(a - a_n)\|{\cdot}\|b_n\| + \|(b - b_n)\|{\cdot}\|a_n\|) = 0 \Rightarrow \lim_{n\to\infty} a_n b_n = ab$.

□

**Corollary 1:** *Consider a linear algebra* $\boldsymbol{A}$ *equipped with a norm (also called normed algebra) satisfying the two elements* $a,\ b \in \boldsymbol{A}$ *and* $e \in \boldsymbol{A}$*:*

(i) $\|ab\| \le \|a\|{\cdot}\|b\|$
(ii) $\|e\| = 1$*, where e is the multiplication identity element* $a{\cdot}e = e{\cdot}a = a$

*then* $\boldsymbol{A}$ *is also considered to be* ***complete with its norm****, thus forming an* $\boldsymbol{A}$*-Banach algebra. It holds, by the definition of norm and property (i), that:*

1. $\|a^2\| \le \|a\|^2$;
2. *And generalizing* $\|a^n\| \le \|a\|^n$

**Corollary 2:** *For a linear algebra* $\boldsymbol{A}$ *with a norm (normed algebra) and any elements* $a,\ b \in \boldsymbol{A}$*, the following hold:*

1. $\|a + b\| \le \|a\| + \|b\|$;
2. *In general,* $\|a + b\|^n \le \|a\|^n + \|b\|^n$.

**Banach Algebra Example:** *An example of a Banach algebra in physical systems is the* ***convolution operator*** *in telecoms and signal theory. The set* $L^1(\mathbb{R})$ *of integrable functions of the Lebesgue measure on the real axis and the norm [11]:*

$$\|g(x)\| = \int_{-\infty}^{+\infty} g(x)dx$$

*Equipped with the multiplication mapping of convolution,*

$$\forall f, g \in \mathrm{A}, (f, g) \mapsto f * g(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt$$

*is a Banach algebra.*
*Indeed:*

$$\|f * g\| = \int_{-\infty}^{+\infty} |f * g(x)|dx = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} |f(t)g(x-t)|dt \right| dx \le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(t)|\cdot|g(x-t)|dtdx.$$

*Using Fubini's integration:*

$$\|f * g\| \le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(t)|\cdot|g(x-t)|dtdx = \int_{-\infty}^{+\infty} |f(t)| \int_{-\infty}^{+\infty} |g(x-t)|dxdt \Rightarrow \|f * g\| \le \|f\|\cdot\|g\|.$$

**Definition 5:** *Any **A** Banach algebra is called a **C* algebra** [11] if it is equipped with:*

1. *The involution $\{* : \boldsymbol{A} \to \boldsymbol{A}\}$;*
2. *The C* condition: $\forall\ a \in A$, $\|aa^*\| = \|a\|^2$*

*For any **C* algebra**, the following related definitions hold:*

1. *An element $a \in \boldsymbol{A}$, is called **self-adjoint** if $\forall a \in \boldsymbol{A},\ a = a^*$;*
2. *An element $a \in \boldsymbol{A}$, is called **normal** if $\forall a \in \boldsymbol{A},\ a\cdot a^* = a^*\cdot a$;*
3. *An element $a \in \boldsymbol{A}$, is called **unitary** if $\forall a, e \in \boldsymbol{A},\ a\cdot a^* = a^*\cdot a = e$.*

**Corollary 3:** *The identity e in a C* algebra is self-adjoint.*

**Proof of Corollary 3:** Following the property (iii) of Definition 3, $\forall a \in A\ ae^* = (ea^*)^* = (a^*)^* = a$ and $e^*a = (a^*e)^* = (a^*)^* = a$. Hence, $e^*$ is an identity, and since the identity element is always unique, then $e = e^*$. □

## 3. Hilbert Spaces Overview

This section will introduce the reader into the Hilbert spaces' properties, with examples.

### *3.1. Hilbert Spaces Definition*

Hilbert spaces form an important class of Banach spaces where the concept of orthogonality of vectors is defined. From an axiomatic point of view, a Hilbert space ***H*** is an infinite-dimensional inner product space which is a complete metric space with respect to the metric generated by the corresponding inner product. This general axiomatic definition proposes an ***abstract Hilbert space***, with several examples of concrete linear spaces satisfying this definition of the abstract space.

**Definition 6:** *A Hilbert space **H** is an **A**-Banach algebra vector space where its elements are functions (vectors of Hilbert space) $\Psi(r)$, $\Phi(r)$.*

*Furthermore, the Hilbert space is a linear space [12]:*

I. *If $\Psi(r)$, $\Phi(r) \in \boldsymbol{H}$ then $(\Psi(r) + \Phi(r)) \in \boldsymbol{H}$;*

II. *If* $\Psi(r) \in \boldsymbol{H}$ *and* $\forall a \in \mathbb{C}$ , $a\Psi(r) \in \boldsymbol{H}$.

A Hilbert space is a ***complex vector space*** with a ***positive definite inner product***. The inner product for any physical system (telecom, signal processing, or quantum system) function is defined as the following:

$$(\Phi(r), \Psi(r)) = \int \Phi^*(r)\Psi(r)d^3(r) < \infty$$
$$(\Phi(r), \Phi(r)) = \int \Phi^*(r)\Phi(r)d^3(r) = \int |\Phi(r)|^2 d^3(r) < \infty$$

This inner product is anti-linear in its first argument. Indeed, $\forall a, b \in \mathbb{C}$ and $\forall\, \Phi_1, \Phi_2,\ \Psi \in \boldsymbol{H}$ hold:

$$(a\Phi_1 + b\Phi_2, \Psi) = \int (a\Phi_1 + b\Phi_2)^* \Psi d^3(r) = a^*(\Phi_1, \Psi) + b^*(\Phi_2, \Psi)$$

In the same way, this inner product is linear in its second argument, since $\forall a, b \in \mathbb{C}$ and $\forall\, \Phi_1, \Phi_2,\ \Psi \in \boldsymbol{H}$ hold:

$$(\Phi, \alpha\Psi_1 + b\Psi_2) = \int \Phi^*(\alpha\Psi_1 + b\Psi_2)d^3(r) = a(\Phi, \Psi_1) + b(\Phi, b\Psi_2)$$

The ***length*** (**norm**) of a vector $\Psi(r)$ in a physical system is positive definite, and is related to the inner product as [13]:

$$\|\Psi(r)\|^2 = (\Psi(r), \Psi(r)) = \int \Psi^*(r)\Psi(r)d^3(r) = \int |\Psi(r)|^2 d^3(r) = \|\Psi\|^2 = (\Psi, \Psi) \geq 0$$

Along with the conclusion: $\sqrt{\int \Psi^*(r)\Psi(r)d^3(r)} = \sqrt{\int |\Psi(r)|^2 d^3(r)} \geq 0$.
The addition of two vectors in a Hilbert space is a closed function.
Indeed, $\forall\, \Psi_1,\ \Psi_2\ \in \boldsymbol{H}$:

$$(\Psi_1 + \Psi_2, \Psi_1 + \Psi_2) = (\Psi_1, \Psi_1) + (\Psi_2, \Psi_2) + (\Psi_1,\ \Psi_2) + (\Psi_2, \Psi_1)$$
$$(\Psi_1 - \Psi_2, \Psi_1 - \Psi_2) = (\Psi_1, \Psi_1) + (\Psi_2, \Psi_2) - (\Psi_1,\ \Psi_2)$$
$$-(\Psi_2, \Psi_1)\,(\Psi_1 + \Psi_2, \Psi_1 + \Psi_2) + (\Psi_1 - \Psi_2, \Psi_1 - \Psi_2) = 2(\Psi_1, \Psi_1) + 2(\Psi_2, \Psi_2) \Rightarrow$$
$$(\Psi_1 + \Psi_2, \Psi_1 + \Psi_2) = 2(\Psi_1, \Psi_1) + 2(\Psi_2, \Psi_2) - (\Psi_1 - \Psi_2, \Psi_1 - \Psi_2) \Rightarrow$$
$$(\Psi_1 + \Psi_2, \Psi_1 + \Psi_2) \leq 2(\Psi_1, \Psi_1) + 2(\Psi_2, \Psi_2) = 2\int |\Psi_1(r)|^2 d^3(r) + 2\int \Psi_2|(r)|^2 d^3(r) < \infty$$

Some other properties can be proven [13]:

1. $(\Psi, \Phi) = (\Phi, \Psi)^*$, indeed:

$$(\Phi, \Psi)^* = (\Phi(r), \Psi(r))^* = \left(\int \Phi^*(r)\Psi(r)d^3(r)\right)^* = \int \Psi^*(r)\Phi(r)d^3(r) = (\Psi(r), \Phi(r)) = (\Psi, \Phi)$$

2. Special case $(\Psi, \Psi) = (\Psi, \Psi)^*$
3. $|(\Psi, \Phi)|^2 = (\Psi, \Phi)(\Psi, \Phi)^* = (\Phi, \Psi)^*(\Phi, \Psi) = |(\Phi, \Psi)|^2$

**Corollary 4:** *The Schwartz inequality holds in Hilbert spaces for physical wave functions* $\Psi(r),\ \Phi(r)$ *as the following:*

$$(\Psi, \Phi)\cdot(\Phi, \Psi) = (\Phi, \Psi)^*(\Phi, \Psi) = |(\Phi, \Psi)|^2 \leq (\Psi, \Psi)\cdot(\Phi, \Phi)$$

**Proof of Corollary 4:** Consider vector $X \in H$, such that $\forall \lambda \in \mathbb{C}, X = \Phi + \lambda\Psi$. Then it holds that the length of $X \in H$ is positive, that is:

$$(X,X) \geq 0 \Rightarrow (\Phi+\lambda\Psi,\Phi+\lambda\Psi) \geq 0 \Rightarrow (\Phi,\Phi+\lambda\Psi)+\lambda^*(\Psi,\Phi+\lambda\Psi) \geq 0 \Rightarrow$$
$$(\Phi,\Phi)+\lambda(\Phi,\Psi)+\lambda^*(\Psi,\Phi)+\lambda^*\lambda(\Psi,\Psi) \geq 0 \Rightarrow (\Phi,\Phi)+\lambda(\Phi,\Psi)+\lambda^*(\Psi,\Phi)+|\lambda|^2(\Psi,\Psi) \geq 0 \Rightarrow$$

The Schwarz inequality is obviously true if $\Phi = 0$ since:

$$(\Psi,0)\cdot(0,\Psi) = 0 = |(0,\Psi)|^2 = (\Psi,\Psi)\cdot(0,0) = 0$$

Then we consider the general case $\Phi \neq 0$ and by selecting the value $\lambda = -\frac{(\Psi,\Phi)}{(\Psi,\Psi)}$ we get:

$$(\Phi,\Phi) - \frac{(\Psi,\Phi)}{(\Psi,\Psi)}(\Phi,\Psi) - \frac{(\Psi,\Phi)^*}{(\Psi,\Psi)^*}(\Psi,\Phi) + \left|\frac{(\Psi,\Phi)}{(\Psi,\Psi)}\right|^2(\Psi,\Psi) \geq 0 \Rightarrow$$
$$(\Phi,\Phi) - \frac{(\Psi,\Phi)}{(\Psi,\Psi)}(\Phi,\Psi) - \frac{(\Phi,\Psi)}{(\Psi,\Psi)}(\Psi,\Phi) + \left|\frac{(\Psi,\Phi)}{(\Psi,\Psi)}\right|^2(\Psi,\Psi) \geq 0 \Rightarrow$$
$$(\Phi,\Phi) - 2\frac{(\Psi,\Phi)}{(\Psi,\Psi)}(\Phi,\Psi) + \left|\frac{(\Psi,\Phi)}{(\Psi,\Psi)}\right|^2(\Psi,\Psi) \geq 0 \Rightarrow$$
$$(\Phi,\Phi) - 2\frac{(\Psi,\Phi)(\Psi,\Phi)^*}{(\Psi,\Psi)} + \left|\frac{(\Psi,\Phi)}{(\Psi,\Psi)}\right|^2(\Psi,\Psi) \geq 0 \Rightarrow$$
$$(\Phi,\Phi) - 2\frac{|(\Psi,\Phi)|^2}{(\Psi,\Psi)} + \frac{|(\Psi,\Phi)|^2}{(\Psi,\Psi)} \geq 0 \Rightarrow (\Phi,\Phi) - \frac{|(\Psi,\Phi)|^2}{(\Psi,\Psi)} \geq 0 \Rightarrow |(\Psi,\Phi)|^2 \leq (\Phi,\Phi)(\Psi,\Psi)$$

And the closure is proved. □

### 3.2. *$l^2$-Linear Space as a Typical Hilbert Spaces Example*

An example of such concrete linear space, satisfying the Hilbert space definition, is the $l^2$ linear space with elements to be vectors of sequences of real or complex numbers, of great importance in ***physics***, ***signal theory***, ***wireless communications*** and ***physical systems*** in general.

The vectors of the $l^2$ linear space are defined as the following:

$$\forall x_n \in \mathbb{C}, f = \{x_n\},\ n \in (1,2,\ldots\infty) : \sum_{n=1}^{\infty}|x_n|^2 < \infty$$

In such a space, the elements $\{x_n\}$ are called the ***components*** or the ***coordinates*** of the vector $f$ with the extra property of a ***zero vector*** $f_0 = \{x_n\} = (0,0,\ldots\infty) = \{0_n\}$, $\forall n \in (1,2,\ldots\infty)$, with all component elements to be zero numbers.

