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# The Bogomolny–Carioli Twisted Transfer Operators and the Bogomolny–Gauss Mapping Class Group

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**Abstract:** The twisted reflection operators are defined on the hyperbolic plane. They are then specialized in hyperbolic reflections, according to which the desymmetrized $PSL\ (2,\mathbb{Z})$ group is rewritten. The Bogomolny–Carioli transfer operators are newly analytically expressed in terms of the Dehn twists. The Bogomolny–Gauss mapping class group of the desymmetrized $PSL\ (2,\mathbb{Z})$ domain is newly proven. The paradigm to apply the Hecke theory on the CAT spaces on which the Dehn twists act is newly established. The Bogomolny–Gauss map is proven to be one of infinite topological entropy.

**Keywords:** transfer operators; Gauss map; pseudo-Anosov mapping classes; Dehn twists; topological entropy of the Bogomolny–Gauss map; Bogomolny-Carioli Theory

## 1. Introduction

Orientation-preserving homeomorphisms are studied in [1]. From [1], the result of the mapping class groups of a closed oriented surface are recalled from [2], of which their involutions correspond to quadratic surds.

From [3], the pseudo-Anosov mapping classes are studied; its 'measured foliation' corresponds to that admitting the Margulis measure from its $\sigma$-algebra on the corresponding Borel (sub-)sets. From [3], the Hamiltonian flow, which is also autonomous, is studied for pseudo-Anosov diffeomorphisms: the results here apply as the Hamiltonian flow considered is that of a free particle, for which the phase space is invariant under the action of the Hamiltonian flow maps.

It is the purpose of the paper to write the Bogomolny quantum Gauss map as one generated after two generators, which are the Dehn twists.

In several previous approaches [4–6], the dynamics of the geodesics flow of the desymmetrized $PSL\ (2,\mathbb{Z})$ group was performed according to the specification of the suitable grouping of the composition of operators (i.e., the 'words' of the 'language code'). According to the present approach, it will be possible to isolate the composition of the action of two of the generators of the group (i.e., those corresponding to the hyperbolic reflections with respect to the two degenerate geodesics) from the hyperbolic reflection on the non-degenerate geodesics. In the present approach, the Bogomolny-Carioli transfer operators [7] are considered to be applied the Dehn twist.

Furthermore, a new proof is given of the infinite entropy of the Gauss map.

The application of the Dehn twist theory to the Bogomolny–Gauss map is newly achieved. The Bogomolny–Gauss map is applied on a chosen Poincaré surface of a section of the Poincaré return map of the dynamics induced after the Bogomolny–Gauss map; the resulting reduced Birkhoff surface is a manifold of dimension 1 and genus 1: it is an appropriate portion of (degenerate) geodesics delimiting the domain of the desymmetrized



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$PSL\ (2,\mathbb{Z})$ group on the Upper Poincaré Half Plane. The method of considering two (generic) closed curves of the dynamics (i.e., geodesics) that meet transversely is used to simplify the coding of the symbolic dynamics of the suitable composition of the generators of the Bogomolny–Gauss map of the desymmetrized $PSL\ (2,\mathbb{Z})$ group.

The action of the Gauss-Bogomolny map is rewritten in terms of the Dehn-twist theory. The Dehn-twist operators needed for the definition of the suitable composition of the generators of the desymmetrized $PSL\ (2,\mathbb{Z})$ group are, therefore, newly written. The representation of the action of the hyperbolic reflections on the two degenerate geodesics which are two of the sides delimiting the $PSL\ (2,\mathbb{Z})$ domain are in this way contained in the newly-defined Dehn operators. This procedure allows one to isolate the composition of hyperbolic reflections on the two degenerate geodesics from that of the non-degenerate portion of geodesics.

The invariant functions from the Bogomolny transfer operator after the Dehn-twist representation are newly found.

As a further result, the topological entropy of the Gauss map is proven to be infinite: the protocols in [8–10] are intended for surfaces of genus $g \geq 2$; in the present proof, a new paradigm is developed for surfaces of genus $g = 1$.

