International Journal of *Topology*

Article

Johnstone'e Non-Sober Dcpo and Extensions

Dongsheng Zhao

Department of Mathematics and Mathematics Education, National Institute of Education Singapore, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616, Singapore; dongsheng.zhao@nie.edu.sg

Abstract: One classic result in domain theory is that the Scott space of every domain (continuous directed complete poset) is sober. Johnstone constructed the first directed complete poset (dcpo for short) whose Scott space is not sober. This non-sober dcpo has been used in many other parts of domain theory and more properties of it have been uncovered. In this survey paper, we first collect and prove the major properties (some of which are new as far as we know) of Johnstone's dcpo. We then propose a general method of constructing a dcpo from given posets and prove some properties. Some problems are posed for further investigation. This paper can serve as a relatively complete resource on Johnstone's dcpo.

Keywords: directed complete posets; scott topology; sober space; well-filtered space

MSC: 06B35; 06B30; 54A05

check for updates

Academic Editor: Michel Planat

Received: 21 November 2024
Revised: 7 February 2025
Accepted: 19 February 2025
Published: 3 March 2025

Citation: Zhao, D. Johnstone'e Non-Sober Dcpo and Extensions. *Int. J. Topol.* **2025**, *2*, 3. https://doi.org/10.3390/ijt2010003

Copyright: © 2025 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Sobriety is one of the most important properties for non-Hausdorff topological spaces. It is a classic result in domain theory that the Scott space of every domain (continuous directed complete posets) is sober. Johnstone [1] constructed the first dcpo whose Scott space is not sober (this dcpo will be called Johnstone's dcpo). This elegant example inspired many in-depth work on the properties of Scott spaces of posets. For example, Isbell [2] constructed the first complete lattice whose Scott space is not sober. Kou [3] gave the first dcpo whose Scott space is well filtered but not sober (see [4] for another more straightforwardly defined example of such a dcpo), thus answering the problem posed by Heckmann in [5]. Jia [6] found the first countable dcpo with a well-filtered and non-sober Scott space. In [7], Xu, Xi and Zhao constructed a complete Heyting algebra whose Scott space is not sober, thus answering the problem posed by Achim Jung. Applying Johnstone's dcpo, Ho, Goubault-Larrecq, Jung and Xi [8] constructed a dcpo P whose Scott space is not sober and the sobrification of the Scott space of P is the Scott space of a dcpo Q, showing that there are two non-isomorphic dcpos such that their Scott closed set lattices are isomorphic. In [9], Miao, Li, Xi and Zhao constructed a countable complete lattice with a non-sober Scott space. Recently, Miao, Xi, Jia, Li and Zhao found two sober dcpos such that their order product is not sober [10]. In [11], Xi and Lawson proved that the Scott space of every complete lattice is well filtered; thus, the non-sober complete lattice constructed by Isbell was actually the first example of a well-filtered and non-sober dcpo.

In the current paper, we will collect the major properties of Johnstone's dcpo, then suggest a general method of constructing a dcpo from the given posets. Some open problems related to Johnstone's dcpo will be posed for further study.

2. Preliminaries

We quickly recall some basic concepts and results in domain theory that will be used later.

For any subset A of poset P, $\bigvee A$ ($\bigwedge A$, resp.) will denote the supremum (infimum, resp.) of A in P.

A nonempty subset D of a poset is *directed* if every two elements in D have an upper bound in D (for any $x, y \in D$, there is a $d \in D$ such that $x \leq d$ and $y \leq d$).

A subset U of a poset $(P, \leq)$ is *Scott open* if (i) U is an upper set ($U = \uparrow U = \{x \in P : y \leq x \text{ for some } y \in U\}$), and (ii) for any directed subset $D \subseteq P$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. All Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called the *Scott topology* on P. The space $(P, \sigma(P))$ is denoted by ΣP, called the *Scott space* of P. The complements of Scott open sets are called Scott closed sets.

Remark 1. *(1) For any element x of poset P, the closure $cl(\{x\})$ of $\{x\}$ with respect to the Scott topology equals*

$$\downarrow x = \{y \in P : y \leq x\}.$$

(2) A mapping $f : P \longrightarrow Q$ between two posets is continuous with respect to the Scott topologies on P and Q, respectively, if and only if for any directed subset $D \subseteq P$ with $\bigvee D$ existing, it holds that $f(\bigvee D) = \bigvee f(D)$. Such a mapping f is said to be Scott continuous.

(3) A subset $F \subseteq P$ is Scott closed if it is a lower set ($F = \downarrow F = \{x \in P : x \leq y \text{ for some } y \in F\}$) and for any directed set $D \subseteq F$, it holds that $\bigvee D \in F$ whenever $\bigvee D$ exists.

A poset is called a *directed complete poset* (*dcpo* for short) if every directed subset of the poset has a supremum. For more about Scott topology and dcpos, we refer the reader to [12,13].