In such a space, the addition of two vectors is defined as:

$$\forall f = \{x_n\},\ g = \{y_n\},\ f+g = \{x_n+y_n\},\ n \in (1,2,\ldots\infty) : \sum_{n=1}^{\infty}|x_n+y_n|^2 < \infty$$

The condition of the vector addition follows from the Cauchy inequality:

$$\sum_{n=1}^{\infty}|x_n+y_n|^2 = \sum_{n=1}^{\infty}\left(|x_n|^2+2|x_n||x_n|+|y_n|^2\right) \leq \sum_{n=1}^{\infty}|x_n|^2 + \sum_{n=1}^{\infty}|y_n|^2 < \infty$$

In such linear space $l^2$ we define two conditions:

**Condition 1:** *The multiplication of a vector by a number.*

$$\forall\lambda \in \mathbb{C}, \lambda f = \{\lambda x_n\},\ n \in (1,2,\ldots\infty) : \sum_{n=1}^{\infty}|\ \lambda x_n|^2 < \infty$$

**Condition 2:** *The scalar product:*

$\forall\{x_n\},\{y_n\}\in\mathbb{C}$ *with* $\overline{y}_n$ *the conjugate of* $y_n$ *and* $\forall\, f,g\in H$:

$$(f,g)=\langle f,g\rangle=\sum_{n=1}^{\infty}(x_n\overline{y}_n)\in\mathbb{C}$$

*The scalar product converges absolutely, since:*

$$\sum_{n=1}^{\infty}|x_n\overline{y}_n|\leq\frac{1}{2}\sum_{n=1}^{\infty}|x_n|^2+\frac{1}{2}\sum_{n=1}^{\infty}|\overline{y}_n|^2<\infty$$

The scalar product in linear space $l^2$ defines an ***inner product space*** which satisfies the following four properties:

**Property 1:** $(g,f)=\overline{(f,g)}$.
*Indeed, by scalar product definition,* $\forall\{x_n\},\{y_n\}\in\mathbb{C}$:

$$\overline{(f,g)}=\overline{\sum_{n=1}^{\infty}(x_n\overline{y}_n)}=\sum_{n=1}^{\infty}\left(\overline{x}_n\overline{(\overline{y})}_n\right)=\sum_{n=1}^{\infty}(\overline{x}_ny_n)\sum_{n=1}^{\infty}(y_n\overline{x}_n)=(g,f)$$

**Property 2:** *The linearity of the scalar product with respect to its first property and first condition:*

$$\forall\lambda_1,\lambda_2\in\mathbb{C},\forall f_1,f_2,g\in H:\ (\lambda_1f_1+\lambda_2f_2,g)=\lambda_1(f_1,g)+\lambda_2(f_2,g)$$

*Indeed, by scalar product definition:* $\forall\{x_{n1}\},\{x_{n2}\},\{y_n\}\in\mathbb{C}$:

$$(\lambda_1f_1+\lambda_2f_2,g)=\sum_{n=1}^{\infty}([\lambda_1x_{n1}+\lambda_1x_{n1}]\overline{y}_n)=\sum_{n=1}^{\infty}([\lambda_1x_{n1}]\overline{y}_n+[\lambda_2x_{n2}]\overline{y}_n)=$$
$$=\sum_{n=1}^{\infty}(\lambda_1x_{n1}\overline{y}_n)+\sum_{n=1}^{\infty}(\lambda_2x_{n2}\overline{y}_n)=\lambda_1\sum_{n=1}^{\infty}(x_{n1}\overline{y}_n)+\lambda_2\sum_{n=1}^{\infty}(x_{n2}\overline{y}_n)==\lambda_1(f_1,g)+\lambda_2(f_2,g)$$

**Property 3:** *The reverse linearity of the scalar product, with respect to its first property and first condition:*

$$\forall\lambda_1,\lambda_2\in\mathbb{C},\forall f,g_1,g_2\in H:\ (f,\lambda_1g_1+\lambda_2g_2)=\overline{\lambda}_1(f,g_1)+\overline{\lambda}_2(f,g_2)$$

*Indeed, by scalar product definition*: $\forall\{x_n\},\{y_{n1}\},\{y_{n2}\}\in\mathbb{C}$:

$$(f,\lambda_1g_1+\lambda_2g_2)=\sum_{n=1}^{\infty}\left(x_n\left(\overline{\lambda_1y_1+\lambda_2y_2}\right)\right)=\sum_{n=1}^{\infty}\left(x_n\left(\overline{\lambda_1y_1}+\overline{\lambda_2y_2}\right)\right)=$$
$$=\sum_{n=1}^{\infty}\left(x_n\left(\overline{\lambda_1y_1}\right)\right)+\sum_{n=1}^{\infty}\left(x_n\left(\overline{\lambda_2y_2}\right)\right)=\overline{\lambda}_1\sum_{n=1}^{\infty}(x_n\overline{y}_1)+\overline{\lambda}_2\sum_{n=1}^{\infty}(x_n\overline{y}_2)=\overline{\lambda}_1(f,g_1)+\overline{\lambda}_2(f,g_2)$$

**Property 4:** $\forall f\in H$ *the* $(f,f)\geq 0$
*which follows directly from the definition of the scalar product:*

$$\forall\{x_n\}\backslash\{\mathbf{0}\}:\ \{x_n\}\in\mathbb{C},0\leq\sum_{n=1}^{\infty}|x_n|^2\leq\infty$$

Under the restriction of $\sum_{n=1}^{\infty}|x_n|^2=0$ iff $\forall x_n\in\mathbb{C},\{x_n\}=\{\mathbf{0}\}$.

The positive square root $\|f\| = \sqrt{(f,f)} = \sqrt{\sum_{n=1}^{\infty}(x_n\overline{x}_n)} = \sqrt{\sum_{n=1}^{\infty}|x_n|^2} \geq 0$ is known as the norm of the element vector $f$, with the condition $\|f\| = 0$ iff $\forall x_n \in \mathbb{C}, \{x_n\} = \{\mathbf{0}\}$.

Moreover $\forall\lambda \in \mathbb{C}$ and $\forall\, f \in H$:

$$\|\lambda f\| = \sqrt{(\lambda f, \lambda f)} = \sqrt{\sum_{n=1}^{\infty}\left(\lambda x_n \overline{\lambda}\overline{x}_n\right)} = \sqrt{\sum_{n=1}^{\infty}|\lambda|^2|x_n|^2} = |\lambda|\sqrt{\sum_{n=1}^{\infty}|x_n|^2} = |\lambda|\|f\| \geq 0$$

The scalar product in $l^2$ space follows the ***Cauchy–Schwarz inequality***, i.e.,:

$$|(f,g)| \leq \|f\|\|g\| \Rightarrow \left|\sum_{n=1}^{\infty} x_n\overline{y}_n\right| \leq \sqrt{\sum_{n=1}^{\infty}|x_n|^2}\sqrt{\sum_{n=1}^{\infty}|y_n|^2}$$

Indeed, $\forall\lambda \in \mathbb{R}, \forall f,g \in H,\ (f,g) > 0$:

$$\exists k\left(\left(k = \frac{(f,g)}{|(f,g)|} = \frac{\sum_{n=1}^{\infty}(x_n\overline{y}_n)}{\left|\sum_{n=1}^{\infty} x_n\overline{y}_n\right|}\right) \wedge \left(k\overline{k} = \frac{(f,g)}{|(f,g)|}\frac{\overline{(f,g)}}{|(f,g)|} = \frac{|(f,g)|^2}{|(f,g)|^2} = 1\right)\right)$$

And from the fourth property the following inequality holds:

$$\begin{gathered}\left(\overline{k}f + \lambda g, \overline{k}f + \lambda g\right) \geq 0 \ \Rightarrow \left(\overline{k}f, \overline{k}f\right) + \left(\overline{k}f, \lambda g\right) + \left(\lambda g, \overline{k}f\right) + (\lambda g, \lambda g) \geq 0 \Rightarrow \\ \overline{k}k(f,f) + \overline{k}\lambda(f,g) + \overline{(kf, \lambda g)} + \lambda^2(g,g) \geq 0 \Rightarrow \\ (f,f) + \lambda\left(\overline{k}f, g\right) + \lambda\overline{(kf,g)} + \lambda^2(g,g) \geq 0 \Rightarrow \\ (f,f) + \lambda\left[\overline{k}(f,g) + \overline{k(f,g)}\right] + \lambda^2(g,g) \geq 0 \Rightarrow (f,f) + 2\lambda Re\left(\overline{k}|(f,g)|\right) + \lambda^2(g,g) \geq 0\end{gathered}$$

where the quadratic form in $\lambda \in \mathbb{R}$ is non-negative, implying that:

$$|(f,g)|^2 \leq (f,f)(g,g) \Rightarrow |(f,g)| \leq \|f\|\|g\|$$

where the special condition $|(f,g)| = \|f\|\|g\|$ holds only if the quadratic has a double root, implying that $\overline{k}f + \lambda g = 0$, i.e., the two vectors $f, g$ are linearly dependent.

**Corollary 5:** *The $l^2$ space furthermore satisfies the **triangle inequality**:*

$$\|f+g\| \leq \|f\| + \|g\| \Rightarrow \left\|\sqrt{\sum_{n=1}^{\infty}|x_n|^2} + \sqrt{\sum_{n=1}^{\infty}|y_n|^2}\right\| \leq \left\|\sqrt{\sum_{n=1}^{\infty}|x_n|^2}\right\| + \left\|\sqrt{\sum_{n=1}^{\infty}|y_n|^2}\right\|$$

**Proof of Corollary 5:**

$$\begin{gathered}\|f+g\|^2 = (f+g, f+g) = (f,f) + (f,g) + (g,f) + (g,g) \Rightarrow \\ \|f+g\|^2 = \|f\|^2 + (f,g) + (g,f) + \|g\|^2\end{gathered}$$

From the Cauchy–Bunyakovski inequality, it follows that
$\|f+g\|^2 \leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2 \Rightarrow \|f+g\| \leq \|f\| + \|g\|$,
with the equality condition $\|f+g\| = \|f\| + \|g\|$ iff $(g,f) = \|f\|\|g\|$

The scalar product defines the orthogonality of two vectors $f, g$: $(f,g) = (g,f) = 0\ iff\ f \perp g$ which implies that $\|f+g\|^2 = \|f\|^2 + (f,g) + (g,f) + \|g\|^2 \Rightarrow \|f+g\|^2 = \|f\|^2 + \|g\|^2$ and the proof is completed.

The $l^2$ inner-product space becomes $l^2$ ***metric space*** if the distance between two vectors points is defined as:

$$\forall f, g \in H,\ D[f,g] = \|f - g\| = \left\| \sqrt{\sum_{n=1}^{\infty} |x_n|^2} - \sqrt{\sum_{n=1}^{\infty} |y_n|^2} \right\|$$

Along with the following properties (given here without proof):

1. $\forall f, g \in H : f \neq g,\ D[f,g] = D[g,f] > 0$;
2. $\forall f \in H,\ D[f,f] = 0$;
3. The triangle inequality: $\forall f, g \in H : f \neq g,\ D[f,g] \leq D[f,h] + D[h,g]$.
   □

## 4. Functionals and Normal Operators in Hilbert Spaces

### 4.1. Functionals in Hilbert Spaces

Let $D \subset H$ denote a subset of a Hilbert space H. For any point $f \in D$ a function $\varphi \in H :\ \forall f \in D,\ \varphi(f) \to \mathbb{C}$ is called a ***functional in space H*** with domain $D_T$ [14].

Suppose two functionals $\varphi_A$ with domain $D_A$ and $\varphi_B$ with domain $D_B$, fulfilling the extra condition $D_B \subset D_A$. If $\varphi_A(f) = \varphi_B(f)$ then $\varphi_A$ is known as the extension of $\varphi_B$, $\varphi_B \subset \varphi_A$. A functional $\varphi_A$ is continuous at a point $f_1 \in D_A$ if: $\forall f \in D_A : \lim\limits_{f \to f_1} \varphi_A(f) = \varphi_A(f_1)$.
Moreover, a functional $\varphi$ is said to be linear if:

1. Its domain D is linear manifold, i.e., $\forall f, g \in D$ and $\forall a, b \in \mathbb{C}$, $af + bg \in D$.
2. $\forall f, g \in D$ and $\forall a, b \in \mathbb{C}$ $\varphi(af + bg) = a\varphi(f) + b\varphi(g)$.
3. The norm of the functional $\varphi$, $\|\varphi\|_D$ satisfies $\sup\limits_{f \in D,\ |f| \leq 1} |\varphi(f)| < \infty$ with $\left|\varphi\left(\frac{f}{\|f\|}\right)\right| \leq \|\varphi\|_D$.

The following theorem, known as the Riesz Representation Theorem for Hilbert spaces, plays a central result in functional analysis since it establishes a correspondence between bounded linear functionals on a Hilbert space and vectors in that space.

**Theorem 2 (Riesz representation theorem without proof):** *Let A be a Hilbert space. Then $\forall x, y \in A$, there is a bounded linear functional on A with $f(x) = (x,y) = \langle x,y \rangle$ with the norm $\|f\| = \|y\|$. Conversely, for every bounded linear functional $f \in X^*$ there exists a unique vector $y \in A$, such that $\forall x \in A$ the equality holds $f(x) = \langle x,y \rangle$ along with the norm $\|f\| = \|y\|$*

**Proof of Theorem 2:** Let A be a Hilbert space and let $x, y \in A$, and $a, b \in \mathbb{C}$. Define a functional $f : A \mapsto \mathbb{C}$, by: $f(x) = (x,y) = \langle x,y \rangle$, where $\langle \cdot,\cdot \rangle$ denotes the inner product on A.

Since $f(ax_1 + bx_2) = (ax_1 + bx_2, y) = \langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle = af(x_1) + bf(x_2)$, the functional $f : A \mapsto \mathbb{C}$ is linear. Moreover, it is bounded since following the Cauchy–Schwarz inequality $|f(x)| = |\langle x,y \rangle| \leq \|x\|\|y\| = c\|x\|$, where there is a constant $c = \|y\| :\ |f(x)| \leq c\|x\|$ with the norm $\|f\| = \underbrace{sup}_{\|x\| \leq 1} |f(x)| = \underbrace{sup}_{\|x\| \leq 1} |\langle x,y \rangle| = \|y\|$.

Conversely, let $f \in A^*$ be a bounded linear functional on A. Then $\exists y \in A : \forall x \in A,\ f(x) = (x,y) = \langle x,y \rangle$. Using the Riesz representation theorem, $\forall f \in A^*, \exists! y \in A :\ (x) = (x,y) = \langle x,y \rangle$ where the norm of the functional equals $\|f\| = \|y\|$, and the theorem is proved. □

### 4.2. Operators in Hilbert Spaces

In a similar way, any function $T \in H :\ \forall f \in D\ and\ g \in H,\ T(f) = Tf = g$ is called an ***operator in H*** with its domain as D. Consider two operators T with domain $D_T$ and $S$

with domain $D_S$, satisfying the additional condition $D_S \subset D_T$. If $Tf = Sf = g$, then T is known as the extension of S, or $S \subset T$.

In a Hilbert space there also exists:

- An identity operator $1f = f$ mapping each vector function $f \in \boldsymbol{D}$ into itself, that is the domain of 1 operator is the $n^{th}$ dimension Hilbert space $H$ with range $\boldsymbol{D} \subseteq \boldsymbol{H}$.
- A null operator $0f = 0$ mapping every vector function $f \in \boldsymbol{D}$ into zero, that is the domain of 0 operator is the $n^{th}$ dimension Hilbert space $H$ with range $D = \{0\}$.