One of the advantages of the presented derivations consists in not using the continued-fraction decomposition of the coordinates of the endpoints that parameterize the geodesics, i.e., differently from [11,12].

Moreover, the symbolic dynamics are codified in the Dehn-twist operator rather than in the 'language-code decomposition' as in [13].

The new paradigm finds use in the evaluation of the multiplicities of the lengths of periodic (closed) geodesics.

One of the advantages is the enormous simplifications in the analysis of the dynamical systems deriving from the Gauss map.

More in detail, it is now possible to apply the Markov partitions to the Anosov flows instead.

The paper is organized as follows:

In Section 2, the desymmetrized $PSL\ (2,\mathbb{Z})$ group is introduced.

In Section 3, the twisted reflection operators are defined on the hyperbolic plane.

In Section 4, the twisted reflection operators are defined on the desymmetrized $PSL\ (2,\mathbb{Z})$ group.

In Section 5, the Bogomolny–Gauss mapping class group of the desymmetrized $PSL\ (2,\mathbb{Z})$ domain is newly proven.

In Section 6, the invariant functions from the Bogomolny transfer operator after the Dehn-twist representation are newly studied.

In Section 7, the Dinaburg Theorem is newly extended to spaces on which the Dehn-twist operators act.

In Section 8, the paradigm to apply Hecke theory on generic CAT(K) spaces is newly established, and the instance of the Bogomolny–Gauss map is newly studied.

Prospective studies are presented in Section 9, where the adjoint of the Gauss-map operator is proven to be the operator of the operator of the Anosov flow: the two operators are proven to admit the same spectra.

In Appendix A, the topological entropy of the Gauss map is newly proven to be infinite.

In Appendix B, complements of Mathematical Cosmology are provided, as one of the congruence subgroups of the desymmetrized $PSL\ (2,\mathbb{Z})$ group is one used to describe the evolution of the Universe in the vicinity of the cosmological singularity when the time derivatives dominate the spatial gradients in the Einstein Field Equations.

## 2. The Desymmetrized *PSL* (2, $\mathbb{Z}$) Group

The Upper Poincaré Half Plane is coordinatized after the abscissa $u$ and the ordinate axis $v$, such that the geodesics are half-circumferences centered on the $u$ axis. The absolute of the Upper Poincaré Half Plane consists of the $u$ axis plus one point at infinity.

The desymmetrized $PSL\ (2, \mathbb{Z})$ is defined on the Upper Poincaré Half plane after the domain after the domain described within the portions of geodesics

$$u = 0, \tag{1a}$$

$$u = -\frac{1}{2}, \tag{1b}$$

$$u^2 + v^2 = 1. \tag{1c}$$

More in detail, the domain of the desymmetrized $PSL\ (2, \mathbb{Z})$ group contains one cusp point, which coincides with the point at infinity of the absolute.

The generators of the group are the three hyperbolic reflections on the geodesic sides of the domain.

The Hamiltonian geodesic flow is parameterized according to the oriented endpoints of the geodesics, whose portions constitute the trajectories according to which the closed curves are considered within the group domain. The Hamiltonian flow can be parameterized according to unit-velocity geodesics, whose time orientation is one determined according to the intersections with the absolute; the time evolution is dictated as starting from the endpoint $u^-$ and is directed towards the endpoint $u^+$ according to the half-circumferences of radius $r$ and of center $u^0$ as

$$\frac{\mid u^+ \mid + \mid u^- \mid}{2} = r \tag{2}$$

and

$$\frac{\mid u^+ \mid - \mid u^- \mid}{2} = u^0. \tag{3}$$

The following schematizations find use also in the study of $PSL\ (2, \mathbb{Z})$.

*The Congruence Subgroups*

The group is taken as a work protocol as it is the smallest domain, after which the further (non-Hecke) group domains on the Upper Poincaré Half Plane can be constructed.

The congruence subgroups of the desymmetrized $PSL\ (2, \mathbb{Z})$ domain tile the Upper Poincaré Half Plane. The obtained tessellation is constructed after the needed composition of hyperbolic reflections (i.e., with respect to the symmetry lines).

The further congruence subgroups are constructed after the composition of hyperbolic reflections with respect to the chosen symmetry lines needed for the tessellation.

