*Article*

# To the Theory of Unsteady Thermal Conductivity Caused by the Hot Core of the Earth

**Sergey O. Gladkov**

Moscow Aviation Institute, National Research University, 125993 Moscow, Russia; sglad51@mail.ru

**Abstract:** A solution is given to the spatial-temporary distribution of temperature in the volume of the Earth, due to the specified power of the thermal radiation of the hot core. An estimate is made of the maximum possible cooling time of the core.

## 1. Introduction

The question considered in this paper is related to the analysis of the distribution of temperature within the Earth's surface. Ideally, we believe that it should only be caused by thermal radiation coming from the nucleus, the temperature of which, according to some independent, objective sources [1–3], is approximately equal to $5000 \div 6000$ K.

Based on the estimates given, for example, by the authors of [4–6] (see also [7–9]), the size of the inner core may vary slightly, but the approximate value of its average radius can be considered equal to about $r_N = 1000$ km. It is quite clear that the point source model does not "work" here; therefore, the use of the thermal conductivity equation $\dot{Q}_N(t)\delta(\mathbf{r})$, where $\dot{Q}_N(t)$—is a power of thermal radiation core, is not acceptable. This means that to adequately account for the effect of thermal radiation of the core on the spatial-temporal distribution of temperature throughout the internal volume of the Earth $V - V_N$, where $V = \frac{4\pi R^3}{3}$—its volume, $R$—is its radius, and $V_N = \frac{4\pi r_N^3}{3}$—the volume of the spherical core, provided that the surface temperature can be considered constant, we need to consider the final value $r_N$ (see Formula (18)).

Formally, this is not difficult to do if you use any local distribution, which is the most convenient for task at hand. From our perspective, the most suited to such a trial function may well be Gauss's distribution, which, in a three-dimensional symmetrical case, can be presented in the following form

$$\dot{Q}_N = \dot{Q}_0(t)e^{-\frac{r^2}{r_N^2}} \tag{1}$$

where the central distance $r$ is calculated from the center of the core, which we choose as the beginning of the coordinates. That is, the radius-the vector at an arbitrary point of the region $r \in [V - V_N]$ has coordinates $\mathbf{r} = (x, y, z)$. The power of radiation attributed to the unit of the volume of the nucleus, which is marked as $\dot{Q}_0(t)$, is generally a function of time, which will then be accurately taken into account. In accordance with this equation, thermal conductivity can then be presented in the form of

$$c_P \frac{\partial T}{\partial t} = \kappa \Delta T - \dot{Q}_0(t)e^{-\frac{r^2}{r_N^2}} \tag{2}$$

where $c_P$—the average volume of the Earth is its isobaric heat capacity, classified as a unit of volume and $\kappa$—is the average coefficient of thermal conductivity of the Earth. It is

necessary to underline here that the average thermal conductivity of the Earth is a rather complex function of temperature, pressure, and other physical parameters. Note that the temperature appearing in $\kappa$ is the constant temperature of the Earth's crust $T_0$, which does not affect the distribution of temperature over the volume of the Earth according to Equation (2). An analogical approach was used, for example, in the work of [10], where the formation process of thermodynamic equilibrium in solids was investigated, considering the connection with the thermostat, a role played by the external environment.

Note also that for Boltzmann's constant $k_B$ in Equation (2), we used an equal unit. In the final response, this will be taken into account.

The Equation (2) should be decided on the basis of the initial and boundary conditions that can be formulated, for example, as follows. The initial distribution of temperature by the volume of the Earth outside the core will be set in the form of anisotropic law

$$T(\mathbf{r},t)|_{t=0} = \langle T \rangle + T_0 \sin\theta\cos\varphi \tag{3}$$

On the Earth's surface, which we consider spherical, the temperature will be set in the form of

$$T(\mathbf{r},t)|_{r=R} = \langle T \rangle + T_1(R) - (T_0\sin\theta\cos\varphi + T_1(R))e^{-\alpha t} \tag{4}$$

where $\theta$ is the azimuth angle of the spherical coordinate system is, and the $\varphi$—is a polar angle, temperature $\langle T \rangle$ represents a very definite temperature of the countdown, starting from which we can correctly estimate the temperature at any point on the Earth's surface, depending on the azimuth angle $\theta$ and polar $\varphi$ angle.

The simplest dependence on angular variables we choose will have little impact on the subsequent estimate of the coefficient $\alpha$, which will be calculated below, as well as the radial dependence $T_1(r)$ after the equation is solved (2).

## 2. Methods. Solving the Equation (2)

To solve problems (2)–(4), it is convenient to use the decomposition method of the searchable function $T(\mathbf{r},t)$ in Laplace integral over time.