**Definition 7:** *An operator* $\boldsymbol{T}$ is **linear** *if its domain* $D$ *is a linear manifold and* $\forall f, g \in D$ *and* $\forall a, b \in \mathbb{C}$ *it holds:* $T(af + bg) = aTf + bTg$.

*A linear operator is bounded if* $\sup\limits_{f \in D,\ \|f\| < 1} \|Tf\| < \infty$ *where* $\sup\limits_{f \in D,\ \|f\| < 1} \|Tf\| = \|T\|$ *is the norm of linear operator T in D [12].*

**Corollary 6:** *For* $T \in B(H)$ *on the space of bounded linear operators on a Hilbert H space, from the norm definition it implies that* $\|T\| = \|T^*\| = \|TT^*\|^{1/2}$.

**Proof of Corollary 6:**
$\forall x \in H, \|x\| < 1 \Rightarrow \|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^* Tx, x \rangle \leq \|T^* Tx\| \|x\| \leq \|T^* T\| \leq \|T^*\| \|T\|$ and from sup the proof is completed. □

**Proposal 1:** *An operator T is continuous at a point* $f_1 \in D_T$ if $\forall f \in D_T : \lim\limits_{f \to f_1} Tf = Tf_1$

**Proof of Proposal 1**: An operator T is said to be continuous at a point $f_1 \in D_T$ if for any sequence $\{f_n\} \subset D_T : \lim\limits_{n \to \infty} f_n = f_1$ we have $\lim\limits_{n \to \infty} Tf_n = Tf_1$ or $\lim\limits_{f_n \to f_1} Tf_n = Tf_1$

If the operator $T$ is linear and bounded, it satisfies $\|Tf - Tf_1\| \leq \|T\| \|f - f_1\|$, where the operator norm $\|T\| = \underbrace{sup}_{\|f\| \leq 1} \|Tf\|$. Let an arbitrary $\varepsilon > 0$ and choose $\delta = \frac{\varepsilon}{\|T\|}$, then $\forall \|f - f_1\| < \delta \Rightarrow \|Tf - Tf_1\| \leq \|T\| \|f - f_1\| < \|T\| \frac{\varepsilon}{\|T\|} = \varepsilon$, which implies that $\lim\limits_{f \to f_1} Tf = Tf_1$ and the statement is proved. □

**Theorem 3 The Bilinear Functional Representation (General Form):** *If* $\Phi : HxK \mapsto \mathbb{C}$ *is a bounded bilinear (sesquilinear form) functional with bound M, then it has the following representation:*

$$\forall f, g \in H,\ \exists\, T \in B(H, K) :\ \Phi(f, g) = (Tf, g) \equiv \langle Tf, g \rangle$$

*where* $\boldsymbol{T}$ *is a bounded linear operator with domain H, uniquely determined by* $\boldsymbol{\Phi}$ *with the norm property* $\|T\| = \|\Phi\|$

**Proof of Theorem 3:** Following the previous Riesz Theorem 1, $\overline{\Phi(f, g)}$ defines a linear functional in $g \in H$, and $\exists! h_f$, uniquely defined by element $f : \forall g \in H, \overline{\Phi(f, g)} = \left(g, h_f\right)$ or $\Phi(f, g) = \left(h_f, g\right)$.

We further define the mapping $T : H \to H \Big| \ Tf = h_f$. Then, $\Phi(f, g) = (Tf, g) \equiv \langle Tf, g \rangle$

Since $\Phi$ is a bilinear functional, i.e., $\forall f, g, h \in H$: $\Phi(af + bg, h) = a\Phi(f, h) + b\Phi(g, h)$, it holds that $(T(af + bg), h) = a(Tf, h) + b(Tg, h) \Rightarrow ([T(af + bg) - aTf - bTg], h) = 0$ which implies that $T(af + bg) - aTf - bTg = 0 \Rightarrow T(af + bg) = aTf + bTg$, which proves that $T$ is a linear operator in domain H. Finally, $|(Tf, g)| \leq \|Tf\| \|g\| \Rightarrow \|\Phi\| = \sup \frac{|\Phi(f,g)|}{\|f\| \|g\|} = \sup \frac{|(Tf,g)|}{\|f\| \|g\|} \leq \sup \frac{\|Tf\| \|g\|}{\|f\| \|g\|} \Rightarrow \|\Phi\| \leq \sup \frac{\|Tf\|}{\|f\|}$

Moreover, it holds that:

$$\|\Phi\| = \sup\frac{|(Tf,g)|}{\|f\|\|g\|} \geq \sup\frac{|(Tf,Tf)|}{\|f\|\|g\|} = \sup\frac{\|Tf\|}{\|f\|}$$

and the only way to satisfy simultaneously both conditions is $\|\Phi\| = \|T\|$.

Finally, suppose that operator $T$ is not uniquely determined by the bilinear functional $\Phi$. Then, $\forall f,g \in H \,\exists\, T,T' \in B(H,K): \; \Phi(f,g) = (Tf,g) = (T'f,g) \Rightarrow (Tf - T'f,g) = 0 \;\Rightarrow T = T'$ and the operator $T$ is uniquely determined by the bilinear functional $\Phi$, and the theorem is proved. □

**Theorem 4:** *If* $\Phi : H \times K \mapsto \mathbb{C}$ *is a bounded bilinear (sesquilinear) form with bound M, then* $\exists\, \boldsymbol{T} \in B(H,K)$ *and* $S \in B(K,H)$ *such that* $\Phi(f,g) = (\boldsymbol{T}f,g) = \langle \boldsymbol{T}f,g\rangle = (f,\boldsymbol{S}g) = \langle f,\boldsymbol{S}g\rangle$. *Then* $\boldsymbol{S} \in B(K,H)$ *is the* ***adjoint*** $\boldsymbol{T} \in B(K,H)$, *denoted as* $\boldsymbol{S} = \boldsymbol{T}^*$*:* $\langle \boldsymbol{T}f,g\rangle = (f,\boldsymbol{T}^*g)$ *with the extra property that* $S = T^*$ *is also bounded and* $\|T\| = \|S\| = \|T^*\|$.

**Proof of Theorem 4:** From the bilinear functional $\Phi$ follows $|\Phi(f,g)| = |(Tf,g)| \leq \|Tf\|\|g\| \leq \|T\|\|f\|\|g\|$. Letting $M = \|T\|\|f\|$, the $\Phi(f,g)$ is bounded. Following the Riesz representation theorem, there exists a unique mapping $\exists! S \in B(K,H) : \forall g \in H, \Phi(f,g) = (f,Sg)$ and finally $(Tf,g) = (f,Sg) = (f,T^*g)$.

It further follows $(T^*f,g) = (f,Tg) \Rightarrow |(T^*f,g)| = |(f,Tg)| \leq \|f\|\|Tg\| \leq \|f\|\|T\|\|g\|$.

Setting $h = T^*f$, it follows that $\|T^*f\|^2 = (T^*f,\; T^*f) = (f,TT^*f) \leq \|f\|\|TT^*g\| \leq \|f\|\|T\|\|T^*g\|$.

Which implies that: $\|T^*f\| \leq \|f\|\|T\|$, hence $S = T^*$ is bounded with $\|T^*\| \leq \|T\|$.

From the boundedness of $S = T^*$ it follows that $(T^*)^*$ is well defined and from the previous argument it follows that $\|(T^*)^*\| \leq \|T^*\|$, hence $(T^*)^*$ is also bounded. Since $((T^*)^*f,g) = (f,T^*g) = (Tf,g) \Rightarrow \left(\boldsymbol{T}^*\right)^* = \boldsymbol{T}$, it follows that $\|(T^*)^*\| = \|T\| \leq \|T^*\|$.

Which is true if $\|T^*\| = \|T\|$, and the theorem is proved. □

**Remark:** *Following the involution property of* $C^*$ *algebra,* $\forall\, T \in B(H,K), S \in B(K,H)$ *and* $\forall\, a,b \in C$*:*

1. $(aT + bS)^* = \bar{a}T + \bar{b}S$
2. $(TS)^* = S^*T^*$
3. $(T^*)^* = T$

*4.3. Dirac Hilbert Operator Formalism for Physical Systems*

According to Section 4.2 let $D$ denote a subset of an n$^{\text{th}}$ dimension Hilbert space $\boldsymbol{H}$, i.e., $\boldsymbol{D} \subset \boldsymbol{H} \subset \mathbb{R}^{\boldsymbol{n}}$. Assume an element function (vector of vector space) $f \in \boldsymbol{D} \subset \mathbb{R}^{\boldsymbol{n}}$. Hence, any function $\boldsymbol{T}$ which defines a relation for each $f \in \boldsymbol{D}$ to a particular vector function $g = \boldsymbol{T}f \in \boldsymbol{H}$ is called an operator in the Hilbert space $H$, with domain $\boldsymbol{D} \subset \boldsymbol{H}$. The range of the operator $\boldsymbol{T}$ is the set $\Delta_T$ of all vector function elements $g = \boldsymbol{T}f$, where $f$ runs through all subset $\boldsymbol{D} \subset \boldsymbol{H} \subset \mathbb{R}^{\boldsymbol{n}}$. In more formal mathematical language, any function $T/\boldsymbol{D} \mapsto \boldsymbol{H} : \forall f \in \boldsymbol{D}$ and $g \in \boldsymbol{H}, \boldsymbol{T}(f) = \boldsymbol{T}f = g \in \boldsymbol{H}$ is called an operator in $H$ with its domain on $D$ [15,16].

Two operators $\boldsymbol{T},\boldsymbol{F}$ are said to be equivalent if their respective domains $D_T = D_F = D$ coincide, and if $\forall f \in \boldsymbol{D}, \boldsymbol{T}f = \boldsymbol{F}f$.

Let $\boldsymbol{T},\boldsymbol{F}$ be two operators with domains $D_T, D_F$, respectively, and ranges $\Delta_T = H$ and $\Delta_F \cap D_T \neq 0$. Then, we can define the product of the two operators as a single operator $TF : \forall f \in D_F,\; \boldsymbol{F}f \in \Delta_F \cap D_T \neq 0$ or $\forall f \in D_F,\; TFf = T(Ff) \in \Delta_F \cap D_T$.

In the same way, let $F,T$ be two operators with domains $D_F, D_T$, respectively, and ranges $\Delta_F = H$ and $\Delta_T \cap D_F \neq 0$. Where the product of the two operators is a single operator $FT : \forall f \in D_T,\; \boldsymbol{T}f \in \Delta_T \cap D_F \neq 0$ or [15,16], $\forall f \in D_T,\; FTf = F(Tf) \in \Delta_T \cap D_F$.

Since the domains of the two operators $\boldsymbol{TF}$ and $\boldsymbol{FT}$ are deferent, i.e., $\Delta_F \cap D_T \neq \Delta_T \cap D_F$, they are not equivalent, and hence $\forall f \in D, \boldsymbol{TFf} \neq \boldsymbol{FTf} \Rightarrow [\boldsymbol{TF}, \boldsymbol{FT}] \neq 0$ and the operators do not commute. This is crucial, since when two operators in a physical system do not commute, this implies that certain properties or observables associated with the system cannot be measured simultaneously with arbitrary precision. This concept is fundamental in quantum mechanics as well in telecom theory, and has profound implications for our understanding of the behaviour of physical systems. In general, it is worth remembering that the non-commutativity of Hilbert operators in a physical system is of fundamental importance to the algebraic structure of a quantum mechanical system or wireless telecom systems, since it leads to unique properties such as entanglement, superposition, and the probabilistic nature of system measurements.

Following the well-known Bilinear Functional Representation theorem, if $\Phi : H \times K \mapsto \mathbb{C}$ is a bounded bilinear (sesquilinear form), then it has a proper representation $\forall f, g \in \boldsymbol{E_r}$, $\exists\, \boldsymbol{T} \in B(H, K) : \ \Phi(f, g) = (Tf, g) \equiv \langle Tf, g \rangle$, where $\boldsymbol{T}$ is a bounded linear operator with the domain $\boldsymbol{E_r}$, uniquely determined by $\boldsymbol{\Phi}$ with the norm property $\|T\| = \|\Phi\|$ [17].

Back to Dirac's formalism, as a general definition, a linear *operator* $\boldsymbol{T}$ maps vectors to vectors, or in other words a linear operator $\boldsymbol{T}$ maps from the ket space into itself, i.e., $\boldsymbol{T} : \boldsymbol{E_r} \mapsto \boldsymbol{E_r}$. There are two types of operators of interest in physical systems:

- The **linear operator**: $\forall c_1, c_2 \in \mathbb{C}$, and $\Phi_1, \Phi_2 \in \boldsymbol{E_r}$, $\exists \boldsymbol{T}$ :

$$\boldsymbol{T}(c_1 |\Phi_1\rangle + c_2 |\Phi_2\rangle) = c_1 \boldsymbol{T} |\Phi_1\rangle + c_2 \boldsymbol{T} |\Phi_2\rangle$$

- The **antilinear operator**: $\forall c_1, c_2 \in \mathbb{C}$, and $\Phi_1, \Phi_2 \in \boldsymbol{E_r}$, $\exists \boldsymbol{T}'$ :

$$\boldsymbol{T}'(c_1 |\Phi_1\rangle + c_2 |\Phi_2\rangle) = c_1^* \boldsymbol{T}' |\Phi_1\rangle + c_2^* \boldsymbol{T}' |\Phi_2\rangle$$

Operators form a vector space; hence, it follows that [18]:

- The **multiplication property**: $\left(\boldsymbol{TT}'\right)|\Psi\rangle = \boldsymbol{T}\left(\boldsymbol{T}'|\Psi\rangle\right)$; however, the multiplication operation is generally not commutative, i.e., $\left(\boldsymbol{TT}'\right)|\Psi\rangle \neq \left(\boldsymbol{T}'\boldsymbol{T}\right)|\Psi\rangle$ or $\left[\boldsymbol{TT}', \boldsymbol{T}'\boldsymbol{T}\right] \neq 0$ as stated previously.
- The **associative property**: $((AB)C)|\Psi\rangle = (AB)(C|\Psi\rangle) = (A(B(C|\Psi\rangle))) = (A(BC|\Psi\rangle)) = ABC|\Psi\rangle$, hence $(AB)C = A(BC) = ABC$.