As an example, the congruence subgroup $\Gamma_0(PSL\ (2, \mathbb{Z}))$ is one whose domain is constructed after applying the hyperbolic reflection with respect to the side Equation (1b); as a result, the congruence subgroup $\Gamma_0(PSL\ (2, \mathbb{Z}))$ is one defined after the domain

$$u = 0, \tag{4a}$$

$$u = -1, \tag{4b}$$

$$u^2 + v^2 = 1, \tag{4c}$$

$$(u+1)^2 + v^2 = 1. \tag{4d}$$

It is generated after the four hyperbolic reflections with respect to the portions of geodesics enclosing the domain Equations (4a)–(4d).

## 3. The Twisted Reflection Operators

The useful material here was picked from [2].

It will be applied to the Bogomolny Gauss map as generated from the Bogomolny–Carioli operators in the opportune specifications from [14] and from [15] passing through [16].

These concepts will be applied to the Bogomolny transfer operators and to the Gauss map.

*The Bogomolny Transfer Operators*

The Bogomolny Gauss map is generated after the three hyperbolic reflections $\mathcal{A}, \mathcal{B}, \mathcal{C}$ with respect to the domain sides of the triangular domain as

$$z \to z' \equiv \mathcal{A}z = -\bar{z} + 1, \tag{5a}$$

$$z \to z' \equiv \mathcal{B}z = -\bar{z}, \tag{5b}$$

$$z \to z' \equiv \mathcal{C}z = -\frac{1}{\bar{z}}, \tag{5c}$$

respectively.

The Bogomolny Gauss map is defined from [14]. The quantum version of the Gauss map $T$ is newly written: the following new definition is given

**Definition 1.** *Let $gX$ be the s-iterate on the geodesics (where $g$ parameterizes the Hamiltonian flow $X$). The Gauss map is defined as*

$$T_s g(x) = \sum_{n=1}^{n=\infty} \left[ \frac{1}{(x-n)^{2s}} g\left(\frac{1}{x-n}\right) + \epsilon \frac{1}{(1-x+n)^{2s}} g\left(\frac{1}{1-x+n}\right) + \frac{1}{(x-n-1)^{2s}} g\left(\frac{1}{x-n-1}\right) \right] \tag{6}$$

*where $n$ represents the iterates $(\mathcal{B}\mathcal{A})^n$ of the operator $BA$. The value of $\epsilon$ is taken as $\epsilon = 1$ in the case of Neumann boundary conditions, and as $\epsilon = -1$ in the case of the Dirichlet boundary conditions.*

From [15], the following Corollary is newly written from Definition 1 ibidem.

**Corollary 1.** *The semiclassical map acquires the following representation after the Bogomolny transfer operators $\mathbf{T}_E(q, q')$ on the chosen surface of section $\Sigma$ as*

$$\mathbf{T}_E(q, q') \int_{\Sigma} T_E(q, q') \psi(q) d^N q. \tag{7}$$

As a consequence, the $(k)$-dim Schroedinger equation is reconducted to a $(k-1)$-dim quantum map from [17]. The operator $\mathbf{T}$ satisfies the following

**Theorem 1.** *The operator $\mathbf{T}$ admits an invariant function $\tilde{\psi}$, which is written as*

$$\tilde{\psi}(q') \equiv \int_{\Sigma} T_e(q', q) \tilde{\psi}(q) d^N q. \tag{8}$$

**Proof.** The invariant function $\tilde{\psi}$ has its definition dictated after the compatibility condition of the considered Fredholm determinant as

$$det(\hat{1} - \mathbf{T}_E) = 0. \tag{9}$$

□

Accordingly, the following theorem is reported

**Theorem 2.** *The solutions of Equation (9) are proven to coincide with the dynamical zeta function as an infinite product over all the periodic orbits.*

## 4. Rephrasing in Terms of the Dehn Twists

The new protocol is here spelled out to define the Dehn twists for the Bogomolny theory.