For two elements x and y in poset P, x is *way below* y, written as $x \ll y$, if, for any directed set $D \subseteq P$ with $\bigvee D$ existing and $\bigvee D \geq y$, there is some $d \in D$ such that $x \leq d$.

An element $x \in P$ is called *compact* if $x \ll x$. The set of all compact elements of P is denoted by $K(P)$.

A poset L is called *continuous* if, for each $x \in L$, $\{z \in L : z \ll x\}$ is directed and $x = \bigvee \{z \in L : z \ll x\}$.

A continuous dcpo is often called a *domain*.

A poset L is *algebraic* if, for each $x \in L$, $\{z \in L : z \leq x, z \in K(L)\}$ is directed and $x = \bigvee \{z \in L : z \leq x, z \in K(L)\}$.

A nonempty subset F of topological space X is *irreducible* if, for any closed subsets F_1, F_2 of X, $F \subseteq F_1 \cup F_2$ implies $F \subseteq F_1$ or $F \subseteq F_2$.

Clearly, every singleton set is irreducible. Also, the closure of every singleton set is irreducible (see the remark after Definition O-5.5 in [12]). A space X is called *sober* if every irreducible closed set F of X is the closure of a unique singleton set: there is a unique $x \in X$, such that $F = cl(\{x\})$.

A subset A of a topological space is called *saturated* if A equals the intersection of all open sets containing it. A T_0 space X is *well filtered* if, for any open set U and filtered family $\mathcal{F}$ of saturated compact subsets of X, $\bigcap \mathcal{F} \subseteq U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$. Every sober space is well filtered (Theorem II-1.21 of [12]).

The *specialization order* $\leq_\tau$ on a T_0 space (X, τ) is defined by $x \leq_\tau y$ if $x \in cl(\{y\})$ (for $x, y \in X$).

A T_0 space X is called *bounded -sober* if every upper bounded (with respect to the specialization order) irreducible closed set equals the closure of a singleton set [14]. Clearly, every sober space is bounded sober. The converse conclusion is not true.

3. Johnstone's Dcpo

One of the classic results in domain theory is that the Scott space of every domain is sober (Corollary II-1.13 [12]). However, it is not trivial whether the Scott space of every dcpo is sober. Peter Johnstone constructed the first example of a dcpo whose Scott space is not sober.

Example 1 ([1]). *Let $\mathbb{N}$ be the set of all positive integers with the ordinary order $\leq$ of numbers. Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. Extend the order on $\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$ by letting $n < \infty$ for all $n \in \mathbb{N}$.*

Define the order $\leq$ on $\mathbb{J}$ by $(m,n) \leq (m',n')$ if either

(i) $m = m'$ and $n \leq n'$;

(ii) Or $n' = \infty$ and $n \leq m'$.

All the major properties of $\mathbb{J}$ will be numbered as (Jk) with k a number. Among the nine properties listed below, (J5), (J7), (J8) and (J9) are new (as far as we know).

The properties (J1) and (J2) below are from [1].

(**J1**) A subset $D \subseteq \mathbb{J}$ is directed if either it has a largest element, or there is an m such that $D = \{(m, d_i) : i \in I\} \subseteq \downarrow(m, \infty)$.

In the latter case, $\bigvee D = (m, \bigvee\{d_i : i \in I\})$. Here, either $\bigvee\{d_i : i \in I\} = \infty$ or $\bigvee\{d_i : i \in I\} = n$ for some $n \in \mathbb{N}$.

Thus, in both cases, $\bigvee D$ exists, showing that $\mathbb{J}$ is a dcpo.

(**J2**) A nonempty Scott closed set F of $\mathbb{J}$ is irreducible if and only if either $F = \downarrow(m,n)$ for some $(m,n) \in \mathbb{J}$ or $F = \mathbb{J}$.

Clearly, $\mathbb{J} \neq \downarrow(m,n) = cl(\{(m,n)\})$ for any $(m,n) \in \mathbb{J}$; thus, the Scott space $(\mathbb{J}, \sigma(\mathbb{J}))$ is not sober.

However, as $\mathbb{J}$ does not have the top element, we see that $\Sigma\mathbb{J}$ is bounded sober. Hence, $\mathbb{J}$ is a dcpo whose Scott space is bounded sober but not sober.

(**J3**) $\Sigma\mathbb{J}$ is not well filtered.

To see this, for each $n \in \mathbb{N}$, let $A_n = \{(k, \infty) : k \geq n\}$ and $U = \emptyset$. Then, one can verify that each A_n is a compact saturated set, $\{A_n : n \in \mathbb{N}\}$ is a filtered family and $\bigcap\{A_n : n \in \mathbb{N}\} = \emptyset = U$. Since not one of the A_n is contained in $\emptyset = U$, we see that $\Sigma\mathbb{J}$ is not well filtered.