## 5. Projection Operators on Hilbert Spaces

**Definition 8:** *A subset $\boldsymbol{S} \subset \boldsymbol{H}$ **linear manifold** is called a if $\forall f, g \in S$ and $\forall \lambda_1, \lambda_2 \in \mathbb{C}$, with the corresponding linear combination $\lambda_1 f + \lambda_2 g \in S$.*

**Definition 9:** *Consider the finite subset $\boldsymbol{S_M} \subset \boldsymbol{H}$ with all elements $\{f_i\}$, $i \in (1, 2, \ldots k)$ and the corresponding subset $\boldsymbol{S} \subset \boldsymbol{H}$ of all finite linear combinations $\forall \lambda_1, \lambda_2, \ldots \lambda_k \in \mathbb{C}, \lambda_1 f_1 + \lambda_2 f_2 + \ldots \lambda_k f_k$. The subset $\boldsymbol{S} \subset \boldsymbol{H}$ of all finite linear combinations is the smallest linear manifold $S$ which contains a finite set $S_M$. The subset $\boldsymbol{S} \subset \boldsymbol{H}$ is defined as the **linear envelope** of $S_M$ and it is a closed linear envelope since $\boldsymbol{H}$ is a closed metric space* [19].

**Lemma 1:** *Consider a point element $p \in \boldsymbol{H}$ and select a vector element $f \in \boldsymbol{S_M} \subset \boldsymbol{H}$, such that $\varepsilon = \inf_{f \in \boldsymbol{S_M}} \|p - f\|$. There exists at most one single point $q \in \boldsymbol{S_M}$ such that $\varepsilon = \|p - q\|$.*

**Proof of Lemma 1:** $\exists q_1, q_2 \in \boldsymbol{S_M}, \forall \lambda_1, \lambda_2 \in \mathbb{C}, \lambda_1 q_1 + \lambda_2 q_2 \in \boldsymbol{S_M}$ and the following property holds $\left\| p - \frac{q_1+q_2}{2} \right\| \geq \varepsilon$. Furthermore, $\left\| p - \frac{q_1+q_2}{2} \right\| = \left\| \left(p - \frac{q_1}{2}\right) + \left(p - \frac{q_2}{2}\right) \right\| \leq \frac{1}{2}\left\| p - \frac{q_1}{2} \right\| + \frac{1}{2}\left\| p - \frac{q_2}{2} \right\| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which implies that the triangle inequality holds:

$$\left\| p - \frac{q_1+q_2}{2} \right\| = \frac{1}{2}\left\| p - \frac{q_1}{2} \right\| + \frac{1}{2}\left\| p - \frac{q_2}{2} \right\| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Furthermore, $\exists k \geq 0((p - q_1 \neq 0) \Rightarrow p - q_1 = k(p - q_2))$. And for the special case $= 1 \geq 0 \;\; p - q_1 = p - q_2 \Rightarrow q_1 = q_2$.

Finally, $\forall (k \neq 1) \cap k \geq 0 \Rightarrow p - q_1 = k(p - q_2) \Rightarrow p = \frac{(q_2 - kq_1)}{(1-k)}$, which implies that $p \in \boldsymbol{S_M} \subset \boldsymbol{H}$, which finally contradicts our initial assumption of $p \in \boldsymbol{H}$.

Hence, $k = 1$ and $q_1 = q_2$, that is a single point $q$, and the Lemma 1 is proved. □

**Lemma 2:** $\exists \{x_n\}_1^\infty \in \boldsymbol{S_M} \subset \boldsymbol{H} : \lim_{n \mapsto \infty} \|p - x_n\| = \varepsilon.$

**Proof of Lemma 2:** The given infimum $\varepsilon = \inf_{f \in \boldsymbol{S_M}} \|p - f\|$ represents the greatest lower bound of the set of distances $\|p - f\|$ for all vectors $f \in \boldsymbol{S_M} \subset \boldsymbol{H}$. This means that for any $\delta > 0$, there exists a vector $f_\delta \in \boldsymbol{S_M} \subset \boldsymbol{H}$, such that $\|p - f_\delta\| < \varepsilon + \delta$, implying that elements in $\boldsymbol{S_M}$ can be found with norms arbitrarily close to $\varepsilon$. A sequence $\{x_n\}_1^\infty \subset \boldsymbol{S_M}$ can be then constructed such that $\lim_{n \to \infty} \|p - x_n\| = \varepsilon$, or in other words $\|p - x_n\| \to \varepsilon$ as $n \to \infty$.

Using the definition of the infimum $\forall n \in \mathbb{N}, \exists x_n \in \boldsymbol{S_M}$: $\|p - x_n\| < \varepsilon + \frac{1}{n}$.

This choice ensures that $\|p - x_n\|$ is within $\varepsilon$ with an arbitrary small error difference as $n \to \infty$. Since $\|p - x_n\| < \varepsilon + \frac{1}{n} \Rightarrow$ it implies that $\lim_{n \to \infty} sup\|p - x_n\| \leq \varepsilon$.

Additionally, $\forall n \in \mathbb{N}, \|p - x_n\| \geq \varepsilon \Rightarrow \lim_{n \to \infty} inf\|p - x_n\| \geq \varepsilon$.

Combining the two results, the only possible solution for the sequence $\{x_n\}_1^\infty \subset \boldsymbol{S_M}$ is $\lim_{n \to \infty} \|p - x_n\| = \varepsilon$, and the Lemma 2 is proved. □

**Theorem 5:** *For any subspace $\boldsymbol{S_M} \subset \boldsymbol{H}$ with a point element $p \in \boldsymbol{H}$ and a vector element $f \in \boldsymbol{S_M} \subset \boldsymbol{H}$ such that it satisfies the infimum statement $\varepsilon = \inf_{f \in \boldsymbol{S_M}} \|p - f\|$, there exists one single vector $g \in \boldsymbol{S_M} \subset \boldsymbol{H}$ with the property $\|p - g\| = \varepsilon$, or in a more formal way:*

$$\forall \boldsymbol{S_M} \subset \boldsymbol{H}, \forall p \in \boldsymbol{H}, \forall f \in \boldsymbol{S_M} \subset \boldsymbol{H}, \left( \varepsilon = \inf_{f \in \boldsymbol{S_M}} \|p - f\| \Rightarrow \exists! g \in \boldsymbol{S_M} : \|p - g\| = \varepsilon \right)$$

**Proof of Theorem 5:** Following Lemma 2, take two sequence of vectors $\{x_n\}_1^\infty \in \boldsymbol{S_M}$, i.e., $\{x_n\}_1^\infty, \{x_m\}_1^\infty \in \boldsymbol{S_M}$, such that $\lim_{n,m \mapsto \infty} \left\| p - \frac{x_n + x_m}{2} \right\| \leq \varepsilon$ and $\left\| p - \frac{x_n + x_m}{2} \right\| = \left\| \left(p - \frac{x_n}{2}\right) + \left(p - \frac{x_m}{2}\right) \right\| \leq \frac{1}{2}\left\| p - \frac{x_n}{2} \right\| + \frac{1}{2}\left\| p - \frac{x_m}{2} \right\|$, under the extra condition, $\forall \{x_n\}_1^\infty, \{x_m\}_1^\infty \in \boldsymbol{S_M}$, $\left\| p - \frac{x_n + x_m}{2} \right\| \geq \varepsilon$, which finally implies that $\lim_{n,m \mapsto \infty} \left\| p - \frac{x_n + x_m}{2} \right\| = \varepsilon$.

$\forall f, g \in \boldsymbol{S_M}$, recall the Cauchy–Bunyakovski inequality $\|f + g\|^2 \leq (\|f\| + \|g\|)^2 \Rightarrow \|f + g\| \leq \|f\| + \|g\|$, from which the following identity is concluded $\|f + g\|^2 + \|f - g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$.

Substituting the change of variables: $f = p - x_n$ and $g = p - x_m$:

$$\|x_n - x_m\|^2 \le 2\left(\|p - x_n\|^2 + \|p - x_m\|^2\right) - 4\left\|p - \tfrac{x_n + x_m}{2}\right\|^2 \Rightarrow$$
$$\lim_{n,m \mapsto \infty} \|x_n - x_m\|^2 \le \lim_{n,m \mapsto \infty} \left[2\left(\|p - x_n\|^2 + \|p - x_m\|^2\right) - 4\left\|p - \tfrac{x_n + x_m}{2}\right\|^2\right] \Rightarrow$$
$$\lim_{n,m \mapsto \infty} \|x_n - x_m\|^2 \le 4\varepsilon - 4\varepsilon = 0 \Rightarrow \lim_{n,m \mapsto \infty} \|x_n - x_m\| = 0$$

Now $\lim_{n \mapsto \infty} \|p - x_n\| = \varepsilon$ and $\lim_{n \mapsto \infty} \|g - x_n\| = 0$ with the extra property of $\|p - g\| \le \|p - x_n\| + \|g - x_n\| \Rightarrow \lim_{n \mapsto \infty} \|p - g\| \le \lim_{n \mapsto \infty} \|p - x_n\| + \lim_{n \mapsto \infty} \|g - x_n\| = \varepsilon + 0 = \varepsilon$.

From our initial assumption, $\|p - g\| \ge \varepsilon$ implies that $\|p - g\| = \varepsilon$, and the theorem is proved. □

**Theorem 6:** *Consider a point element $p \in \boldsymbol{H}$ with vector point elements $g, g' \in \boldsymbol{S_M} \subset \boldsymbol{H}$ such that $\|p - g\| = \inf_{g' \in S_M} \|p - g'\| = \varepsilon$. Then the vector $(\boldsymbol{p} - \boldsymbol{g}) \perp \boldsymbol{S_M}$* i.e., *the vector $(\boldsymbol{p} - \boldsymbol{g})$ is orthogonal to the subspace $\boldsymbol{S_M} \subset \boldsymbol{H}$ .*

**Proof of Theorem 6:** From the previous Theorem 5 it was proved that the point vector element $g \in \boldsymbol{S_M} \subset \boldsymbol{H}$ is the closest point to $p \in \boldsymbol{H}$. If we assume that the vector $(p - g)$ is not orthogonal to any vector $f \in \boldsymbol{S_M} \subset \boldsymbol{H}$ and consequently to the corresponding subspace $\boldsymbol{S_M} \subset \boldsymbol{H}$, this assumption implies that $\forall f \in \boldsymbol{S_M} \subset \boldsymbol{H} : (p - g, f) = \delta \ne 0$.

Furthermore, consider a vector $g' = g + \frac{\delta}{(f,f)} f \in \boldsymbol{S_M} \subset \boldsymbol{H}$. From the definition of a scalar product:

$$\|p - g'\|^2 = \left(p - g - \tfrac{\delta}{(f,f)} f, p - g - \tfrac{\delta}{(f,f)} f\right) = (p - g, p - g) - \left(p - g, \tfrac{\delta}{(f,f)} f\right) - \left(\tfrac{\delta}{(f,f)} f, p - g\right) + \left(\tfrac{\delta}{(f,f)} f, \tfrac{\delta}{(f,f)} f\right) =$$
$$\|p - g\|^2 - \overline{\tfrac{\delta}{(f,f)}} (p - g, f) - \tfrac{\delta}{(f,f)} (f, p - g) + \tfrac{|\delta|^2}{|(f,f)|^2} (f,f) = \|p - g\|^2 - \tfrac{\bar{\delta}}{(f,f)} (p - g, f) - \tfrac{\delta}{(f,f)} (f, p - g) + \tfrac{|\delta|^2}{(f,f)} =$$
$$\|p - g\|^2 - \tfrac{|\delta|^2}{(f,f)}$$

Which further implies that $\|p - g'\|^2 = \|p - g\|^2 - \frac{|\delta|^2}{(f,f)} \Rightarrow \|p - g'\| \le \|p - g\|$, which contradicts the initial statement of $\|p - g\| = \inf_{g' \in S_M} \|p - g'\|$

Hence, the vector $(p - g) \perp \boldsymbol{S_M}$ and the theorem is proved. □

**Corollary 7:** *From Theorem 11, it follows that $\forall g \in \boldsymbol{S_M} \subset \boldsymbol{H}, p \in \boldsymbol{H}, \exists f \in \boldsymbol{F} : p = g + f$, where $\boldsymbol{F}$ is the set of all orthogona; vectors $f \in \boldsymbol{F} : f \perp \boldsymbol{S_M}$,* i.e., *orthogonal to $\boldsymbol{S_M} \subset \boldsymbol{H}$.*

According to Lemma 1, the set $\boldsymbol{F}$ is closed and forms a subspace. Furthermore, the $\boldsymbol{F} = \boldsymbol{H} \ominus \boldsymbol{S_M}$ and in general, the Hilbert space H is the direct orthogonal sum (set operation between structured sets) of the two sets $\boldsymbol{S_M}$ and $\boldsymbol{F}$ i.e., $\boldsymbol{H} = \boldsymbol{S_M} \oplus \boldsymbol{F}$.

It follows that $p = g + f \Rightarrow \|p\|^2 = \|g\|^2 + \|f\|^2$.

Finally, the vector $g \in \boldsymbol{S_M}$ is the component or projection of $p \in \boldsymbol{H}$ onto the subspace $\boldsymbol{S_M}$.

**Definition 10:** *There is a unique operator $\boldsymbol{P} = P_{S_M}$ which maps the $p \in \boldsymbol{H}$ into its proper projection $g$ on $\boldsymbol{S_M} \subset \boldsymbol{H}$. This operator is called the projection operator on $\boldsymbol{S_M} \subset \boldsymbol{H}$. Mathematically, it is defined as: $g = \boldsymbol{P}p$.*

**Corollary 8:** *The projection operator $\boldsymbol{P}$ of $p \in \boldsymbol{H}$ into $g \in \boldsymbol{S_M} \subset \boldsymbol{H}$ is bounded with $\|P\| = 1$.*

**Proof of Corollary 8:** $\forall p \in H, g \in S_M, f \in F$, $\|p\|^2 = \|g\|^2 + \|f\|^2 \Rightarrow \|g\| \le \|p\|$, and it follows that $\|g\| = \|Pp\| \le \|p\| \Rightarrow \|P\| \le 1$. Finally, for the $\in \boldsymbol{S_M}$ $g = pand\|P\| = 1$. □

**Corollary 9:** *This follows the projection operator properties:*

1. *$\boldsymbol{P}^2 = \boldsymbol{P}$. Indeed, $\forall p \in \boldsymbol{H}$ the vector $g = Pp \in S_M$ and $Pg = g$ and $Pg = P(Pp) = P^2 p = g$. Hence, the property $\boldsymbol{P}^2 = \boldsymbol{P}$ holds.*

2. *The projection operator is self-adjoint, i.e.,* $\boldsymbol{P}^* = \boldsymbol{P}$. *Indeed* $\forall p_1,\ p_2 \in \boldsymbol{H}$ *holds:*
   $p_1 = g_1 + f_1$, *with* $g_1 = Pp_1$ *and* $p_2 = g_2 + f_2$, *with* $g_2 = Pp_2$
   *Take the inner product* $(g_1, p_2) = (g_1, g_2 + f_2) = (g_1, g_2) + (g_1, f_2) = (g_1, g_2)$
   *And* $(p_1, g_2) = (g_1 + f_1, g_2) = (g_1, g_2) + (f_1, g_2) = (g_1, g_2)$
   *Hence,* $(g_1, p_2) = (p_1, g_2) \Rightarrow (Pp_1, p_2) = (p_1, Pp_2) = (P^* p_1, p_2) \Rightarrow \boldsymbol{P}^* = \boldsymbol{P}$.
3. *From properties 1,2 it follows that:* $(Pp, p) = \left(P^2 p, p\right) = (Pp, P^* p) = (Pp, Pp) \geq 0$.