**Definition 2.** *Let $F_{n,r}$ be an orientable surface of genus n and r boundary components.*

**Definition 3.** *Let $M_{n,r}$ be a mapping class group of $F_{n,r}$ of isotopy classes of orientation-preserving homeomorphisms of $F_{n,r}$ which leave the boundary point-like fixed.*

From [16] (pp. 44–47), $M_{n,r}$ can be generated by $2n+1$ twists with respect to closed curves, which is the minimal number of twist generators.

From [18] (Sections 1.5.3–1.5.7), given a group $G$ acting on a group $A$, the twisted group of $A$ with respect to a cocycle $\psi$, i.e., ${}_{\phi}A$ has the same group structure as $A$ but ${}_{\phi}A$ has a twisted action on $G$.

From [2], for $n = 1$ the group $M$ is generated by twists with respect to the closed curves $\alpha_1$ and $\alpha_2$. The specification is that two closed curves are equal if they are isotopic on $F$.

Homeomorphisms are composed from left to right.

From [19], the following definition is taken

**Definition 4.** *The Dehn twist with respect to a closed curve $\alpha$ depends on the orientation of the surface F but not on the orientation of the closed curve $\alpha$. The twist is indicated as the operator $T^D_\alpha$.*

From [2], the two Dehn generators are taken as

$$\mathcal{S}_n = T^D_{\alpha_1} T^D_{\alpha_2} \dots T^D_{\alpha_n}, \tag{10a}$$
$$\mathcal{R}_n = T^D_{\delta_{n-1}} T^{D}{}^{-1}_{\delta_n}. \tag{10b}$$

Let $G$ be a subgroup of $M$ generated by $\mathcal{R}_n$ and by $\mathcal{S}_n$.

From [2] (p. 379), the following paradigm is newly worked out.

**Proposition 1.** *$G = M$, i.e., the group $M$ is generated by $\mathcal{R}_n$ and by $\mathcal{S}_n$.*

**Theorem 3.** *Let F be a compact orientable surface of genus n, either closed or with one boundary component, and let M be the mapping class group of F. Thus, M can be generated by two elements.*

**Lemma 1.** *Let $\alpha$ be a closed curve, and let h be a homomorphism, then*

$$\alpha' = (\alpha)h \tag{11}$$

*which implies*

$$T^D_{\alpha'} = (T^D_\alpha)h. \tag{12}$$

**Lemma 2.** *Let $\alpha$, $\beta$ be two simple closed curves of F, and let $\alpha$ and $\beta$ meet transversely: then the following composition holds*

$$t^D_\alpha t^D_\beta t^D_\alpha = t^D_\beta t^D_\alpha t^D_\beta, \tag{13}$$

*where $t^D$ is the Dehn operator specified for the case of two curves only.*

**Proof.** The following composition holds

$$t_\alpha^{D-1} t_\beta^D t^D \alpha = t^D \beta t_\alpha^D t_\beta^{D-1}. \quad (14)$$

□

It follows that

**Theorem 4.** *The following definition of the generators holds*

$$(\beta) t_\alpha^D = (\alpha) t_\beta^D. \quad (15)$$

.

**Proof.** By construction the Proof of Lemma 2is used to simplify the composition of the elements of the group. □

## 5. The Bogomolny–Gauss Mapping Class Group of the Desymmetrized $PSL\ (2, \mathbb{Z})$ Domain

From Definition 2, the domain of the desymmetrized $PSL\ (2, \mathbb{Z})$ group is one of genus 0. The Poincaré surface of section of the Poincaré return map is one with one boundary with one hole (i.e., it is a portion of geodesic, i.e., a portion of the degenerate geodesic which is represented as a portion of the vertical line on the Poincaré Upper Half Plane).

As a comment of Definition 3, the Gauss map leaves the Poincaré surface of the section unvaried.

As an example, the composition of homeomorphisms $BA$ of the desymmetrized domain of the $PSL\ (2, F)$ group is obtained as considering the conjugate of $A$ with respect to $B$ as

$$(A)B = B^{-1}AB, \quad (16)$$

from which the following expression is found for the composition $\mathcal{B}\mathcal{A}_n$ contained in the Gauss map

$$A^{-1}B(A)BA = \mathcal{B}\mathcal{A}. \quad (17)$$

The operator $\mathcal{A}\mathcal{B}$ is found as

$$B^{-1}A(B)AB = \mathcal{A}\mathcal{B}_n, \quad (18)$$

where $A \equiv t_{\alpha_1}^D$ and $B = t_{\alpha_2}^D$.