In the following, we shall show some less trivial properties of $\mathbb{J}$. Some of them have been mentioned or proved elsewhere. However, for the reader's convenience, we shall provide a brief explanation of them.

An element a of poset P is a maximal element (or maximal point) if, for any $y \in P$, $a \leq y$ implies $a = y$. The set of all maximal points of P is denoted by Max(P). In the following, the set Max(P) with the relative Scott topology on P will be called the maximal point space of P. By Zorn's Lemma, one deduces that for any dcpo P, $P = \downarrow\text{Max}(P)$; that is, every element of P is below some maximal point(s).

The following result has been known for quite some time; however, we could not find a resource where a complete proof was provided. Here, we give a brief explanation. See the more general Proposition 3 in Section 4.

Proposition 1. *The space Max($\mathbb{J}$) is homeomorphic to $(\mathbb{N}, \tau_{cof})$, where τ_{cof} is the co-finite topology on $\mathbb{N}$ ($U \in \tau_{cof}$ if either $U = \emptyset$ or $\mathbb{N} - U$ is a finite set).*

To see this, first note that Max($\mathbb{J}$) = $\{(m, \infty) : m \in \mathbb{N}\}$.

Let $F \subseteq \mathbb{J}$ be a nonempty Scott closed set. If $F \cap \text{Max}(\mathbb{J})$ is an infinite set, then one can deduce that $F = \mathbb{J}$.

Now, for any finite subset $G \subseteq \text{Max}(\mathbb{J})$, $\downarrow G$ is a Scott closed set of $\mathbb{J}$ and
$G = \downarrow(G \cap \text{Max}(\mathbb{J}))$.

It follows that a subset of Max($\mathbb{J}$) is closed (with respect to the relative Scott topology) if and only if it is either a finite set or the whole set of Max($\mathbb{J}$). That is, the relative Scott topology on Max($\mathbb{J}$) is the co-finite topology.

Clearly, there is a bijection between $\mathbb{N}$ and Max($\mathbb{J}$); thus, the space Max($\mathbb{J}$) is homeomorphic to $(\mathbb{N}, \tau_{cof})$.

A *poset model* of a T_1 space X is a poset P such that X is homeomorphic to Max(P). For any P, Max(P) is T_1 [15]. By the above proposition, we have the following.

(**J4**) The dcpo $\mathbb{J}$ is a poset model of $(\mathbb{N}, \tau_{cof})$.

Proposition 2. *For any continuous function $f : Max(\mathbb{J}) \longrightarrow Max(\mathbb{J})$, there is a continuous function $F : \Sigma P \longrightarrow \Sigma P$ such that $F_{Max(\mathbb{J})} = f$.*

Proof. Let $p : \mathbb{J} \longrightarrow \mathbb{N}$ be a function defined by $p(m, n) = m$.

Let f : Max($\mathbb{J}$)$\longrightarrow$Max($\mathbb{J}$) be a continuous function. For each $n \in \mathbb{N}$, define

$$\alpha(n) = inf\{p(f(k, \infty)) : n \leq k\}.$$

Then, $\alpha : \mathbb{N} \longrightarrow \mathbb{N}$ is a function and monotone: $n_1 \leq n_2$ implies $\alpha(n_1) \leq \alpha(n_2)$.

(1) If α is a bounded function (there is an l such that $\alpha(n) \leq l$ for all n), then f is a constant function.

In fact, assume that $\alpha(n) \leq l$ holds for all n. Then, for any $n \in \mathbb{N}$, there exists $n' \geq n$ such that $f(n', \infty) \in B = \{(m, \infty) : m \leq l\}$, which is a closed set in Max($\mathbb{J}$). Thus, $f^{-1}(B)$ is an infinite and closed set; hence, $f^{-1}(B) =$ Max($\mathbb{J}$) since the topology on Max($\mathbb{J}$) is the co-finite topology. But then Max($\mathbb{J}$) $= f^{-1}(B) = \bigcup_{m \leq l} f^{-1}(\{(m, \infty)\})$. Note that each $f^{-1}(\{(m, \infty)\})$ is a closed subset of Max($\mathbb{J}$). Observe that Max($\mathbb{J}$) can not be expressed as the union of two proper closed subsets (a proper closed subset must be finite); it follows that Max($\mathbb{J}$) $= f^{-1}(\{(m, \infty)\})$ for some m, showing that f is a constant function.

(2) Assume that f is a constant function and $f(m, \infty) = (m_0, \infty)$ for all m. Define $F : \mathbb{J} \longrightarrow \mathbb{J}$ by $F(m, n) = (m_0, n)$. Then, it is easy to see that F is Scott continuous and an extension of f.

(3) Now assume that f is not a constant function. By (1), the function α is not bounded.

Define $F : \mathbb{J} \longrightarrow \mathbb{J}$ by

$$F(m, n) = \begin{cases} f(m, n), & n = \infty, \\ (p(f(m, \infty)), \alpha(n)), & n \neq \infty. \end{cases}$$

Clearly, F is an extension of f. It remains to show that F is Scott continuous. By its definition, one can easily verify that F is monotone.