Indeed, according to well-known formulas of direct and reverse decomposition [10], we have the following transformation

$$T_p(\mathbf{r}) = \int_0^{+\infty} T(\mathbf{r},t)e^{-pt}dt \tag{5}$$

and

$$T(\mathbf{r},t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} T_p(\mathbf{r})e^{pt}dp \tag{6}$$

where $\sigma > 0$, $T_p(t)$—is a Laplace's image of the function you are after, and Re $p > 0$.

After substitution (6), in Equation (2), we obtain

$$\Delta T_p + \lambda^2 T_p = \frac{\dot{Q}_p}{\kappa}e^{-\frac{r^2}{r_N^2}} \tag{7}$$

where the parameter $\lambda = \sqrt{\frac{p}{\chi}}$, $\chi = \frac{\kappa}{c_P}$—is an average temperature conductivity coefficient of Earth and $\dot{Q}_p$—Laplace–image of the function of the power of thermal radiation $\dot{Q}_0(t)$, that is

$$\dot{Q}_p = \int_0^{+\infty} \dot{Q}_0(t)e^{-pt}dt \tag{8}$$

Equation (7) is easily solved by the Green function method [11,12] (see also the papers [13–17]), which, in the results, provide a solution to the species

$$T_p(\mathbf{r}) = \overline{T}_p(\mathbf{r}) - \frac{\dot{Q}_p}{4\pi\kappa} \int\limits_{V-V_N} \frac{\cos\lambda\left|\mathbf{r}-\mathbf{r}'\right| e^{-\frac{r'^2}{r_N^2}} dV'}{\left|\mathbf{r}-\mathbf{r}'\right|} \tag{9}$$

where $\overline{T}_p(\mathbf{r})$—solves homogeneous Equation (7), i.e.,

$$\overline{T}_p(\mathbf{r}) = \left(C_{1p}r + \frac{C_{2p}}{r^2}\right)\sin\theta\cos\varphi \tag{10}$$

where $C_{1p}, C_{2p}$—are the integration constants.

To convert the internal integral to (9), it is convenient to move to a spherical system of coordinates with a polar axis directed along a fixed radius-a vector $\mathbf{r}$.

As a result, we obtain

$$J_{vn} = \int\limits_{r_N}^{R}\int\limits_{0}^{\pi}\int\limits_{0}^{2\pi} \frac{r'^2 \cos\lambda\left|\mathbf{r}-\mathbf{r}'\right| e^{-\frac{r'^2}{r_N^2}}}{\left|\mathbf{r}-\mathbf{r}'\right|} dr'\sin\theta d\theta d\varphi = 2\pi\int\limits_{r_N}^{R} r'e^{-\frac{r'^2}{r_N^2}}\left[\sin\lambda\left(r+r'\right) - \sin\lambda\left|r-r'\right|\right]dr' \tag{11}$$

Substituting expression (11) in image (9), we find

$$T_p(\mathbf{r}) = \overline{T}_p(\mathbf{r}) - \frac{\dot{Q}_p}{2\kappa}\int\limits_{r_N}^{R} r'e^{-\frac{r'^2}{r_N^2}}\left[\sin\lambda\left(r+r'\right) - \sin\lambda\left|r-r'\right|\right]dr'$$

Performing simple transformations associated with the difference in sinuses, bringing them to a factorized appearance, and applying the acceptance of integration into parts as a result of simple calculations, we find

$$
\begin{aligned}
T_p = \overline{T}_p(\mathbf{r}) - \\
-\frac{\dot{Q}_p r_N^2}{\kappa\lambda r}\left[\frac{\cos\lambda r\sin\lambda r_N}{e} - e^{-\frac{R^2}{r_N^2}}\sin\lambda r\cos\lambda R + \lambda\left(\cos\lambda r\int\limits_{r_N}^{r} e^{-\frac{r'^2}{r_N^2}}\cos\lambda r'dr' - \sin\lambda r\int\limits_{r}^{R} e^{-\frac{r'^2}{r_N^2}}\sin\lambda r'dr'\right)\right].
\end{aligned} \tag{12}
$$

Recall that here $r_N \leq r \leq R$.

After substitution of the image (12) in the conversion (6) with the view (10), we come to the next solution

$$
\begin{aligned}
T(\mathbf{r},t) = \langle T\rangle + \left(C_1(t)r + \frac{C_2(t)}{r^2}\right)\sin\theta\cos\varphi - \frac{1}{2\pi i}\frac{r_N^2}{\kappa\lambda r}\int\limits_{-i\infty+\sigma}^{i\infty+\sigma}\dot{Q}_p\left[\frac{\cos\lambda r\sin\lambda r_N}{e} - e^{-\frac{R^2}{r_N^2}}\sin\lambda r\cos\lambda R+\right. \\
\left. +\lambda\left(\cos\lambda r\int\limits_{r_N}^{r} e^{-\frac{r'^2}{r_N^2}}\cos\lambda r'dr' - \sin\lambda r\int\limits_{r}^{R} e^{-\frac{r'^2}{r_N^2}}\sin\lambda r'dr'\right)\right]e^{pt}dp.
\end{aligned} \tag{13}
$$

To clarify the nature of temperature distribution by Earth volume, we need to know the temporal dependence of the power of the thermal radiation core. As a reasonable assumption, we may choose a function $\dot{Q}_0(t)$ in the form of exponential dependence (see condition (4)) as

$$\dot{Q}_0(t) = \dot{Q}_0 e^{-\alpha t} \tag{14}$$

where $\alpha$—means that there is a small radiation fading factor.