The reflection with respect to the non-degenerate geodesics side of the triangle is obtained after the operator $\mathcal{R}_1$ in Equation (10b).

As a comment to Definition 4, one notices that the Poincaré surface of a section of the Poincaré return map of the Hamiltonian flow (and, in particular, of the closed curves of the Hamiltonian flow) is oriented. The closed geodesics are oriented as well. The orientation is defined after the application of the tools from [20] (Proposition 2.10 item (3)), which allows one to define the orientation of biinfinite geodesics form the notion of boundaries of hyperbolic groups.

The specification of Lemma 2 was chosen because two closed geodesics of the desymmetrized $PSL\ (2, \mathbb{Z})$ domain necessarily meet transversely, i.e., the case of two closed geodesics that overlap is not comprehended, as the closed curves are in $1 - to - 1$ correspondence with the initial conditions, as from the boundary topology (Proposition 2.10 (3) from [20]).

The quantum Dehn-twist version of the Gauss map $T$ of closed geodesics is, therefore, written and the following new definition is obtained

**Definition 5.** *Let $gX$ be the s-iterate on the closed geodesics $gX$ (where $g$ parameterizes the Hamiltonian flow $X$).*

*The quantum Dehn-twist version of the Gauss map $T$ of closed geodesics is*

$$T_s g(x) = [\mathcal{B}\mathcal{A}_n g(z) + \epsilon \mathcal{A}\mathcal{B}_n g(z) + \mathcal{R}_1 g(z)] \tag{19}$$

*where $n$ represents the iterates $(BA)^n$ of the operator $BA$. The value of $\epsilon$ is $\epsilon = 1$ in the case of Neumann boundary conditions, and $\epsilon = -1$ in the case of the Dirichlet boundary conditions.*

## 6. The Invariant Functions from the Bogomolny Transfer Operator After the Dehn-Twist Representation

The following new theorem is given

**Theorem 5.** *The Bogomolny transfer operators on closed geodesics admit the Dehn-twist representation*

$$\mathbf{T} \equiv \mathcal{S}_1 \mathcal{R}_\infty. \tag{20}$$

**Proof.** By construction. The proof by construction consists of constructing the operator **T** after the composition of the hyperbolic reflections $A$ and $B$ from Definition 5, i.e., for a total of $n$ compositions of the operators plus the hyperbolic reflection $\mathcal{R}_1$ on the non-degenerate-geodesics domain side. □

**Remark 1.** *In the case an alternative derivation is looked for, one notices that the Poincaré surface of a section of the map is of genus* 1.

The following new theorem is given

**Theorem 6.** *The operator* **T** *admits an invariant function $\tilde{\psi}$, which is written as*

$$\tilde{\psi}(q') \equiv \mathcal{R}_\infty \mathcal{S}_1 \tilde{\psi}(q). \tag{21}$$

**Proof.** After applying Theorem 4 and after the results of [1] (after which the phase space is left invariant, where the definition of the Anosov flow of this problem is taken from [21]). □

**Remark 2.** *The Anosov flow is associated with the free Hamiltonian with the appropriate boundary conditions, which coincide with infinite-potential walls corresponding to the group domain. The free Hamiltonian with the appropriate boundary conditions leaves the fundamental symplectic form invariant, i.e., the phase space available in the system is invariant under the Hamiltonian flow.*

*Because the Bogomolny–Carioli transfer operators admit the Dehn twist representation (after Theorem 4), the results from [1] can be applied.*

## 7. Discussion

The boundary maps were very recently used in [22] to achieve the pushforward of conformal measures. The Anosov subgroups and some related subgroup structures are considered.

Furthermore, ibidem, the ergodicity of horospherical foliation with respect to the wanted measure generalizes the properties of Borel–Anosov subgroups in higher rank: the procedure can be used to compare with the foliations introduced in [23].