By Remark 1(2), We need to verify that for any directed set $D \subseteq \mathbb{J}$, $F(\bigvee D) = \bigvee F(D)$.

If D has the largest element, say x_0, then $F(\bigvee D) = F(x_0) = \bigvee F(D)$ because F is monotone.

Now assume that D does not have the largest element, then $D = \{(a, n_k) : k = 1, 2, ...\}$ for some fixed $a \in \mathbb{N}$ and we can assume that

$$n_1 < n_2 < \cdots.$$

Now, $F(\bigvee D) = F(a, \infty) = f(a, \infty) = (b, \infty)$ for some $b \in \mathbb{N}$. For any $l \in \mathbb{N}$, as α is not bounded, there is n' such that $\alpha(n') \geq l$. Take one n_k such that $n' \leq n_k$. Then,

$$l \leq \alpha(n') \leq \alpha(n_k).$$

Hence, $(b,l) \leq (b,\alpha(n_k)) = (p(f(a,\infty)),\alpha(n_k)) = F(a,n_k) \leq \bigvee F(D)$. Therefore, $F(\bigvee D) = (b,\infty) = \bigvee\{(b,l) : l \in \mathbb{N}\} \leq \bigvee F(D)$. Since F is monotone, we have $F(\bigvee D) \geq \bigvee F(D)$. At last, we have

$$F(\bigvee D) = \bigvee F(D).$$

The proof is completed.

□

Definition 1. *A poset P is called* continuously extendable *if, for any continuous mapping $f : Max(P) \longrightarrow Max(P)$, there is a continuous mapping of $F : \Sigma P \longrightarrow \Sigma P$, which is an extension of f: for any $x \in Max(P)$, $F(x) = f(x)$.*

Remark 2. *(1) If a poset P has the top element T_P, then Max(P)=$\{T_P\}$. Then, every function $f : Max(P) \to Max(P)$ has the continuous extension $\hat{f} : P \longrightarrow P$, which is the constant function $\hat{f}(x) = T_P$ for all $x \in P$. Thus, P is continuously extendable. In particular, every complete lattice is continuously extendable.*

(2) By Theorem 5.7 of [16], for dcpo $P = \mathbf{I}\mathbb{R}$ of all closed intervals of real numbers with the reverse inclusion order, one can only prove that if the continuous function $f : Max(\mathbf{I}\mathbb{R}) \longrightarrow Max(\mathbf{I}\mathbb{R})$ satisfies some extra conditions, then it has the continuous extension $\hat{f} : \mathbf{I}\mathbb{R} \longrightarrow \mathbf{I}\mathbb{R}$.

By Proposition 2, we have the following.

(**J5**) $\mathbb{J}$ is continuously extendable.

By a classic result (Theorem II-1.14 of [12]), a dcpo P is continuous if and only if the lattice $(\sigma(P), \subseteq)$ of all Scott open sets of P with the set inclusion order is a completely distributive lattice (this is equivalent to that the lattice of all Scott closed sets with the set inclusion order is completely distributive) [12]. In addition, the Scott space of every continuous dcpo is sober [12]. Using these facts, one can deduce the following property of continuous dcpo P, where $C_\sigma(P)$ denotes the lattice of all Scott closed sets of P:

$$\text{For any dcpo } Q, C_\sigma(P) \cong C_\sigma(Q) \text{ implies } P \cong Q, \tag{1}$$

here, $\cong$ is the *isomorphism* relation between posets.

A dcpo P satisfying the property (1) is called C_σ-determined.

For two subsets F and G of a dcpo P, F is *way below* G, written $F \ll G$, if, for any directed subset D, $\bigvee D \in \uparrow G$ implies $D \cap \uparrow F \neq \emptyset$.

A dcpo P is called *quasicontinuous* if, for each $x \in P$, the set family $\{F \subseteq P : F \ll \{x\} \text{ and } F \text{ is finite}\}$ is filtered (for any finite $F_1 \ll \{x\}, F_2 \ll \{x\}$, there is finite $F \ll \{x\}$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$) and

$$\uparrow x = \bigcap \{\uparrow F : F \ll \{x\} \text{ and } F \text{ is finite}\}.$$

Every continuous dcpo is quasicontinuous (Proposition III-3.10 of [12]).

A dcpo P is *locally quasicontinuous (continuous, resp.)* if each subposet $\downarrow x$ ($x \in P$) is quasicontinuous (continuous, resp.) By [17], the dcpo $\mathbb{J}$ is locally quasicontinuous.

Remark 3. *(1) Note that $\mathbb{J}$ is not locally continuous. For instance, the subposet $\downarrow(1,\infty)$ is not continuous.*

(2) The product of any finite number of quasicontinuous dcpos is quasicontinuous [12]. Hence, for any positive integer n, the Cartesian product $\mathbb{J}^n$ is locally quasicontinuous.