**Theorem 7 (general theorem of projection operators):** *For any projection operator* $\boldsymbol{P} \in B(H)$ $\exists \boldsymbol{S_M} \subset \boldsymbol{H}$ *such that* $\boldsymbol{P}$ *is a projection operator on* $\boldsymbol{S_M} \subset \boldsymbol{H}$ *under the following two properties:*

1. $\forall p_1,\ p_2 \in \boldsymbol{H}\ \left(P^2 p_1, p_2\right) = (Pp_1, p_2)$
2. $\forall p_1,\ p_2 \in \boldsymbol{H}\ (Pp_1, p_2) = (p_1, Pp_2)$

**Proof of Theorem 7:** From Property (2) and the Corollary 8, it follows that operator $P \in B(H)$ is bounded. This implies that $\|Pp\|^2 = (Pp, Pp) = \left(P^2 p, p\right) = (Pp, p) \leq \|Pp\| \|p\| \Rightarrow \|Pp\| \leq \|p\|$, which implies that the norm of the operator $P \in B(H)$ is less than one.

Consider the set $\boldsymbol{S_M} \subset \boldsymbol{H}$ of all vectors $g \in H : Pg = g$ defining a linear manifold (see Definition 15). Consider further the sequence of vectors $\{g_n\}_1^\infty \in \boldsymbol{S_M}$ with the property $\lim\limits_{n \mapsto \infty} g_n = g$. Applying the projection operator, $Pg_n = g_n$ and $Pg - g_n = Pg - Pg_n = P(g - g_n)$, which implies that $\|Pg - g_n\| = \|P(g - g_n)\| \leq \|g - g_n\|$, and $\lim\limits_{n \mapsto \infty} \|Pg - g_n\| = \|Pg - g\| = \lim\limits_{n \mapsto \infty} \|g - g_n\| = 0$, which finally implies that $Pg = g \in \boldsymbol{S_M}$, which proves the closure of $\boldsymbol{S_M}$, which eventually implies that $\boldsymbol{S_M}$ is a subspace.

Furthermore, $\forall p, h \in \boldsymbol{H}$ it holds that $P(Pp) = P^2 p = Pp$, which implies that the vector $Pp \in \boldsymbol{S_M}$. Consider a projection operator $\boldsymbol{P_p} \in \boldsymbol{S_M}$, then $(Pp, h) = (p, Ph) = (p, h) = \left(p, P_p h\right) = \left(P_p p, h\right) \Rightarrow \boldsymbol{P_p} = \boldsymbol{P}$, and the theorem is proved. □

**Corollary 10**: *The projection of a vector* $x \in \boldsymbol{H}$ *onto the subspace* $\boldsymbol{S_N} \subset \boldsymbol{H}$ *spanned by the finite set of orthogonal basis vectors* $\{\varphi_n\}_{n-1}^N$ *is given by the projection operator* $\boldsymbol{P} \in \boldsymbol{S_N} \subset \boldsymbol{H}$*:*

$$\boldsymbol{P}x = \sum_{n-1}^{N} \langle x, \varphi_n \rangle \varphi_n$$

**Proof of Corollary 10:** Let $\boldsymbol{H}$ be a Hilbert space of square-integrable functions and let $\boldsymbol{S_N} \subset \boldsymbol{H}$ be a finite-dimensional subspace spanned by the orthogonal basis functions $\{\varphi_n\}_{n-1}^N$, where $\varphi_n \in \boldsymbol{H}$. This means that $\boldsymbol{S_N} = span\{\varphi_n\}_{n-1}^N$.

Given a signal $x \in \boldsymbol{H}$, the projection of $x$ onto the subspace $\boldsymbol{S_N}$ is denoted as $\boldsymbol{P}x \in \boldsymbol{S_N}$, where $\boldsymbol{P}$ is the projection operator. Typically, the projection $\boldsymbol{P}x \in \boldsymbol{S_N}$ is defined as the unique element in $\boldsymbol{S_N}$ that minimises the "distance" from $x$ to the subspace $\boldsymbol{S_N}$. Formally, we need to find $\hat{x} \in \boldsymbol{S_N}$, such that $\|x - \hat{x}\| = \inf\limits_{s \in \boldsymbol{S_N}} \|x - s\|$, where $\|\cdot\|$ denotes the norm induced by the inner product on $\boldsymbol{H}$. The orthogonal projection theorem (***theorem* 7** above) guarantees that such a unique $\hat{x} \in \boldsymbol{S_N}$ exists in $\boldsymbol{S_N} \subset \boldsymbol{H}$ and satisfies the orthogonality condition $\forall n \in \{1, \ldots, N\},\ \langle x - \hat{x}, \varphi_n \rangle = 0$, which implies that the **error** vector $x - \hat{x}$ is orthogonal to every basis function in the subspace, i.e., $(x - \hat{x}) \perp \{\varphi_n\}_{n-1}^N$.

Let us express the projection $\hat{x}$ as a linear combination of the orthogonal basis functions with $c_n$ are the coefficients to be determined:

$$\hat{x} = \sum_{n=1}^{N} c_n \varphi_n$$

Using the orthogonality condition $\langle x - \hat{x}, \varphi_n \rangle = 0$, we obtain:

$$\langle x - \sum_{n=1}^{N} c_n \varphi_n, \varphi_m \rangle = 0, \ \forall m \in \{1, \ldots, N\}$$

Expanding the inner product gives:

$$\langle x, \varphi_m \rangle - \langle \sum_{n=1}^{N} c_n \varphi_n, \varphi_m \rangle = 0 \Rightarrow \langle x, \varphi_m \rangle - \sum_{n=1}^{N} c_n \langle \varphi_n, \varphi_m \rangle = 0$$

Since $\langle \varphi_n, \varphi_m \rangle = \delta_{n,m}$ the equation simplifies to $\langle x, \varphi_m \rangle = c_m \langle \varphi_m, \varphi_m \rangle = c_m \|\varphi_m\|^2 \neq 0$ and solving for $c_m$ we obtain $c_m = \frac{\langle x, \varphi_m \rangle}{\|\varphi_m\|^2}$. If we assume, without loss of generality, that the basis functions are normalised (i.e., the basis functions are ***orthonormal*** $\|\varphi_m\| = 1$), the coefficients become $c_m = \langle x, \varphi_m \rangle$, and the projection $\boldsymbol{P}x \in \boldsymbol{S_N}$ can be expressed as:

$$\boldsymbol{P}x = \sum_{n-1}^{N} c_m \varphi_n = \sum_{n-1}^{N} \langle x, \varphi_n \rangle \varphi_n$$

where the corollary is proved. □

The final step is to prove that the defined operator $\boldsymbol{P}x$ fulfils the projection operator properties, as defined in Hilbert spaces. Indeed:

Linearity: $\forall x, y \in \boldsymbol{H}, \forall a, b \in \mathbb{C}$:

$$\begin{aligned} \boldsymbol{P}(ax + by) &= \sum_{n-1}^{N} \langle ax + by, \varphi_n \rangle \varphi_n = \sum_{n-1}^{N} \langle ax, \varphi_n \rangle \varphi_n + \sum_{n-1}^{N} \langle by, \varphi_n \rangle \varphi_n = \\ &= a \sum_{n-1}^{N} \langle x, \varphi_n \rangle \varphi_n + b \sum_{n-1}^{N} \langle y, \varphi_n \rangle \varphi_n = a\boldsymbol{P}(x) + b\boldsymbol{P}(y) \end{aligned}$$

Idempotence: We need to show that applying $\boldsymbol{P}$ twice results in the same output as applying it once:

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{P}x) &= \boldsymbol{P}\left( \sum_{n-1}^{N} \langle x, \varphi_n \rangle \varphi_n \right) = \sum_{m=1}^{N} \langle \left( \sum_{n-1}^{N} \langle x, \varphi_n \rangle \varphi_n \right), \varphi_m \rangle \varphi_m = \\ &= \sum_{m=1}^{N} \sum_{n-1}^{N} \langle x, \varphi_n \rangle \langle \varphi_n, \varphi_m \rangle \varphi_m = \sum_{m=1}^{N} \sum_{n-1}^{N} \langle x, \varphi_n \rangle \delta_{m,n} \varphi_m = \sum_{n-1}^{N} \langle x, \varphi_n \rangle \varphi_n = \boldsymbol{P}x \end{aligned}$$

Self-Adjointness: We need to show that $\forall x, y \in \boldsymbol{H}$ $\langle \boldsymbol{P}x, y \rangle = \langle x, \boldsymbol{P}y \rangle$:

$$\begin{aligned} \langle \boldsymbol{P}x, y \rangle &= \langle \sum_{n-1}^{N} \langle x, \varphi_n \rangle \varphi_n, y \rangle = \sum_{n-1}^{N} \langle x, \varphi_n \rangle \langle \varphi_n, y \rangle \\ \langle x, \boldsymbol{P}y \rangle &= \langle x, \sum_{m-1}^{N} \langle y, \varphi_m \rangle \varphi_m \rangle = \sum_{m-1}^{N} \langle y, \varphi_m \rangle \langle x, \varphi_m \rangle \end{aligned}$$

Since, in our case of physical systems, the inner product $\langle \cdot, \cdot \rangle$ in a Hilbert space is a map from $\boldsymbol{H} \times \boldsymbol{H} \rightarrow \mathbb{R}$ to the field of scalars, then $\langle y, \varphi_m \rangle = \langle \varphi_n, y \rangle$ due to the commutativity of the inner product in real spaces, and finally $\langle \boldsymbol{P}x, y \rangle = \langle x, \boldsymbol{P}y \rangle$.

## 6. Signals Projection Operators in Hilbert Spaces

Let $\boldsymbol{H}$ be a Hilbert space of square-integrable functions $L^2(\mathbb{R}) \subset \boldsymbol{H}$, where any telecom signal $x(t)$ is represented as an element in this space. As stated previously, a fundamental feature of a Hilbert space is that it has an inner product, denoted by $\langle \cdot, \cdot \rangle$, which allows for defining concepts like orthogonality and projection. In the context of OFDM, the signals are decomposed over a set of orthogonal basis functions $\{\varphi_n(t)\}_{n-1}^{N}$, where $\varphi_n(t)$ corresponds to a sub-carrier out of the available ones in the frequency domain, with $T$ being the OFDM symbol duration, following the representation:

$$\varphi_n(t) = \frac{1}{\sqrt{T}} e^{j\frac{2\pi nt}{T}}$$

The transmitted signal, then, can be expressed as a linear combination of orthogonal subcarrier functions, modulated by a sequence of bits $X_N$:

$$x(t) = \sum_{n=0}^{N} X_N \varphi_n(t) = \sum_{n=0}^{N} \frac{X_N}{\sqrt{T}} e^{j\frac{2\pi nt}{T}}$$

These basis functions are orthonormal, where, according to *Definition 6* on the Hilbert space inner product formalism, it means that $\forall\ \varphi_n(t), \varphi_m(t) \in \boldsymbol{H}$:

$$(\varphi_m(t), \varphi_n(t)) = \langle \varphi_m(t), \varphi_n(t) \rangle = \int_{-\infty}^{+\infty} \varphi_m(t) \varphi_n^*(t) dt = \delta_{m,n}$$

which, indeed, holds for our OFDM sub-carrier's basis function:

$$(\varphi_m(t), \varphi_n(t)) = \int_{-\infty}^{+\infty} \varphi_m(t) \varphi_n^*(t) dt = \frac{1}{T} \int_{-\infty}^{+\infty} e^{j\frac{2\pi(m-n)t}{T}} dt = \frac{1}{T} \delta_{m,n}$$

Following now a more abstract mathematical formalism in our analysis, given a signal $x(t) \in \boldsymbol{H}$, we can always project this signal onto the subspace $\boldsymbol{S_N} \subset \boldsymbol{H}$ spanned by the finite set of orthogonal basis functions $\{\varphi_n(t)\}_{n-1}^{N}$. The projection of $x(t)$ onto this subspace is given by the projection operator $\boldsymbol{P}$, which is defined as:

$$\boldsymbol{P}x(t) = \sum_{n-1}^{N} \langle x(t), \varphi_n(t) \rangle \varphi_n(t) = \sum_{n-1}^{N} (x(t), \varphi_n(t)) \varphi_n(t) = \sum_{n-1}^{N} c_n \varphi_n(t)$$

where $\langle x(t), \varphi_n(t) \rangle = (x(t), \varphi_n(t))$ represents the inner product of the signal $x(t)$ with the basis function $\varphi_n(t)$, which gives the projection coefficient for the $n^{th}$ basis function. The result of this projection is a linear combination of the orthogonal basis functions, each weighted by the corresponding projection coefficient.

The projection coefficients $\langle x(t), \varphi_n(t) \rangle$ can be explicitly written as:

$$c_n = \langle x(t), \varphi_n(t) \rangle = (x(t), \varphi_n(t)) = \int_{-\infty}^{+\infty} x(t) \varphi_n^*(t) dt$$

Thus, the projection of $x(t)$ onto this subspace $\boldsymbol{S_N} \subset \boldsymbol{H}$ is given by:

$$\boldsymbol{P}x(t) = \sum_{n-1}^{N} c_n \varphi_n(t) = \sum_{n-1}^{N} \left( \int_{-\infty}^{+\infty} x(t) \varphi_n^*(t) dt \right) \varphi_n(t)$$

In a mathematical or geometrical representation, this equation represents the decomposition of the signal $x(t)$ into its components, along the orthogonal directions defined by the basis functions $\varphi_n(t)$.

**Theorem 8:** *The projection error* $e(t) = x(t) - \boldsymbol{P}x(t)$ *is orthogonal to the* $\boldsymbol{P}x(t)$ *in subspace* $\boldsymbol{S_N} \subset \boldsymbol{H}$*, i.e.,* $\langle e(t), \varphi_n(t) \rangle = 0,\ \forall \{\varphi_n(t)\}_{n-1}^{N},\ n \in \{1, \ldots, N\}$.