In [24], in the case of strictly negative-curvature spaces, the concept of volume entropy is related to that of topological entropy according to the possibility of defining a metric on the considered manifold: this case is comprehended. As a result, it is possible to extend the proof that Theorem 4 from [25] holds in the here-studied case, i.e., that

**Theorem 7.** *The volume entropy of the metric of the Upper Poincaré Half Plane equals its topological entropy.*

## 8. Application of Hecke Theory on Dehn Twists for CAT(k) Spaces

Application to the schematizations of CAT spaces from [20] is here provided; the purpose of the investigation is to upgrade the discussed measure(s) ibidem to newly assess the connection from the found CAT space with the Maass forms on $\mathcal{L}^2$ with the Hilbert measure.

If the curvature $K$ is negative, the CAT(K) space is hyperbolic; the geodesic line between any two points is unique. From Definition a p. 3 ibidem, a space of curvature $\leq 0$ is a geodesic metric space 'where every point has a convex CAT(K) neighbourhood'. In Definition 2.4 ibidem, the Hausdorff measure is defined. In Definition 2.7 ibidem, the geodesic boundaries are defined for a $\delta$- hyperbolic metric space $(X, d)$; for a point $x \in X$, $x$ is named the 'base point'. The relative geodesic boundaries of $X$ with respect to $x$ are $\mathcal{F}^g_x X = \{ \mid \gamma \mid : \gamma : [0, \infty) \rightarrow X$ is a geodesic ray $\gamma(0) = x$ (with $\mathcal{F}$ denoting the (also generic) boundaries).

In the present case, the $\sigma$-algebra is considered on the Borel (sub-)set(s) on which there are eigenfunctions of the Laplacian (where the Laplacian enjoys the rigidity property); this way, each geodesic distance line connects two points in which the Maass forms are defined: after the application of Hecke theory [26], the eigenfunctions exhibit at most polynomial growth, from which fact the existence of the geodesic boundaries let the announced aims follow. The classical Bogomolny transfer operators (defined in [27]) assure that the found measure is conserved in the iterations of the Bogomolny–Gauss map; the iterations of the Bogomolny–Gauss map are, therefore, newly expressed through the Dehn-twist representation: the new following paradigm holds

**Theorem 8.** *The Dehn twist leaves the geodesic boundaries of the CAT space invariant.*

**Proof.** There exists a one-to-one correspondence between the geodesic boundaries of the CAT(-1)spaces and the Bogomolny–Gauss map expressed in terms of the Dehn twists. □

**Corollary 2.** *The geodesics rays which are defining relative geodesic boundaries are invariant under the Bogomolny–Gauss map.*

**Theorem 9.** *The Mass forms are ensured to be invariant under the Dehn-twists (issued from the Bogomolny–Gauss map).*

**Proof.** The classical Bogomolny transfer operator is chosen in order to make the correspondence hold. □

## 9. Prospective Studies

The features of the dynamical zeta function of the Dehn twist class mapping are still under investigation.

The analysis of [28] is aimed at defining the Fredholm determinants in topology, with emphasis on the graph on which it is indicated; the possibility to describe the closed geodesics from automorphisms on trees and then to apply the Margulis function on the resulting expression should be considered after [1].

For this, it is necessary to remark from [29] that it is possible to define a nuclear operator from $T_s$, from which it is possible to prove that

**Theorem 10.** *$T_s$ and its adjoint (Anosov-flow) operator U admit the same spectra.*

**Proof.** It is calculated straightforwardly that

$$T_s g X = \epsilon g X \tag{22}$$

with $\epsilon = 1$ at Neumann boundary conditions, and $\epsilon = -1$ at Dirichlet boundary conditions. □

The following cases of the Fredholm-related determinants are to be analyzed next:

(a) $det[\hat{I} - T_s] = 0$, and
(b) $det[\hat{I} + T_s] = 0$.