The following result was proved in [18].

Theorem 1. *Every locally quasicontinuous dcpo is C_σ-determined.*

By the above result and Remark 3 (2), one deduces the following result proved in [18] and [17] using different methods.

(**J6**) For any $n \in \mathbb{N}$, the dcpo $\mathbb{J}^n$ is C_σ-determined.

Given a dcpo P and a Scott closed subset G of P, the set $\downarrow(G \cap \mathrm{Max}(P))$ need not be a Scott closed set, although it is a lower set.

Example 2. *Let $P = \mathbb{N} \cup \{a_i : i \in \mathbb{N}\} \cup \{T, b\}$. The partial order $\leq$ on P is generated by*

$$a_i < i, a_i < a_{i+1} < b < T, \text{for each } i \in \mathbb{N}.$$

Then, $(P, \leq)$ is a dcpo, where Max(P)=$\mathbb{N} \cup \{T\}$. The set $G = P - \{T\}$ is a Scott closed set, $\downarrow(G \cap Max(P)) = P - \{T, b\}$, which is not Scott closed because $D = \{a_i : i \in \mathbb{N}\}$ is a directed subset of $P - \{T, b\}$ and $\bigvee D = b \notin P - \{T, b\}$.

However, for the dcpo $\mathbb{J}$, this is true for all Scott closed sets G of $\mathbb{J}$.

As a matter of fact, let $G \subseteq \mathbb{J}$ be a Scott closed set and $G \cap \mathrm{Max}(\mathbb{J}) \neq \emptyset$. Assume that $G \cap \mathrm{Max}(\mathbb{J}) = \{(n_i, \infty) : i \in \mathbb{N}\}$. Then,

$$\downarrow(G \cap \mathrm{Max}(\mathbb{J})) = \bigcup\{\downarrow(n_i, \infty) : i \in \mathbb{N}\} \subseteq G.$$

Let $D \subseteq \downarrow(G \cap \mathrm{Max}(\mathbb{J}))$ be directed. Then, by Remark 1, either D has the largest element or $D = \{(n_{i_0}, m_k) : k \in \mathbb{N}\}$ for some i_0. If D has the largest element, then $\bigvee D \in D \subseteq \downarrow (G \cap \mathrm{Max}(\mathbb{J}))$. Otherwise, $\bigvee D = (n_{i_0}, t) \leq (n_{i_0}, \infty) \in \downarrow(G \cap \mathrm{Max}(\mathbb{J}))$. Since $\downarrow(G \cap \mathbb{J})$ is already a lower set, it is a Scott closed subset of $\mathbb{J}$.

Hence, we have the following property of $\mathbb{J}$.

(**J7**) For any Scott closed subset G of $\mathbb{J}$, $\downarrow(G \cap \mathrm{Max}(\mathbb{J}))$ is a Scott closed subset of $\mathbb{J}$.

A T_0 space X is called a T_D space if, for any $x \in X$, $cl(\{x\}) - \{x\}$ is closed. A T_0 space X is called a $\mathcal{B}$-space if there is T_D space Y such that $C(X) \cong C(Y)$ [19]. This is equivalent to that there is T_D space Y such that the sobrification of X is homeomorphic to the sobrification of Y.

A closed set A of space X is called *completely irreducible* if for any collection $\{F_i : i \in I\}$ of closed sets F_i, $A = cl(\bigcup_{i \in I})F_i$ implies $A = F_i$ for some $i \in I$.

Theorem 2 ([19]). *A T_0 space X is a $\mathcal{B}$-space if and only if every closed set is the supremum of some completely irreducible closed sets: for any closed $F \subset X$, there are completely irreducible closed sets $F_i (i \in I)$ such that*

$$F = \bigvee\{F_i : i \in I\} = cl(\bigcup\{F_i : i \in I\}).$$

If a closed set $F \subseteq \Sigma P$, where P is a poset, is completely irreducible, then, as $F = \bigvee\{\downarrow x : x \in F\}$, we have $F = \downarrow x$ for some $x \in F$.

(**J8**) The space $\Sigma\mathbb{J}$ is a $\mathcal{B}$-space.

For the proof, it is enough to observe that every $\downarrow(m, n)$ $(n \neq \infty)$ is a completely irreducible closed set and every closed set is a supremum of such closed sets.

By Cramer [20], a topological space X is called T^m, where m is a cardinal, if for any $x \in X$, $cl(\{x\}) - \{x\}$ is the union of at most m closed sets. Thus, T^1 spaces are exactly T_D spaces. It is easy to verify the following.

(**J9**) The space $\Sigma\mathbb{J}$ is a $T^{\aleph_0}$ space.

4. Extensions and Problems

In this section, we shall use Johnstone's method to construct new dcpos and investigate their properties. Some open problems for further study will be listed.