**Proof of Theorem 8:** Let $\boldsymbol{H}$ be a Hilbert space, and $\boldsymbol{S_N} \subset \boldsymbol{H}$ be a subspace of $\boldsymbol{H}$. The projection operator $\boldsymbol{P}$ maps any function $x(t) \in \boldsymbol{H}$ to a function $(t) \in \boldsymbol{S_N}$, such that $\boldsymbol{P}x(t)$ is the closest approximation to $x(t)$ in the subspace $\boldsymbol{S_N} \subset \boldsymbol{H}$. If we substitute the projection error $e(t) = x(t) - \boldsymbol{P}x(t)$ into the inner product $\langle e(t), \varphi_n(t)\rangle$ , i.e.,: $\langle e(t), \varphi_n(t)\rangle = \langle x(t) - \boldsymbol{P}x(t), \varphi_n(t)\rangle = \langle x(t), \varphi_n(t)\rangle - \langle \boldsymbol{P}x(t), \varphi_n(t)\rangle$.

By definition, $\boldsymbol{P}x(t)$ is a linear combination of the basis functions, i.e., $\forall m \neq n$:

$$\boldsymbol{P}x(t) = \sum_{n-1}^{N} \langle x(t), \varphi_m(t)\rangle \varphi_m(t)$$

Thus:

$$\langle \boldsymbol{P}x(t), \varphi_n(t)\rangle = \left\langle \sum_{n-1}^{N} \langle x(t), \varphi_m(t)\rangle \varphi_m(t), \varphi_n(t) \right\rangle = \left( \sum_{n-1}^{N} \langle x(t), \varphi_m(t)\rangle \right) \langle \varphi_m(t), \varphi_n(t)\rangle = \left( \sum_{n-1}^{N} \langle x(t), \varphi_m(t)\rangle \right) \delta_{m,n} = \langle x(t), \varphi_n(t)\rangle$$

And substituting the above $\langle e(t), \varphi_n(t)\rangle = \langle x(t), \varphi_n(t)\rangle - \langle x(t), \varphi_n(t)\rangle = 0$, which completes the theorem proof leading us to the next theorem. □

**Theorem 9:** *The projection error $e(t) = x(t) - \boldsymbol{P}x(t)$ minimises the $L^2$- norm, i.e.,:*

$$\|e(t)\|^2 = \|x(t) - \boldsymbol{P}x(t)\|^2 = \text{minimum}$$

**Proof of Theorem 9:** The $L^2$-norm of the error is given by $\|e(t)\|^2 = \langle e(t), e(t)\rangle = \|x(t) - \boldsymbol{P}x(t)\|^2 = \langle x(t) - \boldsymbol{P}x(t), x(t) - \boldsymbol{P}x(t)\rangle$. Assume that there is another approximation $y(t) \in \boldsymbol{S_N}, y(t) \neq x(t)$ within the subspace $\boldsymbol{S_N} \subset \boldsymbol{H}$ which minimises the $L^2$-norm. To compare those two approximations, we mutually subtract them $x(t) - y(t) = (x(t) - \boldsymbol{P}x(t)) + (\boldsymbol{P}x(t) - y(t)) = e(t) + (\boldsymbol{P}x(t) - y(t))$

Taking the $L^2$-norm of this difference

$$\|x(t) - y(t)\|^2 = \langle e(t) + (\boldsymbol{P}x(t) - y(t)), e(t) + (\boldsymbol{P}x(t) - y(t))\rangle = \|e(t)\|^2 + 2\langle e(t), \boldsymbol{P}x(t) - y(t)\rangle + \|\boldsymbol{P}x(t) - y(t)\|^2.$$

Where the term $\langle e(t), \boldsymbol{P}x(t) - y(t)\rangle = 0$ since $e(t)$ is orthogonal to the subspace $\boldsymbol{S_N} \subset \boldsymbol{H}$ and finally $\|x(t) - y(t)\|^2 = \|e(t)\|^2 + \|\boldsymbol{P}x(t) - y(t)\|^2$. This equality implies that $\|x(t) - y(t)\|^2 \geq \|e(t)\|^2$ with the minimum condition of the equality $\|x(t) - y(t)\|^2 = \|e(t)\|^2$ only when $y(t) = \boldsymbol{P}x(t)$ and the theorem is proved. □

## 7. Wireless Communication Channel Operator on Hilbert Spaces

### *7.1. Convolution Operation Representation on Hilbert Spaces*

In signal theory and wireless communications, the convolution is an operator that combines two functions to produce a third function. In wireless communication and signal processing, it's commonly used to describe the output of a linear time-invariant system when it's subjected to an input signal. Mathematically, the convolution of two functions $f(t),\ g(t) \in \boldsymbol{H}$ is an operator $\boldsymbol{L} : \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{n}} \mapsto \mathbb{R}^{\boldsymbol{n}}$ and it represents how the input signal $f(t)$ is modified as it passes through a system (wireless channel) characterised by its impulse response $g(t)$, i.e.,:

$$L(f(t),\ g(t)) = (f *\ g\ )(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = f(t) *\ g(t)$$

Recall that a Hilbert space $\boldsymbol{H}$ is a mathematical concept that generalises the notion of Euclidean space being a complete inner product space, meaning it has an inner product and is complete in the sense that every Cauchy sequence converges to a limit in the space. In the context of Hilbert spaces, convolution can be considered as a linear operator, specifically when dealing with functions on $\mathbb{R}^{n}$ that belong to the Hilbert space $L^2$. Indeed, $\forall f, g, h \in L^2$ *and* $\forall c \in \mathbb{C}$, it satisfies the properties of linearity as follows:

*Additivity property*: Let $f_1(t),\ f_2(t) \in \boldsymbol{H}$ be two functions with the convolution operator defined as:

$$L(f_1(t)+f_2(t),\ g(t)) = ((f_1+f_2) *\ g\ )(t) = \int_{-\infty}^{+\infty} (f_1(\tau)+f_2(\tau))g(t-\tau)d\tau =$$
$$= (f_1(t)+f_2(t)) *\ g(t) = \int_{-\infty}^{+\infty} (f_1(\tau))g(t-\tau)d\tau + \int_{-\infty}^{+\infty} (\ f_2(\tau))g(t-\tau)d\tau =$$
$$= (f_1(t)) *\ g(t) + (f_2(t)) *\ g(t) = L(f_1(t),\ g(t)) + L(f_2(t),\ g(t))$$

*Homogeneity:* Let $f(t) \in \boldsymbol{H}$ be a function and $c$ a scalar within the convolution operator

$$L(cf(t),\ g(t)) = (cf(t) *\ g\ )(t) = \int_{-\infty}^{+\infty} (cf(\tau))g(t-\tau)d\tau =$$
$$= (cf(t)) *\ g(t) = c\int_{-\infty}^{+\infty} (f(\tau))g(t-\tau)d\tau = c(f(t) *\ g(t)) = cL(f(t),\ g(t))$$

**Lemma 3:** *An* ***isometric*** *operator* $\boldsymbol{T}: \boldsymbol{H} \mapsto \boldsymbol{H}$ *is said to be preserve the mathematical operation of the inner product* $(\cdot,\cdot):\ H \times H \to \mathcal{F}$*, if* $\forall x, y \in \boldsymbol{H}, (Tx, Ty) = (x, y)$.

**Proof of Lemma 3:** Operators that satisfy the ***isometric*** property $\forall x \in \boldsymbol{H}, \|Tx\| = \|x\|$ are defined and known as ***isometric*** operators.

Case 1: Considering physical real systems,

$$\forall x, y \in \mathbb{R} \subset \boldsymbol{H}, \forall \boldsymbol{T}: \boldsymbol{H} \mapsto \boldsymbol{H}, (\cdot,\cdot):\ H\ x\ H \to \mathbb{R} \subset \boldsymbol{H},\ \|T(x+y)\|^2 = \|x+y\|^2 \Rightarrow (T(x+y), T(x+y)) = (x+y, x+y).$$

Expanding the right inner product part $(x+y, x+y) = (x,x) + (x,\mathrm{y}) + (y,x) + (y,y)$ and considering the conjugate Property 4 from Definition 2 of the inner product in Hilbert spaces, $(y,x) = \overline{(x,y)}$ then $(x+y, x+y) = (x,x) + (x,\mathrm{y}) + (y,x) + (y,y) = (x,x) + (x,\mathrm{y}) + \overline{(x,y)} + (y,y)$

For real Hilbert spaces $\overline{(x,y)} = (x,\mathrm{y}) \in \mathbb{R}$ hence $(x+y, x+y) = (x,x) + (x,\mathrm{y}) + (x,\mathrm{y}) + (y,y) = (x,x) + (y,y) + 2\Re(x,y)$, where $\Re$ corresponds to a real part.

In the same way, $(T(x+y), T(x+y)) = (Tx, Tx) + (Ty, Ty) + 2\Re(Tx, Ty) \Rightarrow$

$$(Tx, Tx) + (Ty, Ty) + 2\Re(Tx, Ty) = (x,x) + (y,y) + 2\Re(x,y) \Rightarrow (\boldsymbol{T}x, \boldsymbol{T}y) = (x,y)$$

Case 2: Considering general systems,
$\forall x, y \in \mathbb{C} \subset \boldsymbol{H}, \forall \boldsymbol{T}: \boldsymbol{H} \mapsto \boldsymbol{H}, (\cdot,\cdot):\ H \times H \to \mathbb{C} \subset \boldsymbol{H}$

$$\|T(x+\mathrm{i}y)\|^2 = \|x+\mathrm{i}y\|^2 \Rightarrow (T(x+\mathrm{i}y), T(x+\mathrm{i}y)) = (x+\mathrm{i}y, x+\mathrm{i}y)$$

Expanding the right product $(x+\mathrm{i}y, x+\mathrm{i}y) = (x,x) + \mathrm{i}(x,\mathrm{y}) - \mathrm{i}(y,x) + (y,y)$ and considering again the conjugate Property 4 from Definition 2 of the inner product in Hilber spaces, $(y,x) = \overline{(x,y)}$ then $(x+\mathrm{i}y, x+\mathrm{i}y) = (x,x) + (y,y) + \mathrm{i}\Big((x,\mathrm{y}) - \overline{(x,y)}\Big)$

Note that since $x, y \in \mathbb{C}$ then the $(x,\mathrm{y}) = \alpha + i\beta \in \mathbb{C}$ $\alpha$ and $\overline{(x,y)} = \alpha - i\beta \in \mathbb{C}$. Then, $\Big((x,\mathrm{y}) - \overline{(x,y)}\Big) = \alpha + i\beta - (\alpha - i\beta) = 2i\beta = 2Im(x,\mathrm{y})$ which is a pure imaginary number resulting in $(x+\mathrm{i}y, x+\mathrm{i}y) = (x,x) + (y,y) + 2Im(x,\mathrm{y})$.

Expand, then, the left part in a similar way:

$$(T(x+\mathrm{i}y), T(x+\mathrm{i}y)) = (Tx, Tx) + (Ty, Ty) + \mathrm{i}\Big((Tx, T\mathrm{y}) - \overline{(Tx, Ty)}\Big) =$$
$$= (Tx, Tx) + (Ty, Ty) + 2Im(Tx, T\mathrm{y})$$

where

$$(T(x+\mathrm{i}y), T(x+\mathrm{i}y)) = (x+\mathrm{i}y, x+\mathrm{i}y) \Rightarrow (Tx, Tx) + (Ty, Ty) + 2Im(Tx, Ty) = (x, x) + (y, y) + 2Im(x, \mathrm{y}$$

and the Lemma is proved. □

The linear operators in Hilbert spaces maintain the properties of linearity, just like in finite-dimensional vector spaces, and preserve the inner product structure, i.e., they do not change the "angle" or "length" of the vectors in the Hilbert space. Indeed:

The **norm** of a vector $x \in \boldsymbol{H}$ is defined by $\|x\| = \sqrt{(x, x)}$, which can be interpreted as the "length" of the vector. Following the previous Lemma 3, $\forall \boldsymbol{T} : \boldsymbol{H} \mapsto \boldsymbol{H}$ which preserves the inner product (isometric operator), it also preserves the norm or length of the vector $\|Tx\| = \sqrt{(Tx, Ty)} = \sqrt{(x, y)} = \|x\|$.

The **angle** between two vectors $x, y \in \boldsymbol{H}$ are related to the inner product $(x, y)$ through the Cauchy–Schwarz inequality $|(x, y)| \leq \|x\|\|y\| \Rightarrow cos(\theta) = \frac{|(x,y)|}{\|x\|\|y\|}$.

For an isometric operator $\boldsymbol{T} : \boldsymbol{H} \mapsto \boldsymbol{H}$, in the same way the angle between two vectors $\boldsymbol{T}x, \boldsymbol{T}y \in \boldsymbol{H}$ are related to the inner product $(\boldsymbol{T}x, \boldsymbol{T}y)$ through the Cauchy–Schwarz inequality $|(\boldsymbol{T}x, \boldsymbol{T}y)| \leq \|\boldsymbol{T}x\|\|\boldsymbol{T}y\| \Rightarrow cos(\varphi) = \frac{|(\boldsymbol{T}x,\boldsymbol{T}y)|}{\|\boldsymbol{T}x\|\|\boldsymbol{T}y\|} = \frac{|(x,y)|}{\|x\|\|y\|} = cos(\theta)$, which implies the logical conclusion that $cos(\varphi) = cos(\theta) \Rightarrow \varphi = \theta + 2k\pi,\ k = 0, 1, 2, \ldots$, hence the angle is preserved.

It is well noted that these properties are crucial in many applications, such as quantum mechanics and signal processing, where preserving geometric relationships is essential. The proof shows that inner product preservation implies that such operators are **isometries**, maintaining the fundamental geometric structure of the Hilbert space.