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## Appendix A. The Closed Geodesics of the Bogomolny Map Have Infinite Topological Entropy

The aim of the present Appendix is to review the phenomena which lead to the need to find an alternative approach to that of homology classes of negative-curvature surfaces, due to the fact that the topological entropy of the problem addressed in the present paper is infinite; moreover, it is here stressed that the most useful approach is not the Lorentz-gas approach [30] but the approach proposed in [31] for the determination of the domain on which the Hamiltonian flow is considered. Indeed, in the definition of [31], the criteria for the definition of the domain of the billiard approach are dictated; differently, in [30], part of the billiard domain is considered for the geodesics to stay, in which the Gauss map does not apply.

In [8], after [23] and after [9], a countable infinity of closed geodesics is proven to exist with asymptotic formulas of the lengths. From Theorem pag 379 ibidem, one states that

**Theorem A1.** *There existsthe C constant, and there exists $h > 0$ such that*

$$lim_{t\to\infty} \frac{\pi(t,\alpha)}{Ce^{ht}/t^{g+1}} \to 1 \tag{A1}$$

*with $\pi(t,\alpha)$ being the number of closed geodesics $\gamma$ for which the length description $l[\gamma] \leq t$, $[\gamma] = \alpha$ length; $[\gamma] \in \mathcal{H}(s,\mathbb{Z})$ is the associated homology class.*

From [32], and from [10], the curvature is equal to $-1$ and the following properties hold for a surface of genus $g \geq 2$:

$$h = 1, \tag{A2}$$

$$C = (g-1)^g. \tag{A3}$$

In the present paper, the surface considered is the surface of a section of the Poincaré return map, which has a genus $g = 1$.

Topological entropy for geodesics flows is described in [33].

It could be possible to scrutinize also the result from [10]. Ibidem $\mathcal{M}$ is a compact Riemann surface of negative curvature which is considered. The number $\pi(x)$ of closed geodesics on $\mathcal{M}$ whose length is at most $x$ is counted for $L$ length $L \leq x$ as

$$\pi(x) \sim \frac{Ce^{hx}}{x} \tag{A4}$$

$h$ is the topological entropy of the geodesics flow and $C$ a suitable constant.

From [8], the Jacobian torus of closed geodesics with the (flat metric) induced after the harmonic one-forms on $\mathcal{M}$ is deduced. For distinct homology classes, the equidistribution property holds as

$$lim_{x\rightarrow\infty}\frac{\pi(x,\alpha)}{\pi(x,\beta)} = 1. \tag{A5}$$

The projection of the fundamental group onto the first homology group is considered in [10].

It is our aim to recall from [23] that the topological entropy for the here-chosen problem is infinite.

The tools from [23] are here briefly recalled.

Let $b(R)$ be the number of closed geodesics whose length does not exceed $k$ on an m-dimensional manifold $\mathcal{M}^m$; let $D_R(y)$ be the volume of a spherical neighborhood of radius $R$ whose center is at $y$. The geodesics flux $T_X$ is Anosov.

From Th. 1 from [23], $\exists$ continuous positive functions $c_1$, $c_2$ on $\mathcal{M}^m$ and $d$ a constant, from which, from Remark 1 ibidem,

$$D_R(y) \sim c_1\frac{e^{dR}}{d}. \tag{A6}$$

From Th. 3 from [23], $\exists$ a constant $c$ such that

$$b(R) \sim c\frac{e^{dR}}{R}. \tag{A7}$$

Theorem 4 from [23] defines that, for a manifold of curvature $K$, $b(R)$ is calculated as

$$b(R) \sim \frac{e^{R(m-1)K}}{R(m-1)K}. \tag{A8}$$

Theorem 4 from [23] admits the new development

**Theorem A2.** *The topological entropy for the free group of the Hamiltonian flow on the chosen surface of the section is infinite*

**Proof.** The $m = 1$ manifold $\mathcal{M}^1$ is chosen, such that Equation (A8) implies that $b(R) \sim \infty$; from the comparison with Equation (A4), one has that the topological entropy $h$ is infinite. □

Furthermore, the new determination is found

**Theorem A3.** *The constant C from Equation (A4) is finite.*

**Proof.** *From Equation (A8).* □

It is possible to remark that the result Theorem A3 agrees with the Main Theorem from [34].