As a result of the substitution of dependence (14) in the definition for the image of Laplace (8), we find that

$$\dot{Q}_p = \frac{\dot{Q}_0}{\alpha + p}$$

Consequently, the general expression (13) becomes quite specific and is easily calculated by the theory of deductions, according to which the only simple pole of under integration expression is located on the actual axis of a complex argument $p$ in the point $p = -\alpha$.

Thus, in accordance with (13), we come to the next analytical answer to the question posed at the beginning of the article regarding the spatial-temporal distribution of temperature in the volume of the Earth

$$
\begin{aligned}
T(\mathbf{r}, t) = \langle T \rangle + \left( C_1(t)r + \frac{C_2(t)}{r^2} \right) \sin\theta \cos\varphi - \\
- \frac{r_N^2 \dot{Q}_0 e^{-\alpha t}}{r\kappa} \sqrt{\frac{\chi}{\alpha}} \left[ \frac{ch\left(r\sqrt{\frac{\alpha}{\chi}}\right) sh\left(r_N\sqrt{\frac{\alpha}{\chi}}\right)}{e} - e^{-\frac{R^2}{r_N^2}} ch\left(R\sqrt{\frac{\alpha}{\chi}}\right) sh\left(r\sqrt{\frac{\alpha}{\chi}}\right) + \right. \\
\left. + \sqrt{\frac{\alpha}{\chi}} \left( ch\left(r\sqrt{\frac{\alpha}{\chi}}\right) \int_{r_N}^{r} e^{-\frac{r'^2}{r_N^2}} ch\left(r'\sqrt{\frac{\alpha}{\chi}}\right) dr' + sh\left(r\sqrt{\frac{\alpha}{\chi}}\right) \int_{r}^{R} e^{-\frac{r'^2}{r_N^2}} ch\left(r'\sqrt{\frac{\alpha}{\chi}}\right) dr' \right) \right].
\end{aligned}
\tag{15}
$$

Accounting for the boundary condition (4), decision (15) should be rewritten as

$$
\begin{aligned}
T(\mathbf{r}, t) = \langle T \rangle + T_0(r, t) \sin\theta \cos\varphi + \\
+ \frac{r_N^2 \dot{Q}_0 (1 - e^{-\alpha t})}{r\kappa} \sqrt{\frac{\chi}{\alpha}} \left[ \frac{ch\left(r\sqrt{\frac{\alpha}{\chi}}\right) sh\left(r_N\sqrt{\frac{\alpha}{\chi}}\right)}{e} - e^{-\frac{R^2}{r_N^2}} ch\left(R\sqrt{\frac{\alpha}{\chi}}\right) sh\left(r\sqrt{\frac{\alpha}{\chi}}\right) + \right. \\
\left. + \sqrt{\frac{\alpha}{\chi}} \left( ch\left(r\sqrt{\frac{\alpha}{\chi}}\right) \int_{r_N}^{r} e^{-\frac{r'^2}{r_N^2}} ch\left(r'\sqrt{\frac{\alpha}{\chi}}\right) dr' + sh\left(r\sqrt{\frac{\alpha}{\chi}}\right) \int_{r}^{R} e^{-\frac{r'^2}{r_N^2}} ch\left(r'\sqrt{\frac{\alpha}{\chi}}\right) dr' \right) \right].
\end{aligned}
\tag{16}
$$

### 3. Results

*Assessment of the Time of the "Life" of the Earth*

The fading rate $\alpha$ can be calculated based on the condition

$$\lim_{t \to \infty} T(\mathbf{r}, t)\big|_{r=R} = \langle T \rangle \tag{17}$$

where the right part of this equality, according to decision (16), has a very transparent physical meaning, as it represents the difference between stationary temperatures, namely

$$\langle T \rangle = T_0(R) - T_1(R, \infty)$$

The solution (16) can be reduced to the expression

$$\langle T \rangle = T_0 + \frac{r_N^2 \dot{Q}_0}{R\kappa} ch\left(R\sqrt{\frac{\alpha}{\chi}}\right) \int_{r_N}^{R} e^{-\frac{r'^2}{r_N^2}} ch\left(r'\sqrt{\frac{\alpha}{\chi}}\right) dr',$$