Let Q be a poset, P a dcpo and $\phi : Q \longrightarrow (P \setminus \mathrm{Max}(P))$ a monotone mapping. We define the partial order $\leq_\phi$ on $Q \times P$ by $(q_1, p_1) \leq_\phi (q_2, p_2)$ if

(i) either $q_1 = q_2$ and $p_1 \leq p_2$;

(ii) or $p_2 \in \mathrm{Max}(P)$ and $p_1 \leq \phi(q_2)$.

It is easy to see that $\leq_\phi$ is a partial order. We shall use $Q \times P_\phi$ to denote the poset $(Q \times P, \leq_\phi)$.

Remark 4. *(1) As in the case of $\mathbb{J}$, a subset $D \subseteq Q \times P$ is directed if either there is a $q \in Q$ such that $D = \{(q, p_i) : i \in I\}$ with $\{p_i : i \in I\}$, a directed subset of P, or D has the largest element (if there are $(q_1, p_1), (q_2, p_2) \in D$ with $q_1 \neq q_2$). If D has the largest element, it would be the supremum of D. If $D = \{(q, p_i) : i \in I\}$ with $\{p_i : i \in I\}$, a directed subset of P, then $\bigvee D = (q, \bigvee\{p_i : i \in I\})$. Hence, $Q \times P_\phi$ is a dcpo.*

(2) $Max(Q \times P_\phi) = Q \times Max(P)$.

Example 3. *The subsequent examples serve to illustrate this concept.*

(1) *Let $Q = (\mathbb{N}, \leq)$ be the poset of all positive integers with the ordinary order of numbers and $P = (\mathbb{N} \cup \{\infty\}, \leq)$ be the extension of Q with the top element ∞ added. Then, $Q \times P_{id} = \mathbb{J}$, where $id : Q \longrightarrow P$ is the embedding mapping: $id(m) = m$ for all $m \in Q$.*

(2) *Let $Q = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$ and $P = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ be the given sets of real numbers with the ordinary order of numbers. Let $id : Q \longrightarrow P$ be the embedding mapping. Then, the poset $Q \times P_{id}$ has $[0, 1) \times [0, 1]$ as underlying set and $(x, y) \leq (x', y')$ holds if and only if either $x = x'$ and $y \leq y'$ or $y' = 1$ and $y \leq x'$.*

(3) *In general, let P be a dcpo, $Q = P - Max(P)$ and $id : Q \longrightarrow P$ be the identity mapping: $x \mapsto x (\forall x \in Q)$. Then, we have a natural dcpo, which will be denoted by $J(P)$. In particular, if $\mathbb{N}^T$ is the complete chain $\mathbb{N} \cup \{\infty\}$ with ∞ as the top element, then $\mathbb{J} = J(\mathbb{N}^T)$.*

(4) *Let $P = \mathbb{N}^T$ be the dcpo defined in (3) and $Q = [0, +\infty)$ with the usual order of numbers. Define $\phi : Q \longrightarrow P$ by $\phi(x) = [x]$, the integer part of x for each $x \in Q$. Then, $Q \times P_\phi = \{(x, n) : x \in [0, +\infty), n \in \mathbb{N}^T\}$ and $(x, n) \leq_\phi (y, m)$ if and only if either $x = y$ and $n \leq m$ or $n \leq [y]$ and $m = \infty$.*

Proposition 3. *Let P be a chain that is a dcpo and has the top element T_P.*

(1) *A Scott closed subset $F \subseteq J(P)$ is irreducible if and only if either $F = \downarrow(x, y)$ for some $(x, y) \in J(P)$ or $F = J(P)$. Thus, the Scott space of $J(P)$ is not sober.*

(2) *For any Scott closed set G of $J(P)$, $\downarrow(G \cap Max(J(P)))$ is a Scott closed subset of $J(P)$.*

(3) *$Max(J(P))$ is homeomorphic to $(P \setminus \{T_P\}, \tau_{cob})$, where $U \in \tau_{cob}$ if and only if either $U = \emptyset$ or $P - (U \bigcup \{T_P\})$ has an upper bound in $P \setminus \{T_P\}$.*

(4) *$J(P)$ is C_σ-determined.*

Proof. The proofs of (1), (2) and (4) are similar to that of $\mathbb{J}$, we skip the details.

We now prove (3). By the definition of τ_{cob}, $H \subseteq P - \{T_P\}$ is closed in (P, τ_{cob}) if and only if either $H = P - \{T_P\}$, or H has an upper bound in $P - \{T_P\}$.

Note that $\mathrm{Max}(J(P)) = \{(p, T_P) : p \in P - \{T_P\}\}$.

Let $\phi : \mathrm{Max(J(P))} \longrightarrow (P \setminus \{T_P\})$ be the mapping defined by $\phi(p, T_P) = p$. Then, clearly, ϕ is a bijection.