**Corollary 11:** *The convolution operator is not an **isometric** operator, i.e., it **does not preserve any inner product** in Hilbert spaces.*

**Proof of Corollary 11:** According to *Definition 6* on the Hilbert space inner product formalism, the inner product in a Hilbert space of functions that $f(t),\ g(t) \in \boldsymbol{H}$:

$$(f(t),\ g(t)) = \langle f(t),\ g(t) \rangle = \int_{-\infty}^{+\infty} f(t) g^*(t) dt$$

For an operator $L$ to preserve the inner product, it must satisfy (see Lemma 3):

$$(\boldsymbol{L}(f_1(t), g(t)), \boldsymbol{L}(f_2(t), g(t))) = (f_1(t), f_2(t))$$

Hence,

$$(L(f_1(t), g(t)),\ L(f_2(t), g(t))) = \int_{-\infty}^{+\infty} (f_1(t) * g(t))((f_2(t) * g(t)))^* dt =$$
$$\int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f_1(\tau) g(t-\tau) d\tau \right) \left( \left( \int_{-\infty}^{+\infty} f_2(\tau) g(t-\tau) d\tau \right) \right)^* dt \neq (f_1(t), f_2(t)) == \int_{-\infty}^{+\infty} f_1(t)(f_2(t))^* dt$$

Which is obviously different to the inner product:

$$(f_1(t), f_2(t)) = \int_{-\infty}^{+\infty} f_1(t)(f_2(t))^* dt$$

And, ultimately, this leads to the logical conclusion $(\boldsymbol{L}(f_1(t), g(t)),\ \boldsymbol{L}(f_2(t), g(t))) \neq (f_1(t), f_2(t))$, and the corollary is proved. □

As a conclusion, although convolution is a linear operator in Hilbert space, it may not be commonly referred to as a "***Hilbert operator***"; rather, it is indeed a ***linear operator in the context of*** $\boldsymbol{L^2}$ ***functions, which form a Hilbert space***.

A ***unitary*** operator $\boldsymbol{U}$ on a complex Hilbert space $\boldsymbol{H}$ is an ***isometric*** operator, preserving the inner product and the norm, also satisfying the following condition $U^*U = UU^* = \boldsymbol{I}$, where $U^*$ is the adjoint of $U$, such that $\forall x, y \in \boldsymbol{H}, (Ux, y) = (x, U^*y)$ and $\boldsymbol{I}$ is the identity operator. In other words, a unitary operator is a bijective (one-to-one and onto) isometry.

Since unitary operators are isometric, they preserve the inner product, i.e., $\forall x, y \in \boldsymbol{H}, (Ux, Uy) = (x, y)$ and the norm $\|Ux\| = \|x\|$. Unitary operators are always invertible, with the inverse given by the ***adjoint*** operator, $U^{-1} = U^*$. Unitary operators generalise the concept of rotations in complex vector spaces, where they represent transformations that preserve the geometry of the space without altering lengths or angles.

**Corollary 12:** *The convolution operator* $\boldsymbol{L} : \boldsymbol{H} \mapsto \boldsymbol{H}$ *has an adjoint operator.*

**Proof of Corollary 12:** The adjoint $\boldsymbol{L}^*$ of a linear operator $\boldsymbol{L} : \boldsymbol{H} \mapsto \boldsymbol{H}$ on a Hilbert space $\boldsymbol{H}$ satisfies the property $(Lf, h) = (f, L^*h)$. Then $\forall f, g, h \in L^2(\mathbb{R}) \subset \boldsymbol{H},\ \boldsymbol{L} : \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{n}} \mapsto \mathbb{R}^{\boldsymbol{n}},$ the inner products, along with the definition of the convolution operator, are defined as

$$(L(f,g),h) = \int_{-\infty}^{+\infty} (f*g)(t)\overline{h(t)}dt = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau\right)\overline{h(t)}dt = $$
$$= \int_{-\infty}^{+\infty} f(\tau)\left(\int_{-\infty}^{+\infty} g(t-\tau)\overline{h(t)}dt\right)d\tau$$

Setting a change of variable $u = t - \tau \Rightarrow t = u + \tau$,

$$(L(f,g),h) = \int_{-\infty}^{+\infty} f(\tau)\left(\int_{-\infty}^{+\infty} g(u)\overline{h(u+\tau)}du\right)d\tau = \int_{-\infty}^{+\infty} f(\tau)\overline{L^*(h)(\tau)}d\tau = \left(f, \overline{L^*h}\right)$$

Which leads us to the adjoint operator function to satisfy the condition adjoint condition

$$\overline{L^*h} = \int_{-\infty}^{+\infty} g(u)\overline{h(u+\tau)}du \Rightarrow L^*h = \int_{-\infty}^{+\infty} \overline{g(u)}h(u+\tau)du$$

□

**Corollary 13:** *The convolution operator can be an isometric operator under certain conditions.*

**Proof of Corollary 13:** Let an operator $\boldsymbol{L} : \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{n}} \mapsto \mathbb{R}^{\boldsymbol{n}}$ be a convolution operator of two functions $f(t),\ g(t) \in L^2(\mathbb{R}) \in \boldsymbol{H}$, defined as:

$$L(f(t),\ g(t)) = (f *\ g\ )(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = f(t) *\ g(t)$$

Following Corollary 11, the convolution operator preserves the inner product and the norm if the kernel $g(t-\tau)$ is a Dirac delta function, i.e., that $g(t-\tau) = \delta(t-\tau)$, where:

$$(L(f_1(t), g(t)),\ L(f_2(t), g(t))) = \int_{-\infty}^{+\infty}$$
$$(f_1(t) * g(t))((f_2(t) * g(t)))^* dt = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f_1(\tau) g(t-\tau) d\tau \right) \left( \left( \int_{-\infty}^{+\infty} f_2(\tau) g(t-\tau) d\tau \right) \right)^* dt =$$
$$\int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f_1(\tau) \delta(t-\tau) d\tau \right) \left( \left( \int_{-\infty}^{+\infty} f_2(\tau) \delta(t-\tau) d\tau \right) \right)^* dt =$$
$$= \int_{-\infty}^{+\infty} f_1(t)(f_2(t))^* dt = (f_1(t), f_2(t)) =$$

In this case, the convolution operator $\boldsymbol{L} : \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{n}} \mapsto \mathbb{R}^{\boldsymbol{n}}$ acts as the **identity operator** on the Hilbert space $\boldsymbol{H}$. That is, $\boldsymbol{L}$ maps any function $f(t) \in \boldsymbol{H}$ to itself, $\boldsymbol{L} f(t) = f(t)$. The kernel $g(t, \tau) = \delta(t - \tau)$ essentially means that there is a direct correspondence between the input signal at time $\tau$ and the output signal at time $t$, with no alteration, scaling, or delay applied to the input. Hence, since $\boldsymbol{L}$ is the identity operator, it preserves the geometry of the space. The norm and inner product of any vector (function) $f$ or $f(t)$ remain unchanged under $\boldsymbol{L}$. This corresponds to a "trivial transformation" that leaves all elements of the space unchanged. In the telecom theory of wireless channels signal processing, if the convolution kernel is a Dirac delta function, the output signal is exactly the same as the input signal. There is no filtering, distortion, or modification applied to the input; the system behaves as a perfect pass-through filter. □

**Corollary 14:** *The convolution operator can be a unitary operator under certain conditions.*

**Proof of Corollary 14:** Following Corollary 13, the convolution operator is isometric under the assumption that its kernel is a δ-Dirac function, and the convolution operator becomes an identity operator. In this special scenario, the adjoint operator is the same as the identity operator, since $(Lf, h) = (f, h) = (f, L^* h)$. □

*7.2. Convolution Operation in the Dirac Notation*

In Dirac notation (bra-ket notation), we represent states in terms of vectors in a Hilbert space.

**Lemma 4:** *In Dirac notation, where functions are represented as **ket** vectors, the convolution operation can be as following:*

$$L(\,|f\,\rangle,\ |g\,\rangle) = (f *\ g\ )(x) = \int_{-\infty}^{+\infty} \langle \boldsymbol{x} | \boldsymbol{f} \rangle \langle \boldsymbol{y} | \boldsymbol{g} \rangle dy = f(t) *\ g(t)$$

*where **bra-ket** the notation $\langle \boldsymbol{x} | \boldsymbol{f} \rangle$ and $\langle \boldsymbol{y} | \boldsymbol{g} \rangle$ respectively represent the inner products (system state) of the ket vectors $|f\,\rangle$, $|g\,\rangle$ with the position eigenkets $|x\,\rangle$, $|y\,\rangle$ respectively. Hence, $\langle \boldsymbol{x} | \boldsymbol{f} \rangle$ represents the projection (inner product) of the state $|f\,\rangle$ onto the position eigenstate $|x\,\rangle$ . In other words, $\langle \boldsymbol{x} | \boldsymbol{f} \rangle$ gives the value of the function $f$ at the position $\boldsymbol{x}$, i.e., $f(x)$. In a similar way, $\langle \boldsymbol{y} | \boldsymbol{g} \rangle$ represents the value of the function $g$ at the position $\boldsymbol{y}$, i.e., $g(y)$.*

**Proof of Lemma 4:** From signal processing point of view, the convolution $(f *\ g\ )(x)$ can be interpreted as the overlap of the function $f$, shifted by $\boldsymbol{x}$, with the function $g$. In the Dirac notation provided, the expression for the convolution can be rewritten in terms of the projections $\langle \boldsymbol{x} | \boldsymbol{f} \rangle$ and $\langle \boldsymbol{y} | \boldsymbol{g} \rangle$. We want to show that the given expression for $L(\,|f\,\rangle,\ |g\,\rangle)$ in bra-ket notation corresponds to the usual definition of convolution.

Starting with the given Dirac form:

$$L(|f\rangle,\ |g\rangle) = \int_{-\infty}^{+\infty} \langle \boldsymbol{x}|\boldsymbol{f}\rangle\langle \boldsymbol{y}|\boldsymbol{g}\rangle dy$$

we can interpret $\langle \boldsymbol{x}|\boldsymbol{f}\rangle$ as the function $f(x)$, and $\langle \boldsymbol{y}|\boldsymbol{g}\rangle$ as the function $g(y)$, meaning:

$$L(|f\rangle,\ |g\rangle) = \int_{-\infty}^{+\infty} f(x)g(y)dy \neq \int_{-\infty}^{+\infty} f(y)g(x-y)d\tau$$

It is obvious, then. that this does not yet resemble the typical convolution form. To match the traditional convolution definition, we need to . The issue arises because $f(x)$ and $g(y)$ are evaluated independently, rather than relating $g$ to a shifted argument as in the convolution formula. To reconcile these, we need to express the convolution in terms of inner products in the Dirac formalism. Consider the state $|y-x\rangle$ and then use the following interpretation $\langle \boldsymbol{x}|\boldsymbol{f}\rangle = f(x)$ and $\langle \boldsymbol{y}-\boldsymbol{x}|\boldsymbol{g}\rangle = g(x-y)$.

Then, the convolution could be totally harmonised in the Dirac formalism:

$$L(|f\rangle,\ |g\rangle) = (f * g)(x) = \int_{-\infty}^{+\infty} \langle \boldsymbol{x}|\boldsymbol{f}\rangle\langle \boldsymbol{y}-\boldsymbol{x}|\boldsymbol{g}\rangle dy = f * g$$

And the Lemma 4 is proved. □

*7.3. Wireless Channel Impulse Response as a Hilbert Operator*

In a time invariant channel, i.e., a channel considered to change slower than the observation instances produced by the action of an operator, the impulse response can be simply written as:

$$h(t) = \sum_{k=1}^{\nu} A_k e^{j\varphi_k}\delta(t-\tau_k)$$

where $t$ is the time observation reference, $\tau$ is the delay due to a multipath, $\tau_k$ is the delay of each different $k \in \{1, 2, \dots v\}$ out of $v$ paths over the wireless channel, $\varphi_k(t)$ is the different phase contribution of each path which is to be determined and $A_k(t)$ is the time dependant amplitude contribution of each path. Consider further a general base-band signal s(t), modulating an analog carrier $e^{j2\pi f_c t}$ resulting in the complex signal $x(t)$= Re$\left\{s(t)e^{j2\pi f_c t}\right\}$, being the input to the wireless channel *h*(*t*). The received signal will be the linear convolution of the input signal and the impulse response of the wireless channel [14]:

$$y(t) = L(|x\rangle,\ |h\rangle) = \int_{-\infty}^{+\infty} \langle \boldsymbol{t}|x\rangle\langle \boldsymbol{t}-\boldsymbol{\tau}|h\rangle d\tau = x(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau =$$

$$= \int_{-\infty}^{+\infty} \left[\left(\sum_{k=1}^{\nu} A_k e^{j\varphi_k}\delta(\tau-\tau_k)\right)\left(s(t-\tau)e^{j2\pi f_c(t-\tau)}\right)\right] d\tau =$$

$$= \sum_{k=1}^{\nu} A_k e^{j\varphi_k}$$

$$\int_{-\infty}^{+\infty} s(t-\tau)e^{j2\pi f_c(t-\tau)}\delta(\tau-\tau_k)d\tau = \sum_{k=1}^{\nu} A_k e^{j\varphi_k} s(t-\tau_k)e^{j2\pi f_c(t-\tau_k)} =$$

$$= \sum_{k=1}^{\nu} (A_k e^{j\varphi_k} s(t-\tau_k)e^{-j2\pi f_c \tau_k})e^{j2\pi f_c t}$$

This convolution can also be viewed as the action of a linear operator $\boldsymbol{T}$ on the Hilbert space $\boldsymbol{H}$, where $\boldsymbol{T}$ is defined by:

$$(\boldsymbol{T}x)(t) = \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau$$

The operator $\boldsymbol{T}$ acts as a kernel operator with kernel function $h(t,\tau)$, which characterises the relationship between the input signal at time τ and the output signal at time t.

The operator $\boldsymbol{T}$ should be bounded to ensure that the channel does not amplify the input signal unboundedly. Indeed, the integral operator $\boldsymbol{T}$ maps a function $x \in L^2(\mathbb{R})$ to another function in $L^2(\mathbb{R})$. We need to show that there exists a constant $\boldsymbol{C}$, such that $\forall x \in L^2(\mathbb{R}) \|Tx\| \leq C\|x\|$. Start by considering the $L^2$- norm of $(\boldsymbol{T}x)(t)$:

$$\|\boldsymbol{T}x\|^2 = \int_{-\infty}^{+\infty} |(\boldsymbol{T}x)(t)|^2 dt = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau \right|^2 dt$$

And from the Cauchy–Schwarz inequality:

$$\left| \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau \right|^2 \leq \left( \int_{-\infty}^{+\infty} |h(t-\tau)|^2 d\tau \right) \left( \int_{-\infty}^{+\infty} |x(\tau)|^2 d\tau \right)$$

which implies that:

$$\|\boldsymbol{T}x\|^2 \leq \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} |h(t-\tau)|^2 d\tau \right) \|x\|^2 dt$$

Define:

$$K = \sup_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} |h(t-\tau)|^2 d\tau \;\Rightarrow \|\boldsymbol{T}x\|^2 \leq K\|x\|^2 \Rightarrow \|Tx\| \leq C\|x\|,\ C = \sqrt{K}$$

The Rellich–Kondrachov theorem provides conditions under which certain embeddings of Sobolev spaces into $L^2$ spaces are compact. We state a version of the theorem applicable to our case:

**Rellich–Kondrachov Theorem:** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. The embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, where $H^1(\Omega)$ denotes the Sobolev space of functions with square-integrable first derivatives.*

To connect this to our integral operator, we should make some considerations. First, we have to localise the problem. Since $\mathbb{R}$ is unbounded, we consider the restriction of the integral operator to a bounded domain $\Omega \subset \mathbb{R}$. We truncate the kernel function $h(t,\tau)$ to have compact support within $\Omega$. The kernel $h(t,\tau)$ is further approximated by smoother functions in $H^1(\mathbb{R})$, where $H^1(\mathbb{R})$ is the Sobolev space of functions with square-integrable first derivatives. This approximation allows us to use the Rellich–Kondrachov theorem for compact embeddings. Hence, the approximated kernel function and the operator still satisfy the boundedness criteria.