From which, we can see that

$$\alpha = \frac{\chi}{(R - r_N)^2} \ln^2\left( \frac{R\kappa(\langle T \rangle - T_0)}{\dot{Q}_0 r_N^3} \right) \tag{18}$$

Magnitude $\dot{Q}_0$ is worth further discussion. The energy of radiant radiation according to the Planck formula, related to the unit of volume, is given by a known dependence [18]

$$\varepsilon = \sigma T^4,$$

where $\sigma = \frac{k_B^4 \pi^2}{30(\hbar c)^3}$ is a Stefan-Boltzmann's constant; $c$—speed of light; $\hbar$—Planck's constant. On this basis, it is possible to write down that $\dot{Q}_0 = \frac{4\pi r_N^2 c\varepsilon}{V_N} = \frac{3c\varepsilon}{r_N}$ or

$$\dot{Q}_0 = \frac{3c\sigma T_N^4}{r_N} \tag{19}$$

Substituting (19) in (18) obtains

$$\alpha = \frac{\chi}{(R - r_N)^2} \ln^2 \left( \frac{R\kappa(\langle T \rangle - T_0)}{3c\sigma T_N^4 r_N^2} \right) \tag{20}$$

It should be noted here that radiant heat exchange is a rather slow process. If there are other efficient mechanisms for the return of thermal energy from the core, Formula (20) can only estimate the upper limit of the cooling time of the Earth's core, which will lead to a natural stop of its rotation around its axis; for example, endothermic chemical reactions occurring in the Earth's core or the purely hydrodynamic inhibition of the nucleus in the surrounding melt. However, both of these effects can only lead to a slight change in attenuation coefficient $\alpha$.

To numerically estimate the maximum "lifetime" of the core, which is obviously determined by reverse dependence $\tau_N = \frac{1}{\alpha}$, we can use numerical values, as given, for example, in [19].

Indeed, believing that $\chi = 1.4 \cdot 10^{-3}$ cm$^2$/s, $R = 6.4 \cdot 10^8$ cm, $r_N \sim 1000$ km $= 10^8$ cm, $c = 3 \cdot 10^{10}$ cm/s, $\sigma \approx 10^{-15}$ SGS, $T_N = 5000$ K, $\kappa = 3 \cdot 10^{19} 1/$cm·s, $\langle T \rangle \sim T_0 \sim 300$ K $\sim 4 \cdot 10^{-14}$ erg.

After their substitution in Formula (20), we find

$$\frac{1}{\tau_N} = \frac{\chi}{(R - r_N)^2} \ln^2 \left( \frac{R\kappa(\langle T \rangle - T_0)}{3c\sigma T_N^4 r_N^2} \right) \sim \frac{1.4 \cdot 10^{-3}}{25 \cdot 10^{16}} \ln^2 \left( \frac{6.4 \cdot 10^8 \cdot 3 \cdot 10^{19} \cdot 3 \cdot 10^{-14}}{9 \cdot 10^{10} \cdot 10^{-15} \cdot 625 \cdot 10^{12} \cdot 10^{16}} \right) \approx 2 \cdot 10^{-18} \ (1/\text{s}) \tag{21}$$

That is, the maximum allotted "time of life" of the Earth's core should be approximately $\tau_N = 5 \cdot 10^{17}$ s. In terms of years, this will be about 16 billion years.

## 4. Discussion

The analytical approach proposed in the paper can be used as the basis for experimental verification of the results to obtain more accurate information about the cooling time of the Earth's core.

The calculated value of the cooling time of the core showed a quite optimistic and reasonable value (about 16 billion years), which correlates with the results of other authors, according to which this time ranges from 12 to 20 billion years.

For example, in the work of [5], the mobility of the Earth's core was investigated. According to the results obtained by the authors, the attenuation time of the nuclei oscillations correlates well with the above estimate (21). The authors of [6] studied the dynamics of the nucleus in the surrounding melt, and, according to the authors, the time they obtained is also consistent with the result (21).

It is worth paying attention, however, to the fact that the analytical approach proposed above, based on an alternative method of calculation, is based not on the evaluative, as in many authors, but on the exact solution to the problem. This, in our opinion, is very important from a methodological perspective.

Moreover, it should be underlined that the resulting assessment does not contradict the results of hydrocarbon analysis (see, for example, [1,2,20,21]).

## 5. Conclusions

At the end of the article, it is worth paying attention to several important results:

1. An analytical description of the distribution of temperature by Earth volume, caused by thermal radiation of the hot core, is proposed.
2. The distribution of temperature by Earth volume, in the form of a function from the radial coordinate *r* and the moment of time *t*, was found.
3. A numerical estimate is made of the upper limit of the Earth's "life" time, which, according to the above result, may correspond to about sixteen billion years.

**Conflicts of Interest:** The authors declare no conflict of interest.

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