(1) Let $F \subseteq \mathrm{Max}(J(P))$ be a closed set in $\mathrm{Max}(J(P))$. Then, there is a Scott closed set $\hat{F}$ of $J(P)$, such that $F = \hat{F} \cap \mathrm{Max}(J(P))$.

If $\phi(F)$ is not upper bounded in $P - \{T_P\}$, then as in the proof of the Proposition 1, one deduces that $\hat{F} = J(P)$; thus, $\phi(F) = \phi(J(P) \cap \mathrm{Max}(J(P))) = \phi(J(P)) = P \setminus \{T_P\}$. Otherwise, $\phi(F)$ has an upper bound in $P - \{T_P\}$. Thus, ϕ sends each closed set of $\mathrm{Max}(J(P))$ to a closed set in $(P - \{T_P\}, \tau_{cob})$. Therefore, ϕ is a closed mapping.

(2) Now assume that $H \subseteq P \setminus \{T_P\}$ is a closed set in $(P \setminus \{T_P\}, \tau_{cob})$. If $H = P \setminus \{T_P\}$, then $\phi^{-1}(H) = \mathrm{Max}(J(P))$, which is closed in $\mathrm{Max}(J(P)$. If $H \neq P - \{T_P\}$, then H has an upper bound, say w in $P \setminus \{T_P\}$. Hence, $w < T_P$ holds in P.

Let

$$\underline{H} = \{(x, p) \in J(P) : x \notin H \text{ and } p \leq w\} \cup \{(h, p) \in J(P) : h \in H \text{ and } p \in P\}.$$

(i) $\underline{H}$ is a lower set in $J(P)$

In fact, assume that $(z, u) \leq (x, p)$ for some $(x, p) \in \underline{H}$. If $x \notin H$, then $p \leq w < T_P$. Hence, $(z, u) \leq (x, p)$ implies $z = x, u \leq p \leq w$. So, $(z, u) \in \underline{H}$. Let $x \in H$. If $z = x$, then $(z, u) = (x, u) \in \underline{H}$. If $z \neq x$, by $(z, u) \leq (x, p)$, we have $p = T_p$ and $u \leq x \leq w$, again implying $(z, u) \in \underline{H}$. Hence, $\underline{H}$ is a lower set.

(ii) Assume that $D \subseteq \underline{H}$ is a directed subset.

If D has the largest element d_0, then $\bigvee D = d_0 \in D \subseteq \underline{H}$, implying $\bigvee D \in \underline{H}$. If D does not have the largest element, by Remark 4(1), there is $x \in P - \{T_P\}$ and directed set $\{d_i : i \in I\} \subseteq P$ such that $D = \{(x, d_i) : i \in I\}$.

If $x \in H$, then $\bigvee D = (x, \bigvee\{d_i : i \in I\}) \in \underline{H}$. If $x \notin H$, then $d_i \leq w$ for all $i \in I$, so $\bigvee\{d_i : i \in I\} \leq w$ holds in P. Thus, $\bigvee D = (x, \bigvee\{d_i : i \in I\}) \in \underline{H}$.

Therefore, $\underline{H}$ is a Scott closed set of $J(P)$.

Since $w \neq T_P$, we see that $\underline{H} \cap \mathrm{Max}(J(P)) = \{(h, T_P) : h \in H\}$ is a closed set of $\mathrm{Max}(J(P)))$. In addition, $\phi^{-1}(H) = \underline{H} \cap \mathrm{Max}(J(P))$. Hence, $\phi : \mathrm{Max}(J(P)) \longrightarrow P - \{T_P\}$ is continuous.

The combination of (1) and (2) proves that ϕ is a homeomorphism.

The proof is completed. □

Example 4. *Let W_1 be the set of all countable ordinals plus the smallest non-countable ordinal ω_1. Then, W_1 is a chain with ω_1 as the top element. A subset $A \subseteq W_1 - \{\omega_1\}$ has an upper bound in $W_1 - \{\omega_1\}$ if and only if it is a countable set. Hence, by Proposition 3, Max$(J(W_1))$ is homeomorphic to $(W_1 - \{\omega_1\}, \tau_{cc})$, where τ_{cc} is the co-countable topology: $U \in \tau_{cc}$ if and only if either $U = \emptyset$ or U^c is countable. Thus, we deduce the result in [4] as a special case. It was proved in [4] that $J(W_1)$ is well filtered.*

Example 5. *Let $P = [0, 1]$ be the set of all real numbers in the unit interval. Then, 1 is the top element in chain P and $P - \{1\} = [0, 1)$. By Proposition 3, Max$(J([0, 1])$ is homeomorphic to $([0, 1), \tau)$, where $F \subseteq [0, 1)$ is a closed set if and only if either $F = [0, 1)$ or $F \subseteq [0, b]$ for some $0 \leq b < 1$. One can check that the Scott space $\Sigma J([0, 1])$ is not well filtered.*

Example 6. *Let $\mathbb{N}^T = \mathbb{N} \cup \{\infty\}$ be chain $\mathbb{N}$ with the top element ∞ added. Max$(\mathbb{N}^T)$ is homeomorphic to $(\mathbb{N}, \tau)$, where $U \in \tau$ if and only if either $U = \emptyset$ or $\mathbb{N} - U$ is upper bounded, which is equivalent to the notion that $\mathbb{N} - U$ is a finite set. Hence, $\tau = \tau_{cof}$. Thus, we re-deduce (J4) as a special case.*

The following are some problems related to dcpo $\mathbb{J}$.