As a conclusion, the integral operator $\boldsymbol{T}$, defined by kernel function $h(t,\tau)$, is both bounded and compact on $L^2(\mathbb{R})$, provided that the kernel satisfies the conditions mentioned above.

### 7.4. Hilbert Operators for Wireless Channel Signal Degradation Representation

Phase misalignment in 5G radio communications is a critical issue that arises primarily due to synchronization clock drifts, impacting the overall system performance. In 5G networks, precise synchronization is essential for various operations, such as beamforming, Multiple-Input Multiple-Output (MIMO) techniques [20], and time-division duplexing (TDD). The network components, including the gNB (base station) and user equipment (UE), rely on accurate clocks to maintain time and frequency alignment. However, even minor clock drifts can lead to phase misalignment, causing destructive interference, degraded beamforming gains, and reduced throughput.

Synchronization clock drifts are typically caused by discrepancies in oscillator frequencies over time, temperature variations, and imperfections in hardware. These drifts can accumulate, leading to phase errors between transmitted and received signals. The phase misalignment can degrade the quality of channel estimation and reciprocity-based beamforming, where the transmitter uses the estimated channel state information for optimal beam adjustment. In a TDD system, where the same frequency band is used for both uplink and downlink transmissions, clock drift can severely impact the channel estimation accuracy, as it assumes reciprocity between the two transmission directions.

Besides clock drifts, ***the wireless radio channel itself can also contribute to phase misalignment***. The radio channel is subject to various effects, such as multipath fading, Doppler shifts due to relative movement between the transmitter and receiver, and path delays. These phenomena introduce ***frequency-dependent phase shifts*** and ***time-varying changes in the channel response***. While the channel-induced phase variations are not directly linked to clock drifts, they can exacerbate phase misalignment by introducing additional randomness to the signal's phase.

In summary, while synchronization clock drifts are a primary cause of phase misalignment in 5G systems, the wireless radio channel's response can also contribute to this problem. A comprehensive approach to addressing phase misalignment must account for both clock stabilization techniques and robust channel estimation algorithms that can mitigate the combined effects of clock drifts and channel variations.

In the context of OFDM [21,22], as stated in previous section, the signals are decomposed over a set of orthogonal basis functions $\{\varphi_n(t)\}_{n-1}^{N}$, where $\varphi_n(t)$ corresponds to a sub-carrier out of the available ones in the frequency domain, with $T$ being the OFDM symbol duration, following the representation:

$$\varphi_n(t) = \frac{1}{\sqrt{T}} e^{j\frac{2\pi nt}{T}}$$

The transmitted signal, then, can be expressed as a linear combination of orthogonal subcarrier functions [22]:

$$x(t) = \sum_{n=0}^{N} X_N \varphi_n(t)$$

Clock misalignment can be modelled as a phase rotation applied to each subcarrier. In the Hilbert space framework, this corresponds to a **rotation operator** $\boldsymbol{U_\theta}$ defined by:

$$U_\theta x(t) = \sum_{n=0}^{N} X_N e^{j\theta_n} \varphi_n(t)$$

where $\theta_n$ represents the phase offset introduced for the $k^{th}$ subcarrier.

**Lemma 5:** *The* ***rotation operator*** $\mathbf{U_\theta}$ *is a* ***linear unitary operator*** *in Hilbert space, since it satisfies both additivity and homogeneity properties:*

*Additivity*: Let $x_1(t)$ and $x_1(t)$ be two signals in the Hilbert space, with corresponding coefficients $X_{N,1}$ and $X_{N,2}$. Then:

$$U_\theta(x_1(t)+x_2(t)) = \sum_{n=0}^{N}(X_{N,1}+X_{N,2})e^{j\theta_n}\varphi_n(t) = \sum_{n=0}^{N}X_{N,1}e^{j\theta_n}\varphi_n(t) + \sum_{n=0}^{N}X_{N,2}e^{j\theta_n}\varphi_n(t) = U_\theta x_1(t) + U_\theta x_2(t)$$

*Homogeneity*: Let $x_1(t) \in \boldsymbol{H}$ be a signal in the Hilbert space with corresponding coefficient $X_{N,1}$ and $c \in \mathbb{C}$ a scalar coefficient, then:

$$U_\theta(cx(t)) = \sum_{n=0}^{N} cX_N e^{j\theta_n}\varphi_n(t) = c\sum_{n=0}^{N} X_N e^{j\theta_n}\varphi_n(t) = cU_\theta x(t)$$

Moreover, the operator $U_\theta$ is unitary preserving the norm, since:

$$(U_\theta x(t), U_\theta x(t)) = \|U_\theta x(t)\|^2 = \left(\sum_{n=0}^{N} X_N e^{j\theta_n}\varphi_n(t), \sum_{m=0}^{M} X_M e^{j\theta_m}\varphi_m(t)\right) = \\ = \sum_{n=0}^{N} X_N X_M^* e^{j(\theta_n-\theta_m)}(\varphi_n(t),\varphi_m(t)) = \sum_{n=0}^{N} X_N X_M^* e^{j(\theta_n-\theta_m)}\delta_{n,m} = \sum_{n=0}^{N} X_N X_M^* = \\ \sum_{n=0}^{N}|X_N|^2 = = \left(\sum_{n=0}^{N} X_N\varphi_n(t), \sum_{m=0}^{M} X_M\varphi_m(t)\right) = (x(t),x(t)) = \|x(t)\|^2 \Rightarrow \|\boldsymbol{U_\theta x(t)}\|^2 = \|\boldsymbol{x(t)}\|^2$$

However, despite $\boldsymbol{U_\theta}$ preserving the norm per basis function, the orthogonality of any signal spanned by the basis $\{\varphi_n(t)\}_{n-1}^{N}$ functions is disturbed, since the rotation phase is not the same per basis vector, thus causing signal degradation by a time-dependent factor $e^{j(\theta_n(t)-\omega_n(t))}$:

$$(U_\theta x(t), U_\theta x(t)) = \left(\sum_{n=0}^{N} X_N e^{j\theta_n(t)}\varphi_n(t), \sum_{m=0}^{M} X_M e^{j\omega_m(t)}\varphi_m(t)\right) = \\ = \sum_{n=0}^{N} X_N X_M^* e^{j(\theta_n(t)-\omega_m(t))}(\varphi_n(t),\varphi_m(t)) = \sum_{n=0}^{N} X_N X_M^* e^{j(\theta_n(t)-\omega_m(t))}\delta_{n,m} = \\ = \sum_{n=0}^{N}|X_N|^2 e^{j(\theta_n(t)-\omega_n(t))} \neq \sum_{n=0}^{N}|X_N|^2 = (x(t),x(t))$$

Combining the rotation operator and the convolution operators together, the received signal will be the linear convolution of the input signal and the impulse response of the wireless channel:

$$y(t) = L(U_\theta|x\rangle,\ |h\rangle) = \int_{-\infty}^{+\infty}\langle \boldsymbol{t}|U_\theta|x\rangle\langle \boldsymbol{t}-\boldsymbol{\tau}|h\rangle d\tau = U_\theta x(t)\ * h(t) = \int_{-\infty}^{+\infty} h(\tau)U_\theta x(t-\tau)d\tau = \\ = \int_{-\infty}^{+\infty} \\ \left[\left(\sum_{k=1}^{\nu} A_k e^{j\varphi_k}\delta(\tau-\tau_k)\right)\left(\sum_{n=0}^{N}\frac{X_N}{}\sqrt{T}e^{j\theta_n}e^{j\frac{2\pi n(t-\tau)}{}T}\right)\right]d\tau = \\ = \sum_{k=1}^{\nu}\sum_{n=0}^{N}\frac{X_N}{\sqrt{T}}A_k e^{j\varphi_k}e^{j\theta_n} \\ \int_{-\infty}^{+\infty} e^{j\frac{2\pi n(t-\tau)}{}T}\delta(\tau-\tau_k)d\tau = \\ \sum_{k=1}^{\nu}\sum_{n=0}^{N}\frac{X_N A_k}{\sqrt{T}}e^{j(\varphi_k+\theta_n+\frac{2\pi n(t-\tau_k)}{}T)} = \\ = \sum_{k=1}^{\nu}\sum_{n=0}^{N}\left(\frac{X_N A_k}{\sqrt{T}}e^{j(\varphi_k+\theta_n-\frac{2\pi n\tau_k}{}T)}\right)e^{j(\frac{2\pi nt}{T})} = \sum_{n=0}^{N}\frac{X_N}{\sqrt{T}}\left(\sum_{k=1}^{\nu}\left(A_k e^{j(\varphi_k+\theta_n-\frac{2\pi n\tau_k}{}T)}\right)\right)e^{j(\frac{2\pi nt}{T})} = \\ = \sum_{n=0}^{N}\frac{X_N}{\sqrt{T}}\boldsymbol{h_{k,n}}e^{j(\frac{2\pi nt}{T})} =$$

This reveals that each subcarrier not only undergoes a phase rotation due to $\theta_n$ but also experiences amplitude degradation as well as mixing with other subcarriers (intercarrier interference, ICI) due to the frequency-shifting $e^{j(-\frac{2\pi n \tau_k}{T})} = e^{j(-2\pi n f \tau_k)}$.

**Funding:** This research received no external funding.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References


1. Akhiezer, N.I.; Glazman, I.M. *Theory of Linear Operators in Hilbert Space*; Dover Books on Mathematics: Mineola, NY, USA, 1993; ISBN 0486318656/9780486318653.
2. Mohan, K.; Narayanamoorthy, J. A Review Note on the Applications of Linear Operators in Hilbert Space. In *Structure Topology and Symplectic Geometry*; IntechOpen: London, UK, 2021. [CrossRef]
3. Yang, H.-C.; Alouini, M.-S. Digital communications over fading channels. In *Order Statistics in Wireless Communications: Diversity, Adaptation, and Scheduling in MIMO and OFDM Systems*; Cambridge University Press: Cambridge, UK, 2011; pp. 7–39.
4. Argoty, C.; Berenstein, A. Hilbert spaces expanded with a unitary operator. *Math. Log. Q.* **2009**, *55*, 37–50. [CrossRef]
5. Lusternik, L.; Sobolev, V. *Elements of Functional Analysis*; Hindustan Publ. Co.: Delhi, India, 1974.
6. Kennedy, R.A.; Sadeghi, P. *Hilbert Space Methods in Signal Processing*; Cambridge University Press: Cambridge, UK, 2013; ISBN 978-0-511-84451-5. [CrossRef]
7. Curtis, C.W.; Reiner, I. *Representation Theory of Finite Groups and Associative Algebras*; Interscience: New York, NY, USA, 1962.
8. Pontryagin, L.S. *Topological Groups*; Princeton University Press: Princeton, NJ, USA, 1958.
9. Jain, P.K.; Ahmad, K. 5.1 Definitions and basic properties of inner product spaces and Hilbert spaces. In *Functional Analysis*, 2nd ed.; New Age International: Delhi, India, 1995; p. 203. ISBN 81-224-0801-X.
10. Axler, S. *Linear Algebra Done Right*, 2nd ed.; Springer: Berlin, Germany, 1997; ISBN 978-0-387-98258-8.
11. Bauer, H. *Measure and Integration Theory*; De Gruyter Studies in Mathematics 26; De Gruyter: Berlin, Germany, 2001; p. 236. ISBN 978-3-11-016719-1.
12. Dales, H.G.; Aeina, P.; Eschmeier, J.; Laursen, K.; Willis, G.A. *Introduction to Banach Algebras, Operators and Harmonic Analysis*; Cambridge University Press: Cambridge, UK, 2003; ISBN ISBN 0-521-53584-0. [CrossRef]
13. Murphy, G.J. *C*-Algebras and Operator Theory*; Academic Press: Cambridge, MA, USA, 1990; ISBN 0-12-511360-9.
14. Louvros, S.; Kougias, I. C-Algebra Applications in Physical Systems. In Proceedings of the 2018 International Conference on Topology and its Applications, Nafpaktos, Greece, 7–11 July 2018; pp. 121–131.
15. Gustafson, S.J.; Sigal, I.M. *Mathematical Concepts of Quantum Mechanics*; Springer: Berlin, Germany, 2003.
16. Hilbert, D.; Courant, R. *Methods of Mathematical Physics*; Interscience: New York, NY, USA, 1953; Volume 1, pp. 501–508.
17. Hall, B.C. *Quantum Theory for Mathematicians*; Graduate Texts in Mathematics; Springer: New York, NY, USA, 2013; Volume 267.
18. Cohen-Tannoudji, C.; Diu, B.; Laloë, F. *Quantum Mechanics; Volume 2: Angular Momentum, Spin, and Approximation Methods*; Wiley-VCH Verlag GmbH & Co. KGaA: Weinheim, Germany, 2020; ISBN 978-3-527-82272-0.
19. Isac, G.; Cojocaru, M. The projection operator in a Hilbert space and its directional derivative. Consequences for the theory of projected dynamical systems. *J. Funct. Spaces Appl.* **2004**, *2*, 71–95. [CrossRef]
20. Zhao, Y.; Häggman, S.G. An Analysis of Orthogonal Frequency Division Multiplexing for Mobile Radio Applications. In Proceedings of the IEEE 44th Vehicular Technology Conference, Stockholm, Sweden, 8–10 June 1994; Volume 3.
21. Liu, H.; Li, G. *OFDM-Based Broadband Wireless Networks*; Design and Optimisation; John Wiley & Sons: Hoboken, NJ, USA, 2005.
22. Louvros, S.; Paraskevas, M.; Kougias, I. Heisenberg Group Operators for Multi-Carrier Signal Transmission over Wireless Channels. In Proceedings of the 2018 International Conference on Topology and its Applications, Nafpaktos, Greece, 7–11 July 2018; pp. 132–139.