Moreover, the following Theorem is given

**Theorem A4.** *The ratio $c_2$ from Theorem 2 from [23] is finite.*

The Hamiltonian flow is characterized as in the following

**Theorem A5.** *The ratio $e^{\mu_X(T^i(x)}/e^{\mu_X(x)}$ is vanishing; therefore, the constant c from Theorem 5 from [23] is $c \equiv -\infty$.*

Therefore,

**Theorem A6.** *The ratio $\alpha_{n_1,n_2}$ from Theorem 6 from [23] is vanishing.*

## Appendix B. The Lifshitz–Khalatnikov–Sinai-Kanin–Shur Map

The Lifshitz–Khalatnikov–Sinai–Kanin–Shur (LKSKS) map [35,36] is developed in order to construct a map also for the oriented endpoint $u^-$ of the geodesics associated to the Gauss map of $u^+$ for the congruence subgroup $\Gamma_2(PSL\ (2,\mathbb{Z}))$ of the desymmetrized $PSL\ (2,\mathbb{Z})$ domain, where the oriented endpoints are those parameterised in Equation (2) and in Equation (3) as from [37]. More in detail, the LKSKS map is one aimed at describing the time evolution of the variable $u^-$. It was pointed out that the time evolution of the $u^+$ variable within the $\Gamma_2$ congruence subgroup corresponds to the domain of the projection of the solution of the Einstein Field Equations within the paradigm in which the time derivatives dominate the spatial gradients (the so-called Belinski–Khalatnikov–Lifshits (BKL) regime [38–42]) after projections on the unit disc, and therefore, on the Upper Poincaré Half Plane of the dynamics which takes place in the Physical spacetime; within this interpretation, the variable $u^+$ is interpreted as the initial conditions of the solutions of the Einstein Field Equations. The degree of freedom offered after the variable $u^-$ is not comprehended within the initial conditions of the Field Equations; it has been commented to be related to initial degrees of anisotropies. The geodesics of the trajectories of the desymmetrized $PSL\ (2,\mathbb{Z})$ group consist of periodic geodesics, non-periodic non-singular chaotic-regime geodesics, and singular geodesics; more in detail, the singular geodesics are geodesics whose future-oriented endpoint is mapped in the point at infinity (the cusps) of the domain; their initial conditions consist of a nonnull set of the unit interval. It can be studied according to the Gauss–Kuzmin Theorem, but it has not been investigated yet.

The LKSKS map(s) are, therefore, maps that relate the Gauss map of the $u^-$ variable to that of the $u^+$ variable; more in detail, the double sequence of the entries that characterize the continued-fraction decomposition of the variables describing the oriented endpoints is built as the $u^-$ decomposition inheriting one entry at each step of the $u^+$-variable evolution.

More specifically, a recurrence formula is worked out for the statistical properties of the stationary limit and a double-infinite sequence is worked out and is uniform in its statistical properties. The evolution formula of the $u^-$ variable in its continued-fraction decomposition consists of a retrograde sequence of the denominators (which are inherited after the application of the map to the $u^+$ variable, as already anticipated).
This construction is apt for the description of the dynamics of the universe evolution close to the cosmological-singularity limit, as it is explained after considering the tiling of the Upper Poincaré Half Plane according to the desymmetrized $PSL\ (2,\mathbb{Z})$ domain to the $\Gamma_2$ congruence subgroup based on the hyperbolic reflection is equivalent to the tiling of the portion of geodesics connecting the sides of the $PSL\ (2,\mathbb{Z})$ desymmetrized domain according to the Kasner transformations [43]; the result is of use in the study of $PSL\ (2,\mathbb{Z})$.

The phase space available for the dynamics of the $\Gamma_2$ congruence subgroup is described in [43] as the available regions of the coordinate space plus the region of the momentum space, which acquires the shape of a 'slab' [43,44]. According to this analysis of the complete phase space, the Hamiltonian flow can be analyzed according to its Anosov properties, i.e., the definition of reduced Birkhoff surfaces is apt to reconduct to the Poincaré surface of a section of the Gauss map to be framed within the properties of the surfaces of the complete phase space [21].

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