Problem 1. Is the countable self-product $\mathbb{J}^{\mathbb{N}}$ of $\mathbb{J}$ characterized as a C_σ-determined dcpo?

Problem 2. Does every T_1 space have a continuously extendable dcpo model?

Note that it was proved in [15] that every T_1 space has a dcpo model.

Problem 3. In which context does dcpo P exhibit the property that its justification $J(P)$ is well filtered?

Note that $\mathbb{J}$ is not well filtered and $J(W_1)$ is well filtered.

Funding: This research receives no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Johnstone, P. Scott is not always sober. In *Continuous Lattices*; Lecture Notes in Math; Springer: Berlin/Heidelberg, Germany, 1981; Volume 871, pp. 282–283.
2. Isbell, J. Completion of a construction of Johnstone. *Proc. Am. Math. Soci.* **1982**, *85*, 333–334. [CrossRef]
3. Kou, H. U_k-admitting dcpos need not be sober. In *Domains and Processes, Semantic Structure on Domain Theory*; Kluwer: Alphen aan den Rijn, The Netherlands, 2001; Volume 1, pp. 41–50.
4. Zhao, D.; Xi, X.; Chen, Y. A new dcpo whose Scott topology is well-filtered but not sober. *Topol. Appl.* **2019**, *252*, 97–102.
5. Heckmann, R. An upper power domain construction in terms of strongly compact sets. In *Lecture Notes in Computer Science*; Springer: Berlin/Heidelberg, Germany, 1992; Volume 598, pp. 272–293.
6. Jia, X. Meet-Continuity and Locally Compact Sober Dcpos. Ph.D. Thesis, University of Birmingham, Birmingham, UK, 2018.
7. Xu, X.; Xi, X.; Zhao, D. A complete Heyting algebra whose Scott space is non-sober. *Fund. Math.* **2021**, *252*, 315–323. [CrossRef]
8. Ho, W.K.; Goubault-Larrecq, J.; Jung, A.; Xi, X. The Ho-Zhao problem. *Log. Methods Comput. Sci.* **2018**, *14*, 7.
9. Miao, H.; Li, Q.; Xi, X.; Zhao, D. Not every countable complete distributive lattice is sober. *Math. Struct. Comp. Sci.* **2023**, *33*, 809–831. [CrossRef]
10. Miao, H.; Xi, X.; Jia, X.; Li, Q.; Zhao, D. Products of two sober dcpo's need not be sober. *arXiv* **2024**, arXiv:2408.08587.
11. Xi, X.; Lawson, J. On well-filtered spaces and ordered sets. *Topol. Appl.* **2017**, *228*, 139–144. [CrossRef]
12. Gierz, G.; Hofmann, K.; Keimel, K.; Lawson, J.; Mislove, M.; Scott, D. *Continuous Lattices and Domains*; Cambridge University Press: Cambridge, UK, 2003.
13. Goubault-Larrecq, J. *Non-Hausdorff topology and Domain Theory*; Cambridge University Press: Cambridge, UK, 2013.
14. Zhao, D.; Fan, T. Dcpo-completion of posets. *Theor. Comp. Sci.* **2010**, *411*, 2167–2173. [CrossRef]
15. Zhao, D.; Xi, X. Directed complete poset models of T_1 spaces. *Math. Proc. Camb. Phil. Soc.* **2018**, *164*, 125–134. [CrossRef]
16. He, Z.; Yang, Z.; Zhao, D. Maximal point spaces of posets of intervals. *Top. Appl.* **2022**, *322*, 108314. [CrossRef]
17. Xu, L.; Zhao, D. C_σ-unique Dcpos and Non-maximality of the Class of Dominated Dcpos Regarding Γ-faithfulness. *Electron. Notes Theor. Comput. Sci.* **2019**, *345*, 249–260. [CrossRef]
18. Zhao, D.; Xu, L. Uniqueness of directed complete posets based on Scott closed set lattices. *Log. Method Comput. Sci.* **2018**, *14*, 1–12.
19. Hoffmann, R. On the sobrification remainder ${}^sX - X$. *Pac. J. Math.* **1979**, *83*, 145–156. [CrossRef]
20. Cramer, T. A definition of separation axioms. *Can. Math. Bull.* **1974**, *17*, 485–491. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.